June 4, 2014

Seidel Triangle Sequences and Bi-Entringer Numbers

Dominique Foata and Guo-Niu Han

En hommage à Pierre Rosenstiehl,
Lui, qui dirige avec grand style,
Ce journal de combinatoire,
Mais sait aussi à l'occasion
Nous raconter une belle histoire:
Fil d'Ariane et boustrophédon.

Abstract. This Seidel Triangle Sequence Calculus makes it possible to derive several three-variate generating functions, in particular for the Bi-Entringer numbers, which count the alternating permutations according to their lengths, first and last letters. The paper has been motivated by this suprising observation: the number of alternating permutations, whose last letter has a prescribed value and is greater than its first letter, is equal to the Poupard number.

1. Introduction

As they have been reinterpreted in our previous paper [FH13a], the *Poupard numbers* $g_n(k)$ and $h_n(k)$ for $n \ge 1, 1 \le k \le 2n-1$ can be defined as the coefficients in the following expansions

$$(1.1)\quad 1+\sum_{n\ge1}\sum_{1\le k\le 2n+1} g_{n+1}(k)\frac{x^{2n+1-k}}{(2n+1-k)!}\frac{y^{k-1}}{(k-1)!}=\frac{\cos(x-y)}{\cos(x+y)};$$

$$(1.2)\quad 1+\sum_{n\ge1}\sum_{1\le k\le 2n+1} h_{n+1}(k)\frac{x^{2n+1-k}}{(2n+1-k)!}\frac{y^{k-1}}{(k-1)!}=\frac{\cos(x-y)}{\cos^2(x+y)}.$$

They are refinements of the tangent and secant numbers

$$(1.3)\qquad \sum_k g_n(k)=T_{2n-1}\quad (n\ge1),$$

$$(1.4)\qquad \sum_k h_n(k)=E_{2n}\quad (n\ge1),$$

which are themselves the coefficients of the Taylor expansions of $\tan u$ and $\sec u$:

$$\tan u=\sum_{n\ge1}\frac{u^{2n-1}}{(2n-1)!}T_{2n-1}=\frac{u}{1!}1+\frac{u^3}{3!}2+\frac{u^5}{5!}16+\frac{u^7}{7!}272+\frac{u^9}{9!}7936+\cdots$$

$$\sec u=\sum_{n\ge0}\frac{u^{2n}}{(2n)!}E_{2n}=1+\frac{u^2}{2!}1+\frac{u^4}{4!}5+\frac{u^6}{6!}61+\frac{u^8}{8!}1385+\frac{u^{10}}{10!}50521+\cdots$$

(See, e.g., [Ni23, p. 177-178], [Co74, p. 258-259]).

Key words and phrases. Entringer numbers, Poupard numbers, Bi-Entringer numbers, tangent numbers, secant numbers, alternating permutations, Seidel matrices, Seidel triangle sequences, greater neighbor statistic.

Mathematics Subject Classifications. 05A15, 05A30, 11B68, 33B10.

Several combinatorial models have been introduced to interpret the Poupard numbers: see [Po89, FH13a, FH13b, FH13c]. The first numerical values of those numbers are displayed in Fig. 1.1-2.

$k=$	1	2	3	4	5	6	7	Sum
$n=1$	1							1
2	0	2	0					2
3	0	4	8	4	0			16
4	0	32	64	80	64	32	0	272

Fig. 1.1. The Poupard Numbers $g_n(k)$.

$k=$	1	2	3	4	5	6	7	Sum
$n=1$	1							1
2	1	3	1					5
3	5	15	21	15	5			61
4	61	183	285	327	285	183	61	1385

Fig. 1.2. The Poupard Numbers $h_n(k)$.

According to Désiré André [An1879, An1881] each permutation $w = x_1x_2\cdots x_n$ of $12\cdots n$ is said to be *(increasing) alternating* if $x_1 < x_2$, $x_2 > x_3$, $x_3 < x_4$, etc. in an alternating way. Let Alt_n be the set of all alternating permutations of $12\cdots n$. He then proved that $\#\,\mathrm{Alt}_n = T_n$ (resp. $= E_n$), if n is odd (resp. even). Let $\mathbf{F}\,w := x_1$ and $\mathbf{L}\,w := x_n$ be the *first* and *last* letters of a permutation $w = x_1x_2\cdots x_n$ of $12\cdots n$.

The numbers $E_n(m) := \#\{w \in \mathrm{Alt}_n : \mathbf{F}\,w = m\}$, now called *Entringer numbers*, were introduced by Entringer himself [En66], who derived their main combinatorial and arithmetical properties. Those numbers are registered as the $A008282$ sequence in Sloane's On-Line Encyclopedia of Integer Sequences, together with an abundant bibliography [Sl]. They naturally constitute another refinement of the tangent and secant numbers. Their first values are shown in Fig. 1.3.

$m=$	1	2	3	4	5	6	Sum
$n=1$	1						1
2	1						1
3	1	1					2
4	2	2	1				5
5	5	5	4	2			16
6	16	16	14	10	5		61
7	61	61	56	46	32	16	272

Fig. 1.3. The Entringer Numbers $E_n(m)$.

We have been led to introduce the *Bi-Entringer numbers*, defined by

$$E_n(m,k) := \#\{w \in \mathrm{Alt}_n : \mathbf{F}\, w = m,\ \mathbf{L}\, w = k\}, \tag{1.5}$$

first, to see whether we could obtain a closed form for their generating function, second, to understand why, and prove that, over the set Alt_{2n-1} and given the event $\{\mathbf{F} < \mathbf{L}\}$, the conditional probability that $\mathbf{F} = k$ is equal to $g_n(k)/T_{2n-1}$, where $g_n(k)$ is the Poupard number defined in (1.1). In Section 5 we shall give two proofs of the latter statement (see Theorem 1.2), a combinatorial one and also an analytic one using the Laplace transform.

Now, to derive the generating function for the Bi-Entringer numbers a study of the so-called *Seidel Triangle Sequences* is to be made and will be developed in Section 2. Roughly speaking, Seidel's memoir [Se1877], as was superbly reactivated by Dumont [Du82], establishes a connection between several sequences of classical numbers and polynomials, by means of a finite difference calculus displayed in matrix form. The method is to be enlarged when dealing with *sequences* of matrices instead of sequences of numbers. It will be seen that with each Seidel Triangle Sequence can be associated an explicit form for its generating function (Theorem 2.2).

The Bi-Entringer numbers, displayed as entries of matrices $M_n := (E_n(m,k))_{1\le m,k\le n}$ (see Fig. 1.4) give rise to four Seidel Triangle Sequences: the sequences of the upper (resp. lower) triangles of the matrices M_n, for n odd and for n even. It will be shown that each matrix M_n for n odd is symmetric with respect to its diagonal, so that it suffices, when n is odd, to give the expression of the generating function for the upper triangles, as stated in the next theorem.

Theorem 1.1. *The generating functions for the coefficients $E_n(m,k)$ are given by*

$$\sum_{1\le m+1\le k\le 2n-1} E_{2n}(m+1,k+1)\frac{x^{2n-k-1}}{(2n-k-1)!}\frac{y^{k-m-1}}{(k-m-1)!}\frac{z^m}{m!} = \frac{\cos x\cos z}{\cos(x+y+z)}; \tag{1.6}$$

$$\sum_{1\le m+1\le k\le 2n-1} E_{2n}(k+1,m+1)\frac{x^{2n-k-1}}{(2n-k-1)!}\frac{y^{k-m-1}}{(k-m-1)!}\frac{z^m}{m!} = \frac{\sin x\sin z}{\cos(x+y+z)}; \tag{1.7}$$

$$\sum_{1\le m+1\le k\le 2n} E_{2n+1}(m+1,k+1)\frac{x^{2n-k}}{(2n-k)!}\frac{y^{k-m-1}}{(k-m-1)!}\frac{z^m}{m!} = \frac{\sin x\cos z}{\cos(x+y+z)}. \tag{1.8}$$

Theorem 1.1 will be proved in Section 3, once the main arithmetical properties of the Bi-Entringer numbers are given. The first values of the Bi-Entringer numbers are displayed in Fig. 1.4, as entries of the matrices $M_n := (E_n(m,k))_{1\le m,k\le n}$.

$$M_1 = \begin{array}{cc} & 1 \\ 1 & \begin{pmatrix} 1 \end{pmatrix} \end{array};\quad M_2 = \begin{array}{cc} & \begin{array}{cc} 1 & 2 \end{array} \\ \begin{array}{c} 1 \\ 2 \end{array} & \begin{pmatrix} . & 1 \\ . & . \end{pmatrix} \end{array};\quad M_3 = \begin{array}{cc} & \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \end{array} & \begin{pmatrix} . & 1 & . \\ 1 & . & . \\ . & . & . \end{pmatrix} \end{array};$$

$$M_4 = \begin{array}{cc} & \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \end{array} & \begin{pmatrix} . & 0 & 1 & 1 \\ . & . & 1 & 1 \\ . & 1 & . & 0 \\ . & . & . & . \end{pmatrix} \end{array};\quad M_5 = \begin{array}{cc} & \begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array} & \begin{pmatrix} . & 2 & 2 & 1 & . \\ 2 & . & 2 & 1 & . \\ 2 & 2 & . & 0 & . \\ 1 & 1 & 0 & . & . \\ . & . & . & . & . \end{pmatrix} \end{array};$$

$$M_6 = \begin{pmatrix} . & 0 & 2 & 4 & 5 & 5 \\ . & . & 2 & 4 & 5 & 5 \\ . & 2 & . & 4 & 4 & 4 \\ . & 2 & 4 & . & 2 & 2 \\ . & 1 & 2 & 2 & . & 0 \\ . & . & . & . & . & . \end{pmatrix};\quad M_7 = \begin{pmatrix} . & 16 & 16 & 14 & 10 & 5 & . \\ 16 & . & 16 & 14 & 10 & 5 & . \\ 16 & 16 & . & 12 & 8 & 4 & . \\ 14 & 14 & 12 & . & 4 & 2 & . \\ 10 & 10 & 8 & 4 & . & 0 & . \\ 5 & 5 & 4 & 2 & 0 & . & . \\ . & . & . & . & . & . & . \end{pmatrix};$$

$$M_8 = \begin{pmatrix} . & 0 & 16 & 32 & 46 & 56 & 61 & 61 \\ . & . & 16 & 32 & 46 & 56 & 61 & 61 \\ . & 16 & . & 32 & 44 & 52 & 56 & 56 \\ . & 16 & 32 & . & 40 & 44 & 46 & 46 \\ . & 14 & 28 & 40 & . & 32 & 32 & 32 \\ . & 10 & 20 & 28 & 32 & . & 16 & 16 \\ . & 5 & 10 & 14 & 16 & 16 & . & 0 \\ . & . & . & . & . & . & . & . \end{pmatrix};$$

Fig. 1.4. The Bi-Entringer Numbers $E_n(m,k)$.

Further arithmetical properties of the Bi-Entringer numbers, in particular involving binomial coefficients, will be given in Section 4. There is also a *linear* connection between Poupard and Bi-Entringer numbers, as stated in the next theorem, which is proved in Section 5.

Theorem 1.2. *For $2 \le k \le 2n$ we have:*

$$2\sum_{m=1}^{k} E_{2n+1}(m,k) = g_{n+1}(k). \tag{1.9}$$

The specialization of identities (1.6) and (1.8) for $z = 0$ provides an expression for the generating function for the *Entringer numbers* $E_n(m)$ themselves, apparently nowhere obtained, to our knowledge. The calculation is banal: just note that $E_{2n}(1, k+1) = E_{2n-1}(2n-k)$ and $E_{2n+1}(1, k+1) = E_{2n}(k)$.

Corollary 1.3. *We have*

$$\sum_{1\le k\le 2n-1} E_{2n-1}(k)\frac{x^k}{(k-1)!}\frac{y^{2n-k-1}}{(2n-k-1)!} = \frac{\cos x}{\cos(x+y)}; \tag{1.10}$$

$$\sum_{1\le k\le 2n} E_{2n}(k)\frac{x^{2n-k}}{(2n-k)!}\frac{y^{k-1}}{(k-1)!} = \frac{\sin x}{\cos(x+y)}. \tag{1.11}$$

Although the Poupard numbers $h_n(k)$, defined in (1.2), will not be further considered in this paper, it was important to mention that the pairs $(g_n(k), h_n(k))$ and $(E_{2n-1}(k), E_{2n}(k))$ form two refinements of the pairs (T_{2n-1}, E_{2n}) having analogous generating function displayed in (1.1), (1.2), (1.10), (1.11).

2. Seidel Triangle Sequences

Throughout the paper the following exponential generating functions will be attached to each infinite matrix $A = (a(m,k))_{m,k\ge 0}$

$$A(x,y) := \sum_{m,k\ge 0} a(m,k)\frac{x^m}{m!}\frac{y^k}{k!};$$

$$A_{m,\bullet}(y) := \sum_{k\ge 0} a(m,k)\frac{y^k}{k!}; \qquad A_{\bullet,k}(x) := \sum_{m\ge 0} a(m,k)\frac{x^m}{m!};$$

for A itself, its m-th row, its k-th column. As can be found in [Du82], a *Seidel matrix* $A = (a(m,k))$ $(m, k \ge 0)$ is defined to be an infinite matrix, whose entries belong to some ring, and obey the following rules:

(SM1) the sequence of the entries from the top row $a(0,0)$, $a(0,1)$, $a(0,2), \ldots$ is given; it is called the *initial sequence*;

(SM2) for $m \ge 1$ and $k \ge 0$ the following relation holds:

$$a(m,k) = a(m-1,k) + a(m-1,k+1).$$

The entries of the Seidel matrix A can be obtained by applying rule (SM2) inductively, starting with the initial sequence. The leftmost column $a(0,0), a(1,0), a(2,0), \ldots$ is called the *final sequence*. As stated in the next proposition, the exponential generating functions for the final sequence $A_{\bullet,0}(x)$ and for the Seidel matrix itself $A(x,y)$ can be derived from the generating function $A_{0,\bullet}(y)$ for the initial sequence. See, e.g., [Du82, DV80].

Proposition 2.1. *Let $A=(a_{i,j})$ $(i,j\ge 0)$ be a Seidel matrix. Then,*

$$A_{\bullet,0}(x)=e^x A_{0,\bullet}(x) \quad \text{and} \quad A(x,y)=e^x A_{0,\bullet}(x+y).$$

As noted by Dumont [Du82], the following example of a Seidel matrix, denoted by $\overline{H}=(\overline{h}_{i,j})_{i,j\ge0}$, goes back to Seidel himself [Se1877]. The initial sequence consists of the sequence of the coefficients of the Taylor expansion of $1-\tanh u=2/(1+e^{2u})$, that is, 1, -1, 0, 2, 0, -16, 0, 272, 0, ... so that

$$\begin{aligned}\overline{H}_{0,\bullet}(y)=1-\tanh y&=1+\sum_{n\ge1}\frac{y^{2n-1}}{(2n-1)!}(-1)^n T_{2n-1}\\ &=1-\frac{y}{1!}1+\frac{y^3}{3!}2-\frac{y^5}{5!}16+\frac{y^7}{7!}272-\frac{y^9}{9!}7936+\cdots\end{aligned}$$

It follows from Proposition 2.1 that

$$\overline{H}_{\bullet,0}(x)=\frac{1}{\cosh x}=\frac{2\,e^x}{1+e^{2x}};\quad \overline{H}(x,y)=\frac{2\,e^x}{1+e^{2x+2y}}; \tag{2.1}$$

and the matrix $\overline{H}$ itself reads:
(2.2)

$$\overline{H}=\begin{pmatrix}1&-1&0&2&0&-16&0&272&0&\cdots\\ 0&-1&2&2&-16&-16&272&272&&\\ -1&1&4&-14&-32&256&544&&&\\ 0&5&-10&-46&224&800&&&&\\ 5&-5&-56&178&1024&&&&&\\ 0&-61&122&1202&&&&&&\\ -61&61&1324&&&&&&&\\ 0&1385&&&&&&&&\\ 1385&&&&&&&&&\\ \vdots&&&&&&&&&\end{pmatrix}.$$

The Entringer numbers $E_n(m)$ mentioned in the introduction appear as entries of the matrix $\overline{H}$, displayed along the counter-diagonals with a given sign. In fact, we have the relation

$$\overline{h}_{i,j}=\begin{cases}(-1)^n\,E_{i+j+1}(j+1), & \text{if } i+j=2n;\\ (-1)^n\,E_{i+j+1}(i+1), & \text{if } i+j=2n-1;\end{cases} \tag{2.3}$$

as can be verified by induction, or still

$$E_{2n+1}(j+1)=(-1)^n\,\overline{h}_{2n-j,j}\quad(0\le j\le 2n); \tag{2.4}$$

$$E_{2n}(i+1)=(-1)^n\,\overline{h}_{i,2n-1-i}\quad(0\le i\le 2n-1). \tag{2.5}$$

The matrix $\overline{H}$ will be given a key role in Section 3.

We now come to the main definition of this section. A sequence of square matrices (A_n) $(n \geq 1)$ is called a *Seidel triangle sequence* if the following three conditions are fulfilled:

(STS1) each matrix A_n is of dimension n;

(STS2) each matrix A_n has null entries along and below its diagonal; let $(a_n(m,k)$ $(0 \leq m < k \leq n-1)$ denote its entries strictly above its diagonal, so that

$$A_1 = (\,\cdot\,)\,; \quad A_2 = \begin{pmatrix} \cdot & a_2(0,1) \\ \cdot & \cdot \end{pmatrix}; \quad A_3 = \begin{pmatrix} \cdot & a_3(0,1) & a_3(0,2) \\ \cdot & \cdot & a_3(1,2) \\ \cdot & \cdot & \cdot \end{pmatrix}; \; \ldots \;;$$

$$A_n = \begin{pmatrix} \cdot & a_n(0,1) & a_n(0,2) & \cdots & a_n(0,n-2) & a_n(0,n-1) \\ \cdot & \cdot & a_n(1,2) & \cdots & a_n(1,n-2) & a_n(1,n-1) \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \cdot & \cdot & \cdot & \cdots & a_n(n-3,n-2) & a_n(n-3,n-1) \\ \cdot & \cdot & \cdot & \cdots & \cdot & a_n(n-2,n-1) \\ \cdot & \cdot & \cdot & \cdots & \cdot & \cdot \end{pmatrix};$$

the dots "$\cdot$" along and below the diagonal referring to null entries.

(STS3) for each $n \geq 2$, the following relation holds:

$$a_n(m,k) - a_n(m,k+1) = a_{n-1}(m,k) \quad (m < k).$$

Record the last columns of the triangles A_2, A_3, A_4, A_5, ... , read from top to bottom, namely, $a_2(0,1)$; $a_3(0,2)$, $a_3(1,2)$; $a_4(0,3)$, $a_4(1,3)$, $a_4(2,3)$; $a_5(0,4)$, $a_5(1,4)$, $a_5(2,4)$, $a_5(3,4)$; ... as counter-diagonals of an infinite matrix $H = (h_{i,j})_{i,j \geq 0}$, as shown next:

$$H := \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \\ 4 \\ \\ \end{array} \!\!\begin{array}{c} \begin{array}{cccccc} \;\;0\;\; & \;\;\;\;1\;\;\;\; & \;\;\;\;2\;\;\;\; & \;\;\;\;3\;\;\;\; & \;\;\;\;4\;\;\;\; & \end{array} \\ \begin{pmatrix} a_2(0,1) & a_3(1,2) & a_4(2,3) & a_5(3,4) & a_6(4,5) & \cdots \\ a_3(0,2) & a_4(1,3) & a_5(2,4) & a_6(3,5) & & \\ a_4(0,3) & a_5(1,4) & a_6(2,5) & & & \\ a_5(0,4) & a_6(1,5) & & & & \\ a_6(0,5) & & & & & \\ \vdots & & & & & \end{pmatrix} \end{array}, \tag{2.6}$$

In an equivalent manner, the entries of H are defined by:

$$h_{i,j} = a_{i+j+2}(j, i+j+1). \tag{2.7}$$

The next theorem shows that the three-variable generating function for a Seidel triangle sequence, when suitably normalized, can be expressed in a very closed form.

Theorem 2.2. *The three-variable generating function for the Seidel triangle sequence* $(A_n = (a_n(m,k)))_{n\ge 1}$ *is equal to*

$$\sum_{1\le m+1\le k\le n-1} a_n(m,k)\frac{x^{n-k-1}}{(n-k-1)!}\frac{y^{k-m-1}}{(k-m-1)!}\frac{z^m}{m!} = e^x H(x+y,z),$$

where H is the infinite matrix defined in (2.7).

Proof. We set up a sequence of infinite matrices $(\Omega^{(p)} = ((\omega_{i,j}^{(p)})_{i,j\ge 0}))$ $(p\ge 0)$ that record the *rows* of the matrices A_n in the following manner

$$\Omega^{(p)} = \begin{pmatrix} a_{p+2}(p,p+1) & a_{p+3}(p,p+2) & a_{p+4}(p,p+3) & \cdots \\ a_{p+3}(p,p+1) & a_{p+4}(p,p+2) & & \\ a_{p+4}(p,p+1) & & & \\ \vdots & & & \end{pmatrix},$$

so that the rows labeled p of the triangles A_n, if they exist, are displayed as counter-diagonals in $\Omega^{(p)}$. Alternatively, the coefficients $\omega_{i,j}^{(p)}$ are defined by

$$\omega_{i,j}^{(p)} = a_{p+i+j+2}(p,p+j+1). \tag{2.8}$$

By (2.7) and (2.8) $H(x,z) = \sum_{p\ge 0}\frac{z^p}{p!}H_{\bullet,p}(x) = \sum_{p\ge 0}\frac{z^p}{p!}\Omega_{0,\bullet}^{(p)}(x)$. From rule (STS3) we get $a_{p+k}(p,p+m) - a_{p+k}(p,p+m+1) = a_{p+k-1}(m,p+m)$, so that each matrix $\Omega^{(p)}$ is a Seidel matrix. It follows by Proposition 2.1 that

$$\Omega^{(p)}(x,y) = e^x\Omega_{0,\bullet}^{(p)}(x+y).$$

Define: $\Omega(x,y,z) := \sum_{p\ge 0}\frac{z^p}{p!}\Omega^{(p)}(x,y)$. Then,

$$\Omega(x,y,z) = \sum_{p\ge 0}\frac{z^p}{p!}\Omega^{(p)}(x,y) = \sum_{p\ge 0}\frac{z^p}{p!}e^x\Omega_{0,\bullet}^{(p)}(x+y) = e^x H(x+y,z).$$

On the other hand,

$$\begin{aligned}\Omega(x,y,z) = \sum_{p\ge 0}\frac{z^p}{p!}\Omega^{(p)}(x,y) &= \sum_{i,j,p\ge 0}\frac{z^p}{p!}\frac{x^i}{i!}\frac{y^j}{j!}\omega_{i,j}^{(p)} \\ &= \sum_{i,j,p\ge 0}\frac{z^p}{p!}\frac{x^i}{i!}\frac{y^j}{j!}a_{p+i+j+2}(p,p+j+1).\end{aligned}$$

With the change of variables $p+i+j+2=n$, $p=m$, $p+j+1=k$, we then get

$$\Omega(x,y,z)=\sum_{1\le m+1\le k\le n-1} a_n(m,k)\frac{x^{n-k-1}}{(n-k-1)!}\frac{y^{k-m-1}}{(k-m-1)!}\frac{z^m}{m!}.$$

This completes the proof of Theorem 2.2. $\square$

3. The Bi-Entringer numbers

Before proving Theorem 1.1 we give a list of properties involving the Bi-Entringer numbers. The celebrated identities *à la boustrophedon* (see [MSY96], and [Ro13, p. 95-101] for a more literary approach) satisfied by the Entringer numbers $E_n(m)$ can be extended over to the Bi-Entringer numbers $E_n(m,k)$, as stated in relations (3.1)—(3.4) below.

Proposition 3.1. *We have:*
(i) $E_n(m,m)=0$ *for all* m *and* $E_n(n,k)=0$ *for all* k *and* $n\ge 2$;
(ii) for n *odd and* $1\le k,m\le n$

$$E_n(m,k)=\sum_{j=k}^{n-1}E_{n-1}(m,j)\quad \textit{if } m<k; \tag{3.1}$$

$$E_n(m,k)=\sum_{j=k}^{n-1}E_{n-1}(m-1,j)\quad \textit{if } m>k. \tag{3.2}$$

In particular, $E_n(n-2,n-1)=0$ *for* $n\ge 5$; $E_n(m,n)=0$ *for all* n,m.

(iii) For n *even and* $1\le m,k\le n$ *we further have:*

$$E_n(m,k)=\sum_{j=1}^{k-1}E_{n-1}(m,j)\quad \textit{if } m<k; \tag{3.3}$$

$$E_n(m,k)=\sum_{j=1}^{k-1}E_{n-1}(m-1,k)\quad \textit{if } m>k. \tag{3.4}$$

In particular, $E_n(n-1,n)=0$ *when* $n\ge 4$; $E_n(i,1)=0$ *for all* i.
(iv) Each matrix M_n *is symmetric with respect to its diagonal (resp. its counter-diagonal), whenever* n *is odd (resp. even), that is,*

$$E_n(m,k)=\begin{cases}E_n(k,m), & \textit{when } n \textit{ is odd,}\\ E_n(n+1-k,n+1-m), & \textit{when } n \textit{ is even.}\end{cases} \tag{3.5}$$

Moreover,

$$(3.6)\quad \sum_k E_n(m,k) = E_n(m) \quad (n \ge 1);$$

$$\sum_m E_n(m,k) = \begin{cases} E_n(k), & \textit{when } n \textit{ is odd;} \\ E_n(n+1-k), & \textit{when } n \textit{ is even.} \end{cases}$$

The proofs of all those properties are easy, by simple manipulations; in particular, (3.5) by using the basic dihedral transformations on alternating permutations. They are omitted.

Proposition 3.2 (The finite difference relations). *We have:*

$$(3.7)\quad E_n(m,k) - E_n(m,k+1) = (-1)^{n-1} E_{n-1}(m,k), \qquad \textit{if } 1 \le m < k \le n-1;$$

$$(3.8)\quad E_n(m,k) - E_n(m,k+1) = (-1)^{n-1} E_{n-1}(m-1,k), \qquad \textit{if } 2 \le k+1 < m \le n.$$

Proof. The two identities can be proved by simple iterations of (3.1)-(3.4). Alternatively, we can also proceed as follows. Let $m \,\mathrm{Alt}_n\, k$ (resp. $m \,\mathrm{Alt}_n\, l\, k$) designate the number of all σ from Alt_n starting with m and ending with k (resp. ending with the right factor $l\,k$). We have:

$$m \,\mathrm{Alt}_n\, k - m \,\mathrm{Alt}_n\, (k+1) = \begin{cases} -m \,\mathrm{Alt}_n\, k\,(k+1), & \text{if } n \text{ is even;} \\ m \,\mathrm{Alt}_n\, (k+1)\,k, & \text{if } n \text{ is odd.} \end{cases}$$

Next, if n is even,

$$-m \,\mathrm{Alt}_n\, k\,(k+1) = \begin{cases} -m \,\mathrm{Alt}_{n-1}\, k, & \text{if } 1 \le m < k \le n-1; \\ -(m-1)\,\mathrm{Alt}_{n-1}\, k, & \text{if } 2 \le k+1 < m \le n; \end{cases}$$

and if n is odd,

$$m \,\mathrm{Alt}_n\, (k+1)\,k = \begin{cases} m \,\mathrm{Alt}_{n-1}\, k, & \text{if } 1 \le m < k \le n-1; \\ (m-1)\,\mathrm{Alt}_{n-1}\, k, & \text{if } 2 \le k+1 < m \le n. \end{cases} \qquad \square$$

Now, let the sequence of matrices $(W_n) = (e_n(m,k))$ be obtained from the matrices $(M_n) = (E_n(m,k))$ by making the following modifications:

(W1) $W_1 := (0)$;

(W2) $W_n := M_n$ for $n \equiv 2, 3 \pmod 4$ and $n \ge 2$;

(W3) $W_n := (-1) M_n$ for $n \equiv 0, 1 \pmod 4$;

(W4) delete the lower triangle from each matrix W_n;

(W5) make the labels start from $0, 1, 2, \ldots$

In other words, for $m < k$ define the *normalized Bi-Entringer Numbers* $e_n(m,k)$ to be:

$$e_{2n}(m,k) := (-1)^{n+1} E_{2n}(m+1,k+1); \tag{3.9}$$

$$e_{2n+1}(m,k) := (-1)^{n+1} E_{2n+1}(m+1,k+1). \tag{3.10}$$

Their first values appear in Fig. 3.1.

$$W_1 = \begin{array}{c} \\ 0 \end{array}\!\!\begin{array}{c} 0 \\ (\,.\,) \end{array}; \quad W_2 = \begin{array}{c|cc} & 0 & 1 \\ \hline 0 & . & 1 \\ 1 & . & . \end{array}; \quad W_3 = \begin{array}{c} \\ 0 \\ 1 \\ 2 \end{array}\!\!\begin{array}{c} \begin{array}{ccc} 0 & 1 & 2 \end{array} \\ \begin{pmatrix} . & 1 & . \\ . & . & . \\ . & . & . \end{pmatrix} \end{array};$$

$$W_4 = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \end{array}\!\!\begin{array}{c} \begin{array}{cccc} 0 & 1 & 2 & 3 \end{array} \\ \begin{pmatrix} . & 0 & -1 & -1 \\ . & . & -1 & -1 \\ . & . & . & 0 \\ . & . & . & . \end{pmatrix} \end{array}; \quad W_5 = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \\ 4 \end{array}\!\!\begin{array}{c} \begin{array}{ccccc} 0 & 1 & 2 & 3 & 4 \end{array} \\ \begin{pmatrix} . & -2 & -2 & -1 & . \\ . & . & -2 & -1 & . \\ . & . & . & 0 & . \\ . & . & . & . & . \\ . & . & . & . & . \end{pmatrix} \end{array};$$

$$W_6 = \begin{pmatrix} . & 0 & 2 & 4 & 5 & 5 \\ . & . & 2 & 4 & 5 & 5 \\ . & . & . & 4 & 4 & 4 \\ . & . & . & . & 2 & 2 \\ . & . & . & . & . & 0 \\ . & . & . & . & . & . \end{pmatrix}; \quad W_7 = \begin{pmatrix} . & 16 & 16 & 14 & 10 & 5 & . \\ . & . & 16 & 14 & 10 & 5 & . \\ . & . & . & 12 & 8 & 4 & . \\ . & . & . & . & 4 & 2 & . \\ . & . & . & . & . & 0 & . \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \end{pmatrix}.$$

Fig. 3.1. The normalized Bi-Entringer Numbers $e_n(m,k)$.

By Proposition 3.2, the sequence (W_n) is a Seidel triangle sequence, and the corresponding matrix H, defined by (2.6)–(2.7), is equal to

$$H = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ \\ \end{array}\!\!\begin{array}{c} \begin{array}{cccccc} 0 & 1 & 2 & 3 & 4 & 5 \end{array} \\ \begin{pmatrix} e_2(0,1) & \cdot & e_4(2,3) & \cdot & e_6(4,5) & \cdot & \cdots \\ \cdot & e_4(1,3) & \cdot & e_6(3,5) & \cdot & & \\ e_4(0,3) & \cdot & e_6(2,5) & \cdot & & & \\ \cdot & e_6(1,5) & \cdot & & & & \\ e_6(0,5) & \cdot & & & & & \\ \cdot & & & & & & \\ \vdots & & & & & & \end{pmatrix} \end{array},$$

the "dots" being written in place of 0's. Note that all the counter-diagonals $e_{2n+1}(0,2n)$, $e_{2n+1}(1,2n)$, ... , $e_{2n+1}(2n-1,2n)$, which are equal to the last columns of the W_{2n+1}'s, have null entries.

It then remains to evaluate $e_{2n}(m,2n-1)$ for $0 \le m \le 2n-1$. By (3.9) we have: $e_{2n}(m,2n-1) = (-1)^{n+1} E_{2n}(m+1,2n)$. But $E_{2n}(m+1,2n)$ is equal to the Entringer number $E_{2n-1}(m+1)$, as each alternating permutation σ from Alt_{2n} such that $\mathbf{F}\,\sigma = m+1$ and $\mathbf{L}\,\sigma = 2n$ can be mapped onto a permutation from Alt_{2n-1} starting with $(m+1)$ by simply deleting the last letter $(2n)$, and this in a bijective manner.

Now, by (2.4), $E_{2n-1}(m+1) = E_{2(n-1)+1}(m+1) = (-1)^{n-1}\,\overline{h}_{2n-2-m,m}$ for $0 \le m \le 2n-2$. Altogether,

$$e_{2n}(m, 2n-1) = \overline{h}_{2n-2-m,m} \quad (0 \le m \le 2n-2), \tag{3.11}$$

that is, by (2.2),

$$H = \begin{pmatrix} 1 & \cdot & 0 & \cdot & 0 & \cdot & 0 & \cdot & \cdots \\ \cdot & -1 & \cdot & 2 & \cdot & -16 & \cdot & & \\ -1 & \cdot & 4 & \cdot & -32 & \cdot & & & \\ \cdot & 5 & \cdot & -46 & \cdot & & & & \\ 5 & \cdot & -56 & \cdot & & & & & \\ \cdot & -61 & \cdot & & & & & & \\ -61 & \cdot & & & & & & & \\ \cdot & & & & & & & & \\ \vdots & & & & & & & & \end{pmatrix}.$$

Thus, H is obtained from the matrix $\overline{H}$, displayed in (2.2), by replacing all the entries $\overline{h}_{i,j}$ such that $i+j$ is odd by zero. By (2.1) we have $\overline{H}(x,y) = \dfrac{2e^x}{1+e^{2x+2y}}$, so that

$$H(x,y) = \frac{\overline{H}(x,y) + \overline{H}(-x,-y)}{2} = e^x \frac{1+e^{2y}}{1+e^{2x+2y}}. \tag{3.12}$$

Let

$$\Omega(x,y,z) = \sum_{1 \le m+1 \le k \le n-1} e_n(m,k) \frac{x^{n-k-1}}{(n-k-1)!} \frac{y^{k-m-1}}{(k-m-1)!} \frac{z^m}{m!}. \tag{3.13}$$

be the three-variate generating function for the $e_n(m,k)$'s. By Theorem 2.2

$$\Omega(x,y,z) = e^x H(x+y, z) = e^{2x+y} \frac{1+e^{2z}}{1+e^{2x+2y+2z}}. \tag{3.14}$$

With $I := \sqrt{-1}$ equation (3.13) reads:

$$\Omega(xI, yI, zI) = \sum_{1 \le m+1 \le k \le n-1} I^{n-2} e_n(m,k) \frac{x^{n-k-1}}{(n-k-1)!} \frac{y^{k-m-1}}{(k-m-1)!} \frac{z^m}{m!}.$$

But, by (3.9) and (3.10)

$$\begin{gathered} I^{2n-2} e_{2n}(m,k) = E_{2n}(m+1, k+1); \\ I^{2n-1} e_{2n+1}(m,k) = I E_{2n+1}(m+1, k+1). \end{gathered}$$

Therefore,

$$\begin{aligned}\Omega(xI,yI,zI) = &\sum_{1\le m+1\le k\le 2n-1} E_{2n}(m+1,k+1)\frac{x^{2n-k-1}}{(2n-k-1)!}\frac{y^{k-m-1}}{(k-m-1)!}\frac{z^m}{m!}\\ &+ I\sum_{1\le m+1\le k\le 2n} E_{2n+1}(m+1,k+1)\frac{x^{2n-k}}{(2n-k)!}\frac{y^{k-m-1}}{(k-m-1)!}\frac{z^m}{m!}.\end{aligned}$$

Next, (3.14) becomes:

$$\begin{aligned}\Omega(xI,yI,zI) &= e^{2xI+yI}\frac{1+e^{2zI}}{1+e^{2xI+2yI+2zI}}\\ &= e^{xI}\frac{e^{-zI}+e^{zI}}{e^{-xI-yI-zI}+e^{xI+yI+zI}}\\ &= (\cos(x)\cos(z)+I\sin(x)\cos(z))/\cos(x+y+z).\end{aligned}$$

By comparing the above two identities for $\Omega(xI,yI,zI)$, we obtain (1.6) and (1.8) in Theorem 1.1.

To prove (1.7) in Theorem 1.1 let $(W'_n) = (e'_n(m,k))$ be the sequence of matrices obtained from $(M_n) = (E_n(m,k))$ by the following modifications:

(W′1) $W'_1 := (0)$;

(W′2) $W'_n := M^r_n$ for $n \equiv 1,2 \pmod 4$ and $n \ge 2$, where M^r_n is obtained from M_n by performing a rotation by 180^o about its center;

(W′3) $W'_n := (-1)M^r_n$ for $n \equiv 0,3 \pmod 4$;

(W′4) delete the lower triangle of each matrix W'_n;

(W′5) start labelling from $0,1,2,\ldots$

In other words, for $m<k$ define the *normalized Bi-Entringer numbers* $e'_n(m,k)$ to be:

$$e'_{2n}(m,k) = (-1)^{n+1}E_{2n}(2n-m,2n-k); \tag{3.15}$$

$$e'_{2n+1}(m,k) = (-1)^{n}E_{2n+1}(2n+1-m,2n+1-k). \tag{3.16}$$

The first values of the W'_n are shown in Fig. 3.2.

$$W'_1 = \begin{array}{c} 0 \\ 0\ \begin{pmatrix} . \end{pmatrix}\end{array};\quad W'_2 = \begin{array}{c|cc} & 0 & 1\\ \hline 0 & . & 0\\ 1 & . & .\end{array};\quad W'_3 = \begin{array}{c|ccc} & 0 & 1 & 2\\ \hline 0 & . & 0 & .\\ 1 & . & . & -1\\ 2 & . & . & .\end{array};$$

$$W'_4 = \begin{array}{c|cccc} & 0 & 1 & 2 & 3\\ \hline 0 & . & 0 & 0 & 0\\ 1 & . & . & -1 & 0\\ 2 & . & . & . & 0\\ 3 & . & . & . & .\end{array};\quad W'_5 = \begin{array}{c|ccccc} & 0 & 1 & 2 & 3 & 4\\ \hline 0 & . & . & . & . & .\\ 1 & . & . & 0 & 1 & 1\\ 2 & . & . & . & 2 & 2\\ 3 & . & . & . & . & 2\\ 4 & . & . & . & . & .\end{array};$$

$$W'_6=\begin{pmatrix} \cdot & 0 & 0 & 0 & 0 & 0\\ \cdot & \cdot & 2 & 2 & 1 & 0\\ \cdot & \cdot & \cdot & 4 & 2 & 0\\ \cdot & \cdot & \cdot & \cdot & 2 & 0\\ \cdot & \cdot & \cdot & \cdot & \cdot & 0\\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{pmatrix};\ W'_7=\begin{pmatrix} \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot\\ \cdot & \cdot & 0 & -2 & -4 & -5 & -5\\ \cdot & \cdot & \cdot & -4 & -8 & -10 & -10\\ \cdot & \cdot & \cdot & \cdot & -12 & -14 & -14\\ \cdot & \cdot & \cdot & \cdot & \cdot & -16 & -16\\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & -16 \end{pmatrix}.$$

Fig. 3.2. The normalized Bi-Entringer Numbers $e'_n(m,k)$.

By Proposition 3.2, (W'_n) is a Seidel triangle sequence, and the corresponding matrix H defined by (2.6)–(2.7), we shall denote by $H'=(h'_{i,j})$, is equal to:

$$H' = \begin{array}{c} \\ 0\\ 1\\ 2\\ 3\\ 4\\ 5\\ \\ \end{array} \begin{array}{c} \begin{array}{cccccc} 0 & 1 & 2 & 3 & 4 & 5 \end{array}\\ \begin{pmatrix} \cdot & e'_3(1,2) & \cdot & e'_5(3,4) & \cdot & e'_7(5,6) & \cdots\\ e'_3(0,2) & \cdot & e'_5(2,4) & \cdot & e'_7(4,6) & & \\ \cdot & e'_5(1,4) & \cdot & e'_7(3,6) & & & \\ e'_5(0,4) & \cdot & e'_7(2,6) & & & & \\ \cdot & e'_7(1,6) & & & & & \\ e'_7(0,6) & & & & & & \\ \vdots & & & & & & \end{pmatrix} \end{array}.$$

Its counter-diagonals $e'_{2n}(0,2n-1), e'_{2n}(1,2n-1),\ldots, e'_{2n}(2n-2,2n-1)$ have null entries, as it is so for the last columns of the W'_{2n}'s. Furthermore, $e'_{2n+1}(m,2n)=\overline{h}_{2n-m-1,m}$ $(0\le m\le 2n-1)$ by (2.5) and (3.16), so that

$$H' = \begin{pmatrix} \cdot & -1 & \cdot & 2 & \cdot & -16 & \cdot & \cdots\\ 0 & \cdot & 2 & \cdot & -16 & \cdot & & \\ \cdot & 1 & \cdot & -14 & \cdot & & & \\ 0 & \cdot & -10 & \cdot & & & & \\ \cdot & -5 & \cdot & & & & & \\ 0 & \cdot & & & & & & \\ \cdot & & & & & & & \\ \vdots & & & & & & & \end{pmatrix},$$

which is derived from $\overline{H}$ by replacing the entries $\overline{h}_{i,j}$ such that $i+j$ is even by 0. Therefore,

$$H'(x,y)=\frac{\overline{H}(x,y)-\overline{H}(-x,-y)}{2}=e^x\,\frac{1-e^{2y}}{1+e^{2x+2y}}.$$

Let

$$(3.17)\quad \Omega'(x,y,z)=\sum_{1\le m+1\le k\le n-1} e'_n(m,k)\frac{x^{n-k-1}}{(n-k-1)!}\frac{y^{k-m-1}}{(k-m-1)!}\frac{z^m}{m!}.$$

be the generating function for the $e'_n(m,k)$'s. By Theorem 2.2

$$(3.18)\quad \Omega'(x,y,z)=e^x H'(x+y,z)=e^{2x+y}\,\frac{1-e^{2z}}{1+e^{2x+2y+2z}}.$$

Then,

$$\Omega'(xI, yI, zI) = \sum_{1\le m+1\le k\le n-1} I^{n-2} e_n(m,k) \frac{x^{n-k-1}}{(n-k-1)!} \frac{y^{k-m-1}}{(k-m-1)!} \frac{z^m}{m!}.$$

By (3.15) and (3.16)

$$\begin{aligned} I^{2n-2} e'_{2n}(m,k) &= E_{2n}(2n-m, 2n-k);\\ I^{2n-1} e'_{2n+1}(m,k) &= -I\, E_{2n+1}(2n+1-m, 2n+1-k). \end{aligned}$$

Therefore,

$$\begin{aligned} &\Omega'(xI, yI, zI)\\ &= \sum_{1\le m+1\le k\le 2n-1} E_{2n}(2n-m, 2n-k) \frac{x^{2n-k-1}}{(2n-k-1)!} \frac{y^{k-m-1}}{(k-m-1)!} \frac{z^m}{m!}\\ &- I \sum_{1\le m+1\le k\le 2n} E_{2n+1}(2n+1-m, 2n+1-k) \frac{x^{2n-k}}{(2n-k)!} \frac{y^{k-m-1}}{(k-m-1)!} \frac{z^m}{m!}. \end{aligned}$$

On the other hand, (3.18) becomes:

$$\begin{aligned} \Omega'(xI, yI, zI) &= e^{2xI+yI} \frac{1-e^{2zI}}{1+e^{2xI+2yI+2zI}}\\ &= e^{xI} \frac{e^{-zI}-e^{zI}}{e^{-xI-yI-zI}+e^{xI+yI+zI}}\\ &= (\sin(x)\sin(z) - I\cos(x)\sin(z))/\cos(x+y+z). \end{aligned}$$

Compare the above two identities for $\Omega(xI, yI, zI)$ and use (3.6). This proves (1.8) in Theorem 1.1, and yields another proof of the identity $E_{2n+1}(m,k) = E_{2n+1}(k,m)$.

4. Row sums with binomial coefficients

We next show that the closed forms for the generating functions for the Bi-Entringer numbers derived in (1.6)–(1.8) provide several identities for the numbers themselves, all involving binomial coefficients.

Proposition 4.1. *We have:*

$$\sum_{k=m}^{2n} (-1)^k \binom{2n}{k} \binom{k}{m} E_{2n+2}(m+1, k+2) = (-1)^n \chi(m=0).$$

Proof. Identity (1.6) may be rewritten as:

$$\sum_{0\le m\le k\le 2n} \frac{z^m}{m!} \frac{x^{2n-k}}{(2n-k)!} \frac{y^{k-m}}{(k-m)!} E_{2n+2}(m+1, k+2) = \frac{\cos(x)\cos(z)}{\cos(x+y+z)}. \tag{4.1}$$

With $y = -x$ and $z = -xz$ identity (4.1) becomes:

$$(4.2)\ \sum_{0\le m\le k\le 2n} \frac{z^m}{m!}\frac{x^{2n}}{(2n-k)!}\frac{(-1)^k}{(k-m)!}E_{2n+2}(m{+}1,k{+}2) = \frac{\cos(x)\cos(-xz)}{\cos(-xz)}.$$

Let $\alpha_n(m)$ be the left-hand side of the identity to prove. Then, (4.2) can be expressed as: $\displaystyle\sum_{n,m\ge 0} z^m \frac{x^{2n}}{(2n)!}\alpha_n(m) = \cos(x)$. ∎

Example. $n = 2$.

$m = 1$:

$$-\binom{4}{1}\binom{1}{1}2+\binom{4}{2}\binom{2}{1}4-\binom{4}{3}\binom{3}{1}5+\binom{4}{4}\binom{4}{1}5 = -8+48-60+20 = 0;$$

$m = 0$:

$$+\binom{4}{0}0-\binom{4}{1}2+\binom{4}{2}4-\binom{4}{3}5+\binom{4}{4}5 = 0-8+24-20+5 = 1.$$

Proposition 4.2. *We have*

$$\sum_{k=m}^{2n}(-1)^k\binom{2n}{k}\binom{k}{m}E_{2n+2}(k+2,m+1) = (-1)^{n-(m-1)/2}\binom{2n}{m}T_m,$$

with the convention that $T_m = 0$, if m is even, and equal to the tangent number T_m otherwise.

Proof. Rewrite identity (1.7) as

$$(4.3)\ \sum_{0\le m\le k\le 2n} \frac{z^m}{m!}\frac{x^{2n-k}}{(2n-k)!}\frac{y^{k-m}}{(k-m)!}E_{2n+2}(k{+}2,m{+}1) = \frac{\sin(x)\sin(z)}{\cos(x+y+z)},$$

and let $y = -x$ and $z = -xz$ in (4.3), to get:

$$(4.4)\ \sum_{0\le m\le k\le 2n} \frac{z^m}{m!}\frac{x^{2n}}{(2n-k)!}\frac{(-1)^k}{(k-m)!}E_{2n+2}(k{+}2,m{+}1) = \frac{\sin(x)\sin(-xz)}{\cos(-xz)}.$$

Let $\alpha_n(m)$ be the left-hand side of the identity to be proved. Then,

$$\sum_{n,m\ge 0} z^m\frac{x^{2n}}{(2n)!}\alpha_n(m) = -\sin(x)\tan(xz) = -\sin(x)\sum_m T_m\frac{(xz)^m}{m!}.$$

Hence,

$$\sum_{n\ge 0}\frac{x^{2n}}{(2n)!}\alpha_n(m) = -\sin(x)T_m\frac{x^m}{m!} = \sum_{k\ge 0}(-1)^{k+1}\frac{x^{2k+1}}{(2k+1)!}T_m\frac{x^m}{m!}.$$ ∎

Example. $n = 3$.

$m = 1$:

$$-\binom{6}{1}\binom{1}{1}16+\binom{6}{2}\binom{2}{1}16-\binom{6}{3}\binom{3}{1}14+\binom{6}{4}\binom{4}{1}10+\binom{6}{5}\binom{5}{1}5=-6;$$

$m = 2$:

$$+\binom{6}{2}\binom{2}{2}32-\binom{6}{3}\binom{3}{2}28+\binom{6}{4}\binom{4}{2}20-\binom{6}{5}\binom{5}{2}10=0.$$

Proposition 4.3. *We have, for $m \geq 1$*

$$\sum_{k=m}^{2n-1}(-1)^k\binom{2n-1}{k}\binom{k}{m}E_{2n+1}(m+1,k+2)=(-1)^{n+1}\chi(m=0).$$

Proof. Rewrite identity (1.8) as

$$\sum_{0\leq m\leq k\leq 2n-1}\frac{z^m}{m!}\frac{x^{2n-1-k}}{(2n-1-k)!}\frac{y^{k-m}}{(k-m)!}E_{2n+1}(m+1,k+2)=\frac{\sin(x)\cos(z)}{\cos(x+y+z)};\tag{4.5}$$

and let $y = -x$ and $z = -xz$ in (4.5), to get:

$$\sum_{0\leq m\leq k\leq 2n-1}\frac{z^m}{m!}\frac{x^{2n-1}}{(2n-1-k)!}\frac{(-1)^k}{(k-m)!}E_{2n+1}(m+1,k+2)=\sin(x).$$

Then,

$$\sum_{0\leq m\leq k\leq 2n-1}\frac{z^m}{m!}\frac{1}{(2n-1-k)!}\frac{(-1)^k}{(k-m)!}E_{2n+1}(m+1,k+2) = (-1)^{n+1}\frac{1}{(2n-1)!}.\tag{4.6}$$

Let $\alpha_n(m)$ be the left-hand side of the identity to prove. By (4.6), $\alpha_n(0) = (-1)^{n+1}$ and $\alpha_n(m) = 0$ for $m \geq 1$. ∎

Example. $n = 3$.

$m = 0$:

$$\binom{5}{0}\binom{0}{0}16-\binom{5}{1}\binom{1}{0}16+\binom{5}{2}\binom{2}{0}14-\binom{5}{3}\binom{3}{0}10+\binom{5}{4}\binom{4}{0}5=1;$$

$m = 1$:

$$-\binom{5}{1}\binom{1}{1}16+\binom{5}{2}\binom{2}{1}14-\binom{5}{3}\binom{3}{1}10+\binom{5}{4}\binom{4}{1}5=0.$$

5. Proofs of Theorem 1.2

In section 4 we have derived several identities for the Bi-Entringer numbers all involving binomial coefficients, in contrast to identity (1.9) that linearly relates Poupard numbers to tangent numbers and Bi-Entringer numbers. The next analytical proof makes use of the Laplace transform

$$\mathcal{L}(f(x), x, s) := \int_0^\infty f(x)e^{-xs}\, dx,$$

which, in particular, maps $x^k/k!$ onto $1/s^{k+1}$:

$$\mathcal{L}(\frac{x^k}{k!}, x, s) = \frac{1}{s^{k+1}}.$$

To illustrate this Laplace transform method we first give a proof of (1.3). Apply the Laplace transform twice to the left-hand side of (1.1), first, with respect to x, s, then to y, t. We get

$$1 + \sum_{n\ge 1}\sum_{1\le k\le 2n+1} g_{n+1}(k)\frac{1}{t^{2n+2-k}}\frac{1}{s^k},$$

an expression which becomes

$$1 + \sum_{n\ge 1}\sum_{1\le k\le 2n+1} g_{n+1}(k)\frac{1}{s^{2n+2}}$$

for $s = t$. We need prove that

$$1 + \sum_{n\ge 1}\sum_{1\le k\le 2n+1} g_{n+1}(k)\frac{1}{s^{2n+2}} = 1 + \sum_{n\ge 1} T_{2n+1}\frac{1}{s^{2n+2}},$$

which is equivalent to

$$\int_0^\infty\int_0^\infty \frac{\cos(x-y)}{\cos(x+y)}\, e^{-t(x+y)} dx\, dy = \int_0^\infty \tan(x)\, e^{-tx} dx.$$

But this identity is true, since by letting $r = x + y$:

$$\begin{aligned}\int_0^\infty\int_0^\infty \frac{\cos(x-y)}{\cos(x+y)} e^{-t(x+y)} dx\, dy &= \int_0^\infty\int_0^r \frac{\cos(r-2y)}{\cos(r)} e^{-tr} dy\, dr\\ &= \int_0^\infty \frac{e^{-tr}}{\cos(r)}\int_0^r \cos(r-2y) dy\, dr\\ &= \int_0^\infty \frac{e^{-tr}}{\cos(r)}\sin(r) dr\\ &= \int_0^\infty \tan(r) e^{-tr} dr,\end{aligned}$$

so that identity (1.3) is proved.

Analytical proof of Theorem 1.2. Start with identity (4.5), which is another form of (1.8) and apply the Laplace transform to its left-hand side three times with respect to (x,s), (y,t), (z,u), respectively. We get

$$\sum_{0\le m\le k\le 2n-1} \frac{1}{u^{m+1}}\frac{1}{s^{2n-k}}\frac{1}{t^{k-m+1}}E_{2n+1}(m+1,k+2),$$

which becomes

$$\sum_{0\le m\le k\le 2n-1} \frac{1}{s^{2n+2}}\frac{1}{t^{k+2}}E_{2n+1}(m+1,k+2), \tag{5.1}$$

when $t\leftarrow st$ and $u\leftarrow st$. Apply the Laplace transform to the right-hand side of (4.5) three times with respect to (x,s), (y,t), (z,u), respectively, and let $t\leftarrow st, u\leftarrow st$. With $r=y+z$ we get:

$$\begin{aligned}
&\int_0^\infty\int_0^\infty\int_0^\infty \frac{\sin(x)\cos(z)}{\cos(x+y+z)}e^{-xs-yst-zst}dx\,dy\,dz\\
&=\int_0^\infty\int_0^\infty\int_0^r \frac{\sin(x)\cos(z)}{\cos(x+r)}e^{-xs-rst}dz\,dr\,dx\\
&=\int_0^\infty\int_0^\infty \frac{\sin(x)\sin(r)}{\cos(x+r)}e^{-xs-rst}dr\,dx\\
&=\int_0^\infty\int_0^\infty \frac{1}{2}\Bigl(\frac{\cos(x-r)}{\cos(x+r)}-1\Bigr)e^{-xs-rst}dr\,dx\\
&=\frac{1}{2}\int_0^\infty\int_0^\infty \Bigl(\sum_{n\ge1}\sum_{1\le k\le 2n+1} g_{n+1}(k)\frac{x^{2n+1-k}}{(2n+1-k)!}\frac{r^{k-1}}{(k-1)!}\Bigr)e^{-xs-rst}dr\,dx\\
&=\frac{1}{2}\sum_{n\ge1}\sum_{1\le k\le 2n+1} g_{n+1}(k)\frac{1}{s^{2n+2-k}}\frac{1}{(st)^k}=
\end{aligned}$$

$$\frac{1}{2}\sum_{n\ge1}\sum_{1\le k\le 2n+1} g_{n+1}(k)\frac{1}{s^{2n+2}}\frac{1}{t^k}. \tag{5.2}$$

Then, (1.9) is a consequence of the identity (5.1)=(5.2). ∎

Combinatorial proof of Theorem 1.2. We make use of the *greater neighbor statistic* "grn," which was defined in our previous paper [FH13a] as follows: let $\sigma=\sigma(1)\sigma(2)\cdots\sigma(n)$ be an alternating permutation from Alt_n, so that $\sigma(i)=n$ for a certain i $(1\le i\le n)$. By convention, let $\sigma(0)=\sigma(n+1):=0$. Then, its definition reads:

$$\mathrm{grn}(\sigma) := \max\{\sigma(i-1),\sigma(i+1)\}.$$

Let $\mathfrak{A}_{2n+1}$ be the set of all *decreasing* alternating permutations of $\{1, 2, \cdots, 2n+1\}$, i.e., permutations $w = x_1 x_2 \cdots x_{2n+1}$ such that $x_1 > x_2$, $x_2 < x_3$, $x_3 > x_4$, etc. and let $\mathfrak{A}_{2n+1,k} := \{\sigma \in \mathfrak{A}_{2n+1} : \operatorname{grn}(\sigma) = k\}$. It was proved in [FH13a] (Theorem 1.4) that

$$g_{n+1}(k+1) = \# \mathfrak{A}_{2n+1,k}\,. \tag{5.3}$$

For $n \geq 1$ each set $\mathfrak{A}_{2n+1,k}$ can be split into two subsets of the same cardinality $\mathfrak{A}^{<}_{2n+1,k} + \mathfrak{A}^{>}_{2n+1,k}$, depending on whether the greater neighbor k is on the left, or on the right of $(2n+1)$.

On the other hand, let

$$G_{2n+1,k} := \{\sigma \in \mathrm{Alt}_{2n+1} : \mathbf{L}\,\sigma = k > \mathbf{F}\,\sigma\}$$

be the set of all *increasing* alternating permutations from Alt_{2n+1} having their last letter equal to k and greater than their first letter. Then, each permutation

$$\sigma = x_1 \cdots x_{j-1}(2n+1)x_{j+1} \cdots x_{2n+1}$$

from $\mathfrak{A}^{<}_{2n+1,k}$, which is such that $x_{j-1} = k > x_{j+1}$ can be mapped onto the permutation $\tau = (x_{j+1}+1) \cdots (x_{2n+1}+1)\,1\,(x_1+1) \cdots (x_{j-1}+1)$ from $G_{2n+1,k+1}$ in a bijective manner. Thus, $\#G_{2n+1,k+1} = \# \mathfrak{A}^{<}_{2n+1,k}$ and $2\,\#G_{2n+1,k+1} = \# \mathfrak{A}^{<}_{2n+1,k} + \# \mathfrak{A}^{>}_{2n+1,k} = \# \mathfrak{A}_{2n+1,k}$, so that

$$g_{n+1}(k+1) = 2\,\#G_{2n+1,k+1} = 2 \sum_{l \leq k} E_{2n+1}(k+1, l),$$

as $E_{2n+1}(k+1, k+1) = 0$. This implies identity (1.9) since $E_{2n+1}(m, l) = E_{2n+1}(l, m)$ for all m, l (see Proposition 3.1(iv)). $\square$

Dumont [Du14] drew our attention to the following relation between Poupard and Bi-Entringer numbers, namely,

$$g_n(k) = 2\,E_{2n}(k, k+1) = 2\,E_{2n}(k+1, k). \tag{5.4}$$

Before giving a combinatorial proof of that identity we state and prove a property on alternating permutations, both increasing and decreasing, that involves the statistics $\mathbf{F}$, $\mathbf{L}$, and also two other statistics attached to the *left* $\mathbf{f}$ and *right* $\mathbf{l}$ neighbors of the maximum. Each permutation σ of $12 \cdots n$ may be written

$$\sigma = x_1 x_2 \cdots x_n = \mathbf{F}(\sigma) \cdots \mathbf{f}(\sigma) \max(\sigma)\, \mathbf{l}(\sigma) \cdots \mathbf{L}(\sigma), \tag{5.5}$$

where $\mathbf{F}(\sigma) = x_1$, $\mathbf{L}(\sigma) = x_n$ and, if $\max(\sigma) := n = x_k$, then $\mathbf{f}(\sigma) = x_{k-1}$, $\mathbf{l}(\sigma) = x_{k+1}$, where the convention $x_0 = x_{n+1} = 0$ still holds. For each finite word $w = y_1 y_2 \cdots y_m$, whose letters are integers, it is convenient to use the notation: $(w+1) := (y_1+1)(y_2+1) \cdots (y_m+1)$, when w is not the empty word e, and e when it is.

Property 5.1. *Let* $\sigma = w_1 \max(\sigma) w_2$ *be a permutation of* $12\cdots(2n-1)$, *so that* $\max(\sigma) = 2n-1$. *Then, the mapping*

$$\phi_1 : w_1 \max(\sigma)\, w_2 \mapsto w_2 \max(\sigma)\, w_1 \tag{5.6}$$

is a bijection of Alt_{2n-1} *onto itself having the property*

$$(\mathbf{f}, \mathbf{l})\sigma = (\mathbf{L}, \mathbf{F})\phi_1(\sigma), \tag{5.7}$$

while the mapping

$$\phi_2 : w_1 \max(\sigma)\, w_2 \mapsto (w_2+1)\, 1\, (w_1+1) \tag{5.8}$$

is a bijection of $\mathfrak{A}_{2n-1}$ *onto* Alt_{2n-1} *having the property*

$$(\mathbf{f}, \mathbf{l})\sigma = (\mathbf{L}-1, \mathbf{F}-1)\phi_2(\sigma). \tag{5.9}$$

The proof of Property 5.1 is straightforward. Just mention three examples: (i) $\phi_1(3427561) = 5617342$ and $(\mathbf{f}, \mathbf{l})\,(34\mathbf{2}\,7\,\mathbf{5}61) = (2, 5) = (\mathbf{L}, \mathbf{F})\,(\mathbf{5}61\,7\,34\mathbf{2})$; (ii) $\phi_2(5471326) = 2437165$ and $(\mathbf{f}, \mathbf{l})(54\,7\,\mathbf{1}326) = (4, 1) = (\mathbf{L}-1, \mathbf{F}-1)(\mathbf{2}\,43716\,\mathbf{5})$; (iii) $\phi_2(7461325) = 5724361$ and $(\mathbf{f}, \mathbf{l})(7\,\mathbf{4}61325) = (0, 4) = (\mathbf{L}-1, \mathbf{F}-1)(\mathbf{5}\,72436\,\mathbf{1})$.

It follows from (5.6)–(5.9) that the product $\phi_1 \circ \phi_2 : \mathfrak{A}_{2n-1} \to \mathrm{Alt}_{2n-1}$ has the property that for every σ from $\mathfrak{A}_{2n-1}$ we have: $(\mathbf{f}-1, \mathbf{l}-1)(\phi_1 \circ \phi_2)(\sigma) = (\mathbf{L}-1, \mathbf{F}-1)\phi_2(\sigma) = (\mathbf{f}, \mathbf{l})\sigma$. In view of (5.3) and since $\mathrm{grn} = \max(\mathbf{l}, \mathbf{f})$, this implies the identity:

$$g_n(k) = \#\,\mathfrak{A}_{2n-1,k-1} = \#\,\mathrm{Alt}_{2n-1,k}\,. \tag{5.10}$$

For $n \geq 1$ each set $\mathrm{Alt}_{2n-1,k}$ can be split into two subsets of the same cardinality $\mathrm{Alt}^{<}_{2n-1,k} + \mathrm{Alt}^{>}_{2n-1,k}$, depending on whether the greater neighbor k is on the left, or on the right of $(2n+1)$. Now, with each permutation

$$\sigma = x_1 x_2 \cdots x_{j-2}\, k\, (2n-1) x_{j+1} x_{j+2} \cdots x_{2n-1}$$

from $\mathrm{Alt}^{<}_{2n-1,k}$ associate the permutation

$$\phi_3(\sigma) := \, k\, x'_{j-2} \cdots x'_2 x'_1\, (2n)\, x'_{2n-1} \cdots x'_{j+2} x'_{j+1} (k+1),$$

where

$$x'_j = \begin{cases} x_j, & \text{if } x_j \leq k; \\ x_j + 1, & \text{if } x_j > k. \end{cases}$$

It is obvious that ϕ_3 is a bijection of $\mathrm{Alt}_{2n-1,k}$ onto the set $\mathfrak{E}_{2n,k}$ of all permutations τ from Alt_{2n} such that $\mathbf{F}\,\tau = k$, $\mathbf{L}\,\tau = k+1$. As $\#\mathfrak{E}_{2n,k} = E_{2n}(k, k+1)$, it follows that

$$\#\,\mathrm{Alt}_{2n-1,k} = 2\#\,\mathrm{Alt}^{<}_{2n-1,k} = 2E_{2n}(k, k+1). \tag{5.11}$$

This proves identity (5.2) in view of (5.10) and (5.11).

Final Remarks. The Seidel Matrix method developed in this paper can also be used to derive the three-variable generating functions obtained in our previous two papers [FH13b] and [FH13c]. It is also the main tool in our next paper [FH13d].

References

[An1879] Désiré André. Développement de $\sec x$ et $\tan x$, *C. R. Math. Acad. Sci. Paris*, vol. **88**, 1879, p. 965–979.

[An1881] Désiré André. Sur les permutations alternées, *J. Math. Pures et Appl.*, vol. **7**, 1881, p. 167–184.

[Co74] Louis Comtet. *Advanced Combinatorics*. D. Reidel/Dordrecht-Holland, Boston, 1974.

[Du82] Dominique Dumont. Matrices d'Euler-Seidel, *Séminaire Lotharingien de Combinatoire*, B05c (1981), 25 pp. [Formerly: Publ. I.R.M.A. Strasbourg, 1982, 182/S-04, p. 59-78.] `http://www.mat.univie.ac.at/~slc/`.

[Du14] Dominique Dumont. Private communication, 2014.

[DV80] Dominique Dumont; Gérard Viennot. A combinatorial interpretation of the Seidel generation of Genocchi numbers, *Combinatorial mathematics, optimal designs* [J. Srivastava, ed., Fort Collins. 1978], p. 77–87. Amsterdam, North-Holland, 1980 (*Annals of Discrete Math.* **6**).

[En66] R. C. Entringer. A combinatorial interpretation of the Euler and Bernoulli numbers, *Nieuw. Arch. Wisk.*, vol. **14**, 1966, p. 241–246.

[FH13a] Dominique Foata; Guo-Niu Han. Finite Difference Calculus for Alternating Permutations, *Journal of Difference Equations and Applications,*, vol. **19**, 2013, p. 1952–1966.

[FH13b] Dominique Foata; Guo-Niu Han. Tree Calculus for Bivariate Difference Equations, to appear in *Journal of Difference Equations and Applications*, 36 pages, 2013.

[FH13c] Dominique Foata; Guo-Niu Han. Secant Tree Calculus, to appear in *Central Europ. J. of Math.*, 24 pages, 2013.

[FH13d] Dominique Foata; Guo-Niu Han. Three-variable generating function for Poupard-Entringer Matrices, 20 pages, 2013.

[MSY76] J. Millar; N. J. A. Sloane; N. E. Young. A New Operation on Sequences: The Boustrophedon Transform, *J. Combin. Theory, Ser. A*, vol. **76**, 1996, p. 44-54.

[Ni23] Niels Nielsen. *Traité élémentaire des nombres de Bernoulli*. Paris, Gauthier-Villars, 1923.

[Po89] Christiane Poupard. Deux propriétés des arbres binaires ordonnés stricts, *Europ. J. Combin.*, vol. **10**, 1989, p. 369–374.

[Ro13] Pierre Rosenstiehl. *Le labyrinthe des jours ordinaires*. Paris, La librairie du XXI^e siècle, Éditions du Seuil, 2013.

[Se1877] L. Seidel. Über eine einfache Enstehungsweise der Bernoullischen Zahlen und einiger verwandten Reihen, *Sitzungberichte der Münch. Akad. Math. Phys. Classe*, 1877, p. 157–187.

[Sl] N.J.A. Sloane. On-line Encyclopedia of Integer Sequences, `http://oeis.org/`.

Dominique Foata
Institut Lothaire
1, rue Murner
F-67000 Strasbourg, France
foata@unistra.fr

Guo-Niu Han
I.R.M.A. UMR 7501
Université de Strasbourg et CNRS
7, rue René-Descartes
F-67084 Strasbourg, France
guoniu.han@unistra.fr