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Hankel Determinant Calculus for the Thue-Morse and related sequences

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Abstract. The Hankel determinants of certain automatic sequences f are evaluated, based on a calculation modulo a prime number. In most cases, the Hankel determinants of automatic sequences do not have any closed-form expressions; the traditional methods, such as LU-decomposition and Jacobi continued fraction, cannot be applied directly. Our method is based on a simple idea: the Hankel determinants of each sequence g equal to f modulo p are equal to the Hankel determinants of f modulo p. The clue then consists of finding a nice sequence g, whose Hankel determinants have closed-form expressions.

Several examples are presented, including a result saying that the Hankel determinants of the Thue-Morse sequence are nonzero, first proved by Allouche, Peyrière, Wen and Wen using determinant manipulation. The present approach shortens the proof of the latter result significantly. We also prove that the corresponding Hankel determinants do not vanish when the powers 2^n in the infinite product defining the ± 1 Thue–Morse sequence are replaced by 3^n.

1. Introduction

Let x be a parameter. We identify a sequence $\mathbf{a} = (a_0, a_1, a_2, \ldots)$ and its generating function $f = f(x) = a_0 + a_1x + a_2x^2 + \cdots$. Usually, $a_0 = 1$. For each $n \geq 1$ and $k \geq 0$ the Hankel determinant of the series f (or of the sequence $\mathbf{a}$) is defined by

$$H_n^{(k)}(f) := \begin{vmatrix} a_k & a_{k+1} & \cdots & a_{k+n-1} \\ a_{k+1} & a_{k+2} & \cdots & a_{k+n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{k+n-1} & a_{k+n} & \cdots & a_{k+2n-2} \end{vmatrix}. \tag{1.1}$$

Let $H_n(f) := H_n^{(0)}(f)$, for short; the *sequence of the Hankel determinants* of f is defined to be:

$$H(f) := (H_0(f) = 1, H_1(f), H_2(f), H_3(f), \ldots).$$

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In some cases Hankel determinants can be evaluated by using basic determinant manipulation, LU-decomposition, or Jacobi continued fraction (see, e.g., [Kr98, Kr05, Fl80, Wa48, Mu23]). However, the Hankel determinants of several power series f related to automatic sequences do not seem to have closed-form expressions, as will be seen in this paper. The following result by Allouche, Peyrière, Wen and Wen [APWW] in 1998, has strongly motivated the present paper.

Theorem 1.1 [APWW]. *Let $P_2 = P_2(x) = \prod_{k=0}^{\infty}(1 - x^{2^k})$ be the ± 1 Thue-Morse sequence. Then $H_n(P_2) \neq 0$ for every positive integer n.*

The first values of the coefficients and Hankel determinants of $P_2(x)$ are:

$$\begin{aligned} P_2 &= (1, -1, -1, 1, -1, 1, 1, -1, -1, 1, 1, -1, 1, -1, -1, 1, -1, \ldots) \\ H(P_2) &= (1, 1, -2, 4, 8, -16, -32, -64, 128, -256, -1536, -3072, \ldots) \end{aligned}$$

A combinatorial proof of Theorem 1.1 was recently derived by Bugeaud and the author [BH13].

Let $\mathbf{u} = (u_1, u_2, \ldots)$ and $\mathbf{v} = (v_0, v_1, v_2, \ldots)$ be two sequences. Recall that the *Jacobi continued fraction* attached to $(\mathbf{u}, \mathbf{v})$, or *J-fraction*, for short, is a continued fraction of the form

$$\frac{v_0}{1 + u_1 x - \cfrac{v_1 x^2}{1 + u_2 x - \cfrac{v_2 x^2}{1 + u_3 x - \cfrac{v_3 x^2}{\ddots}}}}, \tag{1.2}$$

also denoted by

$$\mathbf{J}\begin{bmatrix} \mathbf{u} \\ \mathbf{v} \end{bmatrix} = \mathbf{J}[\mathbf{u}/\mathbf{v}] = \mathbf{J}\begin{bmatrix} u_1, u_2, \cdots \\ v_0, v_1, v_2, \cdots \end{bmatrix}.$$

The basic properties on J-fractions, we now recall, can be found in [Fl80, Wa48, Vi83]. The J-fraction of a given power series f exists (i.e., $f = \mathbf{J}[\mathbf{u}/\mathbf{v}]$) if and only if all the Hankel determinants $H_n(f)$ of f are nonzero. The first values of the coefficients u_n and v_n in the J-fraction expansion can be calculated by the *Stieltjes Algorithm*. Also, Hankel determinants can be calculated from the J-fraction by means of the following fundamental relation:

$$H_n\left(\mathbf{J}\begin{bmatrix} u_1, u_2, \cdots \\ v_0, v_1, v_2, \cdots \end{bmatrix}\right) = v_0^n v_1^{n-1} v_2^{n-2} \cdots v_{n-2}^2 v_{n-1}. \tag{1.3}$$

Conversely, the coefficients u_n and v_n in the J-fraction can be calculated using the Hankel determinants by means of the following relations, when all denominators are nonzero.

$$u_n = -\frac{1}{H_{n-1}^{(1)}}\Big(\frac{H_{n-1}H_n^{(1)}}{H_n} + \frac{H_n H_{n-2}^{(1)}}{H_{n-1}}\Big), \qquad (n \geq 2) \tag{1.4}$$

$$v_n = \frac{H_n H_{n-2}}{(H_{n-1})^2}. \qquad (n \geq 2) \tag{1.5}$$

Relation (1.3) is an efficient method for evaluating Hankel determinants.

Let us try to evaluate the Hankel determinants for the Thue-Morse sequence by using the J-fraction. By the Stieltjes algorithm, we get

$$P_2(x) = \mathbf{J}\begin{bmatrix} \mathbf{u} \\ \mathbf{v} \end{bmatrix} = \mathbf{J}\begin{bmatrix} 1, -1, 1, -1, 1, -1, 1, -1, 1, -1, 1, -1, \cdots \\ 1, -2, 1, -1, -1, -1, 1, -1, 1, -3, \frac{1}{3}, -\frac{1}{3}, -3, \cdots \end{bmatrix}.$$

The top coefficients u_n seem to be very simple. However, we are not able to guess any closed-form expression for the bottom coefficients v_n, which are even rational numbers. Therefore, we cannot prove anything about the Hankel determinants.

Coons [Co13], using the method described in [APWW], proved the following theorem.

Theorem 1.2 [Coons]. *Let*

$$S_2 = S_2(x) = \frac{1}{x}\sum_{n=0}^{\infty} \frac{x^{2^n}}{1 - x^{2^n}}.$$

Then $H_n(S_2) \equiv 1 \pmod 2$.

Again, we are not able to guess any closed-form expression for the Hankel determinants of S_2, as the first values of the coefficients of the series, the Hankel determinants and the J-fraction of S_2 read:

$$S_2 = (1, 2, 1, 3, 1, 2, 1, 4, 1, 2, 1, 3, 1, 2, 1, 5, 1, 2, 1, 3, 1, 2, 1, 4, \ldots)$$
$$H(S_2) = (1, 1, -3, -1, 21, 1, -3, -9, 945, 9, -3, -1, 21, 9, -243, \ldots)$$
$$S_2 = \mathbf{J}\begin{bmatrix} -2, \frac{7}{3}, \frac{23}{3}, -\frac{167}{21}, -\frac{169}{21}, 7, 7, -\frac{629}{105}, -\frac{631}{105}, 7, 7, -\frac{57}{7}, -\frac{55}{7}, \cdots \\ 1, -3, -\frac{1}{9}, -63, -\frac{1}{441}, -63, -1, -35, -\frac{1}{11025}, -35, -1, -63, \cdots \end{bmatrix}$$

The main idea to solve the problem is to proceed as follows:

Let p be a prime number and f a sequence. We want to prove that $H_n(f) \not\equiv 0 \pmod p$; if, apparently, there is no closed-form for the coefficients in the J-fraction of f, we try to find a sequence $g \equiv f \pmod p$, such that the Hankel determinants of g have a closed form. As it is easy to prove that $H_n(f) \equiv H_n(g) \pmod p$, it is very likely that some properties on the Hankel determinants of f can be established.

Question. *How to find a nice sequence g such that $g \equiv f$ for which each coefficient in the J-fraction of g has a closed-form expression?*

By observing the occurrences of the factor 2 in the coefficients of the J-fraction of S_2 given in Theorem 1.2, we guess the following "nice" sequence

$$g = \mathbf{J}\begin{bmatrix} 0, 1, 1, 1, 1, 1, \dots \\ 1, 1, 1, 1, 1, 1, 1, \dots \end{bmatrix},$$

whose Hankel determinant is $H_n(g) = 1 \neq 0$. For proving Theorem 1.2, it remains to prove that $S_2 \equiv g \pmod 2$. For Theorem 1.1, it is more complicated; we need the so-called *grafting technique.* The proofs of Theorems 1.1-2 are given in Section 2 with further examples. In Section 3 we derive two J-fractions by using the chopping method (Proposition 3.2$''$ and Theorem 3.3) and prove that the Hankel determinant sequences of several power series are periodic (Propositions 3.6-8).

On the one hand, we provide short proofs of results established in the papers [APWW, Co13], on the other hand, we obtain several new results. In particular, we should like to single out the following theorem.

Theorem 1.3. *Let $P_3 = P_3(x) = \prod_{k\geq 0}(1 - x^{3^k})$. Then $H_n(P_3) \equiv (-1)^n \pmod 3$ for every positive integer n.*

Notice that the sequence P_3 is obtained from the Thue-Morse sequence P_2 by modifying the exponent of x from 2 to 3. It is worth mentioning that, when $m \geq 4$, the Hankel determinants for the sequence $\prod_{k\geq 0}(1-x^{m^k})$ are not all nonzero. A self-contained and short proof of Theorem 1.3 is found in Section 4. The following result, that could be called "*one sequence, two modulos*", is also proved in Section 4.

Theorem 1.4. *We have*

$$\sqrt{\frac{1}{(1-x)(1+3x)}} \equiv \prod_{k=0}^{\infty}(1 - x^{2^k}) \pmod 4,$$
$$\sqrt{\frac{1}{(1-x)(1+3x)}} \equiv \prod_{k=0}^{\infty}(1 - x^{3^k}) \pmod 3.$$

2. Hankel determinants modulo p and the grafting technique

Let p be a prime number. For a given power series f we present some methods for guessing *and* calculating the J-fraction of f, and also proving properties mod p for its Hankel determinants. An ultimately periodic sequence is written in contracted form by using the star sign. For instance, the sequence $\mathbf{a} = (1, (3, 0)^*)$ represents $(1, 3, 0, 3, 0, 3, 0, \ldots)$, that is, $a_0 = 1$ and $a_{2k+1} = 3, a_{2k+2} = 0$ for each positive integer k. Two sequences $\mathbf{a}$ and $\mathbf{b}$ are said to be *congruent modulo* p if $a_k \equiv b_k \pmod{p}$ for all k. For each integer z we have $(x+z)^p \equiv x^p + z^p \pmod{p}$ and derive the following lemma.

Lemma 2.1. *Let $f(x)$ be a power series with integral coefficients. Then*

$$f(x)^p \equiv f(x^p) \pmod{p}. \tag{2.1}$$

Let a_1, b_1, a_2, b_2 be four integers such that $(p, b_1) = 1$ and $(p, b_2) = 1$. The two fractions a_1/b_1 and a_2/b_2 are said to be *congruent modulo* p if $a_1 b_2 \equiv a_2 b_1 \pmod{p}$. We write $a_1/b_1 \equiv a_2/b_2 \pmod{p}$. This fractional congruence is closed under addition and multiplication. Let $a_1/b_1 \equiv c_1 \pmod{p}$ and $a_2/b_2 \equiv c_2 \pmod{p}$, then $a_1/b_1 + a_2/b_2 \equiv c_1 + c_2 \pmod{p}$ and $a_1/b_1 \times a_2/b_2 \equiv c_1 c_2 \pmod{p}$. The fractional congruence for power series is also closed under addition and multiplication. Also, the ring of formal power series with rational coefficients modulo p is an integral domain.

Lemma 2.2. *Let f and $\hat{f}$ be two power series with rational coefficients and $\mathbf{J}[\mathbf{u}, \mathbf{v}] = f$, $\mathbf{J}[\hat{\mathbf{u}}, \hat{\mathbf{v}}] = \hat{f}$ be their J-fraction expansions. Then*

(1) If $f \equiv \hat{f} \pmod{p}$, then $H(f) \equiv H(\hat{f}) \pmod{p}$.

(2) If $\mathbf{u} \equiv \hat{\mathbf{u}} \pmod{p}$ and $\mathbf{v} \equiv \hat{\mathbf{v}} \pmod{p}$, then $f \equiv \hat{f} \pmod{p}$.

(3) If $\mathbf{v} \equiv \hat{\mathbf{v}} \pmod{p}$, then $H(f) \equiv H(\hat{f}) \pmod{p}$.

Proof. (1) The Hankel determinants are expressed in terms of the coefficients of the power series by using only addition and multiplication. (2) The coefficients of the power series are expressed in terms of the coefficients in the J-fraction by using only addition and multiplication. (3) By the fundamental relation (1.3). □

Remark. The converse of (1) is not true. A counter-example is the following pair with $p = 2$:

$$f = \frac{1 - \sqrt{1 - \frac{4x^2}{1-x}}}{2x^2} \quad \text{and} \quad \hat{f} = \frac{1 - \sqrt{1 - \frac{4x}{1-x}}}{2x}.$$

Let f be a power series and g be a J-fraction. If the two sequences $\mathbf{u}$ and $\mathbf{v}$ in g are ultimately periodic with the same period, we can check that f and g are equal or not. For example, we claim that the J-fraction of the power series

$$f = \frac{(1-x)(1+2x) - \sqrt{(1-x)(1-2x)(1+3x)(1+2x-4x^2)}}{4x^2(1-x)}$$

is equal to

$$g = \mathbf{J}\begin{bmatrix} (-\frac{1}{2}, -\frac{1}{2}, 2)^* \\ 1, (\frac{1}{4}, 2, 2)^* \end{bmatrix}.$$

To see this, we check that f verifies the following quadratic functional equation

$$f = \cfrac{1}{1 - \frac{1}{2}x - \cfrac{\frac{1}{4}x^2}{1 - \frac{1}{2}x - \cfrac{2x^2}{1 + 2x - 2x^2 f}}}.$$

Moreover, the first values of f and g are the same, namely, $(1, \frac{1}{2}, \frac{1}{2}, \ldots)$. Hence, the two power series f and g are equal. Later in the paper this kind of proof will not be reproduced, as it can be done automatically: the sentence *"we can prove"* replaces the full proof.

Proof of Theorem 1.2 [Coons]. We have

$$(xS_2(x))^2 \equiv x^2 S_2(x^2) = \sum_{n=1}^{\infty} \frac{x^{2^n}}{1 - x^{2^n}} = xS_2(x) - \frac{x}{1-x} \pmod 2$$

so that

$$xS_2(x)^2 \equiv S_2(x) - \frac{1}{1-x} \equiv S_2(x) - \frac{1}{1+x} \pmod 2$$

and

$$\left(S_2(x) - \frac{1 + \sqrt{\frac{1-3x}{1+x}}}{2x}\right)\left(S_2(x) - \frac{1 - \sqrt{\frac{1-3x}{1+x}}}{2x}\right) \equiv 0 \pmod 2.$$

We get

$$S_2(x) \equiv \frac{1 - \sqrt{\frac{1-3x}{1+x}}}{2x} \pmod 2.$$

Let g be the right-hand side of the above equation. We can prove

$$g = \mathbf{J}\begin{bmatrix} 0, (-1)^* \\ (1)^* \end{bmatrix}.$$

Hence, $H_n(g) \equiv 1 \pmod 2$, so does $H_n(S_2)$ by Lemma 2.2(1). $\square$

Proof of Theorem 1.3. We successively have

$$\begin{gathered} P_3(x) = (1-x)P_3(x^3) \equiv (1-x)P_3(x)^3 \pmod 3, \\ P_3(x)(1-(1-x)P_3(x)^2) \equiv 0 \pmod 3, \\ 1-(1-x)P_3(x)^2 \equiv 0 \pmod 3, \\ P_3(x)^2 \equiv \frac{1}{1-x} \pmod 3, \\ \Big(P_3(x) - \sqrt{\frac{1}{1-x}}\Big)\Big(P_3(x) + \sqrt{\frac{1}{1-x}}\Big) \equiv 0 \pmod 3, \\ P_3(x) \equiv \sqrt{\frac{1}{1-x}} \pmod 3. \end{gathered}$$

Notice that $P_3(x)$ has integral coefficients, but $\sqrt{\frac{1}{1-x}}$ has rational coefficients. We can prove that

$$\sqrt{\frac{1}{1-x}} = \mathbf{J}\begin{bmatrix} (-1/2)^* \\ 1, 1/8, (1/16)^* \end{bmatrix}.$$

The above J-fraction itself is congruent to

$$g = \mathbf{J}\begin{bmatrix} (1)^* \\ 1, -1, (1)^* \end{bmatrix}$$

modulo 3, by Lemma 2.2(2), knowing that $1/2 \equiv -1 \pmod 3$. We have $H_n(g) = (1,-1)^*$. Hence, $H_n(P_3) \equiv H_n(g) \equiv (1,-1)^* \pmod 3$. $\square$

There is also a proof without using fractional congruence, see Section 4. Notice that $H(P_3(x)) = H(P_3(-x))$ by (1.3). That means $H(g) \neq 0$ for $g = \prod_{k\geq 0}(1 + x^{3^k})$.

For proving Theorem 1.1, we need a technique, called "*grafting*". Let $F(x)$ and $G(x)$ be two J-fractions

$$F(x) = \mathbf{J}\begin{bmatrix} u_1, u_2, u_3, \cdots \\ v_0, v_1, v_2, v_3, \cdots \end{bmatrix} \quad \text{and} \quad G(x) = \mathbf{J}\begin{bmatrix} a_1, a_2, a_3, \cdots \\ b_0, b_1, b_2, b_3, \cdots \end{bmatrix}$$

such that $b_0 = 1$. For each $k \in \mathbf{N}$ the *grafting* of $G(x)$ into $F(x)$ of order k, denoted by $F(x)|^r G(x)$, is defined to be the following J-fraction

$$F(x)|^k G(x) = \mathbf{J}\begin{bmatrix} u_1, u_2, \cdots, u_k, a_1, a_2, a_3, \cdots \\ v_0, v_1, v_2, \cdots, v_k, b_1, b_2, b_3, \cdots \end{bmatrix}.$$

Let $F|G := F|^1G$ and $F||G := F|^2G$, for short.

If $u_i, v_i \pmod p$ exists and $v_i \not\equiv 0 \pmod p$ for all $i \geq k+1$, we define

$$\bar{G} := \mathbf{J}\left[\begin{matrix} u_{k+1} \pmod p, u_{k+2} \pmod p, u_{k+3} \pmod p, \cdots \\ 1, v_{k+1} \pmod p, v_{k+2} \pmod p, v_{k+3} \pmod p, \cdots \end{matrix}\right]$$

and $\bar{F} = F|^r\bar{G}$. Then the Hankel determinants of F and $\bar{F}$ have the following relation

$$\frac{H_n(F)}{H_n(\bar{F})} \equiv 1 \pmod p \tag{2.2}$$

in view of the fundamental relation (1.3).

For instance, the first values of the J-fraction of the Thue-Morse sequence P_2 are

$$P_2 = \mathbf{J}\left[\begin{matrix} (1,-1)^* \\ 1, -2, 1, -1, -1, -1, 1, -1, 1, -3, \frac{1}{3}, -\frac{1}{3}, -3, 1, -1, 1, 1, -3, \cdots \end{matrix}\right].$$

We see that the previous sequences $\mathbf{u}, \mathbf{v}$ contain only one even number, -2, and it occurs at position v_1. Delete (v_1, u_1), which means that we define the following J-fraction g

$$g = \mathbf{J}\left[\begin{matrix} (-1,1)^* \\ 1, 1, -1, -1, -1, 1, -1, 1, -3, \frac{1}{3}, -\frac{1}{3}, -3, 1, -1, 1, 1, -3, \cdots \end{matrix}\right],$$

so that all the Hankel determinants of g are odd fractional numbers by (1.3).

Proof of Theorem 1.1. Define the sequence g by

$$P_2 = \frac{1}{1 + x + 2x^2 g},$$

or

$$g = \frac{1}{2x^2}\left(\frac{1}{P_2} - 1 - x\right).$$

By Theorem 1.4 the following identities hold:

$$\begin{aligned} 1/P_2 &\equiv \sqrt{(1-x)(1+3x)} \pmod 4, \\ g &\equiv \frac{1}{2x^2}\left(1 + x - \sqrt{(1-x)(1+3x)}\right) \pmod 2. \end{aligned}$$

We can prove that the right-hand side $\bar{g}$ of the above equation has a simple J-fraction

$$\bar{g} = \mathbf{J}\left[\begin{matrix} (1)^* \\ (1)^* \end{matrix}\right].$$

Let $\bar{P}_2$ be the grafting of $\bar{g}$ into P_2

$$\bar{P}_2 = P_2|\bar{g} = \frac{1}{1+x+2x^2\bar{g}},$$

so that $H_n(\bar{P}_2) = (-2)^{n-1}$ from (1.3). Hence, $H_n(P_2)/2^{n-1} \equiv 1 \pmod 2$ by (2.2). $\square$

Let $P_2 = \sum_{n=0}^{\infty} \eta_n x^n$ be the Thue-Morse sequence. We now evaluate the Hankel determinants of the following two sequences

$$\delta_n = (\eta_n - \eta_{n+1})/2, \tag{2.3}$$

$$\gamma_n = (\eta_n - \eta_{n+2})/2. \tag{2.4}$$

The following result was proved in [APWW, Proposition 2.2(2)].

Proposition 2.3. *The Hankel determinants of the sequence* $(\delta_n)_{n=0,1,2,\ldots}$ *are odd integral numbers.*

Proof. The generating function for the sequence (γ_n) is equal to

$$f = \frac{1-(1-x)P_2}{2x},$$

which is congruent to

$$g := \frac{1-(1-x)\sqrt{\frac{1}{(1-x)(1+3x)}}}{2x} \pmod 2$$

by Theorem 1.4. We can prove that g has the following J-fraction expansion

$$g = \frac{1-\sqrt{\frac{1-x}{1+3x}}}{2x} = \mathbf{J}\begin{bmatrix} 2, (1)^* \\ (1)^* \end{bmatrix}.$$

Hence, $H_n(g) = 1$ and $H_n(f) \equiv 1 \pmod 2$. $\square$

Proposition 2.4. *The Hankel determinants of the sequence* $(\gamma_n)_{n=0,1,2,\ldots}$ *are odd integral numbers.*

Proof. The generating function for the sequence (γ_n) is equal to

$$f = \frac{1-x-(1-x^2)P_2}{2x^2},$$

which is congruent to

$$g := -\frac{1-x-(1-x^2)\sqrt{\frac{1}{(1-x)(1+3x)}}}{2x^2} \pmod 2$$

by Theorem 1.4. We can prove that g has the following J-fraction expansion

$$g = -\frac{1-x-(1+x)\sqrt{\frac{1-x}{1+3x}}}{2x^2} = \mathbf{J}\begin{bmatrix} (3,-1)^* \\ 1, (-1)^* \end{bmatrix}.$$

Hence, $H_n(g) \equiv 1 \pmod 2$ and $H_n(f) \equiv 1 \pmod 2$. $\square$

Proposition 2.5. *Let*

$$f = 3\prod_{n=1}^{\infty}(1-x^{3^n}) - \frac{2}{1-x}.$$

Then, $H_k(f) \neq 0$ for all k.

Remark. When replacing the factor $1 - x^{3^n}$ by $1 + x^{3^n}$ in the above formula, experimental calculation of the first values suggests that all the Hankel determinants are still nonzero. However, we are not able to prove that the latter Hankel determinants do not vanish.

Proof. We have

$$f = \mathbf{J}\left[\begin{matrix} 2, -7/2, 7/10, 32/65, -187/26, 259/34, -49/272, 241/16, \dots \\ 1, -6, -5/4, -26/25, 10/169, -221/4, 64/289, -17/256, \dots \end{matrix}\right].$$

The factor 3 occurs only once, at position v_1. We use the grafting technique. Define

$$f = \frac{1}{1 + 2x + 6x^2 g}, \tag{2.5}$$

or

$$(1 + 2x + 6x^2 g)\big(3\prod_{n=0}^{\infty}(1-x^{3^n}) - 2\big) = 1 - x. \tag{2.6}$$

By (1.3) we have

$$H_n(f) = (-6)^{n-1} H_{n-1}(g). \tag{2.7}$$

for all n. By Theorem 1.4 identity (2.6) becomes

$$(1 + 2x + 6x^2 g)\Big(3\sqrt{\frac{1}{(1-x)(1+3x)}} - 2\Big) \equiv 1 - x \pmod 9$$

or

$$g \equiv \frac{(1+2x)\sqrt{\frac{1}{(1-x)(1+3x)}} - 1 - x}{x^2} \pmod 3.$$

Let h be the right-hand side of the above equation. Then,

$$h = \mathbf{J}\left[\begin{matrix} (-1)^* \\ 1, (4, -1/2, -1/2)^* \end{matrix}\right],$$

so that $H_n(h) \equiv 1 \pmod 3$. Hence, $H_n(g) \equiv 1 \pmod 3$. By (2.7) we have $H_n(f) \neq 0$. ∎

Proposition 2.6. *Let f be the sequence obtained from P_3 by deleting the first term, i.e., $f = (1 - P_3)/x$. Then, $H(f) \equiv (1)^* \pmod 3$.*

Proof. By Theorem 1.4,

$$f \equiv \frac{1 - \sqrt{\frac{1}{(1-x)(1+3x)}}}{x} \pmod 3.$$

Let g be right-hand side of the above equation. Then, g has the J-fraction

$$g = \mathbf{J}\begin{bmatrix} 3, (-2, 5/2, 5/2)^* \\ 1, (-2, -2, 1/4)^* \end{bmatrix} \equiv \mathbf{J}\begin{bmatrix} 0, (-1, 1, 1)^* \\ (1)^* \end{bmatrix} \pmod 3.$$

Hence, $H(g) \equiv (1)^* \pmod 3$, so does $H(f)$. □

Theorem 2.7. *Let*

$$f = f(x) = \prod_{k\geq 0} (1 - x^{3^k} - x^{2\cdot 3^k}). \tag{2.8}$$

Then $H_n(f) \neq 0$.

Proof. We successively have

$$\begin{aligned} f(x^3) &= \prod_{k\geq 1} (1 - x^{3^k} - x^{2\cdot 3^k}); \\ f = f(x) &= \prod_{k\geq 0} (1 - x^{3^k} - x^{2\cdot 3^k}) = (1 - x - x^2) f(x^3); \\ f &\equiv (1 - x - x^2) f^3 \pmod 3; \\ 1 &\equiv (1 - x - x^2) f^2 \pmod 3; \\ f &\equiv \sqrt{1/(1 - x - x^2)} \pmod 3. \end{aligned}$$

The right-hand side of the above equality has the following J-fraction expansion:

$$\sqrt{\frac{1}{1 - x - x^2}} = \mathbf{J}\begin{bmatrix} (-1/2)^* \\ 1, 5/8, (5/16)^* \end{bmatrix} \equiv \mathbf{J}\begin{bmatrix} (1)^* \\ 1, 1, (-1)^* \end{bmatrix} \pmod 3,$$

so that

$$H(f) \equiv (1, (1, 1, 2, 2)^*) \pmod 3. \quad \square \tag{2.9}$$

Remark. The sequence f defined in (2.8) is a $\{1, -1\}$-sequence.

3. Continued fraction and the chopping method

When the two coefficients in the J-fraction are ultimately periodic with the same period, the corresponding power series is easy to obtained. However this is not always the case, as shown in the following Proposition.

Proposition 3.1. *Let*

$$f(x) = \frac{1 - \sqrt{1 - \frac{4x^4}{1-x^2}}}{2x^4}.$$

Then

$$f = \mathbf{J}\Big[\begin{matrix} (0)^* \\ 1, 1, 1, -1, -1, 2, 1/2, -1/2, -2, 3, 1/3, -1/3, -3, \ldots \end{matrix}\Big].$$

In other words, if $f = \mathbf{J}[\mathbf{u}/\mathbf{v}]$, *then* $u_k = 0$ *and* $v_{4k+1} = k, v_{4k+2} = 1/k, v_{4k+3} = -1/k, v_{4k+4} = -k$ *for every positive integer* k.

The proof of Proposition 3.1 is based on the following generalization with one more parameter z. Proposition 3.1′ becomes Proposition 3.1 when $z = 1$.

Proposition 3.1′. *Let*

$$f = f(x; z) = \frac{1 - (2z-1)x^2 - \sqrt{(1-x^2)(1-x^2-4x^4)}}{2x^2((1-z) + (1-z+z^2)x^2 - x^4)}.$$

Then

$$f = \mathbf{J}\Big[\begin{matrix} (0)^* \\ 1, z, 1/z, -1/z, -z, z+1, 1/(z+1), -1/(z+1), -(z+1), \ldots \end{matrix}\Big].$$

In other words, if $f = \mathbf{J}[\mathbf{u}/\mathbf{v}]$, *then* $u_k = 0$ *and* $v_{4k+1} = z + k, v_{4k+2} = 1/(z+k), v_{4k+3} = -1/(z+k), v_{4k+4} = -(z+k)$ *for every positive integer* k.

Proof. We need to check that $f(x; z)$ verifies the following functional equation:

$$f(x; z) = \cfrac{1}{1 - \cfrac{zx^2}{1 - \cfrac{\frac{1}{z}x^2}{1 + \cfrac{\frac{1}{z}x^2}{1 + zx^2 f(x; z+1)}}}}. \qquad \square$$

Let us explain how to get Proposition 3.1′ from Proposition 3.1. Let

$$f_1 = \frac{1 - \sqrt{1 - \frac{4x^4}{1-x^2}}}{2x^4}$$

and

$$f_1 = \mathbf{J}\left[\begin{array}{c} (0)^* \\ 1, 1, 1, -1, -1, 2, 1/2, -1/2, -2, 3, 1/3, -1/3, -3, \ldots \end{array}\right].$$

Define f_2 by deleting the first four pairs u_i, v_i $(i = 1, 2, 3, 4)$ from the J-fraction of f_1. In other words,

$$f_2 = \mathbf{J}\left[\begin{array}{c} (0)^* \\ 1, 2, 1/2, -1/2, -2, 3, 1/3, -1/3, -3, \ldots \end{array}\right].$$

By the very definition of the continued fraction we get the first values of f_2

$$f_2 = (1, 0, 2, 0, 5, 0, 12, 0, 30, 0, 75, 0, 190, 0, 483, 0, 1235, 0, 3167, \ldots).$$

With the help of a computer algebra system (see [Ru06] for example), we observe that f_2 satisfies the equation

$$(x^6 - 3x^4 + x^2)f_2^2 + (-3x^2 + 1)f_2 - 1 = 0.$$

Define f_3 by deleting the first four pairs u_i, v_i $(i = 1, 2, 3, 4)$ from the J-fraction of f_2 and repeat these steps, we successively get

$$\begin{gathered}(x^6 - 7x^4 + 2x^2)f_3^2 + (-5x^2 + 1)f_3 - 1 = 0, \\ (x^6 - 13x^4 + 3x^2)f_4^2 + (-7x^2 + 1)f_4 - 1 = 0, \\ \cdots\end{gathered}$$

and guess the general equation valid for every z

$$(x^6 - (z^2 - z + 1)x^4 + (z - 1)x^2)f_z^2 + (-(2z - 1)x^2 + 1)f_z - 1 = 0.$$

Solving the above equation yields the series $f(x; z)$, defined in Proposition 3.1′. The above procedure of finding generalization of J-fraction will be called the *chopping method.*

Proposition 3.2. *Let*

$$f(x) = \frac{1 - \sqrt{1 - \frac{4x^4}{1+x}}}{2x^4}.$$

Then

$$H(f(x)) = (1, 1, 0, 0, -1, -1, 0, 0)^*$$

As $H_3(f(x)) = 0$, the traditional method fails. We then have to find a polarization, as stated in the following example, which becomes Proposition 3.2 when $y = -1$ and $z = 0$. Notice that Proposition 3.1 is also a special case of Proposition 3.2′ by taking $y = 0$ and $z = 1$.

Proposition 3.2′. *Let*

$$f(x; y, z) = \frac{1 - \sqrt{1 - \frac{4x^4}{1-yx-zx^2}}}{2x^4}.$$

Then

$$f(x; y, z) = \mathbf{J}\begin{bmatrix} (-y, 0, 0, 0)^* \\ 1, z, 1/z, -1/z, -z, 2z, 1/(2z), -1/(2z), -(2z), \ldots \end{bmatrix}.$$

By Proposition 3.2′ and the fundamental relation (1.3), the Hankel determinants of $f(x; y, z)$ are

$$H(f(x; y, z)) = (1, 1, z, z, -1, -1, -2z, -2z, 1, 1, 3z, 3z, \ldots).$$

When $z = 0$ and $y = -1$ we get

$$H(f(x; -1, 0)) = (1, 1, 0, 0, -1 - 1, 0, 0)^*.$$

Proposition 3.2 is proved.

However we are not able to prove Proposition 3.2′ directly. By using the chopping method we find and prove the following generalization of Proposition 3.2′. Letting $t = 0$ in Proposition 3.2″ we get Proposition 3.2′.

Proposition 3.2″. *Let*

$$f(x) = -\frac{2\,ztx^2 + zx^2 + yx - 1 + \sqrt{(4\,x^4 + yx - 1 + zx^2)\,(yx - 1 + zx^2)}}{2x^2\,(-zx^4 - x^3y + x^2 + x^2z^2t + x^2z^2t^2 + yztx - zt)}$$

Then

$$f = \mathbf{J}\begin{bmatrix} (-y, 0, 0, 0)^* \\ 1, (t+1)z, \frac{1}{(t+1)z}, -\frac{1}{(t+1)z}, -(t+1)z, (t+2)z, \frac{1}{(t+2)z}, \ldots \end{bmatrix}.$$

By using the chopping methods, we derive the following continued fraction.

Theorem 3.3. *Let*

$$g = \frac{-2zx^2 - (sx - x^2y - 1) - \sqrt{(sx - x^2y - 1)^2 - (2x^2)^2}}{2x^2(x^2 + x^2z^2 + z(sx - x^2y - 1))}.$$

Then

$$g = \mathbf{J}\left[\begin{matrix} (-s,0)^* \\ v_0, v_1, v_2, \ldots \end{matrix}\right] = \cfrac{1}{1 - sx - \cfrac{\frac{\alpha_1}{\alpha_0}x^2}{1 - \cfrac{\frac{\alpha_0}{\alpha_1}x^2}{1 - sx - \cfrac{\frac{\alpha_2}{\alpha_1}x^2}{1 - \cfrac{\frac{\alpha_1}{\alpha_2}x^2}{\ddots}}}}}$$

where $v_{2k+1} = \alpha_{k+1}/\alpha_k$, $v_{2k+2} = \alpha_k/\alpha_{k+1}$ *and* α_n *is defined by*

$$\sum_n \alpha_n x^n = \frac{1 + zx}{1 + yx + x^2}.$$

Thus the Hankel determinants are

$$H(g) = (\alpha_0, \alpha_0, \alpha_1, \alpha_1, \alpha_2, \alpha_2, \alpha_3, \alpha_3, \ldots).$$

The proof of Theorem 3.3 is based on the following generalization.

Theorem 3.4. *Let* a_n, b_n, d_n, α_n *be numbers defined by the following generating functions*

$$\sum_{n\geq 0} a_n x^n = \frac{(1 - yz + z^2)(1 - (y^2 - 2)x + x^2)}{(1 - x)(1 - (y^2 - 2)x + x^2)},$$

$$\sum_{n\geq 0} b_n x^n = \frac{-z + y(yz - 1)x - z(yz - 1)x^2}{(1 - x)(1 - (y^2 - 2)x + x^2)},$$

$$\sum_{n\geq 0} d_n x^n = \frac{-1 - (1 + z^2 - 2yz)x - z^2x^2}{(1 - x)(1 - (y^2 - 2)x + x^2)},$$

$$\sum_n \alpha_n x^n = \frac{1 + zx}{1 + yx + x^2}$$

and $f_n(x)$ *by*

$$(a_nx^4 - sb_nx^3 + b_nx^2)f_n(x)^2 + ((yd_n - 2b_n)x^2 - sd_nx + d_n)f_n(x) = d_n. \tag{3.1}$$

Then

$$f_n(x) = \cfrac{1}{1 - sx - \cfrac{\frac{\alpha_{n+1}}{\alpha_n}x^2}{1 - \cfrac{\frac{\alpha_n}{\alpha_{n+1}}x^2}{1 - sx - \cfrac{\frac{\alpha_{n+2}}{\alpha_{n+1}}x^2}{1 - \cfrac{\frac{\alpha_{n+1}}{\alpha_{n+2}}x^2}{\ddots}}}}}$$

When $n = 0$, we have $a(0) = 1 - yz + z^2$; $b(0) = -z$; $d(0) = -1$. Solving (3.1) yields Theorem 3.3. For proving Theorem 3.4, we first convert it to Theorem 3.5, in which the coefficients are given by explicit formulas. This conversion is done by the following change of variables:

$$t + t^{-1} = y; \quad t = \frac{y + \sqrt{y^2 - 4}}{2}; \quad K = (-t)^n. \tag{3.2}$$

Theorem 3.5. *Let $a(K), b(K), d(K), \alpha(K)$ be numbers defined by*

$$\begin{aligned} a(K) &= z^2 - (t + t^{-1})z + 1 \\ b(K) &= -\frac{\big(t^2+1\big)\gamma_2\gamma_3}{\gamma_1^2} - \frac{t\gamma_2^2}{K^2\gamma_1^2} - \frac{K^2\gamma_3^2 t}{\gamma_1^2} \\ d(K) &= -\frac{2t\gamma_2\gamma_3}{\gamma_1^2} - \frac{\gamma_2^2}{K^2\gamma_1^2} - \frac{K^2t^2\gamma_3^2}{\gamma_1^2} \\ \alpha(K) &= -\frac{\gamma_2}{(1-t^2)K} - \frac{K\gamma_3 t}{1-t^2} \end{aligned}$$

where $\gamma_1 = (t-1)(t+1)$; $\gamma_2 = zt - 1$; $\gamma_3 = t - z$, and $f(x;K)$ by

$$\begin{aligned} &\big(a(K)x^4 - sb(K)x^3 + b(K)x^2\big)f(x;K)^2 \\ &\quad + \big(((t+t^{-1})d(K) - 2b(K))x^2 - sd(K)x + d(K)\big)f(x;K) = d(K). \end{aligned} \tag{3.3}$$

Then

$$f(x;K) = \cfrac{1}{1 - sx - \cfrac{\frac{\alpha(-tK)}{\alpha(K)}x^2}{1 - \cfrac{\frac{\alpha(K)}{\alpha(-tK)}x^2}{1 - sx - \cfrac{\frac{\alpha(t^2K)}{\alpha(-tK)}x^2}{1 - \cfrac{\frac{\alpha(-tK)}{\alpha(t^2K)}x^2}{\ddots}}}}}.$$

Proof. Solving (3.3) yields

$$f(x;K) = \frac{\left((sx-1)tQ_1 - (t-1)\,(t+1)\,Q_2\,x^2 + Q_1\sqrt{Q_0}\right)Q_1}{2x^2\left(t(1-sx)Q_3\,Q_1 + K^2\,(t-1)^2\,(t+1)^2\,(zt-1)\,(t-z)\,x^2\right)}$$

where

$$\begin{aligned}
Q_0 &= \left(sxt - t - x^2 - 2\,x^2t - x^2t^2\right)\left(sxt - t - x^2 + 2\,x^2t - x^2t^2\right);\\
Q_1 &= -1 + zt + K^2t^2 - K^2zt;\\
Q_2 &= K^2t^2 - zt - K^2zt + 1;\\
Q_3 &= -K^2z + zt^2 - t + K^2t.
\end{aligned}$$

Then, we can verify

$$f(x;K) = \frac{1}{1 - sx - \cfrac{\frac{\alpha(-tK)}{\alpha(K)}x^2}{1 - \frac{\alpha(-tK)}{\alpha(K)}x^2 f(x;-tK)}}. \quad \square$$

Proposition 3.6. *Let*

$$f = f(x) = \frac{1}{x^4}\sum_{k=1}^{\infty} \frac{x^{2^{k+1}}}{1 - x^{2^k}}.$$

Then, $H(f) \equiv (1,1,1,1,1,1,0,0)^* \pmod 2$.

Proof. We successively have

$$x^8 f(x^2) = \sum_{k=2}^{\infty} \frac{x^{2^{k+1}}}{1-x^{2^k}} = \sum_{k=1}^{\infty} \frac{x^{2^{k+1}}}{1-x^{2^k}} - \frac{x^4}{1-x^2} = x^4 f(x) - \frac{x^4}{1-x^2},$$

$$x^4 f(x^2) = f(x) - \frac{1}{1-x^2},$$

$$x^4 f(x)^2 \equiv f(x) - \frac{1}{1-x^2} \pmod 2,$$

$$f(x) \equiv \frac{1 - \sqrt{1 - \frac{4x^4}{1-x^2}}}{2x^4} \pmod 2.$$

Let g be the right-hand side of the above equation. By Proposition 3.1 we have

$$H(g) = (1,1,1,1,-1-1,-2,-2,1,1,3,3,-1,-1,-4,-4,1,1,5,5,\ldots).$$

In other words,

$$H_{4k}(g) = H_{4k+1}(g) = (-1)^k,$$
$$H_{4k+2}(g) = H_{4k+3}(g) = (-1)^k(k+1),$$

so that

$$H(f) \equiv H_n(g) \equiv (1,1,1,1,1,1,0,0)^* \pmod 2. \quad \square$$

Proposition 3.7. *Let*

$$f = f(x) = \frac{1}{x^4}\sum_{k=0}^{\infty}\frac{x^{2^{k+2}}}{1+x^{2^k}}.$$

Then, $H(f) \equiv (1,1,0,0)^* \pmod 2$.

Proof. Using the method described in the proof of Proposition 3.6, we derive

$$f(x) \equiv \frac{1-\sqrt{1-\frac{4x^4}{1+x}}}{2x^4} \pmod 2.$$

Let g be the right-hand side of the above equation. By Proposition 3.2

$$H(g) = (1,1,0,0,-1,-1,0,0)^*.$$

In other words,

$$H_{4k}(g) = H_{4k+1}(g) = (-1)^k,$$
$$H_{4k+2}(g) = H_{4k+3}(g) = 0.$$

Hence,

$$H(f) \equiv H_n(g) \equiv (1,1,0,0)^* \pmod 2. \quad \square$$

Proposition 3.8. *Let*

$$f = f(x) = \frac{1}{x^2}\sum_{k=0}^{\infty}\frac{x^{2^{k+1}}}{1+x^{2^{k+1}}}.$$

Then $H(f) \equiv (1,1,0,0,1,1)^* \pmod 2$.

Proof. Using the method described in the proof of Proposition 3.6, we derive

$$f(x) \equiv \frac{1-\sqrt{1-\frac{4x^2}{1+x^2}}}{2x^2} \pmod 2.$$

Let g be the right-hand side of the above equality and let α_n be defined by

$$\sum_n \alpha_n x^n = \frac{1-x}{1-x+x^2} = (1,0,-1,-1,0,1)^*.$$

By Theorem 3.3 we have

$$H(g) = (1,1,0,0,-1,-1,-1,-1,0,0,1,1)^*.$$

Hence

$$H(f) \equiv H(g) \equiv (1,1,0,0,1,1)^* \pmod 2. \qquad \square$$

4. One sequence, two modulos

Theorem 1.3 is proved in Section 2 by using the fractional congruence. In fact, the fractional congruence can be avoided.

Proof of Theorem 1.3. We successively have

$$\begin{gathered}
P_3(x) = (1-x)P_3(x^3) \equiv (1-x)P_3(x)^3 \pmod 3,\\
P_3(x)(1-(1-x)P_3(x)^2) \equiv 0 \pmod 3,\\
1-(1-x)P_3(x)^2 \equiv 0 \pmod 3,\\
P_3(x)^2 \equiv \frac{1}{1-x} \pmod 3,\\
P_3(x)^2 \equiv \frac{1}{(1-x)(1+3x)} \pmod 3,\\
\Bigl(P_3(x) - \sqrt{\frac{1}{(1-x)(1+3x)}}\Bigr)\Bigl(P_3(x) + \sqrt{\frac{1}{(1-x)(1+3x)}}\Bigr) \equiv 0 \pmod 3.
\end{gathered}$$

We then have

$$P_3(x) \equiv \sqrt{\frac{1}{(1-x)(1+3x)}} \pmod 3 \tag{4.1}$$

by using the value of $P(0)$. The right-hand side of the above equation has integral coefficients and its J-fraction is equal to:

$$\sqrt{\frac{1}{(1-x)(1+3x)}} = \mathbf{J}\begin{bmatrix} 2, (1)^* \\ (-1)^* \end{bmatrix}, \tag{4.2}$$

so that $H_n(P_3) \equiv 2^{n-1} \equiv (-1)^{n-1} \pmod 3$ by (1.3). $\square$

Next we will prove the "one sequence, two modulos" theorem 1.4. We need the following lemma.

Lemma 4.1. *We have*

$$\sqrt{1-4x} \equiv 1 + 2\sum_{k=0}^{\infty} x^{2^k} \pmod 4.$$

Proof. The following expansion is well known (See [St99, WiCa] for example)

$$\frac{1-\sqrt{1-4x}}{2x} = \sum_{k=0}^{\infty} C_n x^n,$$

where $C_n = \frac{1}{n+1}\binom{2n}{n}$ is the Catalan number. It is easy to see that $C_n \equiv 1 \pmod 2$ if and only if $n = 2^k - 1$ for some integer k [AK73], knowing, for instance, that C_n is the number of binary trees with n vertices. ∎

Proof of Theorem 1.4. We need to prove

$$\sqrt{\frac{1}{(1-x)(1+3x)}} \equiv \prod_{k=0}^{\infty}(1-x^{2^k}) \pmod 4, \tag{4.3}$$

$$\sqrt{\frac{1}{(1-x)(1+3x)}} \equiv \prod_{k=0}^{\infty}(1-x^{3^k}) \pmod 3. \tag{4.4}$$

The second equality is just relation (4.1). For proving the first equality let $f(x)$ be the left-hand side of (4.3). By Lemma 4.1, we get

$$(1-x)f(x) = \sqrt{\frac{1-x}{1+3x}} = \sqrt{1-\frac{4x}{1+3x}} \equiv 1 + 2\sum_{k=0}^{\infty}\Big(\frac{x}{1+3x}\Big)^{2^k} \pmod 4,$$

$$(1-x)f(x) \equiv 1 + 2\sum_{k=0}^{\infty}\Big(\frac{x}{1+x}\Big)^{2^k} \pmod 4,$$

and

$$\begin{aligned}(1-x^2)f(x^2) &\equiv 1 + 2\sum_{k=0}^{\infty}\Big(\frac{x^2}{1+x^2}\Big)^{2^k} \\ &\equiv 1 + 2\sum_{k=0}^{\infty}\Big(\frac{x}{1+x}\Big)^{2^{k+1}} \qquad \text{[By Lemma 2.1]} \\ &\equiv (1-x)f(x) - \frac{2x}{1+x} \pmod 4.\end{aligned} \tag{4.5}$$

Let $P_2(x)$ be the right-hand side of (4.3). Then,

$$(1-x)P_2(x^2) = P_2(x),$$

which implies on one hand $(1-x)(P_2(x))^2 \equiv P_2(x) \pmod 2$, hence

$$P_2(x) \equiv \frac{1}{1+x} \pmod 2,$$

and on the other hand

$$(1-x^2)P_2(x^2) = (1+x)(1-x)P_2(x^2) = (1+x)P_2(x).$$

Hence,

$$(1-x^2)P_2(x^2) - (1-x)P_2(x) \equiv \frac{2x}{1+x} \pmod 4. \tag{4.6}$$

Taking the difference of (4.6) and (4.5) yields

$$(1-x^2)f(x^2) - (1-x^2)P_2(x^2) \equiv (1-x)(f(x)-P_2(x)) \pmod 4,$$

and

$$f(x) - P_2(x) \equiv (1+x)(f(x^2) - P_2(x^2)) \pmod 4. \tag{4.7}$$

By applying (4.7) recursively we get $f(x) - P_2(x) \equiv 0 \pmod 4$, since $f(0) = P_2(0) = 1$. $\square$

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