August 5, 2014

Tree Calculus for Bivariate Difference Equations

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Abstract. Following Poupard's study of strictly ordered binary trees with respect to two parameters, namely, "end of minimal chain" and "parent of maximum leaf" a true Tree Calculus is being developed to solve a partial difference equation system and then make a joint study of those two statistics. Their joint distribution is shown to be symmetric and to be expressed in the form of an explicit three-variable generating function.

1. Introduction

The triangle of numbers

$$f=\begin{array}{ccccccccc}
 & & & & f_0(1) & & & & \\
 & & & f_1(1) & f_1(2) & f_1(3) & & & \\
 & & f_2(1) & f_2(2) & f_2(3) & f_2(4) & f_2(5) & & \\
 & f_3(1) & f_3(2) & f_3(3) & f_3(4) & f_3(5) & f_3(6) & f_3(7) & \\
f_4(1) & f_4(2) & f_4(3) & f_4(4) & f_4(5) & f_4(6) & f_4(7) & f_4(8) & f_4(9)
\end{array}
=\begin{array}{ccccccccc}
 & & & & 1 & & & & \\
 & & & 0 & 1 & 0 & & & \\
 & & 0 & 1 & 2 & 1 & 0 & & \\
 & 0 & 4 & 8 & 10 & 8 & 4 & 0 & \\
0 & 34 & 68 & 94 & 104 & 94 & 68 & 34 & 0
\end{array}$$

Table 1.1. Poupard's triangle.

appears in Sloane's On-Line Encyclopedia of Integer Sequences [Sl06] under reference A008301 and is called Poupard's triangle. As shown by Christiane Poupard [Po89], $f=(f_n(m))$ $(n\ge 0,\ 1\le m\le 2n+1)$ is the unique solution of the finite difference equation system

$$(1.1)\qquad \Delta^2 f_n(m)+2\,f_{n-1}(m)=0\quad (n\ge 1,\ 1\le m\le 2n-1),$$

where Δ stands for the classical finite difference operator (see, e.g., [Jo39])

$$(1.2)\qquad \Delta f_n(m):=f_n(m+1)-f_n(m),$$

so that

$$(1.3)\qquad \Delta^2 f_n(m)=f_n(m+2)-2f_n(m+1)+f_n(m),$$

when taking $f_0(1)=1$; $f_n(1)=0$, and either $f_n(2)=\sum_m f_{n-1}(m)$ $(n\ge 1)$, or $f_n(2n+1)=0$ $(n\ge 1)$ as *initial values*. Note that with the latter set of initial conditions it is immediately seen that the triangle f is symmetric with respect to the column $f_0(1)\ f_1(2)\ f_2(3)\ \ldots\ f_n(n+1)\ \ldots$

Key words and phrases. Tree Calculus, partial difference equations, strictly ordered binary trees, end of minimal chain, parent maximum leaf, bivariate distributions, Poupard triangle, tangent numbers.

Mathematics Subject Classifications. 05A15, 05A30, 11B68, 33B10.

Let

$$\tan u = \sum_{n\ge 1} \frac{u^{2n-1}}{(2n-1)!} T_{2n-1} \tag{1.4}$$
$$= \frac{u}{1!}1 + \frac{u^3}{3!}2 + \frac{u^5}{5!}16 + \frac{u^7}{7!}272 + \frac{u^9}{9!}7936 + \cdots$$

be the Taylor expansion of $\tan u$, the coefficients T_{2n+1} $(n \ge 0)$ being called the *tangent numbers* (see, e.g., [Ni23, p. 177-178], [Co74, p. 258-259]); Poupard further shows that each row sum $f_n(\bullet) := f_n(1) + f_n(2) + \cdots + f_n(2n+1)$ is equal to the *integer* $T_{2n+1}/2^n$ $(n \ge 0)$, that is, reporting to Table 1.1: 1, 1, 4, 34, 496,...

Finally, on the set $\mathfrak{T}_{2n+1}$ of *strictly ordered binary trees* with $(2n+1)$ vertices (see Definition 1.2), she defines two statistics "**eoc**" ("**e**nd **o**f minimal **c**hain") and "**pom**" ("**p**arent **o**f the **m**aximum leaf"), to show that both statistics "eoc" and "pom+1" are equally distributed on each set $\mathfrak{T}_{2n+1}$, and furthermore,

$$\#\{t \in \mathfrak{T}_{2n+1} : \mathrm{eoc}(t) = k+1\} = \#\{t \in \mathfrak{T}_{2n+1} : \mathrm{pom}(t) = k\} = f_n(k) \tag{1.5}$$

for all k; in particular, $\#\mathfrak{T}_{2n+1} = T_{2n+1}/2^n$. Note that a combinatorial proof of the first identity in (1.5) has been given in [FH13].

It was then natural to see whether the *joint* distribution of the pair $(\mathrm{eoc}, \mathrm{pom})$ on each set $\mathfrak{T}_{2n+1}$ could be calculated, no longer by a one-variable system, such as (1.1), but by a system of *partial* finite difference equations (see equations $(R1)$, $(R2)$ below), verifying certain initial conditions. This is the purpose of the paper. To achieve this, we first introduce a sequence $(M_n = (f_n(m,k))$ of $(2n) \times (2n)$-matrices $(n \ge 1)$ with nonnegative integral entries, called a *Delta sequence*, defined by such a system and prove that *each entry $f_n(m,k)$ is equal to the number of trees t from $\mathfrak{T}_{2n+1}$ such that* $\mathrm{eoc}(t) = m$ *and* $\mathrm{pom}(t) = k$. We finally calculate the exponential generating function for the matrices M_n.

For defining the Delta Sequence it is convenient to consider the following four triangles of each square $\{(m,k) : 1 \le m, k \le 2n\}$:

$L_n^{(1)} := \{2 \le k+1 \le m \le 2n-2\}$; $L_n^{(2)} := \{4 \le k+3 \le m \le 2n\}$;

$U_n^{(1)} := \{2 \le m+1 \le k \le 2n-2\}$; $U_n^{(2)} := \{4 \le m+3 \le k \le 2n\}$.

By convention, $f_n(m,k) := 0$ if $(m,k) \notin [1,2n] \times [1,2n]$. The *partial difference operators* $\underset{m}{\Delta}$, $\underset{k}{\Delta}$, act as follows on the entries of the matrices M_n:

$$\underset{m}{\Delta} f_n(m,k) := f_n(m+1,k) - f_n(m,k); \tag{1.6}$$

$$\underset{k}{\Delta} f_n(m,k) := f_n(m,k+1) - f_n(m,k). \tag{1.7}$$

They serve to define the recurrence relations:

$$(R\,1) \qquad \Delta_m^2 f_n(m,k) + 2\,f_{n-1}(m,k) = 0 \quad ((m,k) \in L_n^{(1)});$$

$$(R\,2) \qquad \Delta_k^2 f_n(m,k) + 2\,f_{n-1}(m,k) = 0 \quad ((m,k) \in U_n^{(1)}).$$

Finally, the *row* and *column sums* of $M_n = (f_n(m,k))$ are denoted by

$$f_n(m,\bullet) := \sum_{1\le k\le 2n} f_n(m,k); \quad (1 \le m \le 2n);$$

$$f_n(\bullet,k) := \sum_{1\le m\le 2n} f_n(m,k) \quad (1 \le k \le 2n).$$

Definition 1.1. A sequence of matrices (M_n) $(n \ge 1)$, where each matrix $M_n = (f_n(m,k))$ $(1 \le k,m \le 2n)$ has nonnegative integral entries, having only 0's along its diagonal, and such that $M_1 := \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$, is said to be a *Delta Sequence*, if for $n \ge 2$ both recurrence relations $(R\,1)$ and $(R\,2)$ hold, together with the *initial conditions*:

$(I\,1)$ for $n \ge 2$ the $(2n)$-th column, Col_{2n}, of M_n is the zero-column; its the $(2n-1)$-st column, Col_{2n-1}, is equal to

$$f_{n-1}(1,\bullet),\ f_{n-1}(2,\bullet),\ \ldots,\ f_{n-1}(2n-2,\bullet),\ 0,\ 0,$$

when read from top to bottom.

$(I\,2)$ the $(2n)$-th row of M_n is also equal to

$$f_{n-1}(1,\bullet),\ f_{n-1}(2,\bullet),\ \ldots,\ f_{n-1}(2n-2,\bullet),\ 0,\ 0,$$

when read from left to right; its $(2n-1)$-st row is equal to:

$$f_{n-1}(1,\bullet) + f_{n-1}(\bullet,1),\ f_{n-1}(2,\bullet) + f_{n-1}(\bullet,2),\ \ldots\ ,$$
$$f_{n-1}(2n-3,\bullet) + f_{n-1}(\bullet,2n-3),\ f_{n-1}(2n-2,\bullet) + f_{n-1}(\bullet,2n-2),\ 0,\ 0,$$

when read from left to right.

As shown in Section 2, conditions $(R\,1)$, $(R\,2)$, $(I\,1)$, $(I\,2)$ uniquely determine the Delta Sequence (M_n) $(n \ge 1)$, whose first values are reproduced in Fig. 2.1. As for the "labeled strictly ordered binary trees" and the statistics "eoc," "pom" their definitions are now given.

Definition 1.2. An *n-labeled strictly ordered binary tree* is defined by the following axioms:

(1) it is a *labeled* tree with n nodes, labeled $1, 2, \ldots, n$; the node labeled 1 is called the *root*;

(2) each node has no child (it is then called a *leaf*), or two children, their order being immaterial (it is then called an *internal node*);

(3) when getting along each path from the root to each node, the node labels are in *increasing* order. Let $\mathfrak{T}_{2n+1}$ denote the set of all $(2n+1)$-labeled strictly ordered binary trees.

Let $t \in \mathfrak{T}_{2n+1}$ $(n \ge 1)$. If a node labeled a has two children labeled b and c, define $\min a := \min\{b, c\}$; if it has one child b, let $\min a := b$. The *minimal chain* of t is defined to be the sequence $a_1 \to a_2 \to a_3 \to \cdots \to a_{j-1} \to a_j$, with the following properties: (i) $a_1 = 1$ is the label of the root; (ii) for $i = 1, 2, \dots, j-1$ the $(i+1)$-st term a_{i+1} is the label of an internal node and $a_{i+1} = \min a_i$; (iii) a_j is the node of a leaf. Define the "end of the minimal chain" of t to be $\operatorname{eoc}(t) := a_j$. If the leaf with the maximum label $(2n+1)$ is incident to a node labeled k, define its "parent of the maximum leaf" to be $\operatorname{pom}(t) := k$.

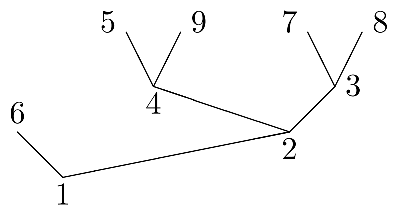


Fig. 1.2. Labeled strictly ordered binary tree.

For example, the minimal chain of the tree t displayed in Fig. 1.2 is $1 \to 2 \to 3 \to 7$, so that $\operatorname{eoc}(t) = 7$ and the parent of its maximum leaf (equal to $2n+1 = 9$) is $\operatorname{pom}(t) = 4$. The main results of this paper are the following theorems.

Theorem 1.1. *Let $(M_n = (f_n(m,k))$ $(n \ge 1)$ be the Delta sequence, as introduced in Definition 1.1. Then, for all $n \ge 1$ and $1 \le m, k \le 2n$*

$$\#\{t \in \mathfrak{T}_{2n+1} : \operatorname{eoc}(t) = m,\ \operatorname{pom}(t) = k\} = f_n(m,k). \tag{1.9}$$

In particular,

$$\sum_{m,k} f_n(m,k) = T_{2n+1}/2^n \quad (n \ge 0). \tag{1.10}$$

Theorem 1.2. *Let $\bigl(M_n = (f_n(m,k))$ $(1 \le m, k \le 2n)\bigr)$ $(n \ge 1)$ be the Delta sequence. Then, the matrices M_n are symmetric with respect to their counter-diagonals:*

$$f_n(m,k) = f_n(2n+1-k, 2n+1-m) \quad (1 \le k, m \le 2n). \tag{1.11}$$

Theorem 1.3. *The triple exponential generating function for the lower triangles of the matrices M_n is given by*

$$\sum_{2 \le k+1 \le m \le 2n} f_n(m,k) \frac{x^{m-k-1}}{(m-k-1)!} \frac{y^{k-1}}{(k-1)!} \frac{z^{2n-m}}{(2n-m)!} = \frac{\cos(\sqrt{2}\,x) + \cos(\sqrt{2}\,y)\,\cos(\sqrt{2}\,z)}{2\,\cos^2\Bigl(\dfrac{x+y+z}{\sqrt{2}}\Bigr)}. \tag{1.12}$$

Theorem 1.4. *The triple exponential generating function for the upper triangles of the matrices M_n is given by*

$$\sum_{2\le m+1\le k\le 2n-1} f_n(m,k)\frac{x^{2n-k}}{(2n-k)!}\frac{y^{k-m-1}}{(k-m-1)!}\frac{z^{m-1}}{(m-1)!} = \sin(\sqrt{2}\,x)\,\sin(\sqrt{2}\,z)\,\frac{1}{2\cos^2\Bigl(\dfrac{x+y+z}{\sqrt{2}}\Bigr)}. \tag{1.13}$$

Theorems 1.1 and 1.2 will be proved in Sections 3–7, where we develop a genuine *Tree Calculus* on the sequence $(\mathfrak{T}_{2n+1})$, which appears to be an alternative for the constructions of several combinatorial bijections. In Theorems 1.3 and 1.4 note that an adequate normalization for the three-variable series is to be found for getting such closed expressions for the generating functions for the $f_n(m,k)$'s.

2. Delta and gamma sequences and organization of the paper

Go back to Definition 1.1 of the Delta Sequence. It is based on the two relations $(R\,1)$, $(R\,2)$ and the two initial conditions $(I\,1)$, $(I\,2)$. Those relations and conditions can be symbolized by the square in Fig. 2.1, as relation $(R\,1)$ (resp. $(R\,2)$) acts on the entries of the lower (resp. upper) entries of the matrix M_n, and initial conditions $(I\,1)$ and $(I\,2)$ refer to the last two columns Col_{2n-1}, Col_{2n} and rows Row_{2n-1}, Row_{2n} of M_n, respectively.

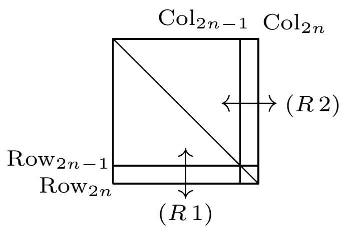


Fig. 2.1: Definition of the Delta Sequence.

In Definition 1.1 of the Delta Sequence the entries of each matrix M_n are derived from M_{n-1} by first applying rules $(I\,1)$ and $(I\,2)$ and letting the diagonal be null; then, starting from $m = 1$ up to $m = 2n-3$, for each k from $2n-3$ down to $m+1$, evaluate $f_n(m,k)$ with equation $(R\,2)$: $f_n(m,k) - 2f_n(m,k+1) + f_n(m,k+2) + 2f_{n-1}(m,k) = 0$, the coefficients $f_n(m,k+1)$, $f_n(m,k+2)$ and $f_{n-1}(m,k)$ being already calculated. Exchanging the roles of m and k the upper entries are obtained by using equation $(R\,1)$. Accordingly, $(R\,1)$, $(R\,2)$, $(I\,1)$, $(I\,2)$ uniquely determine the Delta Sequence (M_n) $(n \ge 1)$.

Calculation of the first matrices. First, $f_1(1,\bullet)=0$, $f_1(2,\bullet)=1$, so that

$$M_2=\begin{pmatrix}0&?&0&0\\ ?&0&1&0\\ 1&1&0&0\\ 0&1&0&0\end{pmatrix}$$

by rules $(I\,1)$ and $(I\,2)$. The remaining entries are obtained by rule $(R\,2)$: $f_2(1,2)-2f_2(1,3)+f_2(1,4)+2f_1(1,2)=f_2(1,2)-2\times 0+0+2\times 0=0$, so that $f_2(1,2)=0$; then, by rule $(R\,1)$: $f_2(2,1)-2f_2(3,1)+f_2(4,1)+2f_1(2,1)=f_2(2,1)-2\times 1+0+2\times 1=0$, so that $f_2(2,1)=0$. Thus,

$$M_2=\begin{pmatrix}0&0&0&0\\ 0&0&1&0\\ 1&1&0&0\\ 0&1&0&0\end{pmatrix}$$

and $f_2(1,\bullet)=0$, $f_2(2,\bullet)=1$, $f_2(3,\bullet)=2$, $f_2(4,\bullet)=1$. The next matrices are displayed in Fig. 1.2.

$$M_3=\begin{pmatrix}0&0&0&0&0&0\\ 0&0&1&2&1&0\\ 1&1&0&4&2&0\\ 2&3&4&0&1&0\\ 1&3&3&1&0&0\\ 0&1&2&1&0&0\end{pmatrix}\quad M_4=\begin{pmatrix}0&0&0&0&0&0&0&0\\ 0&0&4&8&10&8&4&0\\ 4&4&0&16&20&16&8&0\\ 8&12&16&0&28&20&10&0\\ 10&18&24&28&0&16&8&0\\ 8&18&24&24&16&0&4&0\\ 4&12&18&18&12&4&0&0\\ 0&4&8&10&8&4&0&0\end{pmatrix}.$$

$$M_5=\begin{pmatrix}0&0&0&0&0&0&0&0&0&0\\ 0&0&34&68&94&104&94&68&34&0\\ 34&34&0&136&188&208&188&136&68&0\\ 68&102&136&0&274&296&262&188&94&0\\ 94&162&222&274&0&352&296&208&104&0\\ 104&198&276&330&352&0&274&188&94&0\\ 94&198&282&330&330&274&0&136&68&0\\ 68&162&240&282&276&222&136&0&34&0\\ 34&102&162&198&198&162&102&34&0&0\\ 0&34&68&94&104&94&68&34&0&0\end{pmatrix}$$

Fig. 2.2: the first matrices M_n.

Other initial conditions could be stated; they will be mentioned in Section 9. At this point we just describe a second one, materialized by the square in Fig. 2.3

Definition 2.1. A sequence of matrices (M_n) $(n \geq 1)$, where each matrix $M_n = (f_n(m,k))$ $(1 \leq k, m \leq 2n)$ has nonnegative integral entries, having only 0's along its diagonal, and such that $M_1 := \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$, is said to be a *Gamma Sequence*, if for $n \geq 2$ both recurrence relations

$$(R\,3) \qquad \underset{m}{\Delta}^2 f_n(m,k) + 2\, f_{n-1}(m, k-2) = 0 \quad ((k,m) \in U_n^{(2)});$$

$$(R\,4) \qquad \underset{k}{\Delta}^2 f_n(m,k) + 2\, f_{n-1}(m-2, k) = 0 \quad ((k,m) \in L_n^{(2)});$$

hold, together with the *initial conditions*:

$(I\,3)$ for $n \geq 2$ the first row, Row_1, is the zero-row; the second row, Row_2, is equal to

$$0, f_{n-1}(1,\bullet)(=0), f_{n-1}(2,\bullet), \ldots, f_{n-1}(2n-2,\bullet), f_{n-1}(2n-1,\bullet)(=0);$$

when read from left to right.

$(I\,4)$ the first column, Col_1, of M_n is also equal to

$$0, f_{n-1}(1,\bullet)(=0), f_{n-1}(2,\bullet), \ldots, f_{n-1}(2n-2,\bullet), f_{n-1}(2n-1,\bullet)(=0)$$

when read from top to bottom; the second column, Col_2, is equal to

$$0, 0, f_{n-1}(2,\bullet) + f_{n-1}(1,\bullet),\ f_{n-1}(3,\bullet) + f_{n-1}(2,\bullet),\ \ldots$$
$$f_{n-1}(2n-2,\bullet) + f_{n-1}(2n-3,\bullet),\ f_{n-1}(2n-2,\bullet),$$

when read from left to right.

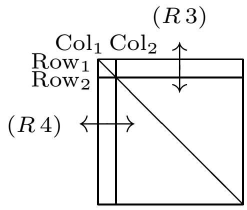


Fig. 2.3: Definition of the Gamma Sequence.

Using the same reasoning as for Definition 1.1 it is seen that the Gamma Sequence is uniquely defined. The fact that Delta and Gamma Sequences are identical will be a consequence of the further theorems (*cf.* Section 6).

Again, go back to Definition 1.1. Each tree from $\mathfrak{T}_{2n+1}$ has an *odd* number of vertices, i.e., $(2n+1)$, with n internal nodes and $(n+1)$ leaves. When giving an orientation (left or right) to each child of each of the n internal nodes, we generate 2^n *planar* strictly ordered binary trees (also called "arbres binaires croissants complets" by Viennot [Vi88, chap. 3, p. 111]). It is known that the latter are equidistributed with the *alternating permutations* of order $(2n+1)$, so that their number is equal to the tangent number T_{2n+1}, a result that goes back to Désiré André [An1879, An1881]. Accordingly,

$$\#\mathfrak{T}_{2n+1} = T_{2n+1}/2^n. \tag{2.1}$$

When dealing with those strictly ordered binary trees, we adopt the following notation and convention: for each triple (n, m, k) let $\mathfrak{T}_{2n+1,m,k}$ (resp. $\mathfrak{T}_{2n+1,m,\bullet}$, resp. $\mathfrak{T}_{2n+1,\bullet,k}$) denote the subset of $\mathfrak{T}_{2n+1}$ of all trees t such that $\operatorname{eoc}(t) = m$ and $\operatorname{pom}(t) = k$ (resp. $\operatorname{eoc}(t) = m$, resp. $\operatorname{pom}(t) = k$). By convention, designate those families *and their cardinalities* by the same symbol and also the matrix of the *integers* $\mathfrak{T}_{2n+1,m,k}$ by $\operatorname{Mat}(\mathfrak{T}_{2n+1})$. Our plan of action will be to show that the sequence $(\operatorname{Mat}(\mathfrak{T}_{2n+1}))$ $(n \geq 1)$ *is identical* to the Delta Sequence.

In Sections 3–6 it will be shown that, when replacing each $f_n(m, k)$ by $\mathfrak{T}_{2n+1,m,k}$ the initial conditions $(I\,1)$ and $(I\,2)$, the two finite difference equations systems $(R\,1)$, $(R\,3)$, the two finite difference equations systems $(R\,2)$, $(R\,4)$ and the initial conditions $(I\,3)$ and $(I\,4)$) hold. This will complete the proofs of Theorems 1.1 and 1.2, as done in Section 7. Further properties of the matrices $\operatorname{Mat}(\mathfrak{T}_{2n+1})$ (and then matrices M_n) will be given in Section 8. Finally, several other equivalent definitions of the Delta sequence will be mentioned in Section 9. Finally, the calculation of the generating functions for the matrices M_n is made in Section 10.

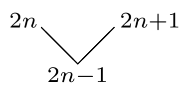
3. The initial conditions $(I\,1)$ and $(I\,2)$

In this section and the next ones we make the convention that whenever a leaf is deleted from a tree, the edge linking the leaf to the tree is also deleted.

For verifying that the matrices $\operatorname{Mat}(\mathfrak{T}_{2n+1})$ have only zero in their diagonals, it suffices to show that $\mathfrak{T}_{2n+1,m,m} = \emptyset$ (or is equal to 0 with our convention). This is true, because if $t \in \mathfrak{T}_{2n+1}$ and $\operatorname{eoc}(t) = \operatorname{pom}(t) = m$, the node $(2n + 1)$ has a parent equal to m. Consequently, m cannot be the end of a minimal chain. Hence, the previous subset is empty.

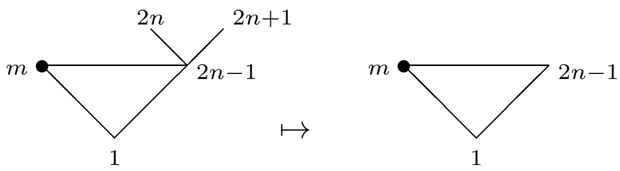
Theorem 3.1. *The initial conditions $(I\,1)$ and $(I\,2)$ hold for the matrices $\operatorname{Mat}(\mathfrak{T}_{2n+1})$ when $f_n(m, \bullet)$ and $f_n(\bullet, k)$ are replaced by $\mathfrak{T}_{2n+1,m,\bullet}$ and $\mathfrak{T}_{2n+1,\bullet,k}$, respectively.*

Proof. $(I\,1)$ First, the $(2n)$-th column of the matrix $\operatorname{Mat}(\mathfrak{T}_{2n+1})$ has zero entries only, as $\operatorname{pom} \leq 2n - 1$. Next, each tree from $\mathfrak{T}_{2n+1,m,2n-1}$ $(1 \leq m \leq 2n - 2)$ must contain the subtree



Hence, $\mathfrak{T}_{2n+1,2n-1,2n-1}$ is empty, for $(2n - 1)$, being an internal node, cannot be the end of the minimal chain. Also, $\mathfrak{T}_{2n+1,2n,2n-1}$ is empty, for the sibling of $(2n-1)$ is necessarily less than $(2n-1)$, so that the minimal chain cannot go through $(2n-1)$ and reach $(2n)$. Furthermore, $\mathfrak{T}_{2n+1,1,2n-1} = \mathfrak{T}_{2n-1,1,\bullet} = 0$, as eoc ≥ 2.

In the remaining cases, that is, $2 \leq m \leq 2n-2$, removing the two leaves $(2n)$, $(2n+1)$ transforms each tree from $\mathfrak{T}_{2n+1,m,2n-1}$ onto a tree from $\mathfrak{T}_{2n-1,m,\bullet}$ in a bijective manner. Such a transformation may be illustrated by the diagram:

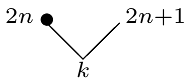


Hence, the $(2n-1)$-st column of the matrix $\mathrm{Mat}(\mathfrak{T}_{2n+1})$ reads:

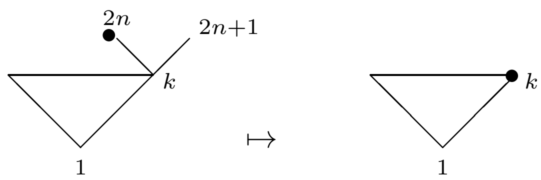
$$\mathfrak{T}_{2n-1,1,\bullet}, \mathfrak{T}_{2n-1,2,\bullet}, \ldots, \mathfrak{T}_{2n-1,2n-2,\bullet},\ 0,\ 0. \tag{3.1}$$

from top to bottom.

$(I\,2)$ For the $(2n)$-th row of the matrix $\mathrm{Mat}(\mathfrak{T}_{2n+1})$ note that $\mathfrak{T}_{2n+1,2n,1} = 0$ for $n \geq 2$. When $k \geq 2$ each tree from $\mathfrak{T}_{2n+1,2n,k}$ must contain the subtree

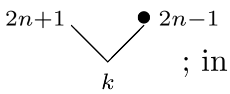


By $(I\,1)$ we then have: $\mathfrak{T}_{2n+1,2n,2n-1} = \mathfrak{T}_{2n+1,2n,2n} = 0$. For the remaining cases $2 \leq k \leq 2n-2$ we can set up a bijection of $\mathfrak{T}_{2n+1,2n,k}$ onto $\mathfrak{T}_{2n-1,k,\bullet}$ by removing the two leaves $2n$ and $(2n+1)$, as illustrated by the next diagram.

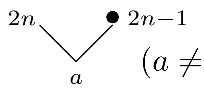


Note that the node k becomes the end of the minimal chain. Thus, the $(2n)$-th row of the matrix $\mathfrak{T}_{2n+1}$ is also equal to (3.1) read from left to right.

Finally, consider the $(2n-1)$-st row of $\mathfrak{T}_{2n+1}$. In an obvious manner, $\mathfrak{T}_{2n+1,2n-1,2n-1} = \mathfrak{T}_{2n+1,2n-1,2n} = 0$. When $1 \leq k \leq 2n-2$, the trees from the *sets* $\mathfrak{T}_{2n+1,2n-1,k}$ fall into two categories $\mathfrak{T}^{I}_{2n+1,2n-1,k}$ and $\mathfrak{T}^{II}_{2n+1,2n-1,k}$. In the first category the trees contain the subtree



; in the second one, the subtree



$(a \neq k)$.

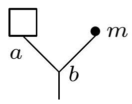
First, note that $\mathfrak{T}^{I}_{2n+1,2n-1,1} = \mathfrak{T}_{2n-1,1,\bullet} = 0$. When $2 \leq k \leq 2n-2$, removing the two leaves $(2n+1)$, $(2n-1)$ and replacing the node label $(2n)$ by $(2n-1)$ maps $\mathfrak{T}^{I}_{2n+1,2n-1,k}$ onto $\mathfrak{T}_{2n-1,k,\bullet}$ in a bijective manner.

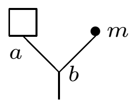
When $1 \le k \le 2n-2$, removing the two leaves $(2n)$, $(2n-1)$, and replacing the node label $(2n+1)$ by $(2n-1)$ maps $\mathfrak{T}^{II}_{2n+1,2n-1,k}$ onto $\mathfrak{T}_{2n-1,\bullet,k}$ in a bijective manner. Thus, the $(2n-1)$-st row of $\mathfrak{T}_{2n+1}$ reads

$$\mathfrak{T}_{2n-1,1,\bullet} + \mathfrak{T}_{2n-1,\bullet,1},\ \mathfrak{T}_{2n-1,2,\bullet} + \mathfrak{T}_{2n-1,\bullet,2},\ \dots\ ,$$
$$\mathfrak{T}_{2n-1,2n-3,\bullet} + \mathfrak{T}_{2n-1,\bullet,2n-3},\ \mathfrak{T}_{2n-1,2n-2,\bullet} + \mathfrak{T}_{2n-1,\bullet,2n-2},\ 0,\ 0. \qquad \square$$

4. Tree Calculus for the relations $(R1)$ and $(R3)$

In the following Tree Calculus subtrees (possibly leaves) are indicated by the symbols "$\bigcirc$," "$\triangledown$", or "$\square$." The end of the minimal chain in each tree is represented by a bullet "$\bullet$." Letters occurring below or next to subtrees are labels of their roots. For instance, the symbols



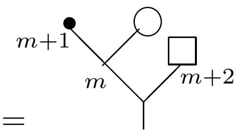
, [

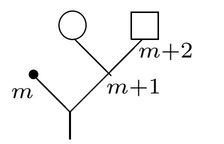
, c]

designate the *families* of all trees t from the underlying set $\mathfrak{T}_{2n+1}$ having a node labeled b [in short, a node b], parent of both a subtree of root a and the leaf m, which is also the end of the minimal chain; moreover, the symbol on the right has the further property that the node labeled c does not belong, *either* to the subtree of root b, *or* to the path going from root 1 to b. In the sequel, the letter "m" is always used to designate the end of the minimal chain, unless explicitly indicated by a letter next to $\bullet$.

Our Tree Calculus consists of two steps: (a) decomposing the sets $\mathfrak{T}_{2n+1,m,k}$ into smaller subsets by considering the mutual positions of the nodes m, $(m+1)$, $(m+2)$ (resp. k, $(k+1)$, $(k+2)$); (b) setting up bijections between those subsets by a simple display of certain subtrees, as done in (4.1).

For instance,

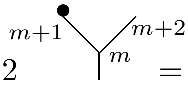
(4.1) $\quad C_3 :=$ 

and $\quad D_1 :=$ 

may be regarded as two subsets of $\mathfrak{T}_{2n+1}$. To each pair $(\underset{m+2}{\square}, \bigcirc)$ there correspond a unique tree from C_3 and a unique tree from D_1, as the nodes of "$\bigcirc$" are all greater than or equal to $(m+2)$. This clearly defines a bijection of C_3 onto D_1.

Those two principles (a) and (b) will be applied in the proofs of the next two theorems 4.1 and 5.1.

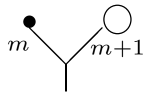
Theorem 4.1. *If (m,k) belongs to $L_n^{(1)} \cup U_n^{(2)} = \{2 \le k+1 \le m \le 2n-2\} \cup \{4 \le m+3 \le k \le 2n\}$, then*

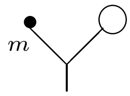
(4.2) $$\Delta_m^2 \, \mathfrak{T}_{2n+1,m,k} + 2$$ 

$$= 0,$$

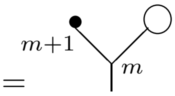
with the understanding that the second term on the left-hand side represents twice the set of all trees from $\mathfrak{T}_{2n+1,m+1,k}$ with the further property that m is the parent of both $(m+1)$ and $(m+2)$.

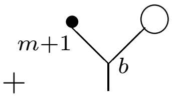
Proof. The decomposition

$$\mathfrak{T}_{2n+1,m,k} =$$ 

$+$ 

means that in each tree from $\mathfrak{T}_{2n+1,m,k}$ the node $(m+1)$ is, or is not, the sibling of the leaf m. In the next decomposition the node m is, or is not, the parent of the leaf $(m+1)$:

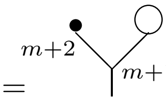
$$\mathfrak{T}_{2n+1,m+1,k} =$$ 

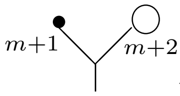
$+$ 

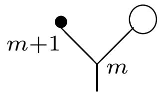
$(b \ne m)$.

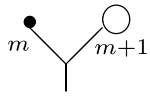
Under the transposition $(m, m+1)$ the node labeled k remains unaffected, because $k \le m-1$ if $(m,k) \in L_n^{(1)}$ and $m+3 \le k$ if $(m,k) \in U_n^{(2)}$, so that the parent of $(2n+1)$ remains k. Thus, the transposition establishes a one-to-one correspondence between the two second terms. Hence,

$$\Delta_m^2 \, \mathfrak{T}_{2n+1,m,k}$$
$$= (\mathfrak{T}_{2n+1,m+2,k} - \mathfrak{T}_{2n+1,m+1,k}) - (\mathfrak{T}_{2n+1,m+1,k} - \mathfrak{T}_{2n+1,m,k})$$

$=$ 

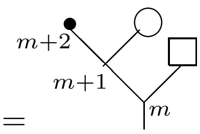
$-$ 

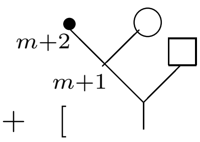
$-$ 

$+$ 

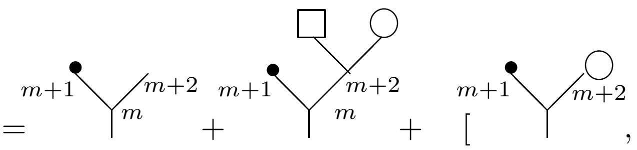
$$:= A - B - C + D.$$

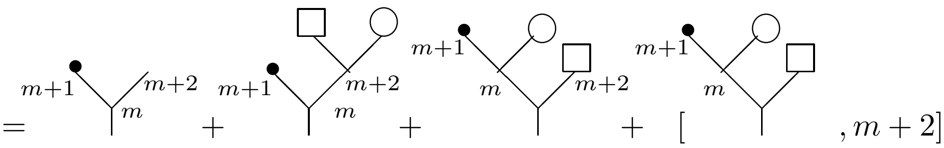
Depending on the mutual positions of nodes m, $(m+1)$ and $(m+2)$ the further decompositions prevail, as again, k remaining still attached to $(2n+1)$:

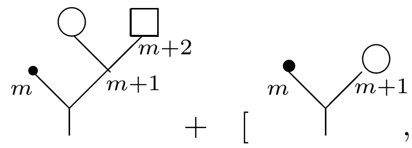
$A =$ 

$+$ $[$ 

$, m] := A_1 + A_2;$

$B =$

$:= B_1 + B_2 + B_3;$

$C =$

$:= C_1 + C_2 + C_3 + C_4;$

$D =$

$:= D_1 + D_2.$

The permutation $\binom{m \;\; m+1 \;\; m+2}{m+2 \;\; m \;\; m+1}$ establishes a one-to-one correspondence between A_2 and C_4 (resp. between B_3 and D_2). On the other hand, $A_1 = B_2 + C_2 = 2\,B_2$, because the two subtrees in $B_2 = C_2$ are incident to the same node, while in A_1 they are incident to two different nodes. Finally, $C_3 = D_1$ as explained in (4.1). Making all the proper cancellations in the sum $\mathop{\Delta^2}\limits_{m} \mathfrak{T}_{2n+1,m,k} = A - B - C + D$ we get identity (4.2). $\square$

Corollary 4.2. *The relations* $(R1)$ *and* $(R3)$ *hold for the matrices* $\mathrm{Mat}(\mathfrak{T}_{2n+1})$*, that is,*

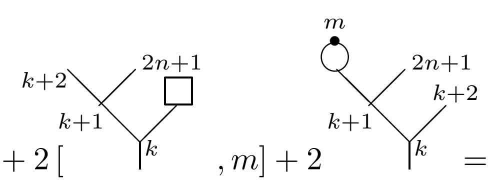
$$\mathop{\Delta^2}\limits_{m} \mathfrak{T}_{2n+1,m,k} + 2\,\mathfrak{T}_{2n-1,m,k} = 0, \quad \textit{if } (m,k) \in L_n^{(1)}; \tag{4.3}$$

$$\mathop{\Delta^2}\limits_{m} \mathfrak{T}_{2n+1,m,k} + 2\,\mathfrak{T}_{2n-1,m,k-2} = 0, \quad \textit{if } (m,k) \in U_n^{(2)}. \tag{4.4}$$

Proof. For (4.3) change the second term in (4.2) as follows: remove the two leaves $(m+1)$, $(m+2)$, and subtract 2 from all the remaining nodes greater than $(m+2)$: the term becomes $\mathfrak{T}_{2n-1,m,k}$, as the node $(2n+1)$ becomes $(2n-1)$ and is still linked to k. For (4.4) do the same changes, but this time, as $m+3 \le k$, the edge going from k to $(2n+1)$ becomes an edge going from $(k-2)$ to $(2n-1)$. $\square$

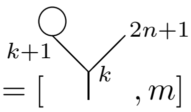
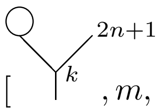
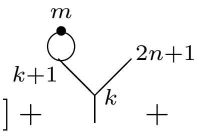
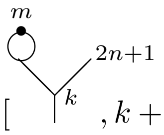
5. Tree Calculus for the relations $(R2)$ and $(R4)$

Theorem 5.1. *If* (m,k) *belongs to* $L_n^{(2)} \cup U_n^{(1)} = \{4 \le k+3 \le m \le 2n\} \cup \{2 \le m+1 \le k \le 2n-2\}$*, then*

(5.1) $\quad \mathop{\Delta^2}\limits_{k} \mathfrak{T}_{2n+1,m,k} + 2\,[$

$= 0,$

with the understanding that the second term on the left-hand side is twice the number of all trees from $\mathfrak{T}_{2n+1,m,k+1}$ with the further property that $(k+2)$ is a leaf incident to $(k+1)$, itself incident to k, the end m of the minimum chain being outside the subtree of root k.

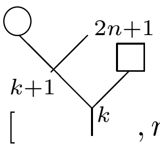
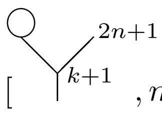
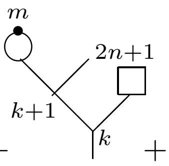
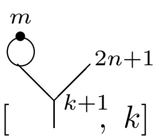
Proof. First,

$\mathfrak{T}_{2n+1,m,k} = [$  $, m] + [$  $, m, k+1] +$  $+ \; [$  $, k+1]$

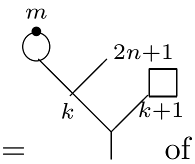
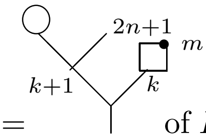
$$:= A_1 + A_2 + A_3 + A_4,$$

meaning that each tree from $\mathfrak{T}_{2n+1,m,k}$ has one of the four forms: either $k+1$ is incident to k, or not, and m is outside or not the subtree of root k; furthermore, the leaf m is the end of the minimal chain.

Using the same dichotomy,

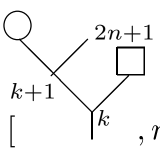
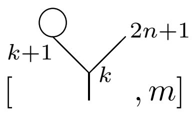
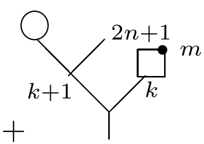
$\mathfrak{T}_{2n+1,m,k+1} = [$  $, m] + [$  $, m, k] +$  $+ \; [$  $, k]$

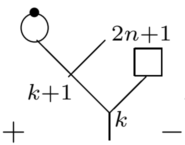
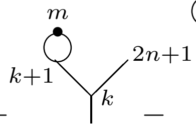
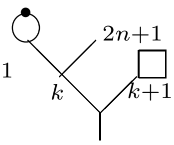
$$:= B_1 + B_2 + B_3 + B_4.$$

Consider the subsets $A_4' :=$  of A_4 and $B_2' :=$  of B_2. The transposition $(k, k+1)$ maps $A_4 \setminus A_4'$ onto B_4 and A_2 onto $B_2 \setminus B_2'$ in a bijective manner.

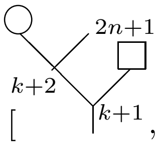
Hence,

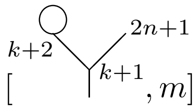
$$\begin{aligned}\mathfrak{T}_{2n+1,m,k+1} - \mathfrak{T}_{2n+1,m,k} &= (B_1 - A_1) + ((B_2 - B_2' - A_2) + B_2') \\ &\quad + (B_3 - A_3) + (B_4 - (A_4 - A_4') - A_4') \\ &= B_1 - A_1 + B_2' + B_3 - A_3 - A_4'\end{aligned}$$

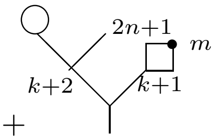
$= [$  $, m] - [$  $, m] \; +$ 

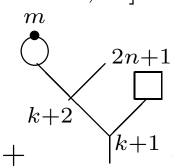
$+$  $-$  $-$  ;

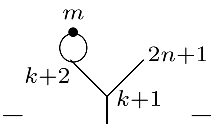
and also

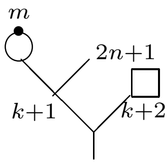
$\mathfrak{T}_{2n+1,m,k+2} - \mathfrak{T}_{2n+1,m,k+1} = [$ 

$, m] - [$ 

$, m] +$ 

$+$ 

$-$ 

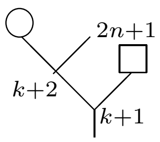
$-$ 

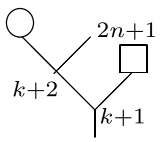
$$:= D_1 - C_1 + D_2' + D_3 - C_3 - C_4'.$$

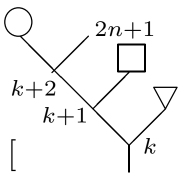
Thus,

$$\begin{aligned}\mathop{\Delta^2}_{k} \mathfrak{T}_{2n+1,m,k} &= \big(\mathfrak{T}_{2n+1,m,k+2} - \mathfrak{T}_{2n+1,m,k+1}\big) - \big(\mathfrak{T}_{2n+1,m,k+1} - \mathfrak{T}_{2n+1,m,k}\big)\\ &= D_1 - C_1 + D_2' + D_3 - C_3 - C_4'\\ &\qquad - B_1 + A_1 - B_2' - B_3 + A_3 + A_4'.\end{aligned}$$

The further decompositions of the components of the previous sum depend on the mutual positions of the nodes k, $(k+1)$, $(k+2)$. First, evaluate the subsum: $S_1 := D_1 - C_1 - B_1 + A_1$ using the decompositions:

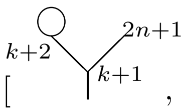
$D_1 = [$ 

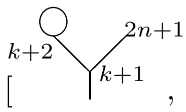
$, m] = [$ 

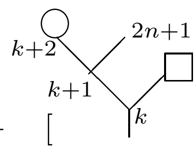
$, m, k\,] + [$ 

$, m]$

$$:= D_{1,1} + D_{1,2};$$

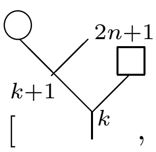
$C_1 = [$ 

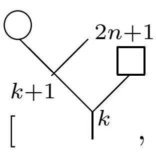
$, m] = [$ 

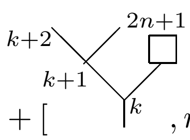
$, m, k\,] + [$ 

$, m]$

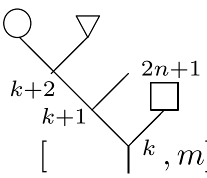
$$:= C_{1,1} + C_{1,2};$$

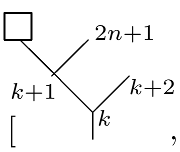
$B_1 = [$ 

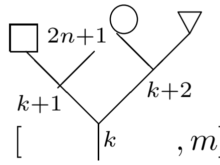
$, m] = [$ 

$, m, k+2] + [$ 

$, m]$

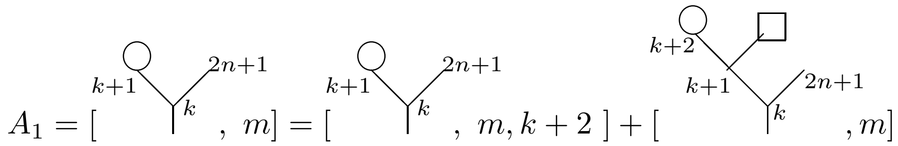
$+ [$ 

$, m] + [$ 

$, m] + [$ 

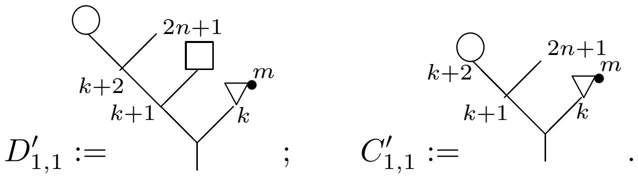
$, m]$

$$\begin{aligned}:= &B_{1,1} + B_{1,2}\\ &+ B_{1,3} + B_{1,4} + B_{1,5};\end{aligned}$$



$$:= A_{1,1} + A_{1,2}.$$

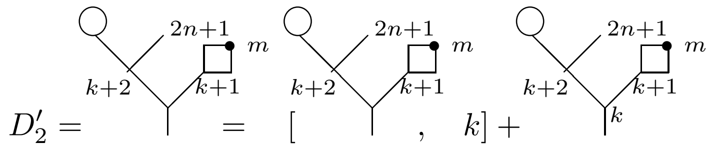
Also, let



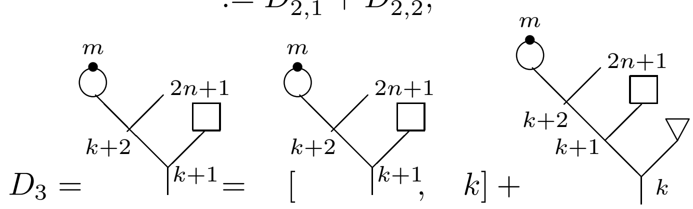
The permutation $\begin{pmatrix} k & k+1 & k+2 \\ k+2 & k & k+1 \end{pmatrix}$ maps $D_{1,1} \setminus D'_{1,1}$ onto $B_{1,1}$ and $C_{1,1} \setminus C'_{1,1}$ onto $A_{1,1}$. Hence, $D_{1,1} = B_{1,1} + D'_{1,1}$, $C_{1,1} = A_{1,1} + C'_{1,1}$. Moreover, $D_{1,2} = 2\,B_{1,3}$, $C_{1,2} = A_{1,2}$, $B_{1,2} = B_{1,4}$, $B_{1,3} = B_{1,5}$. Altogether, $S_1 = D_1 - C_1 - B_1 + A_1 = (B_{1,1} + D'_{1,1} + 2\,B_{1,3}) - (A_{1,1} + C'_{1,1} + A_{1,2}) - (B_{1,1} + B_{1,2} + B_{1,3} + B_{1,2} + B_{1,3}) + (A_{1,1} + A_{1,2})$. Thus,

$$S_1 = -2\,B_{1,2} + D'_{1,1} - C'_{1,1}. \tag{5.2}$$

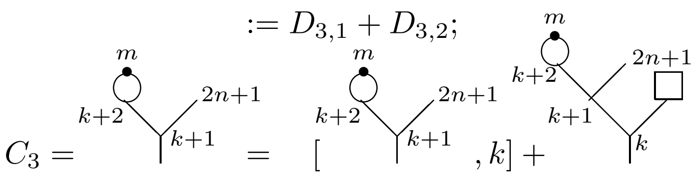
Next, evaluate the sum $S_2 := D'_2 + D_3 - C_3 - C'_4 - B'_2 - B_3 + A_3 + A'_4$ by decomposing each of its components. We have:



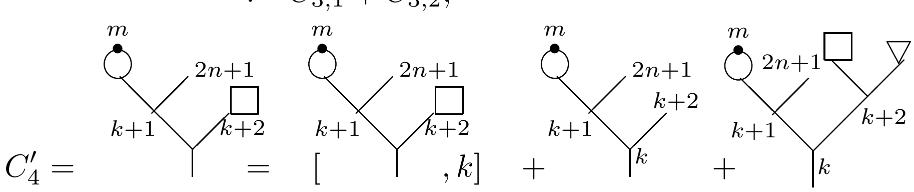
$$:= D'_{2,1} + D'_{2,2};$$



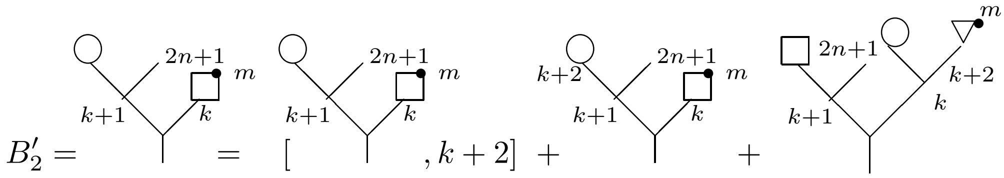
$$:= D_{3,1} + D_{3,2};$$



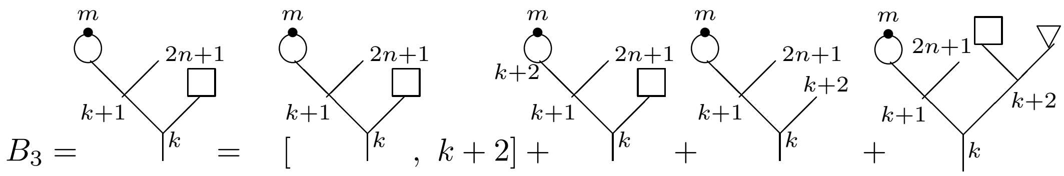
$$:= C_{3,1} + C_{3,2};$$



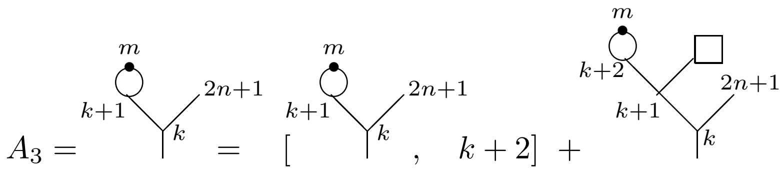
$$:= C'_{4,1} + C'_{4,2} + C'_{4,3};$$



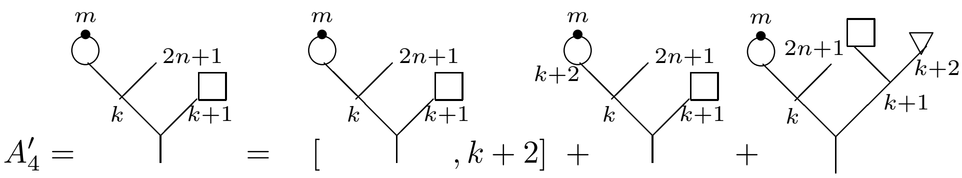
$$:= B'_{2,1} + B'_{2,2} + B'_{2,3};$$



$$:= B_{3,1} + B_{3,2} + B_{3,3} + B_{3,4};$$



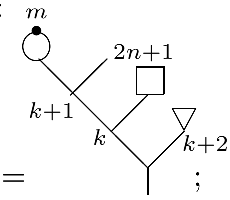
$$:= A_{3,1} + A_{3,2};$$

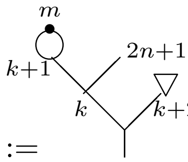


$$:= A'_{4,1} + A'_{4,2} + A'_{4,3}.$$

Within the sum S_2 there are numerous cancellations we now describe.

(a) *Components of the form* $[t, k]$ *or* $[t, k+2]$, *where* t *is a subtree, whose root is labeled.* There are four of them: $D_{3,1}$, $-C_{3,1}$, $-B_{3,1}$, $A_{3,1}$. Consider the subsets:

$B_{3,1,1} :=$ 

; $A_{3,1,1} :=$ 

;

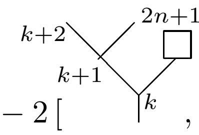
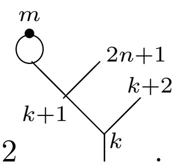
of $B_{3,1}$ and $A_{3,1}$, respectively. The permutation $\begin{pmatrix} k & k+1 & k+2 \\ k+2 & k & k+1 \end{pmatrix}$ maps $D_{3,1}$ onto $B_{3,1} \setminus B_{3,1,1}$ and $C_{3,1}$ onto $A_{3,1} \setminus A_{3,1,1}$. Hence, $D_{3,1} - C_{3,1} - B_{3,1} + A_{3,1} = (B_{3,1} - B_{3,1,1}) - (A_{3,1} - A_{3,1,1}) - B_{3,1} + A_{3,1} = -B_{3,1,1} + A_{3,1,1}$.

(b) *Components of the form* $[t, k]$ *or* $[t, k+2]$, *where* t *is a subtree, whose root is not labeled.* There are four of them: $D'_{2,1}$, $-C'_{4,1}$, $-B'_{2,1}$, $A'_{4,1}$. Again, the permutation $\begin{pmatrix} k & k+1 & k+2 \\ k+2 & k & k+1 \end{pmatrix}$ maps $D'_{2,1}$ onto $B'_{2,1}$, and $C'_{4,1}$ onto $A'_{4,1}$. Hence, $D'_{2,1} - B'_{2,1} = -C'_{4,1} + A'_{4,1} = 0$. Their sum vanish.

(c) *Components represented by a tree* t*, whose root is unlabeled.* There are four of them: $-B'_{2,2}$, $-B'_{2,3}$, $-A'_{4,2}$, $A'_{4,3}$. As $B'_{2,2} = A'_{4,2}$, the contribution of those components to S_2 is then $-B'_{2,3} + A'_{4,3}$.

(d) *Components represented by a tree* t*, whose root is labeled.* There are nine of them: $D'_{2,2}$, $D_{3,2}$, $-C_{3,2}$, $-C'_{4,2}$, $-C'_{4,3}$ $-B_{3,2}$, $-B_{3,3}$, $-B_{3,4}$, $A_{3,2}$. By simply comparing the subtree contents we have: $D'_{2,2} - C_{3,2} = -B_{3,2} + A_{3,2} = 0$, $D_{3,2} - (C'_{4,3} + B_{3,4}) = 0$ and $C'_{4,2} = B_{3,3}$. The contribution of those terms is then $-2\,C'_{4,2}$.

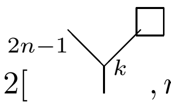
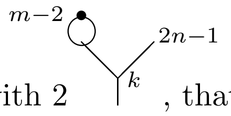
Hence, $S_1+S_2 = (-2\,B_{1,2}+D'_{1,1}-C'_{1,1})+\big((-B_{3,1,1}+A_{3,1,1})+(-B'_{2,3}+A'_{4,3}) + (-2\,C'_{4,2})\big)$. As $D'_{1,1} = B'_{2,3}$, $C'_{1,1} = A_{3,1,1}$ and $B_{3,1,1} = A'_{4,3}$, we get $S_1 + S_2 = -2\,B_{1,2} - 2\,C'_{4,2}$, that is,

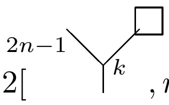
$\Delta^2_k\, \mathfrak{T}_{2n+1,m,k} - 2\,[$  $, m] \quad - 2$  . ∎

Corollary 5.2. *The relations* $(R\,2)$ *and* $(R\,4)$ *hold for the matrices* $\mathrm{Mat}(\mathfrak{T}_{2n+1})$*, that is,*

$$\Delta^2_k\, \mathfrak{T}_{2n+1,m,k} + 2\,\mathfrak{T}_{2n-1,m-2,k} = 0, \quad \textit{if } (m,k) \in L_n^{(2)}; \tag{5.3}$$

$$\Delta^2_k\, \mathfrak{T}_{2n+1,m,k} + 2\,\mathfrak{T}_{2n-1,m,k} = 0, \quad \textit{if } (m,k) \in U_n^{(1)}. \tag{5.4}$$

Proof. When (m,k) belongs to $L_n^{(2)}$, the second term in (5.1) is in bijection with $2[$  $, m-2]$ and the third one with 2  , that is, the set of trees in which the end $m-2$ of the minimal chain is outside (resp. inside) the subtree of root k. The sum of those two terms is then $2\,\mathfrak{T}_{2n-1,m-2,k}$.

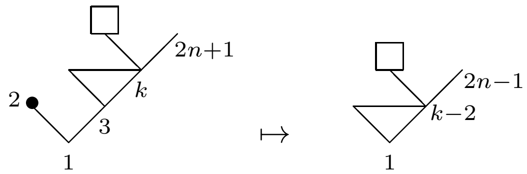
When (m,k) belongs to $U_n^{(1)}$, the third term of (5.1) vanishes and the second one is in bijection with $2[$  $, m]$, equal to $2\,\mathfrak{T}_{2n-1,m,k}$. ∎

6. The initial conditions $(I\,3)$ and $(I\,4)$

Property 6.1. *Initial conditions* $(I\,3)$ *and* $(I\,4)$ *hold for the matrices* $\mathrm{Mat}(\mathfrak{T}_{2n+1})$.

Proof. $(I\,3)$ The first row of each matrix $\mathrm{Mat}(\mathfrak{T}_{2n+1})$ is obviously the zero-row, as 1 can never be the end of the minimal chain. For the second row note that for $n \geq 2$ the set $\mathfrak{T}_{2n+1,2,k}$ is empty when $k = 2$ or $2n$. Also the set $\mathfrak{T}_{2n+1,2,1}$ is empty, for 2 and $(2n+1)$ can be both children of the root only when $n = 1$.

Let $3 \le k \le 2n-1$. As illustrated by the diagram



each tree from $\mathfrak{T}_{2n+1,2,k}$ is transformed into a tree from $\mathfrak{T}_{2n-1,\bullet,k-2}$ by deleting the two nodes 2 and 1 and reducing the remaining nodes by 2. This transformation is obviously a bijection. Thus, $\mathfrak{T}_{2n+1,2,k} = \mathfrak{T}_{2n-1,\bullet,k-2}$ for $3 \le k \le 2n-1$. The second row of $\mathrm{Mat}(\mathfrak{T}_{2n+1})$ is then equal to the sequence

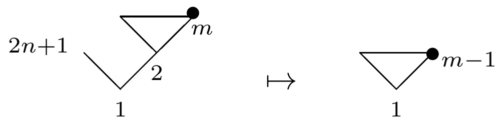
(6.1) $\quad 0, 0, \mathfrak{T}_{2n-1,\bullet,1}, \mathfrak{T}_{2n-1,\bullet,2}, \ldots, \mathfrak{T}_{2n-1,\bullet,2n-3}, \mathfrak{T}_{2n-1,\bullet,2n-2}(=0);$

which is also equal to

(6.2) $\quad 0, \mathfrak{T}_{2n-1,1,\bullet}(=0), \mathfrak{T}_{2n-1,2,\bullet}, \ldots, \mathfrak{T}_{2n-1,2n-2,\bullet}, 0,$

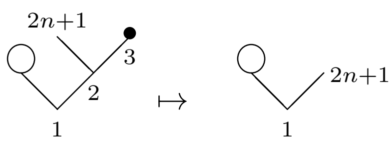
by Poupard's result (1.5) (also by our combinatorial proof in [FH13] mentioned in the Introduction).

(I 4) The set $\mathfrak{T}_{2n+1,m,1}$ is empty when $m = 1, 2$ or $2n$. When $3 \le m \le 2n-1$ the diagram

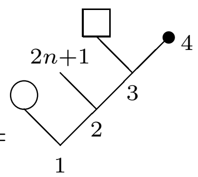


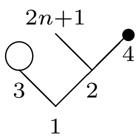
serves to illustrate the transformation that maps each tree from $\mathfrak{T}_{2n+1,m,1}$ onto a tree from $\mathfrak{T}_{2n-1,m-1,\bullet}$, by deleting the two nodes $(2n+1)$ and 1, and reducing the remaining nodes by 1. Thus, the first column of M_n is equal to sequence (6.2), when read from top to bottom.

For the second column we first note that $\mathfrak{T}_{2n+1,m,2}$ is empty when $m = 1, 2$ and $n \ge 2$. When $m = 3$, the mapping

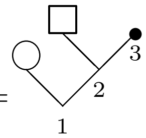
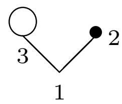


shows that $\mathfrak{T}_{2n+1,3,2}$ is in bijection with $\mathfrak{T}_{2n-1,\bullet,1} = \mathfrak{T}_{2n-1,2,\bullet}$. When $m = 4$, the following decomposition prevails

$\mathfrak{T}_{2n+1,4,2} =$ 

$+$ 

the two sets on the right-hand side being in bijection, respectively, with

$=$  $+$  ,

that is,

$$= \mathfrak{T}_{2n-1,3,\bullet} + \mathfrak{T}_{2n-1,2,\bullet}.$$

To prove the identity $\mathfrak{T}_{2n+1,m,2} = \mathfrak{T}_{2n-1,m-1,\bullet} + \mathfrak{T}_{2n-1,m-2,\bullet} = \mathfrak{T}_{2n+1,m,1} + \mathfrak{T}_{2n+1,m-1,1}$ (by (6.2) and $(I\,4)$) for $5 \le m \le 2n$ proceed by induction on m using relation $(R\,1)$ already proved in (4.3). Write

$$\begin{aligned}
\mathfrak{T}_{2n+1,m,2} &= 2\,\mathfrak{T}_{2n+1,m-1,2} - \mathfrak{T}_{2n+1,m-2,2} - 2\,\mathfrak{T}_{2n-1,m-2,2} && \text{[by (4.3)]}\\
2\,\mathfrak{T}_{2n+1,m-1,2} &= 2\,\mathfrak{T}_{2n+1,m-1,1} + 2\,\mathfrak{T}_{2n+1,m-2,1} && \text{[by induction on } m\text{]}\\
-\,\mathfrak{T}_{2n+1,m-2,2} &= -\mathfrak{T}_{2n+1,m-2,1} - \mathfrak{T}_{2n+1,m-3,1} && \text{[by induction on } m\text{]}\\
-\,2\,\mathfrak{T}_{2n-1,m-2,2} &= -2\,\mathfrak{T}_{2n-1,m-2,1} - 2\,\mathfrak{T}_{2n-1,m-3,1} && \\
& && \text{[by induction on } n \text{ and } m\text{]}\\
2\,\mathfrak{T}_{2n+1,m-1,1} - \mathfrak{T}_{2n+1,m-2,1} - 2\,\mathfrak{T}_{2n-1,m-2,1} &= \mathfrak{T}_{2n+1,m,1} && \text{[by (4.3)]}\\
2\,\mathfrak{T}_{2n+1,m-2,1} - \mathfrak{T}_{2n+1,m-3,1} - 2\,\mathfrak{T}_{2n-1,m-3,1} &= \mathfrak{T}_{2n+1,m-1,1}. && \text{[by (4.3)]}
\end{aligned}$$

By summing those six equations we get $\mathfrak{T}_{2n+1,m,2} = \mathfrak{T}_{2n+1,m,1} + \mathfrak{T}_{2n+1,m-1,1}$, also equal to $\mathfrak{T}_{2n-1,m-1,\bullet} + \mathfrak{T}_{2n-1,m-2,\bullet}$ by (6.2). ∎

Remark. It would be interesting to make up a proof of Property 6.1 that would have no recourse to a recurrence argument for $m \ge 5$ as above.

7. Proofs of Theorems 1.1 and 1.2

Taking Theorem 2.1, Corollaries 4.2 and 5.2, Property 6.1 into account we conclude that the sequence of matrices $\mathrm{Mat}(\mathfrak{T}_{2n+1})$ is both a Delta and a Gamma sequence. Those sequences are then identical and we may write

$$f_n(m,k) = \mathfrak{T}_{2n+1,m,k} \tag{7.1}$$

for all n, m, k. This completes the proof of Theorem 1.1.

We now exploit the properties of the strictly ordered binary trees to prove that the matrices M_n are symmetric with respect to their counter-diagonals (Theorem 1.2). First, the symmetry property is banal for M_1, M_2. For $n \ge 3$ consider the NE- and SW-corners

$$\begin{pmatrix} f_n(1,2n-1) & f_n(1,2n) \\ f_n(2,2n-1) & f_n(2,n) \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ f_n(2,2n-1) & 0 \end{pmatrix}$$

$$\begin{pmatrix} f_n(2n-1,1) & f_n(2n-1,2) \\ f_n(2n,1) & f_n(2n,2) \end{pmatrix} = \begin{pmatrix} f_n(2n-1,1) & f_n(2n-1,2) \\ 0 & f_n(2n,2) \end{pmatrix}$$

of the matrix M_n. As $f_n(2n-1,1) = f_n(2n,2) = \mathfrak{T}_{2n-3} = T_{2n-3}/2^{n-2}$ (by combining (1.11), ($I\,2$), ($I\,3$), (6.1) and (6.2)), both corners are symmetric with respect to their counter-diagonals [in short, counter-symmetric].

Let us prove that the *upper part* of the matrix M_n is counter-symmetric and for $i = 1, 2, \ldots, n-1$ adopt the notation:

$$\begin{aligned}\mathrm{Row}_i &= \{(i,i+1),(i,i+2),\ldots,(i,2n-i),(i,2n-i+1)\};\\ \mathrm{Col}_{2n-i+1} &= \{(i,2n-i+1),(i+1,2n-i+1),\\ &\qquad \ldots,(2n-i-1,2n-i+1),(2n-i,2n-i+1)\}.\end{aligned}$$

Note that Row_i and Col_{2n-i+1} have the cell $(i, 2n-i+1)$, belonging to the counter-diagonal, in common. There is nothing to prove for the pairs (m,k) along the counter-diagonal and also for the entries from Row_1 and Col_{2n}, which are all zero.

Let (m_0, k_0) belong to Row_j for some j such that $2 \le j \le n-1$. Further, assume that (1.11) holds for all $(m,k) \in \mathrm{Row}_1 \cup \cdots \cup \mathrm{Row}_{j-1}$ and all $(m,k) \in \mathrm{Row}_j$ lying on the right of (m_0,k_0), not including (m_0,k_0), that is, $m = j$ and $k > k_0$. By symmetry, (1.11) also holds for all $(m,k) \in \mathrm{Col}_{2n} \cup \cdots \cup \mathrm{Col}_{2n-j+2}$ and all $(m,k) \in \mathrm{Col}_{2n-j+1}$ lying above $(2n+1-k_0, 2n+1-m_0)$ not including the latter pair.

Now, the following relations hold:

$$\begin{aligned}&f_n(m_0,k_0) = 2f_n(m_0,k_0+1) - f_n(m_0,k_0+2) - 2\,f_{n-1}(m_0,k_0) \quad \text{[by } (R\,2)\text{]}\\ &f_{n-1}(m_0,k_0) = f_{n-1}(2n-1-k_0, 2n-1-m_0), \qquad \text{[by induction on } n\text{]}\\ &f_n(m_0,k_0+1) = f_n(2n-k_0, 2n+1-m_0),\\ &f_n(m_0,k_0+2) = f_n(2n-1-k_0, 2n+1-m_0). \text{[both by the local induction]}\end{aligned}$$

Therefore,

$$\begin{aligned}&f_n(m_0,k_0) - 2\,f_n(2n-k_0, 2n+1-m_0)\\ &+ f_n(2n-1-k_0, 2n+1-m_0) + 2\,f_{n-1}(2n-1-k_0, 2n-1-m_0) = 0.\end{aligned}$$

But by $(R\,3)$ written at $(m,k) = (2n-1-k_0, 2n+1-m_0)$ we have:

$$\begin{aligned}&f_n(2n-1-k_0, 2n+1-m_0) - 2\,f_n(2n-k_0, 2n+1-m_0)\\ &\quad - f_n(2n-k_0+1, 2n+1-m_0) + 2\,f_{n-1}(2n-1-k_0, 2n-1-m_0) = 0.\end{aligned}$$

By comparing the last two equations we conclude that

$$f_n(m_0,k_0) = f_n(2n-k_0+1, 2n+1-m_0),$$

which means that (1.11) now holds for $(m,k) = (m_0,k_0)$.

For the entries of M_n lying *below the diagonal* we proceed in the same manner and adopt the notation:

$$\begin{aligned}\mathrm{Row}_{2n+1-i} &= \{(2n+1-i,i),\ldots,(2n+1-i,2n-i)\};\\ \mathrm{Col}_i &= \{(i+1,i),(i+2,i),\ldots,(2n+1-i,i)\};\end{aligned}$$

for $i=1,2,\ldots,n-1$. Again, Row_{2n+1-i} and Col_i have the cell $(2n+1-i,i)$ in common.

Let (m_0,k_0) belong to Col_j for some j such that $1\le j\le n-1$. Further, assume that (1.11) holds for all $(m,k)\in \mathrm{Col}_1\cup\cdots\cup\mathrm{Col}_{j-1}$ and all $(m,k)\in\mathrm{Col}_j$ lying below (m_0,k_0), not including (m_0,k_0), that is, $m>m_0$ and $k=k_0$. By symmetry, (1.11) also holds for all $(m,k)\in \mathrm{Row}_{2n}\cup\cdots\cup\mathrm{Row}_{2n-j+2}$ and all $(m,k)\in\mathrm{Row}_{2n+1-j}$ lying to the left of $(2n+1-k_0,2n+1-m_0)$ not including the latter pair.

Now, the following relations hold:

$$\begin{aligned} f_n(m_0,k_0)&=2f_n(m_0+1,k_0)-f_n(m_0+2,k_0)-2\,f_{n-1}(m_0,k_0),\\ &\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad [\text{by } (R\,1)],\\ f_{n-1}(m_0,k_0)&=f_{n-1}(2n-1-k_0,2n-1-m_0),\\ &\qquad\qquad\qquad\qquad\qquad\qquad\qquad [\text{by induction on } n],\\ f_n(m_0+1,k_0)&=f_n(2n+1-k_0,2n-m_0),\\ f_n(m_0+2,k_0)&=f_n(2n+1-k_0,2n-1-m_0),\\ &\qquad\qquad\qquad\qquad\qquad\qquad [\text{by the local induction}].\end{aligned}$$

Therefore,

$$\begin{aligned} f_n(m_0,k_0)&=2\,f_n(2n+1-k_0,2n-m_0)\\ &\quad-f_n(2n+1-k_0,2n-1-m_0)-2\,f_{n-1}(2n-1-k_0,2n-1-m_0).\end{aligned}$$

But by $(R\,4)$ written at $(m,k)=(2n+1-k_0,2n-1-m_0)$ we have:

$$\begin{aligned} f_n(2n+1-k_0,2n-1-m_0)&=2\,f_n(2n+1-k_0,2n-m_0)\\ &\quad-f_n(2n+1-k_0,2n+1-m_0)-2\,f_{n-1}(2n-1-k_0,2n-1-m_0).\end{aligned}$$

By comparing the last two equations we conclude that

$$f_{2n+1}(m_0,k_0)=f_{2n+1}(2n+1-k_0,2n+1-m_0)),$$

which means that (1.11) now holds for $(m,k)=(m_0,k_0)$. ∎

Define

$$\mathrm{Eoc}(t):=\mathrm{eoc}(t),\quad \text{but}\quad \mathrm{Pom}(t):=2n+1-\mathrm{pom}(t). \tag{7.2}$$

Theorem 7.2. *Let* $\mathfrak{T}_{2n+1}(x,y):=\sum_{t\in\mathfrak{T}_{2n+1}}x^{\mathrm{Eoc}(t)}y^{\mathrm{Pom}(t)}$ *be the generating polynomial for the set* $\mathfrak{T}_{2n+1}$ *by the pair of statistics* (Eoc, Pom). *Then,*

$$\mathfrak{T}_{2n+1}(x,y)=\mathfrak{T}_{2n+1}(y,x). \tag{7.3}$$

Proof. This is a simple consequence of Theorem 1.2: let $g_n(m,k) := \#\{\mathrm{Eoc}=m, \mathrm{Pom}=k\}$. Then,

$$\begin{aligned} g_n(m,k) &= \#\{\mathrm{eoc}=m, \mathrm{pom}=2n+1-k\} = f_n(m, 2n+1-k) \\ &= f_n(k, 2n+1-m) \qquad\qquad\qquad [\text{by } (1.11)] \\ &= g_n(k,m). \end{aligned}$$

Thus,

$$\begin{aligned} \mathfrak{T}_{2n+1}(x,y) &= \sum_{t\in\mathfrak{T}_{2n+1}} x^{\mathrm{Eoc}(t)} y^{\mathrm{Pom}(t)} = \sum_{m,k} g_n(m,k) x^m y^k \\ &= \sum_{m,k} g_n(k,m) x^m y^k = \mathfrak{T}_{2n+1}(y,x). \quad \square \end{aligned}$$

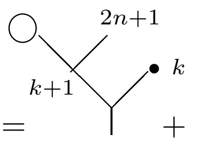
8. Further properties

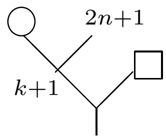
Other properties of the Delta Sequence can be obtained by having a further look at the geometry of the strictly ordered binary trees. The sub- and superdiagonals of the matrices M_n for $n = 2,3,4,5$ are equal, as can be seen in Fig. 2.2. For an arbitrary $n \ge 2$ we have the following.

Property 8.1. *Sub- and super diagonals are equal:*

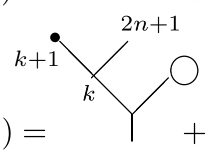
(8.1) $\qquad f_n(k+1,k) = f_n(k,k+1) \qquad (n \ge 2;\ 1 \le k \le 2n-1).$

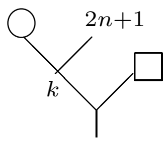
Proof. First, note that k and $(k+1)$ can be siblings in a tree from $\mathfrak{T}_{2n+1,k,k+1}$, but never in a tree from $\mathfrak{T}_{2n+1,k+1,k}$. Second, k can be parent of $(k+1)$ in a tree from the latter set, but never in a tree from the former one. Also, $f_n(2,1) = f_n(1,2) = 0$ for $n \ge 2$ and for $k \ge 2$ we have the decompositions:

$f_n(k,k+1) =$ 

$+ \; [$ 

$, \bullet\, k\ ];$

$f_n(k+1,k) =$ 

$+ \; [$ 

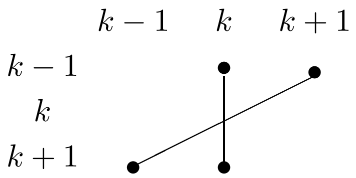
$, \bullet\, (k+1)\ ].$

The first terms in the previous two equations are in bijection, as well as the second ones, the notation "$\bullet\, k$" meaning that k is the end of the minimal chain, following our convention on Tree Calculus (*cf.* Section 4). $\square$

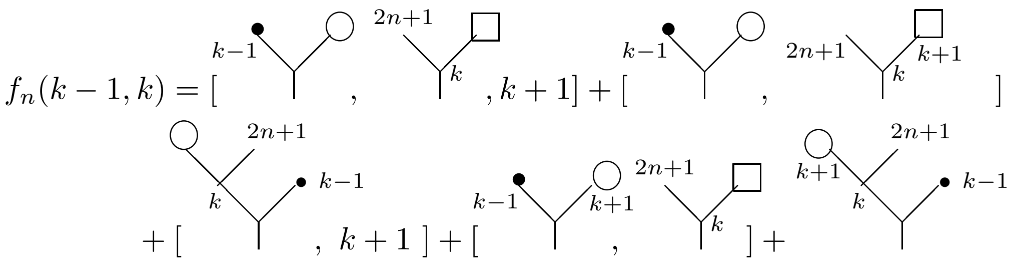
Property 8.2. *We have the crossing equalities:*

$$\begin{aligned} (8.2) \quad f_n(k+1,k-1) + f_n(k-1,k+1) & \\ = f_n(k+1,k) + f_n(k-1,k) & \\ = f_n(k,k+1) + f_n(k,k-1) & \quad (2 \le k \le 2n-1). \end{aligned}$$

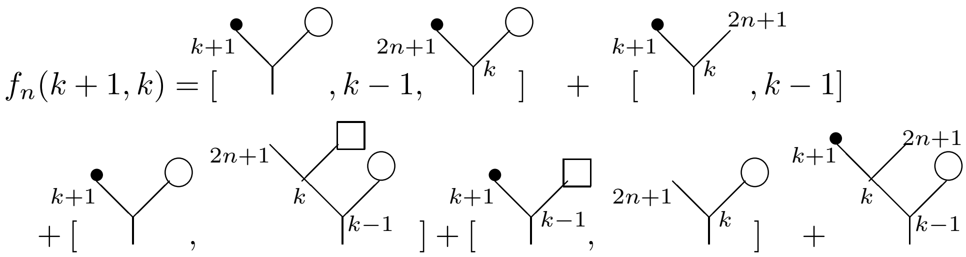
As can be seen in Fig. 2.2, the involved entries in the first identities are located on the four bullets drawn in the following diagram.



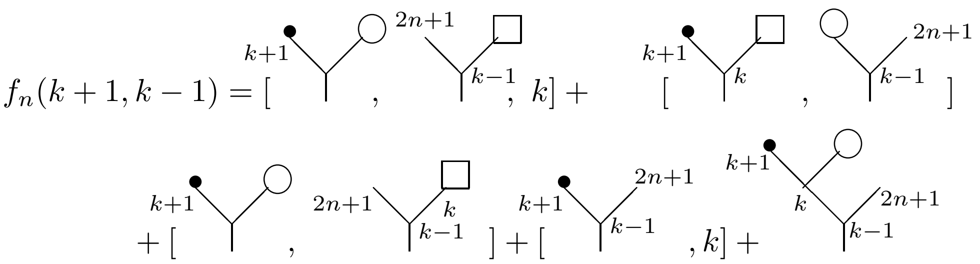
Proof. Let i, j be two different integers from the set $\{(k-1), k, (k+1)\}$. Say that i and j are *connected* in a tree t, if the tree contains the edge i—j, or if i and j are brothers and one of them is the end of the minimal chain of t. Each of the four ingredients of the previous identity is now decomposed into five terms, depending on whether the nodes $(k-1)$, k, $(k+1)$ are connected or not, namely: no connectedness; only $k, (k+1)$ connected; $(k-1), k$ connected; $(k-1), (k+1)$ connected; all connected. Thus,



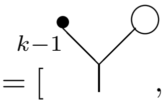
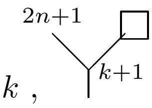
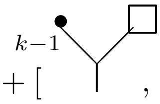
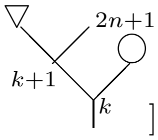
$:= A_1 + A_2 + A_3 + A_4 + A_5;$

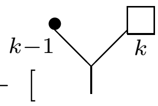
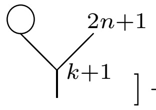
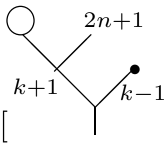
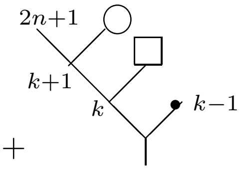


$:= B_1 + B_2 + B_3 + B_4 + B_5;$



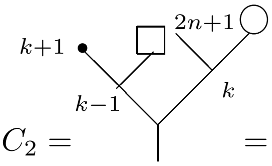
$:= C_1 + C_2 + C_3 + C_4 + C_5;$

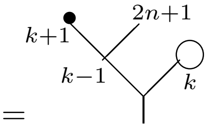
$f_n(k-1, k+1) = [$  $,\ k\ ,$  $] + [$  $,$  $]$

$+\ [$  $,$  $] + [$  $, k\ ] +$ 

$:= D_1 + D_2 + D_3 + D_4 + D_5.$

Now, the following identities hold: $A_1 = C_1$, $A_2 = C_3$, $A_3 = D_4$, $A_4 = D_3$, $2\,A_5 = D_5$; $B_1 = D_1$, $B_3 = D_2$, $B_5 = C_5$. Moreover,

$B_4 - C_2 =$  $= D_5 = 2\,A_5;$

$C_4 - B_2 =$  $= A_5.$

Altogether, $\sum_i (A_i + B_i) - \sum_i (C_i + D_i) = (B_4 - C_2) - (C_4 - B_2) + (A_5 - D_5) = 2\,A_5 - A_5 + (A_5 - 2\,A_5) = 0.$ $\square$

Property 8.3. *The row sums $f_n(m, \bullet)$ form a Poupard Triangle, the initial conditions being: $f_0(1, \bullet) = 1$, $f_n(1, \bullet) = 0$ and $f_n(2, \bullet) = 2\sum_m f_{n-1}(m, \bullet)$ $(n \geq 1)$; and the finite difference system:*

$$\Delta_m^2 f_n(m, \bullet) + 2\,f_{n-1}(m, \bullet) = 0 \quad (1 \leq m \leq 2n-1). \tag{8.3}$$

The column sums $f_n(\bullet, k)$ form a Poupard Triangle, the initial conditions being: $f_0(\bullet, 0) = 1$, $f_n(\bullet, 0) = 0$ and $f_n(\bullet, 1) = 2\sum_k f_{n-1}(\bullet, k)$ $(n \geq 1)$; and the finite difference system:

$$\Delta_k^2 f_n(\bullet, k) + 2\,f_{n-1}(\bullet, k) = 0 \quad (0 \leq k \leq 2n-2). \tag{8.4}$$

There are several proofs of this Property. First, the methods developed in Section 4 can be readapted by disregarding the conditions involving the pom-statistic. Here, we simply work out a specialization of the recurrence relations $(R1)$—$(R4)$, that makes use of the previous two properties. Besides, we only prove the first part of the property that deals with the row sums.

Proof. For $1 \le m \le 2n-2$ we have:

$$\begin{aligned}
\Delta_m^2 f_n(m,\bullet) &= \sum_{1\le k\le 2n} \Delta_m^2 f_n(m,k) \\
&= \sum_{1\le k\le m-1} \Delta_m^2 f_n(m,k) \\
&\qquad + \Delta_m^2 (f_n(m,m) + f_n(m,m+1) + f_n(m,m+2)) \\
&\qquad + \sum_{m+3\le k\le 2n} \Delta_m^2 f_n(m,k) \\
&= \sum_{1\le k\le m-1} -2 f_{n-1}(m,k) \\
&\qquad + f_n(m,m) - 2f_n(m+1,m) + f_n(m+2,m) \\
&\qquad + f_n(m,m+1) - 2f_n(m+1,m+1) + f_n(m+2,m+1) \\
&\qquad + f_n(m,m+2) - 2f_n(m+1,m+2) + f_n(m+2,m+2) \\
&\qquad + \sum_{m+3\le k\le 2n} -2 f_{n-1}(m,k-2).
\end{aligned}$$

In the previous sum the diagonal terms vanish. Also, $f_n(m+1,m) = f_n(m,m+1)$, $f_n(m+2,m+1) = f_n(m+1,m+2)$ by (8.1). The sum of the nine intermediate terms becomes: $-f_n(m+1,m) + f_n(m+2,m) - f_n(m+1,m+2) + f_n(m,m+2)$, which is 0 by (8.2). Hence,

$$\begin{aligned}
\Delta_m^2 f_n(m,\bullet) &= -2 \sum_{1\le k\le m-1} f_{n-1}(m,k) - 2 \sum_{m+1\le k\le 2n-2} f_{n-1}(m,k) \\
&= -2 f_{n-1}(m,\bullet).
\end{aligned}$$

On the other hand, by $(I\,2)$ and (1.10),

$$\begin{aligned}
f_n(2n-1,\bullet) &= \sum_{1\le k\le 2n} f_n(2n-1,k) = \sum_{1\le k\le 2n-2} (f_{n-1}(k,\bullet) + f_{n-1}(\bullet,k)) \\
&= 2\, f_{n-1}(\bullet,\bullet) := 2\,\#\mathfrak{T}_{2n-1} = 2\, T_{2n-1}/2^{n-1}; \\
f_n(2n,\bullet) &= \sum_{1\le k\le 2n} f_n(2n,k) = \sum_{1\le k\le 2n-2} f_{n-1}(k,\bullet) \\
&= f_{n-1}(\bullet,\bullet) = \#\mathfrak{T}_{2n-1} = T_{2n-1}/2^{n-1}; \\
f_n(2n+1,\bullet) &= 0.
\end{aligned}$$

The last three evaluations imply that $\Delta_m^2 f_n(2n-1,\bullet) = -2\, f_{n-1}(2n-1,\bullet) = 0$. ∎

9. Other equivalent definitions for the Delta sequence

Definitions 1.1 and 2.1 have been shown to be equivalent to characterize a Delta Sequence. Other combinations of the recurrence relations $(R\,1)$–$(R\,4)$, together with the initial conditions $(I\,1)$–$(I\,4)$, can be used. We describe them by means of squares and arrows, as was done in Fig. 2.1 and 2.3 (see Fig. 9.1 (a) and (b)). Moreover, further initial conditions can be introduced; they are denoted by (SW), (NE), as they refer only to the South-West and North-East corners of the matrices:

$$(SW)\quad \begin{pmatrix} f_n(2n-1,1) & f_n(2n-1,2) \\ f_n(2n,1) & f_n(2n,2) \end{pmatrix} = \begin{pmatrix} f_{n-1}(\bullet,1) & f_{n-1}(2,\bullet)+f_{n-1}(\bullet,2) \\ 0 & f_{n-1}(\bullet,1) \end{pmatrix};$$

$$(NE)\quad \begin{pmatrix} f_n(1,2n-1) & f_n(1,2n) \\ f_n(2,2n-1) & f_n(2,n) \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ f_{n-1}(2,\bullet) & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ f_{n-1}(2n-2,\bullet) & 0 \end{pmatrix}.$$

When one of those two conditions (SW), (NE) is involved, two recurrence relations among $(R\,1)$–$(R\,4)$ are needed to build up an equivalent definition. In Fig. 9.1 (c) for instance, $(R\,1)$ and $(R\,4)$ are to be associated with (SW). We then get five further equivalent definitions (Fig. 9.1 (c-g)):

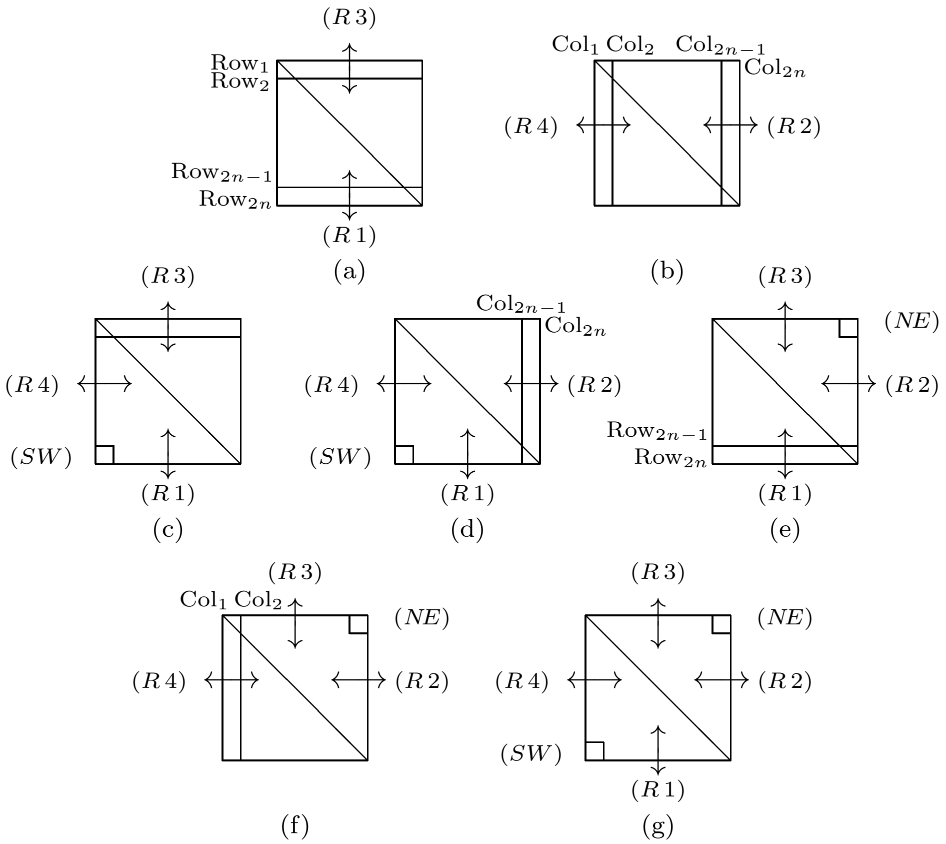


Fig. 9.1 Other definitions of the Delta sequence.

We do not reproduce any proofs for those equivalences, but point out the fact that our Tree Calculus requires that each initial condition be combinatorially interpreted, as was done in Sections 2 and 5.

10. Generating functions for the Delta sequence

1. *Poupard matrices.* Let $G = (g_{i,j})$ $(i \ge 0,\ j \ge 0)$ be an infinite matrix with nonnegative integral entries. Say that G is a *Poupard matrix*, if for every $i \ge 0$, $j \ge 0$ the following identity holds:

$$g_{i,j+2} - 2\,g_{i+1,j+1} + g_{i+2,j} + 2\,g_{i,j} = 0. \tag{10.1}$$

Let $G(x,y) := \sum\limits_{i\ge 0,\, j\ge 0} g_{i,j}\,(x^i/i)\,(y^j/j!)$; $\quad R_i(y) := \sum\limits_{j\ge 0} g_{i,j}\,(y^j/j!)$ $(i \ge 0)$; $C_j(x) := \sum\limits_{i\ge 0} g_{i,j}\,(x^i/i)$ $(j \ge 0)$ be the exponential generating functions for the matrix itself, its rows and columns, respectively.

Proposition 10.1. *The following four properties are equivalent:*
(i) $G = (g_{i,j})$ $(i \ge 0,\ j \ge 0)$ *is a Poupard matrix;*
(ii) $R_i''(y) - 2\,R_{i+1}'(y) + R_{i+2}(y) + 2\,R_i(y) = 0$ *for all* $i \ge 0$*;*
(iii) $C_j''(x) - 2\,C_{j+1}'(x) + C_{j+2}(x) + 2\,C_j(x)$ *for all* $j \ge 0$*;*
(iv) $G(x,y)$ *satisfies the partial differential equation:*

$$\frac{\partial^2 G(x,y)}{\partial x^2} - 2\,\frac{\partial^2 G(x,y)}{\partial x\,\partial y} + \frac{\partial^2 G(x,y)}{\partial y^2} + 2\,G(x,y) = 0. \tag{10.2}$$

Proof. It suffices to write $R_i''(y) - 2\,R_{i+1}'(y) + R_{i+2}(y) + 2\,R_i(y) = \sum\limits_{j\ge 0}(g_{i,j+2} - 2\,g_{i+1,j+1} + 2\,g_{i+2,j} - 2\,g_{i,j})(y^j/j!)$, and, similarly, $C_j''(x) - 2\,C_{j+1}'(x) + C_{j+2}(x) + 2\,C_j(x) = \sum\limits_{i\ge 0}(g_{i+2,j} - 2\,g_{i+1,j+1} + 2\,g_{i,j+2} - 2\,g_{i,j})(x^i/i!)$ to obtain the equivalence between the first three properties. As for the last one, simply note that $G(x,y) = \sum_{i\ge 0} R_i(y)\,x^i/i! = \sum\limits_{j\ge 0} C_j(x)\,y^j/j!$ and make the appropriate derivations. □

Proposition 10.2. *We have*

$$G(x,y) = A(x+y)\,\cos(\sqrt{2}\,y) + B(x+y)\,\sin(\sqrt{2}\,y), \tag{10.3}$$

where $A(x)$ *and* $B(x)$ *are two arbitrary series.*

Proof. Let $\xi := x+y$, $\eta := y$. Then, $\partial G/\partial x = \partial G/\partial \xi$; $\partial G/\partial y = \partial G/\partial \xi + \partial G/\partial \eta$; $\partial^2 G/\partial x^2 = \partial^2 G/\partial \xi^2$; $\partial^2 G/\partial y^2 = \partial^2 G/\partial \xi^2 + 2\,\partial^2 G/\partial \xi\,\partial \eta + \partial^2 G/\partial \eta^2$; $\partial^2 G/\partial x\,\partial y = \partial^2 G/\partial \xi^2 + \partial^2 G/\partial \xi\,\partial \eta$.

Thus, (10.2) can be rewritten as

$$\frac{\partial^2 G}{\partial \eta^2} + 2\,G = 0. \tag{10.4}$$

The ordinary differential equation $G'' + 2G = 0$, whose characteristic polynomial is $r^2+2=0$, has a general solution of the form $A\cos(\sqrt{2}\,\eta)+B\sin(\sqrt{2}\,\eta)$, so that the general solution of (10.2) is exactly given by (10.3). $\square$

The exact expression for the generating function $G(x,y)$ can then be derived, if $A(x+y)$ and $B(x+y)$ can be obtained by an independent calculation, as done in the sequel.

2. *A sequence of Poupard matrices for the lower triangles.* The entries $f_n(m,k)$ $(1\le k<m\le 2n)$ from the lower triangles in the matrices M_n $(n\ge 1)$ are now recorded as entries $\lambda_{i,j}^{(p)}$ $(p\ge 0,\ i\ge 0,\ j\ge 0)$ of infinite matrices $\Lambda^{(p)}=(\lambda_{i,j}^{(p)})$ $(i\ge 0,\ j\ge 0)$ as follows.

Define

$$\lambda_{i,j}^{(p)} := \begin{cases} 0, & \text{if } i+j\equiv p \bmod 2;\\ f_n(m,k), & \text{if } i+j\equiv p+1 \bmod 2;\end{cases} \tag{10.5}$$

with $k:=j+1$, $m:=i+j+2$, $2n:=p+(i+j)+1$. The latter equation makes sense, as $i+j$ and p are of different parity. The mapping $(p,i,j)\mapsto(n,m,k)$ is one-to-one, the reverse mapping being: $p:=2n-m+1$, $i:=m-k-1$, $j:=k-1$. Thus, for $p\ge 0$, the matrix $\Lambda^{(2p+1)}$ reads:

$$\begin{array}{c} \\ 0\\1\\2\\3\\4\\5\\6 \end{array}
\begin{array}{c}
\begin{array}{cccccc} 0 & 1 & 2 & 3 & 4 & 5 \end{array}\\
\begin{pmatrix}
f_{p+1}(2,1) & 0 & f_{p+2}(4,3) & 0 & f_{p+3}(6,5) & 0 & \cdots\\
0 & f_{p+2}(4,2) & 0 & f_{p+3}(6,4) & 0 & \cdots\\
f_{p+2}(4,1) & 0 & f_{p+3}(6,3) & 0 & \cdots\\
0 & f_{p+3}(6,2) & 0 & \cdots\\
f_{p+3}(6,1) & 0 & \cdots\\
0 & \cdots\\
\cdots
\end{pmatrix}
\end{array};$$

and for $p\ge 1$, the matrix $\Lambda^{(2p)}$ is equal to

$$\begin{array}{c} \\ 0\\1\\2\\3\\4\\5\\6 \end{array}
\begin{array}{c}
\begin{array}{cccccc} 0 & 1 & 2 & 3 & 4 & 5 \end{array}\\
\begin{pmatrix}
0 & f_{p+1}(3,2) & 0 & f_{p+2}(5,4) & 0 & f_{p+3}(7,6)\\
f_{p+1}(3,1) & 0 & f_{p+2}(5,3) & 0 & f_{p+3}(7,5) & \cdots\\
0 & f_{p+2}(5,2) & 0 & f_{p+3}(7,4) & \cdots\\
f_{p+2}(5,1) & 0 & f_{p+3}(7,3) & \cdots\\
0 & f_{p+3}(7,2) & \cdots\\
f_{p+3}(7,1) & \cdots\\
\cdots
\end{pmatrix}
\end{array}.$$

Proposition 10.3. *Every matrix $\Lambda^{(p)}$ $(p\ge 0)$ is a Poupard matrix.*

Proof. Using Definition (10.5) we have

$$\begin{aligned}
\lambda_{i,j+2}^{(p)} - 2\,\lambda_{i+1,j+1}^{(p)} + \lambda_{i+2,j}^{(p)} + 2\,\lambda_{i,j}^{(p)}
&= f_{n+1}(m+2,k+2) - 2\,f_{n+1}(m+2,k+1)\\
&\qquad + f_{n+1}(m+2,k) + 2\,f_n(m,k)\\
&= \underset{k}{\Delta}\, f_{n+1}(m+2,k) + 2\,f_n(m,k) = 0,
\end{aligned}$$

by rule $(R\,4)$. $\square$

3. *The first matrix* $\Lambda^{(1)}$. The counter-diagonal of $\Lambda^{(1)}$ at depth $2i$ $(i \ge 0)$ reads

$$\lambda^{(1)}_{2i,0} = f_{i+1}(2i+2,1);\quad \lambda^{(1)}_{2i-1,1} = f_{i+1}(2i+2,2);\quad \cdots$$
$$\cdots\ \lambda^{(1)}_{2i-j,j} = f_{i+1}(2i+2,j+1);\ \cdots\ \lambda^{(1)}_{0,2i} = f_{i+1}(2i+2,2i+1).$$

Those $(2i+1)$ terms are equal to the first $(2i+1)$ entries of the $(2i+2)$-nd row of the matrix M_{i+1}, that is, to the single term $f_1(2,1) = 1$ for $i = 0$ and for $i \ge 1$ to

$$f_i(1,\bullet) = 0,\ f_i(2,\bullet),\ \ldots,\ f_i(j+1,\bullet),\ \ldots,\ f_i(2i,\bullet),\ 0,$$

by virtue of $(I\,2)$. Thus, $\Lambda^{(1)}$ is identical to the Poupard matrix $(f_i(j+1,\bullet))$ $(i,j \ge 0)$:

$$\begin{array}{c} \\ 0\\1\\2\\3\\4\\5\\6 \end{array}
\begin{array}{c}
\begin{array}{ccccccc} 0 & 1 & 2 & 3 & 4 & 5 & 6 \end{array}\\
\begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & f_1(2,\bullet) & 0 & f_2(4,\bullet) & 0 & f_3(6,\bullet) & \\
0 & 0 & f_2(3,\bullet) & 0 & f_3(5,\bullet) & & \\
0 & f_2(2,\bullet) & 0 & f_3(4,\bullet) & & & \\
0 & 0 & f_3(3,\bullet) & & & & \\
0 & f_3(2,\bullet) & & & & & \\
0 & & & & & &
\end{pmatrix}
\end{array},$$

equal to

$$\begin{array}{c} \\ 0\\1\\2\\3\\4\\5\\6\\ \cdot \end{array}
\begin{array}{c}
\begin{array}{cccccccc} 0 & 1 & 2 & 3 & 4 & 5 & 6 & \cdots \end{array}\\
\begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots\\
0 & 1 & 0 & 1 & 0 & 4 & \cdots & \\
0 & 0 & 2 & 0 & 8 & \cdots & & \\
0 & 1 & 0 & 10 & \cdots & & & \\
0 & 0 & 8 & \cdots & & & & \\
0 & 4 & \cdots & & & & & \\
0 & \cdots & & & & & & \\
\cdots & & & & & & &
\end{pmatrix}
\end{array},$$

4. *The first two columns of* $\Lambda^{(p)}$. In the sequel the column labeled j of $\Lambda^{(p)}$ will be denoted by $\Lambda^{(p)}_{\bullet,j}$ and the exponential generating function for that column by $\Lambda^{(p)}_{\bullet,j}(x) = \sum_{i\ge 0} \lambda^{(p)}_{i,j} x^i/i!$. Also, $\Lambda^{(p)}(x,y) := \sum_{j\ge 0} \Lambda^{(p)}_{\bullet,j}(x) y^j/j!$ will be the double exponential generating function for the matrix $\Lambda^{(p)}$.

Proposition 10.4. *The first two columns of each matrix $\Lambda^{(p)}$ $(p\ge 2)$ are related to the columns of $\Lambda^{(1)}$ by the identities:*

$$\Lambda^{(p)}_{\bullet,0}(x)=\Lambda^{(1)}_{\bullet,p-1}(x),\qquad \Lambda^{(p)}_{\bullet,1}(x)=\frac{d}{dx}\Lambda^{(1)}_{\bullet,p-1}(x)+\Lambda^{(1)}_{\bullet,p}(x). \tag{10.6}$$

Proof. For the first identity it suffices to prove $\lambda^{(p)}_{i,0}=\lambda^{(1)}_{i,p-1}$, that is

$$f_n(i+2,1)=f_n(p+i+1,p)$$

when $2n=p+i+1$. This is true by Theorem 1.2. For the second identity it suffices to prove $\lambda^{(p)}_{i,1}=\lambda^{(1)}_{i+1,p-1}+\lambda^{(1)}_{i,p}$, that is

$$f_n(i+3,2)=f_n(p+i+2,p)+f_n(p+i+2,p+1) \tag{10.7}$$

when $2n=p+i+2$. But by $(I\,4)$

$$f_n(i+3,2)=f_n(i+2,1)+f_n(i+3,1),$$

so that by Theorem 1.2, identity (10.7) holds. ∎

5. *Solving the partial differential equation.* By (10.6)

$$\Lambda^{(p)}(x,y)\Big|_{\{y=0\}}=\Lambda^{(p)}_{\bullet,0}(x)=\Lambda^{(1)}_{\bullet,p-1}(x);$$
$$\frac{\partial}{\partial y}\Lambda^{(p)}(x,y)\Big|_{\{y=0\}}=\Lambda^{(p)}_{\bullet,1}(x)=\frac{d}{dx}\Lambda^{(1)}_{\bullet,p-1}(x)+\Lambda^{(1)}_{\bullet,p}(x).$$

As each $\Lambda^{(p)}$ is a Poupard matrix, we can use identity (10.3), so that $\Lambda^{(p)}(x,y)\Big|_{\{y=0\}}=A(x)$ and $\partial\Lambda^{(p)}(x,y)/\partial y=A'(x+y)\cos(\sqrt{2}y)-\sqrt{2}\,A(x+y)\sin(\sqrt{2}y)+B'(x+y)\sin(\sqrt{2}y)+\sqrt{2}\,B(x+y)\cos(\sqrt{2}y)$. Hence,

$$\Lambda^{(p)}(x,y)\Big|_{\{y=0\}}=A(x)=\Lambda^{(1)}_{\bullet,p-1}(x);$$
$$\frac{\partial\Lambda^{(p)}(x,y)}{\partial y}\Big|_{\{y=0\}}=A'(x)+\sqrt{2}\,B(x)=\frac{d}{dx}\Lambda^{(1)}_{\bullet,p-1}(x)+\Lambda^{(1)}_{\bullet,p}(x).$$

Consequently, $A(x)=\Lambda^{(1)}_{\bullet,p-1}(x)$ and $B(x)=\Lambda^{(1)}_{\bullet,p}(x)/\sqrt{2}$ and the general expression for $\Lambda^{p}(x,y)$ reads:

$$\Lambda^{(p)}(x,y)=\Lambda^{(1)}_{\bullet,p-1}(x+y)\cos(\sqrt{2}\,y)+\Lambda^{(1)}_{\bullet,p}(x+y)\sin(\sqrt{2}\,y)/\sqrt{2}. \tag{10.8}$$

This expression still holds for $p=1$. We know that $\Lambda^{(1)}_{\bullet,0}(x)=1$. On the other hand, the coefficient of $x^{2k+1}/(2k+1)!$ $(k\ge 0)$ in $\Lambda^{(1)}_{\bullet,1}(x)$ is $f_{k+1}(2,\bullet)=T_{2k+1}/2^k$. Hence, $\Lambda^{(1)}_{\bullet,1}(x)=\sqrt{2}\,\tan(x/\sqrt{2})$ and

$$\begin{aligned}\Lambda^{(1)}(x,y)&=\cos(\sqrt{2}\,y)+\sqrt{2}\,\tan\Bigl(\frac{x+y}{\sqrt{2}}\Bigr)\frac{\sin(\sqrt{2}\,y)}{\sqrt{2}}\\ &=\cos\Bigl(\frac{x-y}{\sqrt{2}}\Bigr)\Big/\cos\Bigl(\frac{x+y}{\sqrt{2}}\Bigr),\end{aligned} \tag{10.9}$$

a result already obtained by Poupard.

Remark. For getting the solution for $\Lambda^{(1)}(x,y)$ we can also start with the general expression displayed in (10.3) and calculate A and B with the initial conditions $\Lambda^{(1)}(0,y) = \Lambda^{(1)}(x,0) = 1$. We find

$$\begin{aligned}\Lambda^{(1)}(x,y) &= \cos(\sqrt{2}y) + \frac{1-\cos(\sqrt{2}(x+y))}{\sin(\sqrt{2}(x+y))}\sin(\sqrt{2}y)\\ &= \frac{\sin(\sqrt{2}x)+\sin(\sqrt{2}y)}{\sin(\sqrt{2}(x+y))},\end{aligned}\tag{10.10}$$

an expression which is naturally equal to the right-hand side of (10.9) (by a simple trigonometric calculation).

We have not worked out other explicit formulas for $\Lambda^{(p)}(x,y)$ when $p \geq 3$, but only derived the exponential generating function for those series, as explained in the next subsection.

6. *A generating function for the lower triangles.* By (10.9)

$$\begin{aligned}\frac{\partial}{\partial y}\Lambda^{(1)}(x,y) &= \frac{\partial}{\partial y}\cos\Bigl(\frac{x-y}{\sqrt{2}}\Bigr)\Bigm/\cos\Bigl(\frac{x+y}{\sqrt{2}}\Bigr)\\ &= \sin(\sqrt{2}x)\Bigm/\sqrt{2}\,\cos^2\Bigl(\frac{x+y}{\sqrt{2}}\Bigr).\end{aligned}\tag{10.11}$$

Let $\Lambda(x,y,z) := \sum_{p\geq 1}\Lambda^{(p)}(x,y)\dfrac{z^{p-1}}{(p-1)!}$.

Then,

$$\begin{aligned}\Lambda(x,y,z) &= \cos(\sqrt{2}\,y)\sum_{p\geq 1}\frac{z^{p-1}}{(p-1)!}\Lambda^{(1)}_{\bullet,p-1}(x+y)\\ &\qquad + \frac{\sin(\sqrt{2}\,y)}{\sqrt{2}}\sum_{p\geq 1}\frac{z^{p-1}}{(p-1)!}\Lambda^{(1)}_{\bullet,p}(x+y) \qquad \text{[by (10.8)]}\\ &= \cos(\sqrt{2}\,y)\,\Lambda^{(1)}(x+y,z) + \frac{\sin(\sqrt{2}\,y)}{\sqrt{2}}\frac{\partial}{\partial z}\Lambda^{(1)}(x+y,z),\end{aligned}$$

since $\Lambda^{(1)}(x,z) = \sum_{p\geq 0}\dfrac{z^p}{p!}\Lambda^{(1)}_{\bullet,p}(x)$.

By (10.11) we then get:

$$\begin{aligned}\Lambda(x,y,z) &= \cos(\sqrt{2}\,y)\,\cos\Bigl(\frac{x+y-z}{\sqrt{2}}\Bigr)\Bigm/\cos\Bigl(\frac{x+y+z}{\sqrt{2}}\Bigr)\\ &\qquad + \sin(\sqrt{2}\,y)\sin(\sqrt{2}(x+y))\Bigm/2\,\cos^2\Bigl(\frac{x+y+z}{\sqrt{2}}\Bigr)\\ &= \frac{\cos(\sqrt{2}\,x)+\cos(\sqrt{2}\,y)\,\cos(\sqrt{2}\,z)}{2\,\cos^2\Bigl(\dfrac{x+y+z}{\sqrt{2}}\Bigr)}.\end{aligned}$$

Now, express $\Lambda(x,y,z)$ as a series in the $f_n(m,k)$'s. By definition,

$$\Lambda(x,y,z)=\sum_{p,i,j}\lambda_{i,j}^{(p)}\frac{z^{p-1}}{(p-1)!}\frac{x^i}{i!}\frac{y^j}{j!}\quad (p\ge 1,\ i\ge 0,\ j\ge 0);$$

so that by (10.5)

$$\Lambda(x,y,z)=\sum_{k,m,n} f_n(m,k)\frac{x^{m-k-1}}{(m-k-1)!}\frac{y^{k-1}}{(k-1)!}\frac{z^{2n-m}}{(2n-m)!},$$

the latter sum over the set $\{2\le k+1\le m\le 2n\}$. This achieves the proof of Theorem 1.4.

Note that the right-hand side of (1.12) is symmetric in y,z, that is, $\Lambda(x,y,z)=\Lambda(x,z,y)$. The change $y\leftrightarrow z$ in the left-hand side of (1.12) shows that

$$f_n(2n+1-k,2n+1-m)=f_n(m,k),$$

the symmetry proved for the entries $f_n(m,k)$ such that $m\ge k+1$.

Remark. Let $z=0$ in (1.14). We get

$$\sum_{2\le k+1\le 2n} f_n(2n,k)\frac{x^{2n-k-1}}{(2n-k-1)!}\frac{y^{k-1}}{(k-1)!}=\frac{\cos(\sqrt{2}\,x)+\cos(\sqrt{2}\,y)}{2\,\cos^2\Bigl(\dfrac{x+y}{\sqrt{2}}\Bigr)};$$

or,

$$\sum_{i\ge 0,\,j\ge 0}\lambda_{i,j}^{(1)}\frac{x^i}{i!}\frac{y^j}{j!}=\Lambda^{(1)}(x,y)=\frac{\cos(\sqrt{2}\,x)+\cos(\sqrt{2}\,y)}{2\,\cos^2\Bigl(\dfrac{x+y}{\sqrt{2}}\Bigr)};$$

which is another expression for $\Lambda^{(1)}(x,y)$ than (10.9) and (10.10).

7. *A generating function for the upper triangles.* The entries $f_n(m,k)$ $(1\le m<k\le 2n)$ from the upper triangles in the matrices M_n $(n\ge 1)$ are next recorded as entries $\omega_{i,j}^{(p)}$ $(p\ge 0,\ i\ge 0,\ j\ge 0)$ of infinite matrices $\Omega^{(p)}=(\omega_{i,j}^{(p)})$ $(i\ge 0,\ j\ge 0)$ as follows.

Define

$$\omega_{i,j}^{(p)}:=\begin{cases}0, & \text{if } i+j\not\equiv p \bmod 2;\\ f_n(m,k), & \text{if } i+j\equiv p \bmod 2;\end{cases}$$

with $m:=p+1$, $k:=p+j+2$, $2n:=p+i+j+2$. Conversely, $i:=2n-k$, $j:=k-m-1$, $p:=m-1$. Thus, for $p\ge 0$

$$\Omega^{(2p+1)} =$$

$$\begin{array}{cc} & \begin{array}{cccc} 0 & 1 & 2 & 3 \end{array} \\ \begin{array}{c} 0\\1\\2\\3\\4 \end{array} & \begin{pmatrix} 0 & f_{p+2}(2p+2,2p+4) & 0 & \cdots \\ f_{p+2}(2p+2,2p+3) & 0 & f_{p+4}(2p+2,2p+5) & \cdots \\ 0 & f_{p+3}(2p+2,2p+4) & \cdots & \\ f_{p+3}(2p+2,2p+3) & \cdots & & \\ \cdots & & & \end{pmatrix} \end{array}$$

$$\Omega^{(2p)} =$$

$$\begin{array}{cc} & \begin{array}{cccc} 0 & 1 & 2 & 3 \end{array} \\ \begin{array}{c} 0\\1\\2\\3\\4 \end{array} & \begin{pmatrix} f_{p+1}(2p+1,2p+2) & 0 & f_{p+2}(2p+1,2p+4) & 0 \\ 0 & f_{p+2}(2p+1,2p+3) & 0 & \cdots \\ f_{p+2}(2p+1,2p+2) & 0 & \cdots & \\ 0 & \cdots & & \\ \cdots & & & \end{pmatrix} \end{array}$$

Remark. The first rows of all matrices M_n are null, so that $\Omega^{(0)}$ is the infinite matrix with all entries equal to zero!

Also, write
$$\Omega^{(1)} =$$

$$\begin{array}{cc} & \begin{array}{cccccccc} 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 \end{array} \\ \begin{array}{c} 0\\1\\2\\3\\4\\5\\6\\7 \end{array} & \begin{pmatrix} 0 & f_2(2,4) & 0 & f_3(2,6) & 0 & f_4(2,8) & 0 & f_5(2,10) \\ f_2(2,3) & 0 & f_3(2,5) & 0 & f_4(2,7) & 0 & f_5(2,9) & \cdots \\ 0 & f_3(2,4) & 0 & f_4(2,6) & 0 & f_5(2,8) & \cdots & \\ f_3(2,3) & 0 & f_4(2,5) & 0 & f_5(2,7) & \cdots & & \\ 0 & f_4(2,4) & 0 & f_5(2,6) & \cdots & & & \\ f_4(2,3) & 0 & f_5(2,5) & \cdots & & & & \\ 0 & f_5(2,4) & \cdots & & & & & \\ f_5(2,3) & \cdots & & & & & & \end{pmatrix} \end{array}$$

$$= \begin{array}{cc} & \begin{array}{cccccccc} 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 \end{array} \\ \begin{array}{c} 0\\1\\2\\3\\4\\5\\6\\7 \end{array} & \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 4 & 0 & 34 & \cdots \\ 0 & 2 & 0 & 8 & 0 & 68 & \cdots & \\ 1 & 0 & 10 & 0 & 94 & \cdots & & \\ 0 & 8 & 0 & 104 & \cdots & & & \\ 4 & 0 & 94 & \cdots & & & & \\ 0 & 68 & \cdots & & & & & \\ 34 & \cdots & & & & & & \end{pmatrix} \end{array}.$$

Proposition 10.5. *Every matrix $\Omega^{(p)}$ $(p \geq 0)$ is a Poupard matrix.*

Same proof as for Proposition 10.3.

The row labeled i of $\Omega^{(p)}$ will be denoted by $\Omega^{(p)}_{i,\bullet}$ and the exponential generating function for that row by $\Omega^{(p)}_{i,\bullet}(y) = \sum_{j\ge 0}\omega^{(p)}_{i,j}y^j/j!$. Also, $\Omega^{(p)}(x,y) := \sum_{i\ge 0}\Omega^{(p)}_{i,\bullet}(y)x^i/i!$ will be the double exponential generating function for the matrix $\Omega^{(p)}$. As x and y play a symmetric role in (10.2), the solution in (10.2) may also be written

$$G(x,y) = A(x+y)\,\cos(\sqrt{2}\,x) + B(x+y)\,\sin(\sqrt{2}\,x),$$

so that the generating function of each matrix $\Omega^{(p)}$ is of the form

$$\Omega^{(p)}(x,y) = A(x+y)\,\cos(\sqrt{2}\,x) + B(x+y)\,\sin(\sqrt{2}\,x).$$

As the first row of each matrix $\Omega^{(p)}$ is the zero sequence, we have $\Omega^{(p)}(0,y) = A(y) = 0$. Hence, $\Omega^{(p)}(x,y) = B(x+y)\,\sin(\sqrt{2}\,x)$ and

$$\frac{\partial}{\partial x}\Omega^{(p)}(x,y) = \Big(\frac{\partial}{\partial x}B(x+y)\Big)\,\sin(\sqrt{2}x) + \sqrt{2}\;B(x+y)\,\cos(\sqrt{2}x).$$

Therefore,

$$\Omega^{(p)}_{1,\bullet}(y) = \frac{\partial}{\partial x}\Omega^{(p)}(x,y)|\,\Big|_{\{x=0\}} = \sqrt{2}\,B(y)$$

and then

$$\Omega^{(p)}(x,y) = \frac{1}{\sqrt{2}}\sin(\sqrt{2}\,x)\,\Omega^{(p)}_{1,\bullet}(x+y). \tag{10.12}$$

The evaluation of $\Omega^{(1)}_{1,\bullet}(y)$ is easy, as the row labeled 1 of $\Omega^{(1)}$ is $(1,0,1,0,4,0,34,0,\ldots)$, compared with $(0,1,0,1,0,4,0,34,0,\ldots)$, which is the sequence of the coefficients of the Taylor expansion of $\sqrt{2}\,\tan(y/\sqrt{2})$. In fact we have $\omega^{(1)}_{1,2j} = f_{j+2}(2,2j+3) = f_{j+1}(2j+2,\bullet) = T_{2j+1}/2^j$. Thus,

$$\Omega^{(1)}_{1,\bullet}(y) = \frac{d}{dy}\,\sqrt{2}\,\tan\Big(\frac{y}{\sqrt{2}}\Big) = \frac{1}{\cos^2(y/\sqrt{2})};$$

so that

$$\Omega^{(1)}(x,y) = \frac{1}{\sqrt{2}}\sin(\sqrt{2}\,x)\frac{1}{\cos^2\Big(\dfrac{x+y}{\sqrt{2}}\Big)}. \tag{10.13}$$

Proposition 10.6. *For all $p\ge 1$ we have:*

$$\Omega^{(p)}_{1,\bullet}(y) = \Omega^{(1)}_{p,\bullet}(y). \tag{10.14}$$

Same proof as for Proposition 10.4. Now, define:

$$\Omega(x,y,z) := \sum_{p\ge 1}\Omega^{(p)}(x,y)\frac{z^p}{p!} \tag{10.15}$$

and make use of (10.18)—(10.20):

$$\begin{aligned}\Omega(x,y,z) &= \frac{1}{\sqrt{2}}\sin(\sqrt{2}\,x)\sum_{p\ge 1}\Omega^{(p)}_{1,\bullet}(x+y)\frac{z^p}{p!}\\ &= \frac{1}{\sqrt{2}}\sin(\sqrt{2}\,x)\sum_{p\ge 1}\Omega^{(1)}_{p,\bullet}(x+y)\frac{z^p}{p!}\\ &= \frac{1}{\sqrt{2}}\sin(\sqrt{2}\,x)\,\Omega^{(1)}(z,x+y)\\ &= \sin(\sqrt{2}\,x)\,\sin(\sqrt{2}\,z)\,\frac{1}{2\,\cos^2\Bigl(\dfrac{x+y+z}{\sqrt{2}}\Bigr)}.\end{aligned}$$

As all the entries $\omega^{(p)}_{0,j}$ $(p\ge 1,\ j\ge 0)$ are null,

$$\Omega(x,y,z) = \sum_{p,i,j}\omega^{(p)}_{i,j}\frac{z^p}{p!}\frac{x^i}{i!}\frac{y^j}{j!}\quad (p\ge 1,\ i\ge 1,\ j\ge 0);$$

so that by definition of the $\omega^{(p)}_{i,j}$'s

$$\Omega(x,y,z) = \sum_{k,m,n} f_n(m,k)\frac{x^{2n-k}}{(2n-k)!}\frac{y^{k-m-1}}{(k-m-1)!}\frac{z^{m-1}}{(m-1)!}, \tag{10.16}$$

the latter sum over the set $\{2\le m+1\le k\le 2n-1\}$. This achieves the proof of Theorem 1.4.

The right-hand side of (1.13) is symmetric in x,z, that is, $\Omega(x,y,z)=\Omega(z,y,x)$. The change $x\leftrightarrow z$ in the left-hand side of (1.13) shows that

$$f_n(2n+1-k,2n+1-m)=f_n(m,k),$$

the symmetry proved for the entries $f_n(m,k)$ such that $m+1\le k$.

Recently, there has been a revival of studies on arithmetical and combinatorial properties of both tangent and secant numbers. Désiré André's classical model of alternating permutations has been largely used ([KPP94, St10]), in particular by taking the *Entringer recurrence* (see [En66, Po82, Po97, GHZ10]) into account, also by looking at their quadrant marked mesh patterns [KR12]. As developed in the present paper, the model of ordered binary tree, with its natural statistics "eoc" and "pom," gives rise to other refinements of those trigonometric numbers, as was also done in [FH14].

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