André Permutation Calculus: a Twin Seidel Matrix Sequence

Dominique Foata and Guo-Niu Han

Abstract. Entringer numbers occur in the André permutation combinatorial set-up under several forms. This leads to the construction of a matrix refinement of the tangent (respectively secant) numbers. Furthermore, closed expressions for the three-variate exponential generating functions for pairs of so-called Entringerian statistics are derived.

1. Introduction
   1. Entringer numbers
   2. André permutations
   3. Statistics on André permutations
   4. A further bijection between André permutations
   5. The Twin Seidel matrix sequence
   6. Tight and hooked permutations
   7. Trivariate generating functions
2. From Alternating To André Permutations Of The First Kind
   3. The Bijection $\phi$ Between André I And André II Permutations
   4. The Bijection $g$ From The Set Of André I Permutations Onto Itself
   5. The Proof Of Theorem 1.1 (iii) And (iv)
6. Combinatorics Of The Twin Seidel Matrix Sequence
   1. The first evaluations
   2. Tight and not tight André I Permutations
   3. Hooked and unhooked Permutations
   4. A Bijection between $B_n(m,k)$ and $NH_n(m+1,k)$
7. The Making Of Seidel Triangle Sequences
   1. The Seidel tangent-secant matrix
   2. The generating function for the Entringer numbers
   3. Seidel triangle sequences
8. Trivariate Generating Functions
   1. The upper triangles of Twin$^{(1)}$
   2. The upper triangles of Twin$^{(2)}$
   3. The bottom rows of the Matrices $B_n$
   4. The lower triangles of Twin$^{(1)}$
   5. The lower triangles of Twin$^{(2)}$
9. The Formal Laplace Transform
   References

Key words and phrases. Entringer numbers, tangent and secant numbers, alternating permutations, André permutations, Seidel matrix sequence, increasing binary trees, greater neighbor of maximum, spike, pit, tight permutations, hooked permutations, Seidel triangle sequence, formal Laplace transform.

Mathematics Subject Classifications. 05A15, 05A30, 11B68, 33B10.
1. Introduction

The notion of alternating or zigzag permutation devised by Désiré André, back in 1881 [An1881], for interpreting the coefficients \( E(n) \) \((n \geq 0)\) of the Taylor expansion of \( \tan u + \sec u \), the so-called tangent and secant numbers, has remained some sort of a curiosity for a long time, until it was realized that the geometry of these alternating permutations could be exploited to obtain further arithmetic refinements of these numbers. Classifying alternating permutations according to the number of inversions directly leads to the constructions of their \( q \)-analogos (see [AF80, AG78, St76]). Sorting them according to their first letters led Entinger [En66] to obtain a fruitful refinement \( E_n = \sum_m E_n(m) \) that has been described under several forms [OEIS, GHZ11, KPP94, MSY96, St10], the entries \( E_n(m) \) satisfying a simple finite difference equation (see (1.1) below).

In fact, these numbers \( E_n(m) \), called **Entringer numbers** in the sequel, appear in other contexts, in particular when dealing with analytical properties of the André permutations, of the two kinds I and II, introduced by Schützenberger and the first author ([FSch73, FSch71]). For each \( n \geq 1 \) let \( \text{And}_n^I \) (respectively \( \text{And}_n^H \)) be the set of all André permutations of \( 12 \cdots n \) (see §1.2). It was shown that \( \# \text{And}_n^I = \# \text{And}_n^H = E_n \). The first purpose of this paper is to show that there are several natural statistics "stat," defined on \( \text{And}_n^I \) (respectively \( \text{And}_n^H \)), whose distributions are Entringerian, that is, integer-valued mappings "stat," satisfying

\[
\# \{ w \in \text{And}_n^I \text{ (respectively And}_n^H) : \text{stat}(w) = m \} = E_n(m).
\]

The second purpose is to work out a matrix refinement \( E_n = \sum_{m,k} a_n(m,k) \) of the tangent and secant numbers, whose row and column sums \( \sum_k a_n(m,k) \) and \( \sum_m a_n(m,k) \) are themselves refinements of the Entringer numbers. This will be achieved, first by inductively defining the so-called **twin Seidel matrix sequence** \((A_n, B_n) \ (n \geq 2)\) (see §1.5), then by proving that the entries of these matrices provide the joint distributions of pairs of Entringerian statistics defined on André permutations of each kind (Theorem 1.2). See §1.6 for the plan of action.

The third purpose is to obtain analytical expressions for the joint exponential generating functions for pairs of these Entringerian statistics. See §1.7 and the contents of Section 7 and 8. Let us give more details on the notions introduced so far.

1.1. **Entringer numbers.** According to Désiré André [An1879, An1881] each permutation \( w = x_1x_2 \cdots x_n \) of \( 12 \cdots n \) is said to be (increasing) alternating if \( x_1 < x_2, x_2 > x_3, x_3 < x_4, \text{ etc.} \) in an alternating way. Let \( \text{Alt}_n \) be the set of all alternating permutations of \( 12 \cdots n \). He then proved that \( \# \text{Alt}_n = E_n \), where \( E_n \) is the tangent number (respectively secant...
number) when \( n \) is odd (respectively even), these numbers appearing in the Taylor expansions of \( \sec u \) and \( \tan u \):

\[
\tan u = \sum_{n \geq 1} \frac{u^{2n-1}}{(2n-1)!} E_{2n-1} = \frac{u}{1!} + \frac{u^3}{3!} + \frac{u^5}{5!} + \frac{u^7}{7!} + \frac{u^9}{9!} + \cdots,
\]

\[
\sec u = \sum_{n \geq 0} \frac{u^{2n}}{(2n)!} E_{2n} = 1 + \frac{u^2}{2!} + \frac{u^4}{4!} + \frac{u^6}{6!} + \frac{u^8}{8!} + 1385 + \frac{u^{10}}{10!} + \cdots.
\]

(See, e.g., [Ni23, pp. 177–178], [Co74, pp. 258–259]).

Let \( F_w := x_1 \) be the first letter of a permutation \( w = x_1 x_2 \cdots x_n \) of 12· · ··n. For each \( m = 1, \ldots, n \), the Entringer numbers are defined by \( E_n(m) := \#\{w \in \text{Alt}_n : F_w = m\} \), as was introduced by Entringer [En66]. In particular, \( E_n(n) = 0 \) for \( n \geq 2 \). He showed that these numbers satisfy the recurrence

\[
E_1(1) := 1; \quad E_n(n) := 0 \text{ for all } n \geq 2;
\]

\[
\Delta E_n(m) + E_{n-1}(n-m) = 0 \quad (n \geq 2; m = n-1, \ldots, 2, 1);
\]

where \( \Delta \) stands for the classical finite difference operator (see, e.g. [Jo39])

\[
\Delta E_n(m) := E_n(m+1) - E_n(m).
\]

See Fig. 1.1 for the table of their first values. These numbers are registered as the sequence A008282 in Sloane’s On-Line Encyclopedia of Integer Sequences, together with an abundant bibliography [OEIS]. They naturally constitute a refinement of the tangent and secant numbers:

\[
\sum_m E_n(m) = E_n = \begin{cases} 
\text{tangent number}, & \text{if } n \text{ is odd}; \\
\text{secant number}, & \text{if } n \text{ is even}.
\end{cases}
\]

<table>
<thead>
<tr>
<th>( m )</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>Sum</th>
</tr>
</thead>
<tbody>
<tr>
<td>( n = 1 )</td>
<td>1</td>
<td>1</td>
<td>2</td>
<td>3</td>
<td>4</td>
<td>5</td>
<td>6</td>
<td>7</td>
<td>8</td>
<td>9</td>
</tr>
<tr>
<td>2</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>2</td>
<td>5</td>
<td>16</td>
<td>61</td>
<td>272</td>
<td>1385</td>
<td>7936</td>
</tr>
<tr>
<td>3</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>2</td>
<td>5</td>
<td>16</td>
<td>61</td>
<td>272</td>
<td>1385</td>
<td>7936</td>
</tr>
<tr>
<td>4</td>
<td>2</td>
<td>1</td>
<td>0</td>
<td>2</td>
<td>5</td>
<td>16</td>
<td>61</td>
<td>272</td>
<td>1385</td>
<td>7936</td>
</tr>
<tr>
<td>5</td>
<td>5</td>
<td>4</td>
<td>2</td>
<td>0</td>
<td>16</td>
<td>61</td>
<td>272</td>
<td>1385</td>
<td>7936</td>
<td></td>
</tr>
<tr>
<td>6</td>
<td>16</td>
<td>14</td>
<td>10</td>
<td>5</td>
<td>0</td>
<td>61</td>
<td>272</td>
<td>1385</td>
<td>7936</td>
<td></td>
</tr>
<tr>
<td>7</td>
<td>61</td>
<td>56</td>
<td>46</td>
<td>32</td>
<td>16</td>
<td>0</td>
<td>272</td>
<td>1385</td>
<td>7936</td>
<td></td>
</tr>
<tr>
<td>8</td>
<td>272</td>
<td>272</td>
<td>256</td>
<td>224</td>
<td>178</td>
<td>122</td>
<td>61</td>
<td>0</td>
<td>1385</td>
<td></td>
</tr>
<tr>
<td>9</td>
<td>1385</td>
<td>1385</td>
<td>1324</td>
<td>1202</td>
<td>1024</td>
<td>800</td>
<td>544</td>
<td>272</td>
<td>0</td>
<td>7936</td>
</tr>
</tbody>
</table>

Fig. 1.1. The Entringer Numbers \( E_n(m) \)

Now, let \( L_w := x_n \) denote the last letter of a permutation \( w = x_1 x_2 \cdots x_n \) of 12· · ··n. In our previous paper [FH14] we made a full study of the so-called Bi-Entringer numbers defined by

\[
E_n(m, k) := \#\{w \in \text{Alt}_n : F_w = m, L_w = k\},
\]
and showed that the sequence of the matrices \((E_n(m,k))_{1 \leq m,k \leq n}\) \((n \geq 1)\) was fully determined by a partial difference equation system and the three-variable exponential generating function for these matrices could be calculated. As the latter analytical derivation essentially depends on the geometry of alternating permutations, it is natural to ask whether other combinatorial models, counted by tangent and secant numbers, are likely to have a parallel development.

Let \(E(u) := \tan u + \sec u = \sum_{n \geq 0} (u^n / n!) E_n\). Then the first and second derivatives of \(E(u)\) are equal to \(E'(u) = E(u) \sec u\) and \(E''(u) = E(u) E'(u)\), two identities equivalent to the two recurrence relations

\[
(*) \quad E_{n+1} = \sum_{0 \leq 2j \leq n} \binom{n}{2j} E_{n-2j} E_{2j} \quad (n \geq 0), \quad E_0 = 1;
\]

\[
(**) \quad E_{n+2} = \sum_{0 \leq j \leq n} \binom{n}{j} E_j E_{n+1-j} \quad (n \geq 0), \quad E_0 = E_1 = 1.
\]

The first of these relations can be readily interpreted in terms of alternating permutations, or in terms of the so-called Jacobi permutations introduced by Viennot [Vi80]. The second one leads naturally to the model of Andrè permutations, whose geometry will appear to be rich and involves several analytic developments.

1.2. Andrè permutations. These permutations were introduced in [FSch73, FSch71], and further studied in [Str74, FSt74, FSt76]. Other properties have been developed in work by Purtill [Pu93], Hetyei [He96], Hetyei and Reiner [HR98], the present authors [FH01], Stanley [St94], in particular in the study of the \(cd\)-index in a Boolean algebra. More recently, Disanto [Di14] has been able to calculate the joint distribution of the right-to-left minima and left-to-right minima in these permutations.

In the sequel, permutations of a finite set \(Y = \{y_1 < y_2 < \cdots < y_n\}\) of positive integers will be written as words \(w = x_1 x_2 \cdots x_n\), where the letters \(x_i\) are the elements of \(Y\) in some order. The minimum (respectively maximum) letter of \(w\), in fact, \(y_1\) (respectively \(y_n\)), will be denoted by \(\min(w)\) (respectively \(\max(w)\)). When writing \(w = v \min(w) v'\) it is meant that the word \(w\) is the juxtaposition product of the left factor \(v\), followed by the letter \(\min(w)\), then by the right factor \(v'\).

**Definition.** Say that the empty word \(e\) and each one-letter word are both Andrè I and Andrè II permutations. Next, if \(w = x_1 x_2 \cdots x_n\) \((n \geq 2)\) is a permutation of a set of positive integers \(Y = \{y_1 < y_2 < \cdots < y_n\}\), write \(w = v \min(w) v'\). Then, \(w\) is said to be an Andrè I (respectively Andrè II) permutation if both \(v\) and \(v'\) are themselves Andrè I (respectively Andrè II) permutations, and furthermore if \(\max(vv')\) (respectively \(\min(vv')\)) is a letter of \(v'\).
The set of all André I (respectively André II) permutations of \( Y \) is denoted by \( \text{And}_I^Y \) (respectively \( \text{And}_I^H \)) and simply by \( \text{And}_I^n \) (respectively \( \text{And}_I^H \)) when \( Y = \{1, 2, \ldots, n\} \). In the sequel, an André I (respectively André II) permutation, with no reference to a set \( Y \), is meant to be an element of \( \text{And}_I^n \) (respectively \( \text{And}_I^H \)).

Using such an inductive definition, we can immediately see that \( E_n = \# \text{And}_I^n = \# \text{And}_I^H \), the term \( \binom{n}{j} E_j E_{n+1-j} \) in (**) being the number of all André I (respectively André II) permutations of \( x_1 x_2 \cdots x_n + 2 \) such that \( x_j = 1 \). Further equivalent definitions will be given in the beginning of Section 2. The first André permutations from \( \text{And}_I^n \) and \( \text{And}_I^H \) are listed in Table 1.2.

### André permutations of the first kind:

- \( n = 1 \): 1
- \( n = 2 \): 12
- \( n = 3 \): 123, 213
- \( n = 4 \): 1234, 1324, 2314, 2134, 3124
- \( n = 5 \): 12345, 12435, 13425, 23415, 13245, 14235, 31245, 24135, 23145, 21345, 41235, 31425, 21435, 32415, 41325, 31245.

### André permutations of the second kind:

- \( n = 1 \): 1
- \( n = 2 \): 12
- \( n = 3 \): 123, 312
- \( n = 4 \): 1234, 1423, 3412, 4123, 3124
- \( n = 5 \): 12345, 12534, 14523, 34512, 15234, 14235, 34125, 45123, 35124, 51234, 41235, 31245, 51423, 53412, 41523, 31524.

Table 1.2: the first André permutations of both kinds

### 1.3. Statistics on André permutations

The statistics “\( F \)” and “\( L \)” have been previously introduced. Two further ones are now defined: the next-to-last (or the penultimate) letter “\( \text{NL} \)” and greater neighbor of the maximum “\( \text{grn} \)”:

- for \( w = x_1 x_2 \cdots x_n \) and \( n \geq 2 \) let \( \text{NL} w := x_{n-1} \); next, let \( x_i = n \) for a certain \( i \) (\( 1 \leq i \leq n \)) with the convention that \( x_0 = x_{n+1} := 0 \).
- Then, \( \text{grn} w := \max \{x_{i-1}, x_{i+1}\} \).

Let \( (\text{Ens}_n) \) \( (n \geq 1) \) be a sequence of non-empty finite sets and “\( \text{stat} \)” an integer-valued mapping \( w \mapsto \text{stat}(w) \) defined on each \( \text{Ens}_n \).

The pair \( (\text{Ens}_n, \text{stat}) \) is said to be Entringerian, if \( \#\text{Ens}_n = E_n \) and \( \#{w \in \text{Ens}_n : \text{stat}(w) = m} = E_n(m) \) holds for each \( m = 0, 1, \ldots, n \). We also say that “\( \text{stat} \)” is an Entringerian statistic. The pair \( (\text{Alt}_n, F) \) is Entringerian, par excellence, for all \( n \geq 1 \).

**Theorem 1.1.** For each \( n \geq 2 \) the mappings

(i) \( F \) defined on \( \text{And}_I^n \),

(ii) \( n - \text{NL} \) defined on \( \text{And}_I^n \),

(iii) \( (n + 1) - L \) defined on \( \text{And}_I^H \),

(iv) \( n - \text{grn} \) defined on \( \text{And}_I^H \),

are all Entringerian statistics.
Statements (i) and (ii) will be proved in Section 2 by constructing two bijections $\eta$ and $\theta$ having the property

\begin{equation}
\begin{array}{c}
\text{Alt}_n \xrightarrow{\eta} \text{And}_n^I \xrightarrow{\theta} \text{And}_n^I \\
Fw \mapsto w' \mapsto w'' \\
Fw = Fw' = (n - NL)w''
\end{array}
\end{equation}

For proving (iii) and (iv) we use the properties of two new bijections $\phi : \text{And}_n^I \rightarrow \text{And}_n^H$ and $g : \text{And}_n^I \rightarrow \text{And}_n^I$, whose constructions are described in Sections 3 and 4. By means of these two bijections, as well as the bijection $\theta$ mentioned in (1.3), it will be shown in Section 5 that the following properties hold:

\begin{equation}
\begin{array}{c}
\text{And}_n^I \xrightarrow{\phi \circ g} \text{And}_n^H \\
Fw \mapsto w' \mapsto (n + 1 - L)w'
\end{array}
\end{equation}

and

\begin{equation}
\begin{array}{c}
\text{And}_n^I \xrightarrow{\phi \circ \theta} \text{And}_n^H \\
Fw \mapsto w'' \mapsto (n - \text{grn})w''
\end{array}
\end{equation}

thereby completing the proof of Theorem 1.1.

1.4. The fundamental bijection $\phi$. For proving (1.4) and (1.5) and also the next Theorem 1.2, two new statistics are to be introduced, the spike “$\text{spi}$” and the pit “$\text{pit}$”, related to the left minimum records for the former one, and the right minimum records for the latter one. In Section 3 the bijection $\phi$ between $\text{And}_n^I$ and $\text{And}_n^H$ will be shown to have the further property:

\begin{equation}
(F, \text{spi}, NL)w = (\text{pit}, L, \text{grn})\phi(w).
\end{equation}

This implies that

\begin{equation}
\text{for each pair } (m, k) \text{ the two sets } \{w \in \text{And}_n^I : (\text{spi}, NL)w = (m, k)\} \text{ and } \{w \in \text{And}_n^H : (L, \text{grn})w = (m, k)\} \text{ are equipotent.}
\end{equation}

It also follows from Theorem 1.1 that “$(n + 1) - \text{spi}$” on $\text{And}_n^I$ and “$\text{pit}$” on $\text{And}_n^H$ are two further Entringerian statistics.

1.5. The twin Seidel matrix sequence. The next step is to say something about the joint distributions of the pairs $(F, NL)$ on $\text{And}_n^I$ and $(L, \text{grn})$ on $\text{And}_n^H$, whose marginal distributions are Entringerian, as announced in Theorem 1.1. We shall proceed in the following way: first, the notion of twin Seidel matrix sequence $(A_n, B_n)$ ($n \geq 2$) will be introduced
(see Definition below), then the entry in cell \((m, k)\) of \(A_n\) (respectively \(B_n\)) will be shown to be the number of André I (respectively II) permutations \(w\), whose values \((F, NL)w\) (respectively \((L, grn)w\)) are equal to \((m, k)\). The definition involves the partial difference operator \(\Delta\) acting on sequences \((a_n(m, k))\) \((n \geq 2)\) as follows

\[
\Delta_{(1)} a_n(m, k) := a_n(m + 1, k) - a_n(m, k).
\]

The subscript \((1)\) indicates that the difference operator is to be applied to the variable occurring at the first position, which is \('m'\) in the previous equation.

Definition. The twin Seidel matrix sequence \((A_n, B_n)\) \((n \geq 2)\) is a sequence of finite square matrices that obey the following five rules (TS1)–(TS5) (see Diagram 1.3 for the values of the first matrices, where zero entries are replaced by dots):

(TS1) each matrix \(A_n = (a_n(m, k))\) (respectively \(B_n = b_n(m, k))\) \((1 \leq m, k \leq n)\) is a square matrix of dimension \(n\) \((n \geq 2)\) with nonnegative entries, and zero entries along its diagonal, except for \(a_2(1, 1) = 1; \) let \(a_n(m, \bullet) = \sum_k a_n(m, k)\) (respectively \(a_n(\bullet, k) = \sum_m a_n(m, k)\)) be the \(m\)-th row sum (respectively \(k\)-th column sum) of the matrix \(A_n\) with an analogous notation for \(B_n\);

(TS2) for \(n \geq 3\) the entries along the rightmost column in both \(A_n\) and \(B_n\) are zero, as well as the entries in the bottom row of \(A_n\) and the top row of \(B_n\), i.e., \(a_n(\bullet, n) = b_n(\bullet, n) = a_n(n, \bullet) = b_n(1, \bullet) = 0, \) as all the entries are supposed to be nonnegative; furthermore, \(b_n(n, 1) = 0\);

(TS3) the first two matrices of the sequence are supposed to be:

\[
A_2 = \begin{bmatrix} 1 & \cdots & \cdots \\ \vdots & \ddots & \vdots \\ \cdots & \cdots & 1 \end{bmatrix}, \quad B_2 = \begin{bmatrix} \cdots & \cdots & 1 \\ \vdots & \ddots & \vdots \\ 1 & \cdots & \cdots \end{bmatrix}.
\]

(TS4) for each \(n \geq 3\) the matrix \(B_n\) is derived from the matrix \(A_{n-1}\) by means of a transformation \(\Psi : (a_{n-1}(m, k)) \to (b_n(m, k))\) defined as follows

\[
\begin{align*}
(TS4.1) \quad & b_n(n, k) := a_{n-1}(\bullet, k - 1) \quad (2 \leq k \leq n - 1); \\
(TS4.2) \quad & b_n(n - 1, k) := a_{n-1}(\bullet, k) \quad (2 \leq k \leq n - 2);
\end{align*}
\]

and, by induction,

\[
\begin{align*}
(TS4.3) \quad & \Delta_{(1)} b_n(m, k) - a_{n-1}(m, k) = 0 \quad (2 \leq k + 1 \leq m \leq n - 2); \\
(TS4.4) \quad & \Delta_{(1)} b_n(m, k) - a_{n-1}(m, k - 1) = 0 \quad (3 \leq m + 2 \leq k \leq n - 1);
\end{align*}
\]
(TS5) for each $n \geq 3$ the matrix $A_n$ is derived from the matrix $B_{n-1}$ by means of a transformation $\Phi : (b_{n-1}(m, k)) \to (a_n(m, k))$ defined as follows

$$a_n(1, k) := b_{n-1}(1, k - 1) \quad (2 \leq k \leq n - 1);$$

and, by induction,

$$\Delta a_n(m, k) + b_{n-1}(m, k - 1) = 0 \quad (3 \leq m + 2 \leq k \leq n - 1);$$

$$\Delta a_n(m, k) + b_{n-1}(m, k) = 0 \quad (2 \leq k + 1 \leq m \leq n - 1).$$

Diagram 1.3: First values of the twin Seidel matrices

It is worth noting that the twin Seidel matrix sequence involves two infinite subsequences: Twin$^{(1)} = (A_2, A_3, B_3, A_4, B_4, A_5, \ldots)$ and Twin$^{(2)} = (B_2, A_3, B_4, A_5, B_6, \ldots)$. They are independent in the sense that the matrices $A_{2m}$ (respectively $B_{2n}$) depend only on the matrices $B_{2m+1}$ and $A_{2m}$ (respectively $A_{2m+1}$ and $B_{2m}$) with $m < n$, with an analogous statement for the matrices $A_{2n+1}$ (respectively $B_{2n+1}$).

As is easily verified, Rules (TS1)–(TS5) define the twin Seidel matrix sequence by induction in a unique manner. At each step Rules (TS1) and (TS2) furnish all the zero entries indicated by dots, and Rules (TS4.1), (TS4.2), (TS5.1) the initial values. It remains to use the finite difference equations (TS4.3), (TS4.4), (TS5.2), (TS5.3) to calculate the other entries.
Theorem 1.2. The twin Seidel matrix sequence \((A_n = (a_n(m,k)), B_n = (b_n(m,k)))\) \((n \geq 2, 1 \leq m, k \leq n)\) defined by relations (TS1)–(TS5) provides the joint distributions of the pairs \((F, NL)\) on \(\text{And}_n^I\) and \((L, \text{grn})\) on \(\text{And}_n^H\) in the sense that for \(n \geq 2\) the following relations hold:

\[
\begin{align*}
(1.8) \quad a_n(m,k) &= \# \{ w \in \text{And}_n^I : (F, NL)w = (m,k) \}; \\
(1.9) \quad b_n(m,k) &= \# \{ w \in \text{And}_n^H : (L, \text{grn})w = (m,k) \}.
\end{align*}
\]

By Theorems 1.1 and 1.2, the row and column sums of the matrices \(A_n\) and \(B_n\) have the following interpretations:

\[
\begin{align*}
(1.10) \quad a_n(m,\bullet) &= E_n(m), \quad b_n(m,\bullet) = E_n(n+1-m), \quad (1 \leq m \leq n); \\
(1.11) \quad a_n(\bullet,k) &= b_n(\bullet,k) = E_n(n-k), \quad (1 \leq k \leq n);
\end{align*}
\]

and furthermore the matrix-analog of the refinement of \(E_n\) holds:

\[
\sum_{m,k} a_n(m,k) = \sum_{m,k} b_n(m,k) = E_n.
\]

1.6. Tight and hooked permutations. For proving Theorem 1.2, the crucial point is to show that the \(a_n(m,k)\)'s and \(b_n(m,k)\)'s satisfy the partial difference equations (TS4.3), (TS4.4), (TS5.2), (TS5.3), when these numbers are defined by the right-hand sides of (1.8) and (1.9). For each pair \((m,k)\) let

\[
\begin{align*}
A_n(m,k) &= \{ w \in \text{And}_n^I : (F, NL)w = (m,k) \}; \\
B_n(m,k) &= \{ w \in \text{And}_n^H : (L, \text{grn})w = (m,k) \}.
\end{align*}
\]

As the latter set is equinumerous with the set \(\{ w \in \text{And}_n^H : (L, \text{grn})w = (m,k) \}\) by (1.7), we also have \(a_n(m,k) = \#A_n(m,k)\) and \(b_n(m,k) = \#B_n(m,k)\), by (1.8) and (1.9).

For the partial difference equation (TS5.2) (respectively (TS5.3)) the plan of action may be described by the diagram

\[
\begin{align*}
B_{n-1}(m,k-1) & \quad \text{(respectively } B_{n-1}(m,k)) \\
\downarrow \phi & \quad \uplus \quad \text{NT}_n(m,k) \\
A_n(m,k) & \quad \text{(respectively } A_n(m+1,k)) \\
\downarrow f & \\
A_n(m+1,k) &
\end{align*}
\]

This means that the set \(A_n(m,k)\) is to be split into two disjoint subsets, \(A_n(m,k) = T_n(m,k) \uplus \text{NT}_n(m,k)\), in such a way that the first component is in bijection with \(B_{n-1}(m,k-1)\) (respectively \(B_{n-1}(m,k)\)) by using
the bijection $\phi$ defined in (6.6), and the second one with $A_n(m + 1, k)$ by means of the bijection $f$ defined in (6.5). If this plan is realized, the above partial difference equations are satisfied, as

$$
\Delta a_n(m, k) = \#A_n(m + 1, k) - \#A_n(m, k)
= \#NT_n(m, k) - \#A_n(m, k)
= -\#T_n(m, k)
= -\#B_{n-1}(m, k - 1) \text{ (respectively } - \#B_{n-1}(m, k))
= -b_{n-1}(m, k - 1) \text{ (respectively } - b_{n-1}(m, k)).
$$

For the partial difference equation (TS4.3) (respectively (TS4.4)) the corresponding diagram is the following

$$
A_{n-1}(m, k) \quad \text{(respectively } A_{n-1}(m, k - 1) \text{)}
\downarrow \Theta
B_n(m + 1, k) = H_n(m + 1, k) \uplus NH_n(m + 1, k)
\uparrow \beta
B_n(m, k)
$$

where $\Theta$ and $\beta$ are two explicit bijections, defined in (6.8) and (6.13), (6.14), respectively. The elements in $T_n(m, k)$ from (1.13) (respectively in $H_n(m + 1, k)$ from (1.14)) are the so-called tight (respectively hooked) permutations. All details will be given in Section 6 and constitute the bulk of the proof of Theorem 1.2.

1.7. Trivariate generating functions. The final step is to show that the partial difference equation systems (TS4.3), (TS4.4), (TS5.2), (TS5.3) satisfied by the twin Seidel matrix sequence $(A_n, B_n)$ $(n \geq 2)$ make it possible to derive closed expressions for the trivariate generating functions for the sequences $(A_{2n})$, $(A_{2n+1})$, $(B_{2n})$, $(B_{2n+1})$. We list them all in the following theorems. See Section 8 for the detailed proofs.

The calculations are all based on the Seidel triangle sequence technique developed in our previous paper [FH14]. Note that the next generating functions for the matrices $A_n$ do not involve the entries of the rightmost columns and bottom rows, which are all zero; they do not involve the entries of the rightmost columns of the matrices $B_n$ either, since these are also equal to zero, as assumed in (TS2). Finally, the generating functions for the bottom rows of the matrices $B_n$ are calculated separately; see (1.23) and (1.24).

Also, note that the right-hand sides of identities (1.15)–(1.18) are all symmetric with respect to $x$ and $z$, in agreement with their combinatorial
interpretation stated in Theorem 2.4. The property is less obvious for (1.16), but an easy exercise on trigonometry shows that the right-hand side is equal to the fraction \(\frac{\cos x \cos z \sin(x + y + z) - \sin y}{\cos^2(x + y + z)}\). Finally, the summations below are taken over triples \(((m, k, n))\) or pairs \(((k, n))\) for the last two ones; only the ranges of the summations have been written.

**Theorem 1.3** [The sequence \((A_{2n})\) \((n \geq 1)\)]. The generating function for the upper triangles is given by

\[
\sum_{2 \leq m+1 \leq k \leq 2n-1} a_{2n}(m, k) \frac{x^{m-1} y^{k-m-1} z^{2n-k-1}}{(m-1)! (k-m-1)! (2n-k-1)!} = \frac{\cos x \cos z \sin(x + y + z)}{\cos^2(x + y + z)},
\]

and for the lower triangles by

\[
\sum_{2 \leq k+1 \leq m \leq 2n-1} a_{2n}(m, k) \frac{x^{2n-m-1} y^{m-k-1} z^{k-1}}{(2n-m-1)! (m-k-1)! (k-1)!} = \frac{\cos x \sin z}{\cos(x + y + z)} + \frac{\sin x \cos(x + y)}{\cos^2(x + y + z)}.
\]

**Theorem 1.4** [The sequence \((A_{2n+1})\) \((n \geq 1)\)]. The generating function for the upper triangles is given by

\[
\sum_{2 \leq m+1 \leq k \leq 2n} a_{2n+1}(m, k) \frac{x^{m-1} y^{k-m-1} z^{2n-k}}{(m-1)! (k-m-1)! (2n-k)!} = \frac{\cos x \cos z}{\cos^2(x + y + z)}.
\]

and for the lower triangles by

\[
\sum_{2 \leq k+1 \leq m \leq 2n} a_{2n+1}(m, k) \frac{x^{2n-m} y^{m-k} z^{k-1}}{(2n-m)! (m-k)! (k-1)!} = \frac{\cos(x + y) \cos(y + z)}{\cos^2(x + y + z)}.
\]

**Theorem 1.5** [The sequence \((B_{2n})\) \((n \geq 1)\)]. The generating function for the upper triangles is given by

\[
\sum_{2 \leq m+1 \leq k \leq 2n-1} b_{2n}(m, k) \frac{x^{m-1} y^{k-m-1} z^{2n-1-k}}{(m-1)! (k-m-1)! (2n-1-k)!} = \frac{\sin x \cos z}{\cos^2(x + y + z)},
\]

and for the lower triangles by
Theorem 1.6 [The sequence \((B_{2n+1}) (n \geq 1)\). The generating function for the upper triangles is given by

\[
\sum_{2 \leq m+1 \leq k \leq 2n} b_{2n+1}(m, k) \frac{x^{m-1} y^{k-1} z^{2n-k}}{(m-1)! (k-m-1)! (2n-k)!} = \frac{\cos(x+y) \sin(y+z)}{\cos^2(x+y+z)},
\]

and for the lower triangles by

\[
\sum_{2 \leq k+1 \leq m \leq 2n} b_{2n+1}(m, k) \frac{x^{2n-m} y^{m-k-1} z^{k-1}}{(2n-m)! (m-k-1)! (k-1)!} = \frac{-\sin x \sin z \sin(x+y)}{\cos(x+y+z)} + \frac{\cos x \cos(x+y)}{\cos^2(x+y+z)}.
\]

The bivariate generating functions for the bottom rows \(b_n(n, k) (k = 1, 2, \ldots)\) are computed as follows:

\[
\sum_{1 \leq k \leq 2n-1} b_{2n}(2n, k) \frac{x^{2n-k-1} y^{k-1}}{(2n-k-1)! (k-1)!} = \frac{\cos x}{\cos(x+y)};
\]

\[
\sum_{1 \leq k \leq 2n} b_{2n+1}(2n+1, k) \frac{x^{k-1} y^{2n-k}}{(k-1)! (2n-k)!} = \frac{\sin x}{\cos(x+y)}.
\]

The previous generating functions for the matrices \(A_n, B_n\) will be derived analytically in Section 8, from the sole definition of twin Seidel matrix sequence given in §1.5, without reference to any combinatorial interpretation. It will be shown in Section 9 that, conversely, the closed expressions thereby obtained provide an analytical proof of identity (1.12) by means of the formal Laplace transform, that is, the fact that the entries of these matrices furnish a refinement of the tangent and secant numbers.

2. From alternating to André permutations of the first kind

Two further equivalent definitions of André permutations of the two kinds will be given (see Definitions 2.1 and 2.2). They were actually introduced in [Str74, FSt74, FSt76]. First, let \(x\) be a letter of a permutation \(w = x_1 x_2 \cdots x_n\) of a set of positive integers \(Y = \{y_1 < y_2 < \cdots < y_n\}\).
The \(x\)-factorization of \(w\) is defined to be the sequence \((w_1, w_2, x, w_4, w_5)\), where

1. the juxtaposition product \(w_1w_2xw_4w_5\) is equal to \(w\);
2. \(w_2\) is the longest right factor of \(x_1x_2\ldots x_{i-1}\), all letters of which are greater than \(x\);
3. \(w_4\) is the longest left factor of \(x_{i+1}x_{i+2}\ldots x_n\), all letters of which are greater than \(x\).

Next, say that \(x\) is of type I (respectively of type II) in \(w\), if, whenever the juxtaposition product \(w_2w_4\) is non-empty, its maximum (respectively minimum) letter belongs to \(w_4\). Also, say that \(x\) is of type I and II, if \(w_2\) and \(w_4\) are both empty.

**Definition 2.1.** A permutation \(w = x_1x_2\ldots x_n\) of \(Y = \{y_1 < y_2 < \cdots < y_n\}\) is said to be an André permutation of the first kind (respectively of the second kind) [in short, “\(w\) is André I” (respectively “André II”)], if \(x_i\) is of type I (respectively of type II) in \(w\) for every \(i = 1, 2, \ldots, n\).

**Definition 2.2.** A permutation \(w = x_1x_2\ldots x_n\) of \(Y = \{y_1 < y_2 < \cdots < y_n\}\) is said to be an André permutation of the first kind (respectively of the second kind), if it has no double descent (a factor of the form \(x_{i-1} > x_i > x_{i+1}\)) and its troughs (factors \(x_{i-1}x_i x_{i+1}\) satisfying \(x_{i-1} > x_i\) and \(x_i < x_{i+1}\)) are all of type I (respectively of type II). By convention, \(x_{n+1} := 0\).

The following notations are being used. If \(Y = \{y_1 < y_2 < \cdots < y_n\}\) is a finite set of positive integers, let \(\rho_Y\) be the increasing bijection of \(Y\) onto \(\{1, 2, \ldots, n\}\). The inverse bijection of \(\rho_Y\) is denoted by \(\rho_{Y}^{-1}\).

If \(v = y_{i_1}y_{i_2}\ldots y_{i_n}\) is a permutation of \(Y\), written as a word, let \(\rho_Y(v) := \rho_Y(y_{i_1})\rho_Y(y_{i_2})\cdots\rho_Y(y_{i_n}) = i_1i_2\cdots i_n\) be the reduction of the word \(v\), which is then a permutation of \(\{1, 2\ldots n\}\). When dealing with a given word \(v\), the subscript \(Y\) in \(\rho_Y(v)\) may be omitted, so that \(\rho(v) = \rho_Y(v)\). In the same way, the subscript \(Y\) in each composition product \(\rho_Y^{-1}\alpha\rho_Y(v)\) may be omitted, so that \(\rho^{-1}\alpha\rho(v) = \rho_Y^{-1}\alpha\rho_Y(v)\).

Also, let \(e\) be the bijection \(i \mapsto n + 1 - i\) from \(\{1, 2, \ldots, n\}\) onto itself. Furthermore, if \(v = y_{i_1}y_{i_2}\cdots y_{i_n}\) is a permutation of \(Y\), written as a word, let \(C(v) := Y\) and the length of \(v\) be \(|v| = n\). Finally, a left maximum record (respectively left minimum record) of \(v\) is defined to be a letter of \(v\) greater (respectively smaller) than all the letters to its left.

**Proposition 2.1.** Let \(n \geq 2\) and \(w = x_1x_2\ldots x_n\) be André I.

1. In \(w = v \min(w)v'\), both factors \(v\) and \(v'\) are André I.
2. If \(w\) is from \(\text{And}_I^n\), then both permutations \((x_1+1)(x_2+1)\cdots(x_n+1)\) and \(x_1x_2\cdots x_n(n+1)\) belong to \(\text{And}_I^{n+1}\), and \((x_2-1)(x_3-1)\cdots(x_n-1)\) belongs to \(\text{And}_I^{n-1}\) whenever \(x_1 = 1\).
(3) The last letter \( x_n \) is the maximum letter.
(4) Let \( w = w'ywy''x_n \) with \( y \) being the second greatest letter of \( w \). If \( w' \neq e \), then \( Fw'' = \min(w'') \).
(5) For each left maximum record \( y \) of \( v \), less than \( \max(w) \), the two factors \( uy \) and \( u' \) in the factorization \( w = uyu' \) are themselves André I.
(6) Let \( w = vyv' \) be André I. If \( y \) is a left minimum record, then \( v \) is André I.
(7) Let \( w = vyv' \) be an arbitrary permutation with \( y \) a letter. If both factors \( v \) and \( yv' \) are André I and \( y \) is a left minimum record, then \( w \) is André I.

Proof. (1) By the very definition given in Subsection 1.2.

(2) Clear.

(3) Write \( w = v\min(w)v' \). By definition, \( \max(v') = \max(vv') = \max(w) \), and by induction the last letter of \( v' \), which is also the last letter of \( w \), is equal to \( \max(v') = \max(w) \).

(4) If \( w'' \neq e \), let \( x := Fw'' \), and let \((w_1, w_2, x, w_4, w_5)\) be the \( x \)-factorization of \( w \). As \( y \) is the maximum letter of \( w_2 \), the maximum letter of \( w_4 \) must be equal to \( \max(w) \) to make \( x \) of type I. This can be achieved only if \( x \) is the minimum of \( w'' \).

(5) Let \( x := \min(w) \) and \( y \) be a left maximum record less than \( \max(w) \), so that \( w = vxv' = uyu' \) for some factors \( v, v' \neq e, u, u' \neq e \). Two cases are to be considered: (i) \( x \) to the left of \( y \) so that \( w = vxv''uyu' \); (ii) \( y \) to the left of \( x \) so that \( w = uyuy'xv' \) for some factors \( v'', u'' \). In case (i) both factors \( v \) and \( v''uyu' \) are André I, following the definition in §2.1. Now, the letter \( y \) is also a left maximum record of the word \( v''uyu' \). By induction on the length, both \( v''y \) and \( u' \) are André I, so that the two factors \( v \) and \( v''y \) of the word \( uy = vxv''y \) are André I, making the latter word also André I. Thus, both \( uy \) and \( u' \) are André I. In case (ii), the same argument applies: both factors \( uyuy'' \) and \( v' \) are André I, then also \( uy \) and \( u'' \) by induction, as well as the juxtaposition product \( u''xv' \).

(6) If \( y = \min(w) \), then \( v \) is André I by definition. Otherwise, \( y \) is to the left of \( \min(w) \) in \( \{w : w = vyuy \min(w)u'\} \). But \( y \) is also a left minimum record of \( vyu \). By induction on the length, \( v \) is André I.

(7) If \( y = \min(w) \), then \( v' \) is André I by (2). Now, \( v \) and \( v' \) being both André I, the product \( w = vyv' \) is André I by definition. If \( y > \min(w) \), then \( w = vyuy \min(w)u' \). There is nothing to prove if \( v = e \). Otherwise, as \( yv' \) is André I, both factors \( yu \) and \( u' \) are André I. As \( y \) is also a left minimum record of \( vyu \), the juxtaposition product \( vyu \) is André I by induction on the length. Finally, \( w \) itself is André I by definition, as \( u' \) has been proved to be also André I. \( \square \)

In [FSch71] a bijection between \( \text{Andr}_n \) and \( \text{Alt}_n \) was constructed, but did
not preserve the first letter. For proving Theorem 1.1(i), we need construct a bijection

\[(2.1) \quad \eta : w \to \eta(w), \quad \text{such that} \quad Fw = F\eta(w),\]

of \(\text{And}_n^I\) onto the set \(\text{Alt}_n\) of all alternating permutations of length \(n\). For \(n = 1, 2, 3\) it suffices to take: \(1 \mapsto 1, 12 \mapsto 12, 123 \mapsto 132, 213 \mapsto 231\). When \(n \geq 4\), each \(w\) from \(\text{And}_n^I\) can be written \(w = w' 1 w''\), where both factors \(w', w''\) (with \(w'\) possibly empty) are André I.

If \(w' = e\), let \(v' = 1\) and \(v'' := \rho^{-1} c \eta \rho(w'')\);
if \(w' \neq e\), let
\[v' := \rho^{-1} \eta \rho(w');\]
\[v'' := \begin{cases} 
\rho^{-1} \eta \rho(1 w''), & \text{if } |w'| \text{ is even;} \\
\rho^{-1} c \eta \rho(1 w''), & \text{if } |w'| \text{ is odd};
\end{cases}\]
and
\[(2.2) \quad \eta(w) := v' v''.\]

For instance, let \(w = 1234 \in \text{And}_4^I\). Then, \(w' = e, w'' = 234\); hence, \(v' = 1, \rho(w'') = 123, \eta(123) = 132, c(132) = 312, \rho^{-1}(312) = 423 = v''\) and \(\eta(1234) = 1423\).

With \(w = 4361257\) we get \(w' = 436, w'' = 257\); hence, \(\rho(436) = 213, \eta(213) = 231, \rho^{-1}(231) = 463 = v'. \) Also, \(\rho(1257) = 1234, \eta(1324) = 1423, c(1423) = 4132, \rho^{-1}(4132) = 7152 = v''\) and \(\eta(4361257) = 4637152\).

**Theorem 2.2.** The mapping \(\eta\) defined by (2.2) is a bijection between \(\text{And}_n^I\) and \(\text{Alt}_n\) such that \(Fw = F\eta(w)\).

**Proof.** Again, factorize an André I permutation \(w\) in the form \(w = x_1 x_2 \cdots x_n = w' 1 w''\). When \(w' = e\), then \(\rho(w'')\) is an André I permutation starting with \(\rho(x_2)\). By induction, \(\rho(w'')\) is an increasing alternating permutation if \(|w'| \geq 2\). Then \(c \eta \rho(w'')\) will be a falling alternating permutation, as well as the permutation \(v'' = \rho^{-1} c \eta \rho(w'')\), which is also a permutation of \(23\cdots n\). Hence, \(\eta(v) = 1 v''\) will be an alternating permutation starting with 1.

When \(w' \neq e\), then \(v'\) is an alternating permutation of the set \(C(w')\). By induction, it starts with the same letter as the first letter of \(w\), that is, \(x_1\). If \(|w'|\) is even, \(v'' = \rho^{-1} \eta \rho(1 w'')\) is an alternating permutation starting with 1, by induction. The juxtaposition product \(v' v''\) will then be an alternating permutation starting with \(x_1\), as the last letter of \(v'\) is necessarily greater than the first letter of \(v''\). If \(|w'|\) is odd, we just have to verify that \(Lv' < Fv''\). But \(w\), being an André I permutation, ends with its maximum letter \(n\) and so does \(w''\). By induction, \(\eta \rho(1 w'')\) starts with 1, so that \(c \eta \rho(1 w'')\) starts with the maximum letter \(n\). Therefore,
$v''$ is a falling alternating permutation starting with $n$ and $v'v''$ is an alternating permutation starting with $x_1$. \hfill \qed

For each permutation $w = x_1x_2\cdots x_{n-1}x_n$ ($n \geq 2$) the next-to-last letter $\mathbf{NL} w$ of $w$ has been defined as $\mathbf{NL} w := x_{n-1}$. The construction of a bijection $\theta$ from $\text{And}_n^I$ onto itself having the property

\begin{equation}
\mathbf{NL} \theta(w) + \mathbf{F} w = |w| = n
\end{equation}

is quite simple. It suffices to define:

\begin{equation}
\theta(x_1 \cdots x_{n-2}x_{n-1} n) := (n - x_{n-1})(n - x_{n-2}) \cdots (n - x_1) n.
\end{equation}

Property (2.3) is readily seen. It remains to prove that, if $w$ belongs to $\text{And}_n^I$, so does $\theta(w)$. This is the object of the next Proposition.

**Proposition 2.3.** Let $Y = \{y_1 < y_2 < \cdots < y_n\}$ be a finite set of positive integers and $w = x_1x_2\cdots x_n$ be an André I permutation from the set $\text{And}_n^I$. Then $\theta(w) := (x_n - x_{n-1})(x_n - x_{n-2}) \cdots (x_n - x_1)x_n$ is also André I.

**Proof.** Proposition 2.3 is true for $n = 2$, as $\theta(y_1y_2) = (y_2 - y_1)y_2$. For $n = 3$ we have $\theta(y_1y_2y_3) = (y_3 - y_2)(y_3 - y_1)y_3$, $\theta(y_2y_1y_3) = (y_3 - y_1)(y_3 - y_2)y_3$, which are two André I permutations.

For $n \geq 4$ let $w = x_1x_2\ldots x_n \in \text{And}_n^I$ be written $w = w'y_1w''$. If $w' = e$, let $w'' - y_1 := (x_2 - y_1) \cdots (x_{n-2} - y_1)(x_{n-1} - y_1)(y_n - y_1)$. Then, $w''$ is André I by Lemma 2.1 (b), as well as $w'' - y_1$, since $y_1$ is the smallest element of $Y$. By induction,

$$
\theta(w'' - y_1) = (y_n - y_1 - (x_{n-1} - y_1))(y_n - y_1 - (x_{n-2} - y_1))
\quad \cdots (y_n - y_1 - (x_2 - y_1))(y_n - y_1)
= (y_n - x_{n-1})(y_n - x_{n-2}) \cdots (y_n - x_2)(y_n - y_1)
$$

is André I. Therefore,

$$(y_n - x_{n-1})(y_n - x_{n-2}) \cdots (y_n - x_2)(y_n - y_1)y_n
= (x_n - x_{n-1})(x_n - x_{n-2}) \cdots (x_n - x_2)(x_n - x_1)y_n
$$

is André I a fortiori and is precisely the expression of $\theta(w)$ that was wanted. Let $|w'| = k \geq 1$. By induction, both

$$
\theta(w'y_n) = (y_n - x_k) \cdots (y_n - x_2)(y_n - x_1)y_n
$$

and

$$
\theta(y_1w'') = (y_n - x_{n-1}) \cdots (y_n - x_{k+2})(y_n - y_1)y_n
$$
are André I, and also
\[ \hat{\theta}(y_1 w'') := (y_n - x_{n-1}) \cdots (y_n - x_{k+2})(y_n - y_1). \]

The juxtaposition product \( \hat{\theta}(y_1 w'') \theta_n(w'y_n) \) reads
\[ (y_n - x_{n-1}) \cdots (y_n - x_{k+2})(y_n - y_1)(y_n - x_k) \cdots (y_n - x_2)(y_n - x_1) y_n, \]
that is, precisely \( \theta(w) \), since \( y_n - y_1 = y_n - x_{k+1} \).

Now, note that \( x_k \) is the greatest letter of \( w' \) by Lemma 2.1 (3), so that \( (y_n - x_k) \) is the smallest letter of the right factor
\[ (y_n - x_k) \cdots (y_n - x_2)(y_n - x_1) y_n \]
of \( \theta(w) \). On the other hand, \( (y_n - y_1) > (y_n - x_k) \). Thus, \( (y_n - x_k) \) is a trough of \( \theta(w) \); moreover, the \( (y_n - x_k) \)-factorization \( (w_1, w_2, (y_n - x_k), w_4, w_5) \) of \( \theta(w) \) is of type I, since \( w_2 \) contains the letter \( (y_n - y_1) \) and \( w_4 \) the letter \( y_n \), which is greater than \( (y_n - y_1) \). Finally, the \( x \)-factorizations of the other letters \( x \) from \( \hat{\theta}(y_1 w'') \) (respectively from \( \theta(w'y_n) \)) in each of these two factors are identical with their \( x \)-factorizations in \( \hat{\theta}(w) \). They are then all of type I, and \( \theta(w) \) is André I.

By Proposition 2.3 and Identity (2.3) we obtain the following result.

**Theorem 2.4.** The statistics “\( F \)” and “\( (n - NL) \)” are both Entringerian on \( \text{And}_I \). Moreover, the distribution of the bivariate statistic \( (F, n - NL) \) on \( \text{And}_I \) is symmetric.

3. The bijection \( \phi \) between André I and André II permutations

For each permutation \( w = x_1 x_2 \cdots x_n \) of \( 12 \cdots n \) \( (n \geq 2) \) make the convention \( x_{n+1} := 0 \) and introduce the statistic spike of \( w \), denoted by “\( \text{spi} w \),” to be equal to the letter \( x_i \) \((1 \leq i \leq n)\) having the properties
\[ (3.1) \quad x_1 \leq x_1, \; x_1 \leq x_2, \; \ldots, \; x_1 \leq x_i, \; \text{and} \; x_1 > x_{i+1}. \]

The spike statistic may be regarded as the permutation analog of the classical statistic that measures the time spent by a particle starting at the origin and wandering in the \( y > 0 \) part of the \( xy \)-plane, before crossing the \( x \)-axis for the first time. For instance, \( \text{spi}(253416) = 4 \), as all the letters to the left of 4 are greater than or equal to 2, but the letter following 4 is less than 2. Also, \( \text{spi}(425136) = 4 \) and \( \text{spi}(14235) = 5 \).

When \( w \) is an André I permutation and \( \text{spi} w = x_i \), then \( x_i \) is a left maximum record, i.e., greater than all the letters to its left. Otherwise, the minimum trough between the maximum letter within \( x_1 x_2 \cdots x_{i-1} \) would
not be of type I. Accordingly, when \( w \) is an André I permutation, the spike \( x_i \) of \( w \) can also be defined as the smallest left maximum record (or the leftmost one), whose successive letter \( x_{i+1} \) is less than \( x_i \).

For introducing the statistic “\( \text{pit} \)”, we restrict the definition to all permutations \( w = x_1 x_2 \cdots x_n \) of 12\cdots n such that \( n \geq 2 \) and \( x_{n-1} < x_n \). Let \( 1 = a_1 < a_2 < \cdots < x_{n-1} = a_{k-1} < x_n = a_k \) be the increasing sequence of the right minimum records of \( w \), that is to say, the letters which are smaller than all the letters to their right. With the assumption \( x_{n-1} < x_n \), there are always two right minimum records to the right of each letter greater than \( x_n \). If \( x_n = n (= \text{max } w) \), let \( \text{pit } w := 1 (= \text{min } w) \). Otherwise, let \( x_i \) be the rightmost letter greater than \( x_n \) and \( a_j < a_{j+1} \) be the closest pair of right minimum records to the right of \( x_i \). Define \( \text{pit } w := a_{j+1} \).

For instance, \( \text{pit}(451236) = 1 \), as the word ends with the maximum letter 6. In the permutation 614235, the letter 6 is the rightmost letter greater than \( x_n = 5 \), and 1 < 2 is the closest pair of right minimum records to the right of 6, so that \( \text{pit}(614235) = 2 \).

An alternate definition for “\( \text{pit} \)” is the following: if \( w \) ends with \( \text{max } w \), let \( \text{pit } w := \text{min } w \). Otherwise, write \( w = w_1 (\text{min } \text{w}) w_2 \) and define \( \text{pit } w := \text{pit } w_2 \). If \( w_2 \) does not end with the maximum letter, let \( w_2 = w_3 (\text{min } w_2) w_4 \) and define \( \text{pit } w_2 := \text{pit } w_4 \), continue the process until finding a right factor \( w_{2j} \) ending with its maximum letter to obtain \( \text{pit } w := \text{pit } w_2 = \cdots = \text{pit } w_{2j} = \text{min } w_{2j} \). For instance, \( \text{pit}(614235) = \text{pit}(4235) = 2 \).

Remember that we have introduced three other statistics, namely “\( \text{NL} \)” (“next-to-last”), “\( \text{L} \)” (“last”), and “\( \text{grn} \)” (“greater neighbor of the maximum”), and that \( \text{grn } w = \text{NL} w \) whenever \( w \) is an André I permutation. Our goal is to prove the next theorem.

**Theorem 3.1.** The triplets \( (F, \text{spi}, \text{NL}) \) on \( \text{And}^I_n \) and \( (\text{pit}, \text{L}, \text{grn}) \) on \( \text{And}^II_n \) are equidistributed.

Let \( X = \{a_1 < a_2 < \cdots < a_n\} \) be a set of positive integers (or any finite totally ordered set), and let \( \text{And}^I_X \) (respectively \( \text{And}^II_X \)) denote the set of all André I (respectively André II) permutations of \( X \). To prove the previous statement, a bijection

\[
(3.2) \quad \phi : \text{And}^I_X \rightarrow \text{And}^II_X
\]

will be constructed having the property

\[
(3.3) \quad (F, \text{spi}, \text{NL}) w = (\text{pit}, \text{L}, \text{grn}) \phi(w).
\]

When \( n = 0 \), let \( \phi(e) := e \) with \( e \) the empty word. Let \( \phi(a_1) := a_1 \) for \( n = 1 \); \( \phi(a_1 a_2) := a_1 a_2 \) for \( n = 2 \). For \( n \geq 3 \) each permutation \( w \) from
And\textsubscript{1}\textsubscript{X} has one of the two forms:

(i) \( w = v_0 a_1 v_1 a_2 v_2 \);  
(ii) \( w = v_0 a_2 v_2 a_1 v_1 \).

Note that the three factors \( v_0, v_1, v_2 \) and the product \( v_0 a_2 v_2 \) are all André I permutations, and \( v_0 \) is possibly empty. In case (i), \( v_1 \) may be empty, but not \( v_2 \) (which ends with \( a_n \) greater than \( a_2 \)); in case (ii), \( v_2 \) may be empty, but not \( v_1 \) (which ends with \( a_n \)). For both cases (i) and (ii), define

\[
\phi(w) := \phi(v_1) a_1 \phi(v_0 a_2 v_2).
\]

By induction, both factors \( \phi(v_1) \) and \( \phi(v_0 a_2 v_2) \) are André II, as well as \( \phi(w) \), since \( a_2 \) is to the right of \( a_1 \).

Example. Consider the André I permutation

\[
w = 7 8 5 6 9 \, 2 \, 10 \, 1 \, 11 3 12 4 13;
\]

we then have:

\[
\phi(w) \overset{(ii)}{=} \phi(11 3 12 4 13) 1 \phi(7 8 5 6 9 2 10);
\]

\[
\phi(11 3 12 4 13) \overset{(i)}{=} 12 3 \phi(11 4 13);
\]

\[
\phi(11 4 13) \overset{(ii)}{=} 13 4 11;
\]

\[
\phi(7 8 5 6 9 2 10) \overset{(ii)}{=} 10 2 \phi(7 8 5 6 9);
\]

\[
\phi(7 8 5 6 9) \overset{(i)}{=} 5 \phi(7 9 6 9);
\]

\[
\phi(7 8 6 9) \overset{(ii)}{=} 9 6 \phi(7 8) = 9 6 7 8;
\]

so that

\[
\phi(w) = 12 3 13 4 11 1 10 2 5 9 6 7 8.
\]

We can verify that \( (\textbf{F, spi, NL}) w = (\textbf{pit, L, grn}) \phi(w) = (7, 8, 4) \).

With the previous definition of \( \phi \), we see that the maximum letter \( a_n \) of \( X \) occurs in \( \phi(v_0 a_2 v_2) \) (respectively in \( \phi(v_1) \)) when \( w \) is of form (i) (respectively of form (ii)). For constructing the inverse \( \phi^{-1} \) of \( \phi \), this suggests that we start with the factorization

\[
v = w_0 a_1 w_1
\]
of each permutation $v$ from $\text{And}^H_X$ with $\#X \ge 3$, after defining $\phi^{-1}(e) := e$; $\phi^{-1}(a_1) := a_1$ for $n = 1$; $\phi^{-1}(a_1a_2) := a_1a_2$ for $n = 2$. As both $w_0$, $w_1$ are André II permutations with fewer letters, the images $\phi^{-1}(w_0)$, $\phi^{-1}(w_1)$ are defined by induction. Let $v_1 := \phi^{-1}(w_0)$. As the minimum letter of $w_1$ is $a_2$, define $v_0$ and $v_2$ to be the factors in $\phi^{-1}(w_1) := v_0a_2v_2$. Next, let

$$
\begin{align*}
\phi^{-1}(v) := \begin{cases} 
v_0a_1v_1a_2v_2, & \text{if } a_n \text{ is a letter of } w_1; \\
v_0a_2v_2a_1v_1, & \text{if } a_n \text{ is a letter of } w_0.
\end{cases}
\end{align*}

(3.5)$

**Lemma 3.2.** The André II permutation $\phi(w)$ ends with its maximum letter if and only if $w$ starts with its minimum letter $\min w$, and then $\pit \phi(w) = \F w = \min w$.

**Proof.** This is obviously true for $n = 2$. For $n \ge 3$, we have $w = a_1v_1a_2v_2$ and $\phi(w) = \phi(v_1)a_1\phi(a_2v_2)$. As $a_2v_2$ is André I starting with its minimum letter $a_2$, then, by induction, $\phi(a_2v_2)$ ends with its maximum letter, which is equal to $\max w$. Hence, $\pit \phi(w) = \pit(\phi(v_1)a_1\phi(a_2v_2)) = a_1 = \F w$. For the converse take the notation $v = v_0a_1w_1$ of (3.5). When the maximum letter $a_n$ occurs in $w_1$, then $\psi(v) = v_0a_1v_1a_2v_2$ with $v_1 = \psi(w_0)$ and $\psi(w_1) = v_0a_2v_2$. By assumption, $a_n$ occurs at the end of $v$, hence, at the end of $w_1$. By induction, $\psi(w_1) = v_0a_2v_2$ starts with its minimum letter. This can be true only if $v_0 = e$. Therefore, $\psi(v) = a_1v_1a_2v_2$ and starts with its minimum letter $a_1$.

**Theorem 3.3.** The mapping $\phi$ is a bijection between $\text{And}^I_n$ and $\text{And}^H_n$. Moreover, relation (3.3) holds.

**Proof.** The bijectivity is proved by the construction of the inverse $\phi^{-1}$ (see (3.5)). To prove identity (3.3), let $w$ be a André I permutation, either of the form $v_0a_1v_1a_2v_2$, or of the form $v_0a_2v_2a_1v_1$. In both cases,

$$
\pi w = \pi v_0 = \pi(v_0a_2v_2)
\begin{align*}
&= L \phi(v_0a_2v_2) \hspace{2cm} \text{[by induction]} \\
&= L (\phi(v_1)a_1\phi(v_0a_2v_2)) = L \phi(w).
\end{align*}

Next, if $w = v_0a_1v_1a_2v_2$, then

$$
\begin{align*}
\NL w &= \NL(v_0a_2v_2) \\
&= \grn \phi(v_0a_2v_2) \hspace{2cm} \text{[by induction]} \\
&= \grn(\phi(v_1)a_1\phi(v_0a_2v_2)) = \grn \phi(w),
\end{align*}

$$

because the maximum letter $a_n$ is a letter of $v_2$. If $w = v_0a_2v_2a_1v_1$, then $a_1v_1$ has at least two letters and ends with $a_n$, so that $a_1v_1$ is André I of
form (i). Consequently,

\[ NL_w = NL(a_1v_1) \]
\[ = grn(\phi(a_1v_1)) \quad \text{[by induction]} \]
\[ = grn(\phi(v_1)a_1\phi(v_0a_2v_2)) = grn(\phi(w)). \]

When \( w \) does not start with its minimum letter, then \( \phi(w) = \phi(v_1)a_1\phi(v_0a_2v_2) \), and \( \phi(v_0a_2v_2) \) does not end with \( \max w \). Therefore, \( \pit \phi(w) = \pit \phi(v_0a_2v_2) = F v_0a_2v_2 \). As \( v_0 = e \) only in case (ii), we then have: \( \pit \phi(w) = F v_0a_2 = F w \). \[ \Box \]

4. The bijection \( g \) from the set of André I permutations onto itself

When making up the tables of the distribution of the bivariate statistic \((\spi, F)\) on \( \text{And}_n^I \) for \( n = 1, 2, \ldots, 7 \), as shown in Table 4.1, it can be noticed that the matrices are symmetric with respect to their skew-diagonals. The property will hold in general if a bijection \( g \) from \( \text{And}_n^I \) onto itself can be constructed satisfying

\[(4.1) \quad (F, \spi) w = (n + 1 - \spi, n + 1 - F) g(w)\]

for all \( w \) from \( \text{And}_n^I \).

\begin{table}[h]
\centering
\begin{tabular}{c|c|c|c|c|c|c|c|c|c}
\hline
\( \text{F} \) & \( 1 \) & \( 2 \) & \( 3 \) & \( 4 \) & \( 5 \) & \( 1 \) & \( 2 \) & \( 3 \) & \( 4 \) & \( 5 \) & \( 6 \) \\
\hline
\( \text{spi} = 2 \) & 3 & 1 & 4 & 2 & . & & & & & \\
\hline
\( n = 2 \) & 1 & & & & & & & & & & \\
\hline
\end{tabular}
\caption{distribution of \((\spi, F)\) on \( \text{And}_n^I \)}
\end{table}
first letter \( y_1 \) of \( v \) is equal to \( \min X \). Consider an André I permutation \( w = x_1 x_2 \cdots x_n \) from \( \text{And}_n^I \). Let \( 1 = a_1 < a_2 < \cdots < a_r \) (respectively \( 1 = b_1 < b_2 < \cdots < b_s = n \) be the increasing sequence of subscripts such that \( x_{a_1} > x_{a_2} > \cdots > x_{a_r} \) (respectively \( x_{b_1} < x_{b_2} < \cdots < x_{b_s} \)) is the increasing (respectively decreasing) sequence of the left minimum (respectively maximum) records of \( w = x_1 x_2 \cdots x_n \) from \( \text{And}_n^I \).

For the following André I permutation the left minimum (respectively maximum) records are underlined (respectively overlined):

\[
w = \underline{7} \underline{8} 5 6 \underline{9} 2 \underline{10} 1 \underline{1} \underline{1} 3 1 \underline{2} 4 \underline{1} \underline{3}.
\]

Going back to the general case, let

\[
v_1 := x_1 \cdots x_{a_2-1}, \ v_2 := x_{a_2} \cdots x_{a_3-1}, \ \cdots, \ v_r := x_{a_r} \cdots x_n,
\]

so that \( w \) is the juxtaposition product \( v_1 v_2 \cdots v_r \), and the factors \( v_i \) are obtained by cutting the word \( w \) just before each left minimum record. The factorization \( (v_1, v_2, \ldots, v_r) \) is called the canonical factorization of the André I permutation \( w \). Furthermore, the sequence

\[
( (F v_1, L v_1), (F v_2, L v_2), \ldots, (F v_r, L v_r) ),
\]

which is also equal to \( ( (x_1, x_{a_2-1}), (x_{a_2}, x_{a_3-1}), \ldots, (x_{a_r}, x_n) ) \), is called the type of the canonical factorization of \( w \).

In our running example, the canonical factorization reads

\[
w = \underline{7} \underline{8} 5 6 \underline{9} 2 \underline{10} 1 \underline{1} \underline{1} 3 1 \underline{2} 4 \underline{1} \underline{3} \vline \begin{array}{cccc}
v_1 & v_2 & v_3 & v_4 \end{array}
\]

and is of type \( ( (7, 8), (5, 9), (2, 10), (1, 13) ) \).

**Proposition 4.1.** Let \( (v_1, v_2, \ldots, v_r) \) be the canonical factorization of the André I permutation \( w = x_1 x_2 \cdots x_n \) from \( \text{And}_n^I \). Let \( s \) be the number of left maximum records of \( w \). Then:

(i) \( r \leq s \);
(ii) each factor \( v_i \) (\( i = 1, 2, \ldots, r \)) is a simple André I permutation;
(iii) \( L v_i \) is a left maximum record, so that \( L v_1 < L v_2 < \cdots < L v_{r-1} < L v_r = n \) and, of course, \( F v_1 > F v_2 > \cdots > F v_{r-1} > F v_r = 1 \);
(iv) \( (F v_1, L v_1) = (F w, \text{spi} w) \).

Let \( w = x_1 x_2 \cdots x_n \) be an André I permutation from \( \text{And}_n^I \) and \( \overline{w} := \overline{x_n} \overline{x_{n-1}} \overline{x_{n-2}} \cdots \overline{x_1} \) be the permutation defined by \( \overline{x_i} := N - x_{n+1-i} \) (\( i = 1, 2, \ldots, n \)), where \( N \) is some integer greater than \( n \).
Proposition 4.2. If \( w \) is a simple André I permutation, so is \( \overline{w} \).

For constructing the bijection \( g \), let \( n \geq 3 \) and \( N := n + 1 \). If \((-v_1, v_2, \ldots, v_r)\) is the canonical factorization of a permutation \( w \) from \( \text{And}_n^I \), define \( g(w) \) to be the juxtaposition product

\[
(4.1) \quad g(w) := \overline{v}_1 \overline{v}_2 \ldots \overline{v}_r.
\]

Furthermore, if \( \tau = ((p_1, q_1), (p_2, q_2), \ldots, (p_r, q_r)) \) is the canonical factorization type of \( w \), let

\[
\overline{\tau} := ((\overline{q}_1, \overline{p}_1), (\overline{q}_2, \overline{p}_2), \ldots, (\overline{q}_r, \overline{p}_r)).
\]

We then have the following fundamental property of \( g \).

Theorem 4.3. The transformation \( g \) is a bijection from \( \text{And}_n^I \) onto itself. Furthermore, if \( \tau \) is the canonical factorization type of \( w \), then \( \overline{\tau} \) is the canonical factorization type of \( g(w) \). In particular,

\[
(4.2) \quad (\text{F, spi}) g(w) = (n + 1 - \text{spi}, n + 1 - \text{F}) w.
\]

In the running example, where \( n + 1 = 14 \), we get

\[
g(w) = 67 | 589 | 412 | 110211313,
\]

which is of type \((6,7), (5,9), (4,12), (1,13)\).

The proofs of Propositions 4.1, 4.2 and Theorem 4.3 do not present any difficulties and are therefore omitted.

5. The proof of Theorem 1.1 (iii) and (iv)

We reproduce the sequence (1.5) by decomposing the product \( \phi \circ g \):

\[
(5.1) \quad \text{And}_n^I \xrightarrow{g} \text{And}_n^I \xrightarrow{\phi} \text{And}_n^H \xrightarrow{\text{F}} \text{And}_n^H
\]

The first (respectively second) identity \( \text{F} w = n + 1 - \text{spi} g(w) \) (respectively \( \text{spi} g(w) = \text{L} \phi(g(w)) \)) is a specialization of (4.2) (respectively of (3.3)).

Take the example of the previous section: \( w = 78569210111312413 \) and \( g(w) = 67589412110211313 \). By using the definition of \( \phi \) given in (3.4), we get \( \phi(g(w)) = 10111213312489567 \), which belongs to \( \text{And}_{13}^H \), and \( n + 1 - \text{L} \phi(g(w)) = 14 - 7 = 7 = \text{F} w \).
Next, reproduce the sequence (1.4) by decomposing the product $\phi \circ \theta$:

$$
\begin{align*}
\text{And}^I_n & \xrightarrow{\theta} \text{And}^I_n & \xrightarrow{\phi} & \text{And}^H_n \\
& \mapsto & & \mapsto \\
F_w & = n - \text{NL} \theta(w) & = & n - \text{grn} \phi(\theta(w))
\end{align*}
$$

The first (respectively second) identity $F_w = n - \text{NL} \theta(w)$ (respectively $\text{NL} \theta(w) = \text{grn} \phi(\theta(w))$) is a specialization of (2.3) (respectively of (3.3)).

For example, with $w' = 10211312194586713$, we obtain $\theta(w') = 67589412110211313$ and $\phi(\theta(w')) = 10111213312489567$. Thus, $F w' = 10 = n - \text{NL} \theta(w') = 13 - 3 = n - \text{grn} \phi(\theta(w'))$.

The proofs of (iii) and (iv) of Theorem 1.1 are now completed. Another proof of Theorem 1.1 (iii) and (iv) makes use of the properties of a rearrangement group $G_n$, acting on the group $\mathfrak{S}_n$ of all permutations of $\{1, 2, \ldots, n\}$, which were developed in [FSt74, FSt76], and another correspondence $\Gamma$ on binary increasing trees, introduced in [FH13]. They constitute the main ingredients for the constructions of three bijections, $\Gamma$, $\Phi^I$, and $\Phi^H$, appearing in the diagram

$$
\begin{array}{c}
\text{And}^H_n \\
\xrightarrow{\Gamma} \\
\xrightarrow{\Phi^I} \\
\xrightarrow{\Phi^H} \\
\text{And}^H_n
\end{array}
$$

which have the property

$$
L w = 1 + \text{NL} \Phi^I \Gamma(w) = 1 + \text{grn} \Phi^H \Gamma(w).
$$

6. Combinatorics of the twin Seidel matrix sequence

This section is devoted to proving Theorem 1.2. As announced in Subsection 1.5, the question is to show that the integers $a_n(m, k)$ and $b_n(m, k)$, when taken as $a_n(m, k) = \#A_n(m, k)$, $b_n(m, k) = \#B_n(m, k)$ with

$$
\begin{align*}
A_n(m, k) & := \{ w \in \text{And}^I_n : (F, \text{NL})w = (m, k) \}; \\
B_n(m, k) & := \{ w \in \text{And}^I_n : (\text{spi}, \text{grn})w = (m, k) \};
\end{align*}
$$

satisfy all the properties (TS1)–(TS5) stated in Subsection 1.5.

The verifications of properties (TS1), (TS2), (TS3), (TS4.1), (TS4.2), (TS5.1) are easy and given in the next subsection. The proofs of the other properties are much harder and will be developed thereafter.
6.1. The first evaluations. By (1.7), the set $B_n(m, k)$ is equinumerous with

$$B'_n(m, k) := \{ w \in \text{And}_n^H : (L, \text{grn})w = (m, k) \}. \quad (6.3)$$

The evaluations in this subsection are made by using $B'_n(m, k)$ instead of $B_n(m, k)$.

(TS1) Nothing to prove, except for the diagonals of the twin Seidel matrices $A_n$ and $B_n$. They have zero entries when $n \geq 3$, because the first and next-to-last letter of each André I permutation cannot be the same! On the other hand, the identity $Lw = \text{grn}w = m$ would mean that the permutation $w$ from $\text{And}_n^H$ ends with a double descent $n > m > 0$.

(TS2) We have $a_n(k, n) = b_n(k, n) = 0$, because $\text{grn}w \leq n - 1$ for each $w$ from either $\text{And}_n^I$, or $\text{And}_n^H$. Also, $a_n(n, k) = 0$, as each permutation from $\text{And}_n^I$ ends with $n$. Finally, $b_n(1, k) = 0$, because each permutation from $\text{And}_n^H$ cannot end with the letter $1$.

(TS3) We have: $A_2 = 1 \cdot \cdot$ and $B_1 = \cdot 1 \cdot$, because $\text{And}_2^I = \text{And}_2^H = \{12\}$ and $(F, NL, L, \text{grn})(12) = (1, 1, 2, 1)$.

(TS4.1) The entry $b_n(n, k)$ counts the André II permutations $w$ from $\text{And}_n^H$ ending with the two-letter factor $k n$. The deletion of the ending letter $n$ maps $w$ onto an André II permutation $w'$ from $\text{And}_n^H$ ending with $k$ in a bijective manner. Hence, $b_n(n, k) = b_{n-1}(k, *)$, which is equal to $a_{n-1}(*, k - 1)$ by Theorem 1.1 for $1 \leq k \leq n - 1$.

(TS4.2) The entry $b_n(n - 1, k)$ counts the André II permutations $w$ from $\text{And}_n^H$ of the form $w = x_1 \cdots x_{i-2} n x_i \cdots x_{n-1} (n-1)$ with $i \leq n - 1$ and $k$ equal to $x_{i-2}$ or $x_i$. Such a permutation can be mapped onto a permutation $w'$ from $\text{And}_n^H$ defined by

$$w' := x_1 \cdots x_{i-2} (n-1) x_i \cdots x_{n-1}. \quad (6.4)$$

This defines a bijection between the set of all $w$ from $\text{And}_n^H$ such that $(L, \text{grn})w = (n-1, k)$ and the set of all $w'$ from $\text{And}_n^H$ such that $\text{grn}w' = k$ ($1 \leq k \leq n - 2$). Thus, $b_n(n - 1, k) = b_{n-1}(*, k)$, which is also equal to $a_{n-1}(*, k)$ by Theorem 1.1.

(TS5.1) The entry $a_n(1, k)$ counts the permutations $w$ from $A_n^I$ such that $(F, NL)w = (1, k)$. The bijection $1 x_2 \cdots k n \mapsto (x_2 - 1) \cdots (k - 1) (n - 1)$ maps the set of these permutations onto the set of all $w'$ from $A_{n-1}^I$ such that $\text{grn}w' = k - 1$. Hence, $a_n(1, k) = a_{n-1}(*, k - 1)$. 

A detailed explanation of the bijections and the evaluation process is provided for each case, ensuring a comprehensive understanding of the André permutation calculus.
6.2. **Tight André I permutations.** As sketched in Subsection 1.6 and its display (1.13), proving (TS5.2) and (TS5.3) amounts to do the following points:

(a) split each set $A_n(m, k)$ into two disjoint subsets

$$A_n(m, k) = T_n(m, k) \cup NT_n(m, k),$$

in such a way that

(b) when $2 \leq k + 1 \leq m \leq n - 2$ or $3 \leq m + 2 \leq k \leq n - 1$ a bijection

$$f : NT_n(m, k) \rightarrow A_n(m + 1, k);$$

(c) and another bijection

$$\phi : B_n(m, k) \rightarrow T_n(m, k), \quad \text{when } m > k;$$

$$\phi : B_n(m, k - 1) \rightarrow T_n(m, k), \quad \text{when } m < k;$$

can be duly constructed.

Points (a) and (b). Let $f$ be the transposition of the first letter $F w = m$ within a permutation $w$ and the letter equal to $(m + 1)$ $(1 \leq m \leq n - 2)$:

$$f : w = mv(m + 1)v' \mapsto w' = (m + 1)vmv'.$$

If $w$ is an André I permutation, the image $w' = f(w)$ is not always an André I permutation. For example, 423516 belongs to $\text{And}_I^n$, but not $f(w) = 523416$, for the trough 2 is not of type I. However, the reverse transposition

$$f^{-1} : w' = (m + 1)vmv' \mapsto f^{-1}(w') = w = mv(m + 1)v',$$

whenever defined, maps each André I permutation to an André I permutation. The André I permutations $w$, whose images $f(w)$ are not André I permutations, are called **tight**. They are characterized as follows.

**Definition 6.1.** An André I permutation $w = mv(m + 1)v'$ is said to be **tight**, if the following two conditions hold:

(i) either $v = e$, or $v \neq e$ and all its letters are less than $m$;

(ii) either $v' \neq e$ and $F v'$ is less than all the letters of $w$ to its left, or $v' = e$ and necessarily $m = n - 1$.

Let $T_n$ (respectively $NT_n$) be the subset of all André I permutations from $\text{And}_I^n$, which are tight (respectively not tight), and let $T_n(m, k) := T_n \cap A_n(m, k)$, $NT_n(m, k) := NT_n \cap A_n(m, k)$.

Note that the André I permutations from $A_n(1, k)$ are all of the form $1v2v'$ and, either the letters of $v$ are all greater than 2, or $v' \neq e$ but $2 < F v'$, so that at least one of conditions (i), (ii) does not hold. Accordingly, $NT_n(1, k) = A_n(1, k)$ for all $k$, that is, all André I permutations starting with 1 are not tight. Also, note that each André I permutation from $A_n(n - 1, k)$ is of the form $w = (n - 1)vn$ and is necessarily tight, so that $T_n(n - 1, k) = A_n(n - 1, k)$ for all $k$. 

Proposition 6.1. Let \( n \geq 3 \) and let \( w \) be a tight André I permutation from \( \text{Andr}^I_2 \). Then, \( f(w) \) (defined in (6.5)) cannot be an André I permutation.

Proof. Take the notation of (6.5) for \( w \) and \( w' = f(w) \). If \( m = n - 1 \), then \( w' = n v (n-1) \) is not André I. If \( m \leq n - 2, v = e \), and (ii) of Definition 6.1 holds, then \( w' \) contains the double descent \( (m + 1) > m > F v' \), therefore is not André I. If \( m \leq n - 2, v \neq e \) and (ii) of Definition 6.1 holds, let \( x \) be the minimum trough in \( w' \) between \( (m+1) \) and \( m \); then the \( x \)-factorization \( (w_1, w_2, x, w_4, w_5) \) of \( w' \) is such that \( \max w_2 w_4 = m + 1 \) with \( (m + 1) \) a letter of \( w_2 \). Again, \( w' \) cannot be an André I permutation.

Proposition 6.2. If \( 2 \leq k + 1 \leq m \leq n - 2 \) or \( 3 \leq m + 2 \leq k \leq n - 1 \), then \( f \) maps \( NT_n(m, k) \) onto \( A_n(m+1, k) \) in a bijective manner.

Proof. To prove that \( w' \) is André I when \( w \) is not tight, we prove that (i) \( w' \) has no double descent; (ii) all the troughs of \( w' \) are of type I.

(i) The only double descent that could be created when going from \( w \) to \( w' \) is \( (m + 1) > m > F v' \). This could occur only if \( v = e, v' \neq e \) and \( m > F v' \), and this would mean that \( w \) is tight; a contradiction.

(ii) Let \( x_i \) (respectively \( x'_i \)) be the \( i \)-th letter counted from left to right of \( w \) (respectively \( w' \)). Also, let \( (w_1, w_2, x_i, w_4, w_5) \) (respectively \( (w'_1, w'_2, x'_i, w'_4, w'_5) \)) be the \( x_i \)- (respectively \( x'_i \))-factorization of \( w \) (respectively of \( w' \)). Several cases are to be considered.

(1) Suppose that \( x_i \) is to the right of \( (m + 1) \) in \( w \). Then \( x'_i = x_i \). If \( x_i \) is a trough of \( w \), then either \( (m + 1) \) is a letter of \( w_2 \), or not. If it is, then \( w'_2 \) is derived from \( w_2 \) by replacing the letter \( (m + 1) \) by \( m \). Therefore, \( \max w'_2 \leq \max w_2 < \max w_4 = \max w'_4 \), and the \( x'_i \)-factorization remains of type I in \( w' \). If it is not, then \( w'_2 = w_2, w'_4 = w_4 \), and the same conclusion holds.

(2) Now, suppose that \( x_i = (m + 1) \), so that \( x'_i = m \). If \( x_i \) is a trough of \( w \) — this is possible, as \( w \) is supposed to be not tight — then \( v \neq e \), and \( m \) is not a letter of \( w_2 \). Furthermore, \( (m + 1) \) is a letter of \( w'_2 \) only when all the letters between \( m \) and \( (m + 1) \) are greater than \( (m + 1) \). Whatever the particular case may be, we have \( \max w'_2 = \max w_2 < \max w_4 = \max w'_4 \), so that \( x'_i \) is a trough of type I in \( w' \).

(3) Next, let \( x_i \) lie between \( m \) and \( (m + 1) \) in \( w \), so that \( x'_i = x_i \), and suppose that \( x_i \) is trough of \( w \). If \( x_i \) is greater than \( (m + 1) \), then \( w'_2 = w_2 \) and \( w'_4 = w_4 \). Moreover, \( x'_i = x_i \) will be a trough of type I in \( w' \). If \( x_i \) is less than \( m \), the only problem arises when \( m \) and \( (m + 1) \) are the maximum letters of \( w_2 \) and \( w_4 \), respectively. In such a case, all the letters between \( m \) and \( (m + 1) \) are smaller than \( m \) and \( m > F v' \). Hence, \( w \) would be tight, a contradiction.

Thus, the image \( f(w) \) of \( w \), supposed to be not tight, is André I. If
2 ≤ k + 1 ≤ m ≤ n − 2, a fortiori, k < m + 1, so that the next-to-last letter of a permutation \( w \) from \( NT_n(m, k) \), which is equal to \( k \), cannot be equal to \( (m + 1) \). Thus, \( f(NT_n(m, k)) \subset A_n(m + 1, k) \). In the same manner, if 3 ≤ m + 1 < k ≤ n − 1, the inequality \( m + 1 < k \) implies the same inclusion. As \( f \) and \( f^{-1} \) are inverses of each other when applied to the sets \( A_n(m, k) \) and \( A_n(m + 1, k) \), respectively, the restriction of \( f^{-1} \) to \( A_n(m + 1, k) \) is necessarily \( NT_n(m, k) \) by Proposition 6.1. Thus, Proposition 6.2 is proved for 2 ≤ k + 1 ≤ m ≤ n − 2 and 3 ≤ m + 1 < k ≤ n − 1. □

In Table 6.1 the bijection \( f : NT_5(m, k) \rightarrow A_5(m + 1, k) \) is materialized by the vertical arrows. The five tight permutations in And\(_5^f\) are reproduced in boldface. They can only be targets of these arrows. This completes the program of points (a) and (b).

Table 6.1: the bijection \( f : NT_5(m, k) \rightarrow A_5(m + 1, k) \) (\( k \neq m + 1 \)).

\( f : A_5(m, m + 1) \rightarrow A_5(m + 1, m) \)

Remark. When 3 ≤ m + 1 = k ≤ n − 2, in the permutation \( w = m v (m + 1) v' \) from \( A_n(m, m + 1) \) the right factor \( v' \) is equal to the one-letter word \( n \). This implies that \( f \) maps \( A_n(m, m + 1) \) onto \( A_n(m + 1, m) \) in a bijective manner. In particular, \( a_n(m, m + 1) = a_n(m + 1, m) \). This fact is illustrated in Table 6.1 by oblique arrows.

Point (c). Let \( n \geq 3 \), and consider a permutation \( w = x_1 x_2 \cdots x_{n-1} \) from And\(_{n-1}^f\). Let \( x_j = \text{spi} w \). Define \( \phi(w) := x_j x_1' x_2' \cdots x_{n-1}' \), where

\[
(6.6) \quad x'_i := \begin{cases} 
  x_i, & \text{if } x_i \leq x_j - 1; \\
  x_i + 1, & \text{if } x_i \geq x_j. 
\end{cases}
\]

The inverse bijection \( \phi^{-1} \) is defined as follows: let \( w' = x_1' x_2' \cdots x_n' \) belong to \( T_n \); then \( \phi^{-1}(w') := \rho(x_2' \cdots x_n') \), where \( \rho \) is the reduction defined in Section 2.
Theorem 6.3. The mapping $\phi$ is a bijection between $\text{And}^I_{n-1}$ and the set $T_n$ of all tight André I permutations having the following properties:

(i) $\text{spi} w = F \phi(w)$;

(ii) $\text{grn} \phi(w) = \begin{cases} 
\text{grn} w, & \text{if } \text{spi} w > \text{grn} w; \\
\text{grn} w + 1, & \text{if } \text{spi} w < \text{grn} w.
\end{cases}$

Table 6.2: The bijection $\phi : B_5(m, k - 1)$ (respectively $B_5(m, k)$) $\to$ $T_6(m, k)$.

In Table 6.2 the permutations in boldface are the elements of $\text{And}^I_5$. Their images under $\phi$ are the sixteen tight permutations from $T_6$, written in plain below them. The box $(m, k)$ contains the permutations $w$ from $\text{And}^I_5$ such that $\text{spi} w = m$ and $\text{grn} w = k$ (respectively $\text{grn} w = k - 1$) when $m > k$ (respectively when $m < k$). It also contains the elements $w'$ from $T_6$ such that $F w' = m$ and $\text{grn} w' = k$. A hat sign $\hat{\text{ }}$ has been put on the spike of $w$.

Proof of Theorem 6.3. Let $w = x_1 x_2 \cdots x_{n-2} x_{n-1}$ be from $\text{And}^I_{n-1}$. Let $x_j = \text{spi} w$. If $j = 1$, then $x_1 > x_2$ and $\phi(w) = x_1(x_1 + 1)$ $x_2 \cdots x'_{n-2} x'_{n-1}$. Accordingly, $\phi(w)$ is tight. Moreover, $\text{spi} w = x_1 = F \phi(w)$, still since $x_1 > x_2$. Also, either $\text{grn} w = x_{n-2} < x_1 = \text{spi} w$, and then $\text{grn} \phi(w) = x'_{n-2} = x_{n-2} = \text{grn} w$, or $\text{grn} w = x_{n-2} > x_1 = \text{spi} w$, and then $\text{grn} \phi(w) = x'_{n-2} = x_{n-2} + 1 = \text{grn} w + 1$. 

\begin{tabular}{|c|c|c|c|c|}
\hline
$k$ & 1 & 2 & 3 & 4 & 5 \\
\hline
$m = 1$ & - & - & - & - & - \\
\hline
2 & & & $21435$ & $21345$ & \\
& & & 231546 & 231456 & \\
\hline
3 & $32415$ & $31425$ & $23145$ & & \\
& 342516 & 341526 & 321456 & 31245 & 341256 \\
\hline
4 & $23415$ & $34125$ & $24135$ & & \\
& 423516 & 435126 & 425136 & & \\
& $41325$ & $41235$ & $415126$ & & \\
& 451326 & 451236 & 451236 & & \\
\hline
5 & $13425$ & $12435$ & $12345$ & & \\
& 513426 & 512436 & 512346 & & \\
& $14235$ & $13245$ & $13245$ & & \\
& 514236 & 513246 & 513246 & & \\
\hline
\end{tabular}
If $j \geq 2$, we have
\begin{equation}
\phi(w) = x_j x_1 \cdots x_{j-1} (x_j + 1) x_{j+1} x'_{j+2} \cdots x'_{n-1}.
\end{equation}

On the other hand, $\phi(w)$ is André I, because no double descent has been created; furthermore, the new trough $x_1$ is of type I, as the letter $(x_j + 1)$ is to its right. Also, $\phi(w)$ is tight, because $x_{j+1}$ (respectively $(x_j + 1)$) is less (respectively greater) than all the letters to its left. Finally, $\text{spi} w = x_j = F \phi(w)$. Moreover, $\text{grn} \phi(w) = x'_{n-2}$ is equal to $x_{n-2} = \text{grn} w$ or $x_{n-2} + 1 = \text{grn} w + 1$, depending on whether $x_{n-2} = \text{grn} w$ is less than or at least equal to $x_n = \text{spi} w$. \[\square\]

This achieves the program of point (c), by definition of $B_n(m, k)$ given in (6.2).

6.3. Hooked and unhooked permutations. Let $n \geq 3$, and consider the mapping $\Theta$, defined on $\text{And}_n^{I}$ as follows. Let $w = x_1 x_2 \cdots x_{n-1}$ belong to $\text{And}_n^{I}$. Define
\begin{equation}
\Theta(w) := \begin{cases} (x_1 + 1)x_1 x_2^' \cdots x_{n-1}^', & \text{if } x_1 < x_2; \\ x_1(x_1 + 1)x_2^' \cdots x_{n-1}^', & \text{if } x_1 > x_2; \end{cases}
\end{equation}
where $x_i^' := x_i$ (respectively $x_i + 1$) if $x_i < x_1$ (respectively if $x_i > x_1$). Clearly, $\Theta$ is an injection from $\text{And}_n^{I}$ into $\text{And}_n^{I}$. The permutations belonging to the subset $\Theta(\text{And}_n^{I})$ are said to be hooked. Their formal definition is stated next.

\textbf{Definition 6.2.} An André I permutation $w = x_1 x_2 \cdots x_n$ $(n \geq 3)$ from $\text{And}_n^{I}$ is called \textit{hooked}, if $x_1 - 1 = x_2 < x_3$ or $x_1 + 1 = x_2 > x_3$.

Let $H_n$ denote the subset of all the hooked permutations from $\text{And}_n^{I}$. The elements of the set-theoretic difference $NH_n := \text{And}_n^{I} \setminus H_n$ are said to be \textit{unhooked}. Let $H_n(m, k)$ (respectively $NH_n(m, k)$) denote the subset of $H_n$ (respectively of $NH_n$) consisting of all $w$ such that $(\text{spi, grn})w = (m, k)$.

\textbf{Proposition 6.4.} The injection $\Theta$ defined in (6.8) from $\text{And}_n^{I}$ into $\text{And}_n^{I}$ maps $\text{And}_n^{I}$ onto $H_n$. Moreover, for each $w$ from $\text{And}_n^{I}$ we have
\begin{equation}
\text{spi} \Theta(w) = 1 + F w; \\
\text{grn} \Theta(w) = \begin{cases} 1 + \text{grn} w, & \text{if } F w < \text{grn} w; \\ \text{grn} w, & \text{if } F w > \text{grn} w. \end{cases}
\end{equation}

\textbf{Proof.} With the notation of (6.8), we have $\text{spi} \Theta(w) = x_1 + 1$ in both cases. The identity for “grn” follows from the very definition of $\Theta$. \[\square\]
Corollary 6.5. The mapping \( \Theta \) is a bijection between \( A_{n-1}(m,k) \) and \( H_n(m+1,k) \) when \( 1 \leq k < m \leq n-2 \), and onto \( H_n(m+1,k+1) \) when \( 3 \leq m+2 \leq k \leq n-1 \).

In Table 6.3, the sixteen permutations from \( \text{And}_I^5 \) are reproduced in boldface, and below them the hooked permutations from \( H_6 \) that are their images under \( \Theta \). A hat sign \( \hat{\cdot} \) has been put on the spike of each permutation from \( H_6 \).

<table>
<thead>
<tr>
<th>( k = )</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
</tr>
</thead>
<tbody>
<tr>
<td>( m = 1 )</td>
<td>( \cdot )</td>
<td>( \cdot )</td>
<td>( \cdot )</td>
<td>( \cdot )</td>
<td>( \cdot )</td>
</tr>
<tr>
<td>2</td>
<td>( \cdot )</td>
<td>( \cdot )</td>
<td>( 13425 )</td>
<td>( 12435 )</td>
<td>( 12345 )</td>
</tr>
<tr>
<td>3</td>
<td>( \hat{2}14536 )</td>
<td>( \hat{2}13546 )</td>
<td>( \hat{1}4235 )</td>
<td>( \hat{1}3245 )</td>
<td>( \hat{1}2435 )</td>
</tr>
<tr>
<td>3</td>
<td>( 23415 )</td>
<td>( 324516 )</td>
<td>( \hat{2}1435 )</td>
<td>( \hat{2}31546 )</td>
<td>( \hat{2}3145 )</td>
</tr>
<tr>
<td>4</td>
<td>( 32415 )</td>
<td>( 341526 )</td>
<td>( 34125 )</td>
<td>( 35126 )</td>
<td>( \hat{3}25146 )</td>
</tr>
<tr>
<td>4</td>
<td>( \hat{3}25146 )</td>
<td>( \hat{3}25146 )</td>
<td>( \hat{3}25146 )</td>
<td>( \hat{3}25146 )</td>
<td>( \hat{3}25146 )</td>
</tr>
<tr>
<td>5</td>
<td>( 41325 )</td>
<td>( 41325 )</td>
<td>( 41235 )</td>
<td>( 41235 )</td>
<td>( 41235 )</td>
</tr>
</tbody>
</table>

Table 6.3: the bijection \( \Theta : A_5(m,k) \rightarrow H_6(m+1,k) \) (respectively \( H_6(m+1,k+1) \))

Referring to the program displayed in (1.14), the first bijection \( \Theta \) has been constructed. The next step is devoted to the construction of the bijection \( \beta \).

6.4. A bijection between \( B_n(m,k) \) and \( NH_n(m+1,k) \). We go back to the proofs of (TS4.1) and (TS4.2) made in Subsection 6.1. It was shown that \( b_n(n-1,k) = b_n(n,k+1) = b_{n-1}(\bullet,k) \) for \( 1 \leq k \leq n-2 \). By means of the bijections described in Subsection 6.1 and Section 4, and also the bijection \( \phi \) constructed in (5.2), we can set up a bijection between \( B_n(n-1,k) \) and \( B_n(n,k+1) \). We can also proceed directly as follows. Let \( n \geq 3 \) and \( w = x_1 \cdots x_{i-1} \underbrace{\cdot \cdots \cdot \cdot}_{\text{\( n-1 \) \( k \)}} x_{i+1} \cdots k \) be an André I permutation such that \((\text{spi}, \text{grn})w = (n-1,k)\). Then the mapping \( \alpha \) defined by

\[
\alpha(w) := 1(x_1 + 1) \cdots (x_{i-1} + 1) (x_{i+1} + 1) \cdots (k + 1) \hat{n}
\]

fulfills our requirements.
The inverse $\alpha^{-1}$ is easy to find: let $w' = x'_1 \ x'_2 \cdots x'_{n-1} \ n$ be a permutation from $B_n(n, k+1)$, so that $x'_1 = 1$. Then $\alpha^{-1}(w')$ is obtained by first determining the leftmost letter $x'_{i+1}$ less than or equal to $x'_2$ and subsequently letting

$$(6.11) \quad \alpha^{-1}(w') := (x'_2 - 1) \cdots (x'_{i-1} - 1) \ (n - 1) \ (x'_{i+1} - 1) \cdots k \ n.$$ 

The bijection $\alpha$ will be an ingredient for the next bijection, $\beta$, between $B_n(m, k)$ and $NH_n(m + 1, k)$.

First, let

$$(6.12) \quad 2 \leq k + 1 \leq m \leq n - 2 \quad \text{or} \quad 3 \leq m + 2 \leq k \leq n - 1,$$

and partition $B_n(m, k)$ into two subsets $B_n^{(1)}(m, k)$, $B_n^{(2)}(m, k)$ as follows. Note that each permutation $w$ from $B_n(m, k)$ is of the form $w = w_1 m w_2 (m + 1) w_3$ and the factor $w_2$ is never empty, as $m$ is the spike of $w$. Also, $w_3 \neq \epsilon$ because of condition (6.12). Say that an element of $B_n(m, k)$ belongs to $B_n^{(1)}(m, k)$ (respectively to $B_n^{(2)}(m, k)$) if $F w_3$ is not (respectively if $F w_3$ is) a left minimum record, or equivalently, if $\min w_2 < F w_3$ (respectively if $\min w_2 > F w_3$).

Let $w = x_1 x_2 \cdots x_n = w_1 m w_2 (m + 1) w_3$ be from $B_n(m, k)$ with $(m, k)$ satisfying (6.12).

(1) If $w$ belongs to $B_n^{(1)}(m, k)$, define $w' := \beta(w)$ to be the permutation derived from $w$ by transposing the letters $m$ and $(m + 1)$:

$$(6.13) \quad \beta : w = w_1 m w_2 (m + 1) w_3 \mapsto w' = w_1 (m + 1) w_2 m w_3.$$ 

(2) If $w$ belongs to $B_n^{(2)}(m, k)$, consider the factorization $w = v_1 w_3$, where $v_1 = w_1 m w_2 (m + 1)$. Then $v_1$ is André $I$ by Proposition 2.1 (6). Let $n'$ be the length of $v_1$ and $\rho(v_1)$ be the reduction of $v_1$ (by using the increasing bijection from the set $\{x_1, \ldots, m, \ldots, m+1\}$ onto $\{1, 2, \ldots, n'\}$). Thus, $\rho(v_1)$ is an André $I$ permutation from $\And_{n'}^I$ such that $\spi \rho(v_1) = n' - 1$. The bijection $\alpha$, introduced in (6.10), can be applied to $\rho(v_1)$, and the permutation $w' := \beta(w)$ is defined by replacing the left factor $v_1$ of $w$ by $\rho^{-1} \alpha \rho(v_1)$:

$$(6.14) \quad \beta : w = v_1 w_3 \mapsto w' := \rho^{-1} \alpha \rho(v_1) w_3.$$ 

Example. The permutation $w = 4 \ 5 \ 3 \ 8 \ 1 \ 6 \ 7 \ 2 \ 9$ belongs to $B_9^{(1)}(5, 2)$, as $\min w_2 = \min 381 = 1 < 7 = F w_3$. It then suffices to transpose 5 and 6 to get the permutation $w' = 4 \ 6 \ 3 \ 8 \ 1 \ 5 \ 7 \ 2 \ 9$. 
Next, \( w = 356271849 \) belongs to \( B_3^{(2)}(6,4) \), as \( \min w_2 = 2 > 1 = \mathbf{F} w_3 \). Hence, \( v_1 = 35627, \rho(v_1) = 23415, \alpha\rho(v_1) = 13425, \rho^{-1}\alpha\rho(v_1) = 25637 \) and \( w' = 256371849 \).

When \( w \) belongs to \( B_n^{(1)}(m,k) \), the letter \((m + 1)\) occurs to the left of \( m \) in \( w' \). On the other hand, as \( w_2 \) is non-empty and \( m + 1 > \mathbf{F} w_2 \), the permutation \( w' \) is unhooked if \( w_1 = e \). The same conclusion also holds if \( w_1 \neq e \), because \( \mathbf{L} w_1 < m + 1 \) and \( \mathbf{L} w_1 \neq m \). Obviously, \( \text{spi} w' = m + 1 \) and \( \text{grn} w' = k \) by (6.12).

Let us now prove that \( w' \) is André I. Note that the troughs remain the same in both \( w \) and \( w' \). Let \( x \) be a trough within \( w_2 \) and \( (v_1, v_2, x, v_4, v_5) \) (respectively \( (v'_1, v'_2, x, v'_4, v'_5) \)) be the \( x \)-factorization of \( w \) (respectively of \( w' \)). When going from \( w \) to \( w' \), the type of \( x \) is not modified when at least one of the following conditions holds: \( \max v_2 \neq m, \max v_4 \neq m + 1 \). If both were violated for a given \( x \), it would be the case for \( x = \min w_2 \), and all the letters of \( w_2 \) would be less than \( m \). But \( \max v_4 = m + 1 \) implies \( \max v_4 > \mathbf{F} w_3 > \min w_2 \), and \( \mathbf{F} w_2 \) is a trough of \( w \). If \( (v'_1, v'_2, \mathbf{F} w_3, v'_4, v'_5) \) is the \( \mathbf{F} w_3 \)-factorization of \( w \), the word \( \mathbf{F} w_3 v'_4 \) is necessarily a factor of \( v_4 \), as all its letters are greater than \( \min w_2 \). Hence, \( \max v_4 > m + 1 \), a contradiction. Thus, \( w' \) is an unhooked permutation from \( \text{And}_n^I \) such that \( \text{spi} w' = m + 1, \text{grn} w' = k \) with \((m+1)\) to the left of \( m \). In short, \( w' \in \text{NH}_n^{(1)}(m+1,k) \).

As the transposition \( w_1(m+1) w_2 m w_3 \mapsto w_1 m w_2(m+1) w_3 \), when applied to André I permutations with \((m+1)\) to the left of \( m \), always maps an André I onto an André I permutations,

the direct transposition \( \beta \) defined in (6.13) is a bijection between \( B_n^{(1)}(m,k) \) and \( \text{NH}_n^{(1)}(m+1,k) \).

Next, let \( w \) belong to \( B_3^{(2)}(m,k) \), and consider the permutation \( w' = \beta(w) \) defined in (6.14). The left factor \( \rho^{-1}\alpha\rho(v_1) \) of \( \beta(w) \) is André I and ends with \((m+1)\). Therefore, \( \beta(w) \) is of the form \( w'_1 m w'_2(m+1) w_3 \). Again, with the hypothesis (6.12), the letter \( x_{n-1} \), equal to \( k \) in the permutation \( w = x_1 x_2 \cdots k n \), remains untouched when going from \( w \) to \( w' \). Thus, \( \text{grn} w' = \text{grn} w = k \). Next, we get \( \text{spi} \rho(v_1) = n' - 1 \) and \( \text{spi} \alpha\rho(v_1) = n' \); hence, \( \text{spi} \rho^{-1}\alpha\rho(v_1) = m + 1 \). As \( w_3 \) starts with a letter less than all the letters in \( v_1 \), we have \( \text{spi} w' = \text{spi} \rho^{-1}\alpha\rho(v_1) w_3 = m + 1 \). Moreover, \( w_3 \) is André I by Proposition 2.1 (5), so that \( \beta(w) \) is André I by Proposition 2.1 (7). This shows that

the mapping \( \beta \) defined in (6.14) is a bijection between \( B_n^{(2)}(m,k) \) and the set \( \text{NH}_n^{(2)}(m+1,k) \), defined as the set of all unhooked permutations from \( \text{And}_n^I \) such that \( \text{spi} w' = m + 1, \text{grn} w' = k \) with \( m \) to the left of \((m+1)\).
This proves the following theorem.

**Theorem 6.6.** Under condition (6.12), the mapping \( \beta : w \mapsto w' \) defined in (6.13) and (6.14) is a bijection between \( B_n(m, k) = B_n^{(1)}(m, k) \uplus B_n^{(2)}(m, k) \) and \( NH_n(m + 1, k) = NH_n^{(1)}(m + 1, k) \uplus NH_n^{(2)}(m + 1, k) \).

### Table 6.4: the bijection \( \beta : B_6(m, k) \rightarrow NH_6(m + 1, k) \)

*Example.* In Table 6.4, the image \( \beta(w) \) of each André I permutation \( w \) from \( B_6(m, k) \), with \( (m, k) \) satisfying inequalities (6.12) for \( n = 6 \), is indicated by a downarrow. The hooked permutations are reproduced in boldface. Note that they are not bottoms of any downarrows, as \( \beta \) is a bijection between \( B_n(m, k) \) and \( NH_n(m + 1, k) \).

With the construction of the bijection \( \beta : B_n(m, k) \rightarrow NH_n(m + 1, k) \), the program displayed in (1.14) is completed, as

\[
\Delta_n^{(1)} b_n(m, k) = \#B_n(m + 1, k) - \#B_n(m, k) \\
= \#B_n(m + 1, k) - \#NH_n(m + 1, k) \\
= \#H_n(m + 1, k) \\
= \#A_{n-1}(m, k) \quad \text{(respectively = \#A_{n-1}(m, k - 1))}
\]

if \( 1 \leq k < m \leq n - 2 \) (respectively if \( 3 \leq m + 2 \leq k \leq n - 1 \)).
7. The making of Seidel triangle sequences

7.1. The Seidel tangent-secant matrix. In the sequel, three exponential generating functions will be attached to each infinite matrix $A = (a(m, k))_{m,k \geq 0}$,

$$A(x, y) := \sum_{m,k \geq 0} a(m, k) \frac{x^m y^k}{m! k!};$$

$$A_m, \bullet(y) := \sum_{k \geq 0} a(m, k) \frac{y^k}{k!};$$

$$A_{\bullet, k}(x) := \sum_{m \geq 0} a(m, k) \frac{x^m}{m!};$$

for $A$ itself, its $m$-th row, its $k$-th column. Let $\overline{H} = (h_{i,j})$ $(i, j \geq 0)$ be the infinite matrix whose entries are the Entringer numbers $E_n(m)$ displayed along the skew-diagonals with the following sign:

\begin{align*}
(7.1) & \quad h_{i,j} = \begin{cases} (-1)^n E_{i+j+1}(j+1), & \text{if } i + j = 2n; \\ (-1)^n E_{i+j+1}(i+1), & \text{if } i + j = 2n - 1; \end{cases} \\
(7.2) & \quad E_{2n+1}(j+1) = (-1)^n h_{2n-j,j} \quad (0 \leq j \leq 2n); \\
(7.3) & \quad E_{2n}(i+1) = (-1)^n h_{i,2n-1-i} \quad (0 \leq i \leq 2n - 1);
\end{align*}

or still in displayed form:

$$\overline{H} = \begin{pmatrix}
E_1(1) & -E_2(1) & 0 & E_4(1) & 0 & -E_6(1) & 0 & \cdots \\
0 & -E_3(2) & E_4(2) & E_5(4) & -E_6(2) & -E_7(6) & \\
-E_3(1) & E_4(3) & E_5(3) & -E_6(3) & -E_7(5) & \\
0 & E_5(2) & -E_6(4) & -E_7(4) & \\
E_5(1) & -E_6(5) & -E_7(3) & \\
0 & -E_7(2) & \\
-E_7(1) & \\
\vdots & \\
1 & -1 & 0 & 2 & 0 & -16 & 0 & \cdots & \\
0 & -1 & 2 & 2 & -16 & -16 & \\
-1 & 1 & 4 & -14 & -32 & \\
0 & 5 & -10 & -46 & \\
5 & -5 & -56 & \\
0 & -61 & \\
-61 & \\
\vdots & 
\end{pmatrix}.$$
As noted by Dumont [Du82], the definition of such a matrix $H$ goes back to Seidel himself [Se1877]. Entringer [En66] rediscovered the absolute values of the entries, when he classified the alternating permutations according to their first letters. The entries of the top row are the coefficients of the Taylor expansion of $1 - \tanh y = 2/(1 + e^{2y})$:

$$H_{0, \bullet}(y) = 1 - \tanh y = 1 + \sum_{n \geq 1} \frac{y^{2n-1}}{(2n-1)!} (-1)^n E_{2n-1}$$

$$= 1 - \frac{y}{1!} + \frac{y^3}{3!} 2 - \frac{y^5}{5!} 16 + \frac{y^7}{7!} 272 - \frac{y^9}{9!} 7936 + \cdots.$$  

The entries of the leftmost column are the coefficients of the Taylor expansion of $1/\cosh x = 2 e^x / (1 + e^{2x})$, so that

$$H_{\bullet, 0}(x) = \frac{1}{\cosh x} = \sum_{n \geq 0} \frac{x^{2n}}{(2n)!} (-1)^n E_{2n}$$

$$= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} 5 - \frac{x^6}{6!} 61 + \frac{x^8}{8!} 1385 - \cdots.$$  

By means of recurrence (1.1) satisfied by the Entringer numbers and (7.1), we can verify that the entries $h_{i,j}$ obey the following rule: $h_{i,j} = h_{i-1,j} + h_{i-1,j+1}$ for $j \geq 0$, $i \geq 1$, so that the entries $H_{i,j}$ can be obtained by applying such a rule inductively, the entries of the top row being given. Such a matrix is called a Seidel matrix by Dumont [Du82], and its exponential generating function is directly obtained from the exponential generating function for its top row by the formula $H(x, y) = H_{0, \bullet}(x+y) e^x$ (see, e.g., [DV80]). Accordingly,

$$H(x, y) = \frac{2 e^x}{1 + e^{2x+2y}}.$$  

Two further matrices are derived from $H$. The first one, $H_1$, is obtained by replacing all the entries $h_{i,j}$ such that $i + j$ is odd by zero, so that

$$H_1 = \begin{pmatrix} 1 & -1 & 0 & 0 & 0 & \cdots \\ -1 & 4 & -32 & 2 & -16 \\ 0 & 5 & -46 & 5 & -61 \\ -61 \\ \vdots \end{pmatrix}.$$
As $\overline{H}(x, y) = \frac{2e^x}{1 + e^{2x+2y}}$, we get

\begin{equation}
(7.5) \quad \overline{H}_1(x, y) = \frac{H(x, y) + H(-x, -y)}{2} = e^x \frac{1 + e^{2y}}{1 + e^{2x+2y}} = \frac{\cosh y}{\cosh(x + y)}.
\end{equation}

The second one, $\overline{H}_2$, is derived from $\overline{H}$ by replacing the entries $\overline{h}_{i,j}$ such that $i + j$ is even by 0, so that

\begin{equation*}
\overline{H}_2 = \begin{pmatrix}
0 & 2 & -16 & 272 & \\
0 & -10 & 224 & \\
0 & -5 & 178 & \\
0 & 122 & \\
0 & \\
\end{pmatrix}.
\end{equation*}

Therefore,

\begin{equation}
(7.6) \quad \overline{H}_2(x, y) = \frac{H(x, y) - H(-x, -y)}{2} = e^x \frac{1 - e^{2y}}{1 + e^{2x+2y}} = -\frac{\sinh y}{\cosh(x + y)}.
\end{equation}

In the sequel, further matrices will be derived from $\overline{H}_1$ and $\overline{H}_2$, essentially by transposing them and/or removing either their top rows or leftmost columns. The corresponding actions on their respective exponential generating functions $\overline{H}_1(x, y)$ and $\overline{H}_2(x, y)$ are the exchange of the variables $x$ and $y$: $\overline{T}H_1(x, y) := H_1(y, x)$; then, the partial derivatives with respect to $x$ and $y$: $D_x H_i(x, y)$ and $D_y H_i(x, y)$ ($i = 1, 2$).

### 7.2. The generating function for the Entringer numbers

The generating function for the Entringer numbers, already derived in [FH14], can be obtained from relations (7.5) and (7.6). In fact, they are simply equal to $\overline{H}_1(xI, yI)$ and $\overline{T}H_2(xI, yI)$, where $I = \sqrt{-1}$. Thus,

\begin{equation}
(7.7) \quad \sum_{1 \leq k \leq 2n+1} E_{2n+1}(k)x^{2n+1-k}y^{k-1}(2n + 1 - k)! (k - 1)! = \frac{\cos y}{\cos(x + y)};
\end{equation}

\begin{equation}
(7.8) \quad \sum_{1 \leq k \leq 2n} E_{2n}(k)x^{k-1}y^{2n-k}(k - 1)! (2n - k)! = \frac{\sin y}{\cos(x + y)}.
\end{equation}

### 7.3. Seidel triangle sequences

For calculating the generating functions for the twin Seidel matrices, we shall take recourse to the techniques
developed in our previous paper [FH14] for the so-called Seidel triangle sequences. Only definitions will be stated, as well as the main result.

A sequence of square matrices \((C_n)\) \((n \geq 1)\) is called a Seidel triangle sequence if the following three conditions are fulfilled:

(STS1) each matrix \(C_n\) is of dimension \(n\);

(STS2) each matrix \(C_n\) has zero entries along and below its diagonal; let \((c_n(m, k))\) \((0 \leq m < k \leq n - 1)\) denote its entries strictly above its diagonal, so that

\[
C_1 = ( \cdot ); \quad C_2 = \left( \begin{array}{c} \cdot \end{array} \right); \quad C_3 = \left( \begin{array}{ccc} \cdot & c_3(0, 1) & c_3(0, 2) \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \end{array} \right); \quad \ldots ;
\]

\[
C_n = \left( \begin{array}{cccc} \cdot & c_n(0, 1) & c_n(0, 2) & \ldots & c_n(0, n - 2) & c_n(0, n - 1) \\ \cdot & \cdot & c_n(1, 2) & \ldots & c_n(1, n - 2) & c_n(1, n - 1) \\ \cdot & \cdot & \cdot & \ldots & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \end{array} \right);
\]

the dots “.” along and below the diagonal referring to zero entries.

(STS3) for each \(n \geq 3\), the following relation holds:

\[
c_n(m, k) - c_n(m, k + 1) = c_{n-1}(m, k) \quad (m < k).
\]

Record the last columns of the triangles \(C_2, C_3, C_4, C_5, \ldots\), read from top to bottom, namely, \(c_2(0, 1); \quad c_3(0, 2), c_3(1, 2); \quad c_4(0, 3), c_4(1, 3), c_4(2, 3); \quad c_5(0, 4), c_5(1, 4), c_5(2, 4), c_5(3, 4); \ldots\), as skew-diagonals of an infinite matrix \(H = (h_{i,j})_{i,j \geq 0}\), as shown next:

\[
(7.9) \quad H := \begin{pmatrix}
0 & 1 & 2 & 3 & 4 & \ldots \\
1 & c_2(0, 1) & c_3(1, 2) & c_4(2, 3) & c_5(3, 4) & c_6(4, 5) & \ldots \\
2 & c_3(0, 2) & c_4(1, 3) & c_5(2, 4) & c_6(3, 5) & & \\
3 & c_4(0, 3) & c_5(1, 4) & c_6(2, 5) & & \\
4 & c_5(0, 4) & c_6(1, 5) & & \\
\vdots & \vdots & & & & &
\end{pmatrix},
\]

Equivalently, the entries of \(H\) are defined by

\[
(7.10) \quad h_{i,j} = c_{i+j+2}(j, i + j + 1).
\]

The next theorem has been proved in [FH14] and will be of great use in the next sections.
Theorem 7.1. The three-variable generating function for the Seidel triangle sequence \( (C_n = (c_n(m,k)))_{n \geq 1} \) is equal to

\[
\sum_{1 \leq m+1 \leq k \leq n-1} c_n(m,k) \frac{x^{n-k-1}}{(n-k-1)!} \frac{y^{k-m-1}}{(k-m-1)!} \frac{z^m}{m!} = e^x H(x + y, z),
\]

where \( H \) is the infinite matrix defined in (7.10).

With \( I := \sqrt{-1} \), we get

\[
\sum_{1 \leq m+1 \leq k \leq n-1} I^{n-2} c_n(m,k) \frac{x^{n-k-1}}{(n-k-1)!} \frac{y^{k-m-1}}{(k-m-1)!} \frac{z^m}{m!} = e^{ix} H(Ix + Iy, Iz).
\]

8. Trivariate generating functions

Each of the sequences \( \text{Twin}^{(1)} := (A_2, B_3, A_4, B_5, A_6, \ldots) \) and \( \text{Twin}^{(2)} := (B_2, A_3, B_4, A_5, B_6, \ldots) \) (see Diagram 1.3) gives rise to two Seidel triangle sequences, by considering the upper and lower triangles of the matrices.

8.1. The upper triangles of \( \text{Twin}^{(1)} \). The Seidel triangle sequence to be constructed is the following: first, \( C_1 = (\cdot) \), then for \( n \geq 2 \) each \( C_n \) will be derived from the upper triangle of \( A_{n+1} \) (respectively \( B_{n+1} \)) by (i) dropping the rightmost column; (ii) transposing the remaining triangle with respect to its skew-diagonal; (iii) changing the signs of its entries according the following rule. More precisely,

\[
C_n := (-1)^{(n+1)/2} \begin{pmatrix}
\cdot & a_{n+1}(n-1, n) & \cdots & a_{n+1}(2, n) & a_{n+1}(1, n) \\
\cdot & \ddots & \cdots & \vdots & \vdots \\
\cdot & \cdot & \ddots & a_{n+1}(2, 3) & a_{n+1}(1, 3) \\
\cdot & \cdot & \cdot & \ddots & a_{n+1}(1, 2)
\end{pmatrix}
\]

if \( n \) is odd;

\[
C_n := (-1)^{n/2} \begin{pmatrix}
\cdot & b_{n+1}(n-1, n) & \cdots & b_{n+1}(2, n) & b_{n+1}(1, n) \\
\cdot & \ddots & \cdots & \vdots & \vdots \\
\cdot & \cdot & \ddots & b_{n+1}(2, 3) & b_{n+1}(1, 3) \\
\cdot & \cdot & \cdot & \ddots & b_{n+1}(1, 2)
\end{pmatrix}
\]

if \( n \) is even;

By referring to Diagram 1.3, we get \( C_1 = \cdot ; \ C_2 = \begin{pmatrix} 0 \end{pmatrix} ; \ C_3 = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} ; \)
The case when \( n \leq 0 \) is odd can be proved in a similar way.

Therefore,

\[
(8.1) \quad c_n(m, k) = \begin{cases} 
(-1)^{(n+1)/2}a_{n+1}(n - k, n - m), & \text{if } n \text{ is odd}; \\
(-1)^n b_{n+1}(n - k, n - m), & \text{if } n \text{ is even}.
\end{cases}
\]

**Proposition 8.1.** The sequence \( (C_n) (n \geq 1) \) just defined is a Seidel triangle sequence.

**Proof.** We only have to verify that rule (STS3) holds. If \( n \) is odd and \( 0 \leq m < k \leq n - 2 \), then \( 3 \leq m' + 2 := (n - k - 1) + 2 \leq k' := n - m \leq (n + 1) - 1 \) and

\[
c_n(m, k) - c_n(m, k + 1)
= (-1)^{(n+1)/2}(a_{n+1}(n - k, n - m) - a_{n+1}(n - k - 1, n - m))
= (-1)^{(n+1)/2}\Delta (1)_{n+1}(n - k - 1, n - m)
= (-1)^{(n+1)/2}\Delta (1)_{n+1}(m', k')
= (-1)^{(n-1)/2}b_n(m', k' - 1) \quad [\text{by rule (TS5.2)}]
= (-1)^{(n-1)/2}b_n(n - k - 1, n - m - 1)
= (-1)^{(n-1)/2}(-1)^{(n-1)/2}c_{n-1}(m, k) = c_{n-1}(m, k).
\]

The case when \( n \) is even can be proved in a similar way. \( \square \)

The next step is to determine the matrix \( H \), as defined in (7.9), whose skew-diagonals are equal to the rightmost columns of the matrices \( C_n \). For \( n \geq 2 \) the skew-diagonal \( (c_n(0, n - 1), c_n(1, n - 1), \ldots, c_n(n - 2, n - 1)) \) of \( H \), being the rightmost column of \( C_n \), is equal to

\[
\left\{ \begin{array}{l}
(-1)^{(n+1)/2}(a_{n+1}(1, n), a_{n+1}(1, n - 1), \ldots, a_{n+1}(1, 2)), \\
(-1)^n b_{n+1}(1, n), b_{n+1}(1, n - 1), \ldots, b_{n+1}(1, 2)),
\end{array} \right.
\]

which is also equal to
\[
\begin{cases}
(-1)^{(n+1)/2}(b_n(\bullet, n-1), b_n(\bullet, n-2), \ldots, b_n(\bullet, 1)), & \text{if } n \text{ is odd;}
\\
(0, 0, \ldots, 0), & \text{if } n \text{ is even;}
\end{cases}
\]

by Rules (TS5.1) and (TS2); finally, this is equal to
\[
\begin{cases}
(-1)^{(n+1)/2}(E_n(1), E_n(2), \ldots, E_n(n-1)), & \text{if } n \text{ is odd;}
\\
(0, 0, \ldots, 0), & \text{if } n \text{ is even;}
\end{cases}
\]

by (1.11).

Thus,
\[
H = \begin{pmatrix}
0 & E_3(2) & 0 & -E_5(4) & 0 & E_7(6) & 0 & \cdots \\
E_3(1) & 0 & -E_5(3) & 0 & E_7(5) & 0 \\
0 & -E_5(2) & 0 & E_7(4) & 0 \\
-E_5(1) & 0 & E_7(3) & 0 \\
0 & E_7(2) & 0 \\
E_7(1) & 0 \\
0 \\
\vdots
\end{pmatrix}
\]

(8.2)

\[
= \begin{pmatrix}
0 & 1 & 0 & -2 & 0 & 16 & 0 & \cdots \\
1 & 0 & -4 & 0 & 32 & 0 \\
0 & -5 & 0 & 46 & 0 \\
-5 & 0 & 56 & 0 \\
0 & 61 & 0 \\
61 & 0 \\
0 \\
\vdots
\end{pmatrix}
\]

This matrix is to be compared with the matrix \( \overline{H}_1 \) (see §7.1). For getting \( H \), it suffices to delete the top row of \( \overline{H}_1 \) and change the signs of all the entries. As \( \overline{H}_1(x, y) = \cosh y / \cosh(x+y) \) by (7.5), we have

(8.3)

\[
H(x, y) = -D_x \overline{H}_1(x, y) = \frac{\cosh y \sinh(x+y)}{\cosh^2(x+y)}.
\]

Hence, the right-hand side of (7.11) becomes

\[
e^x H(x + y, z) = e^x \frac{\cosh z \sinh(x+y+z)}{\cosh^2(x+y+z)};
\]

and the right-hand side of (7.12) is equal to

\[
e^{Ix} H(Ix + Iy, Iz) = (\cos x + I \sin x) \frac{I \cos z \sin(x+y+z)}{\cos^2(x+y+z)}.
\]

It remains to interpret the left-hand side of identity (7.12) by using (8.1). If \( n = 2l + 1 \), then \( I^{n-2} = (-1)^{l+1} I \) and \( (-1)^{(n+1)/2} = (-1)^{l+1} \).
Thus, $I^{n-2}c_n(m, k) = I a_{n+1}(n-k, n-m)$. The imaginary part of identity (7.12) then reads

$$\sum_{1 \leq m+1 \leq k \leq n-1 \atop n \text{ odd}} a_{n+1}(n-k, n-m) \frac{x^{n-k-1} y^{k-m-1} z^m}{(n-k-1)! (k-m-1)! m!} = \frac{\cos x \cos z \sin(x+y+z)}{\cos^2(x+y+z)}.$$

With the change of variables $n \leftarrow 2n-1$, $n-k \leftarrow m$, $n-m \leftarrow k$, we get (1.15) from Theorem 1.3. Note that the above generating function involves all the matrices $A_4, A_6, \ldots$ of $\text{Twin}^{(1)}$, but not the very first term $A_2 = \left( \begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix} \right)$.

If $n = 2l$, then $I^{n-2} = (-1)^{l-1}$ and $(-1)^{n/2} = (-1)^l$, so that $I^{n-2}c_n(m, k) = -b_{n+1}(n-k, n-m)$. As for the real part, we have

$$\sum_{1 \leq m+1 \leq k \leq n-1 \atop n \text{ even}} b_{n+1}(n-k, n-m) \frac{x^{n-k-1} y^{k-m-1} z^m}{(n-k-1)! (k-m-1)! m!} = \frac{\sin x \cos z \sin(x+y+z)}{\cos^2(x+y+z)}.$$

With the change of variables $n \leftarrow 2n$, $n-k \leftarrow m$, $n-m \leftarrow k$, we get (1.21) from Theorem 1.6.

**8.2. The upper triangles of $\text{Twin}^{(2)}$.** The sequence of triangles to be considered is the following: $C_1 = \cdot$ and for $n \geq 2$

$$C_n := (-1)^{(n-1)/2} \begin{pmatrix} \cdots & \cdots & \cdots & \cdots \\ \cdots & b_{n+1}(n-1, n) \cdots & b_{n+1}(2, n) b_{n+1}(1, n) & \cdots \\ \cdots & \cdots & b_{n+1}(2, 3) b_{n+1}(1, 3) & \cdots \\ \cdots & \cdots & \cdots & b_{n+1}(1, 2) \end{pmatrix} \text{ if } n \text{ is odd;}$$

$$C_n := (-1)^{n/2} \begin{pmatrix} \cdots & \cdots & \cdots & \cdots \\ \cdots & a_{n+1}(n-1, n) \cdots & a_{n+1}(2, n) a_{n+1}(1, n) & \cdots \\ \cdots & \cdots & a_{n+1}(2, 3) a_{n+1}(1, 3) & \cdots \\ \cdots & \cdots & \cdots & a_{n+1}(1, 2) \end{pmatrix} \text{ if } n \text{ is even;}$$

that is, $C_1 = \cdot$, $C_2 = \cdot -1 \cdot \quad 1 \ 2 \ 2$; $C_3 = \cdot \quad 0 \quad \cdot \quad 2 \ 2$; $C_4 = \cdot \quad \cdot \quad \cdot \quad 1 \ 1$. 

\[
C_5 = \begin{array}{cccc}
5 & 4 & 2 & 0 \\
4 & 2 & 0 & \\
1 & 0 & \\
0 & \\
\end{array}; \quad C_6 = \begin{array}{cccc}
-5 & -10 & -14 & -16 \\
-10 & -14 & -16 & -16 \\
-13 & -14 & -14 & \\
-10 & -10 & -5 & \\
\end{array}; \\
C_7 = \begin{array}{cccc}
-61 & -56 & -46 & -32 & -16 & 0 \\
-56 & -46 & -32 & -16 & 0 \\
-41 & -28 & -14 & 0 \\
-20 & -10 & 0 \\
\end{array}; \quad C_8 = \begin{array}{cccc}
61 & 122 & 178 & 224 & 256 & 272 & 272 \\
122 & 178 & 224 & 256 & 272 & 272 \\
173 & 214 & 242 & 256 & 256 \\
194 & 214 & 224 & 224 & \\
\end{array}.
\]

Thus,
\[
(8.6) \quad c_n(m, k) = \begin{cases} 
(-1)^{(n-1)/2}b_{n+1}(n - k, n - m), & \text{if } n \text{ is odd;} \\
(-1)^{n/2}a_{n+1}(n - k, n - m), & \text{if } n \text{ is even.}
\end{cases}
\]

The sequence of triangles \(C_n\) defined by (8.6) is a Seidel triangle sequence (same argument as in the proof of Proposition 8.1). Following the same pattern as in the preceding subsection, we form the matrix \(H\), whose skew-diagonals carry the entries of the leftmost columns of the \(C_n\)'s:

\[
H = \begin{pmatrix}
-1 & 0 & 1 & 0 & -5 & 0 & 61 & \cdots \\
0 & 2 & 0 & -10 & 0 & 122 \\
2 & 0 & -14 & 0 & 178 \\
0 & 0 & -16 & 0 & 224 \\
0 & -16 & 0 & 256 \\
272 \\
\end{pmatrix}.
\]

This matrix is to be compared with the matrix \(\overline{H}_2\) (see Subsection 7.2). We see that \(H\) is obtained from \(\overline{H}_2\) by transposition and deletion of the first row, so that

\[
H(x, y) = D_x\overline{H}_2(y, x) = D_x \left( \frac{-\sinh x}{\cosh(x + y)} \right) \\
= \frac{-\cosh x \cosh(x + y) + \sinh x \sinh(x + y)}{\cosh^2(x + y)} \\
= \frac{-\cosh y}{\cosh^2(x + y)}.
\]
Therefore,

\[ e^x H(x + y, z) = e^x \frac{-\cosh z}{\cosh^2(x + y + z)}; \]

\[ e^{ix} H(ix + Iy, Ix) = (\cos x + I \sin x) \frac{-\cos z}{\cos^2(x + y + z)}. \]

By using (8.6), the left-hand side of identity (7.12) can be computed as follows. If \( n = 2l + 1 \), then \( I^{n-2} = (-1)^{l+1} I \) and \( (-1)^{(n-1)/2} = (-1)^l \). Thus, \( I^{n-2} c_n(m, k) = -I b_{n+1}(n - k, n - m) \). The imaginary part of identity (7.12) reads

\[
\sum_{1 \leq m + 1 \leq k \leq n - 1 \atop n \text{ odd}} b_{n+1}(n - k, n - m) \frac{x^{n-k-1}}{(n-k-1)!} \frac{y^{k-m-1}}{(k-m-1)!} \frac{z^m}{m!} = \frac{\sin x \cos z}{\cos^2(x + y + z)}.
\]

With the change of variables \( n \leftarrow 2n - 1, \ n - k \leftarrow m, \ n - m \leftarrow k \), we get (1.19) from Theorem 1.5.

If \( n = 2l \), then \( I^{n-2} = (-1)^{l-1} \) and \( (-1)^{n/2} = (-1)^l \), so that \( I^{n-2} c_n(m, k) = -a_{n+1}(n - k, n - m) \). As for the real part, we have

\[
\sum_{1 \leq m + 1 \leq k \leq n - 1 \atop n \text{ even}} a_{n+1}(n - k; n - m) \frac{x^{n-k-1}}{(n-k-1)!} \frac{y^{k-m-1}}{(k-m-1)!} \frac{z^m}{m!} = \frac{\cos x \cos z}{\cos^2(x + y + z)}.
\]

With the change of variables \( n \leftarrow 2n, \ n - k \leftarrow m, \ n - m \leftarrow k \), we get (1.17) from Theorem 1.4.

8.3. The bottom rows of the matrices \( B_n \)'s. By Rule (TS4.1) and (2.6), these bottom rows, after discarding the rightmost entry which is always zero, read \( b_2(2, 1) = 1; \ (b_3(3, 1), b_3(3, 2)) = (0, 1), \ (b_4(4, 1), b_4(4, 2), b_4(4, 3)) = (0, 1, 1), \ (b_5(5, 1), b_5(5, 2), b_5(5, 3), b_5(5, 4)) = (0, 1, 2, 2), \ldots \), which are equal to the sequences of the Entringer numbers: \( E_1(1), \ (E_2(2), E_2(1)), \ (E_3(3), E_3(2), E_3(1)), \ (E_4(4), E_4(3), E_4(2), E_4(1)), \ldots \) By (7.7) and (7.8), we recover the two identities (1.23) and (1.24) written at the end of Section 1.

8.4. The lower triangles of Twin\(^{(1)}\). As for the upper triangles, a geometric transformation is to be made to configure these lower triangles into Seidel triangles. The bottom rows of the \( A_n \)'s and \( B_n \)'s being discarded, we form the following sequence of triangles:
\[
C_1 = \cdot 1 0 0 -1 -1 -2 -2 -1 \quad C_2 = \cdot 1 \quad C_3 = \cdot 1 \quad C_4 = \cdot -1 -2 -1 -2 \quad C_5 = \cdot -4 -3 -1 -2
\]

\[
\begin{array}{c}
16 16 14 10 5 0 \\
32 30 24 15 5 \\
44 36 24 10 \\
44 30 14 \\
32 16 \\
16
\end{array}
\]

\[
\begin{array}{c}
8 12 14 \\
14 16 \\
16
\end{array}
\]

Thus, for \(0 \leq m < k \leq n - 1\), we have

\[
(8.9) \quad c_n(m, k) = \begin{cases} ( -1 )^{(n+1)/2} a_{n+1}(k+1, m+1), & \text{if } n \text{ is odd;} \\ ( -1 )^{(n+2)/2} b_{n+1}(k+1, m+1), & \text{if } n \text{ is even.} \end{cases}
\]

The sequence of triangles \( (C_n) \) defined by (8.9) is a Seidel triangle sequence. The corresponding matrix \( H \) reads

\[
H = \begin{pmatrix}
1 & 1 & -2 & -2 & 16 & 16 & \cdots \\
0 & -2 & -2 & 16 & 16 & \\
-1 & -1 & 14 & 14 & \\
0 & 10 & 10 & \\
5 & 5 & \\
0 & \\
\end{pmatrix}
\]

\[
+ \begin{pmatrix}
0 & 1 & -2 & 16 & \cdots \\
0 & -2 & 16 & \\
-1 & -1 & 14 & \\
0 & 10 & \\
0 & 5 & \\
\vdots & \\
\end{pmatrix}
\]

\[
= -D_y \overline{H}_2 - \overline{H}_2.
\]

Thus,

\[
H(x, y) = D_y \frac{\sinh y}{\cosh(x + y)} + \frac{\sinh y}{\cosh(x + y)}
\]

\[
= \frac{\cosh x}{\cosh^2(x + y)} + \frac{\sinh y}{\cosh(x + y)}.
\]
\[
e^x H(x + y, z) = e^x \left( \frac{\cosh(x + y)}{\cosh^2(x + y + z)} + \frac{\sinh z}{\cosh(x + y + z)} \right);
\]
\[
e^{Ix} H(Ix + Iy, Iz) = (\cos x + I \sin x) \left( \frac{\cos(x + y)}{\cos^2(x + y + z)} + \frac{I \sin z}{\cos(x + y + z)} \right).
\]

If \( n = 2l + 1 \), then \( I^{n-2} = (-1)^{l+1}I \) and \(( -1)^{(n+1)/2} = (-1)^{l+1} \). Thus, \( I^{n-2}c_n(m, k) = Ia_{n+1}(k+1, m+1) \). The imaginary part of identity (7.12) becomes

\[
\sum_{\substack{1 \leq m+1 \leq k \leq n-1 \\text{n odd}}} a_{n+1}(k+1, m+1) \frac{x^{n-k-1}}{(n-k-1)!} \frac{y^{k-m-1}}{(k-m-1)!} \frac{z^m}{m!} = \frac{\cos x \sin z}{\cos(x + y + z)} + \frac{\sin x \cos(x + y)}{\cos^2(x + y + z)}.
\]

With the change of variables \( n \leftarrow 2n - 1 \), \( k + 1 \leftarrow m \), \( m + 1 \leftarrow k \), we get (1.16) from Theorem 1.3.

If \( n = 2l \), then \( I^{n-2} = (-1)^{l-1}I \) and \(( -1)^{(n+2)/2} = (-1)^{l+1} \), so that \( I^{n-2}c_n(m, k) = b_{n+1}(k+1, m+1) \). As for the real part, we have

\[
\sum_{\substack{1 \leq m+1 \leq k \leq n-1 \\text{n even}}} b_{n+1}(k+1, m+1) \frac{x^{n-k-1}}{(n-k-1)!} \frac{y^{k-m-1}}{(k-m-1)!} \frac{z^m}{m!} = -\frac{\sin x \sin z}{\cos(x + y + z)} + \frac{\cos x \cos(x + y)}{\cos^2(x + y + z)}.
\]

With the change of variables \( n \leftarrow 2n \), \( k + 1 \leftarrow m \), \( m + 1 \leftarrow k \), we get (1.22) from Theorem 1.6.

8.5. The lower triangles of Twin\(^{(2)}\). Again, the bottom rows of the \( A_n \)'s and \( B_n \)'s having been discarded, the Seidel triangle sequence to be considered is

\[
C_1 = \begin{pmatrix} \cdot & 0 & -1 & \cdot & 0 & 1 \end{pmatrix}, \quad C_2 = \begin{pmatrix} \cdot & 1 \end{pmatrix}, \quad C_3 = \begin{pmatrix} \cdot & -1 \end{pmatrix}, \quad C_4 = \begin{pmatrix} \cdot & 1 & -2 & -1 \end{pmatrix}, \quad C_5 = \begin{pmatrix} \cdot & 1 & 3 & 4 \end{pmatrix}, \quad C_6 = \begin{pmatrix} \cdot & -1 \end{pmatrix}, \quad C_7 = \begin{pmatrix} \cdot & 0 & 1 & 2 & 2 \end{pmatrix}.
\]

\[
C_8 = \begin{pmatrix} \cdot & 0 & 1 & 2 & 2 \end{pmatrix}, \quad C_9 = \begin{pmatrix} \cdot & 0 & 1 & 2 & 2 \end{pmatrix}, \quad C_{10} = \begin{pmatrix} \cdot & 0 & 1 & 2 & 2 \end{pmatrix}, \quad C_{11} = \begin{pmatrix} \cdot & 0 & 1 & 2 & 2 \end{pmatrix}, \quad C_{12} = \begin{pmatrix} \cdot & 0 & 1 & 2 & 2 \end{pmatrix}.
\]
the general formula being

\[
(8.13) \quad c_n(m, k) = \begin{cases} 
(-1)^{(n-1)/2}b_{n+1}(k + 1, m + 1), & \text{if } n \text{ is odd;} \\
(-1)^{(n-2)/2}a_{n+1}(k + 1, m + 1), & \text{if } n \text{ is even.}
\end{cases}
\]

Next, form the matrix \( H \), whose skew-diagonals carry the entries of the rightmost columns of the \( c_n \)'s, and write it as the sum of the two matrices

\[
H = \begin{pmatrix}
1 & -1 & -1 & 5 & 5 & -61 & -61 & \cdots \\
-1 & -1 & 5 & 5 & -61 & -61 & & \\
0 & 4 & 4 & -56 & -56 & & & \\
2 & 2 & -46 & -46 & & & & \\
0 & -32 & -32 & & & & & \\
-16 & -16 & & & & & & \\
0 & & & & & & & \\
\vdots & & & & & & & \\
\end{pmatrix}
\]

\[
= \begin{pmatrix}
1 & \cdot & -1 & \cdot & 5 & \cdot & -61 & \cdots \\
\cdot & -1 & \cdot & 5 & \cdot & -61 & & \\
0 & \cdot & 4 & \cdot & -56 & & & \\
\cdot & 2 & \cdot & -46 & & & & \\
0 & \cdot & -32 & & & & & \\
\cdot & -16 & & & & & & \\
0 & & & & & & & \\
\vdots & & & & & & & \\
\end{pmatrix}
\]

\[
+ \begin{pmatrix}
\cdot & -1 & \cdot & 5 & \cdot & -61 & \cdots \\
-1 & \cdot & 5 & \cdot & -61 & & \\
\cdot & 4 & \cdot & -56 & & & \\
\cdot & 2 & \cdot & -46 & & & \\
\cdot & -32 & & & & & \\
\cdot & -16 & & & & & \\
\vdots & & & & & & & \\
\end{pmatrix}
\]

These matrices are to be compared with the matrix \( \overline{H} \) (see Section 7). Clearly, \( K_2 \) can be obtained from \( \overline{H}_1 \) by deleting the top row and then transposing the matrix, so that \( K_2(x, y) = TD_x\overline{H}_1(x, y) \). Also, \( K_1 = T\overline{H}_1 \), and then \( K_1(x, y) = \overline{H}_1(y, x) \). As \( \overline{H}_1(x, y) = \cosh y/\cosh(x + y) \), we get

\[
H(x, y) = TD_x\overline{H}_1(x, y) + \overline{H}_1(y, x)
\]

\[
e^xH(x + y, z) = e^x \left( -\frac{\cosh(x + y)\sinh(x + y + z)}{\cosh^2(x + y + z)} + \frac{\cosh(x + y)}{\cosh(x + y + z)} \right);
\]

\[
e^{Ix}H(Ix + Iy, Iy) = (\cos x + I \sin x)
\]

\[
\times \left( -\frac{I \cos(x + y)\sin(x + y + z)}{\cos^2(x + y + z)} + \frac{\cos(x + y)}{\cos(x + y + z)} \right).
\]

If \( n = 2l + 1 \), then \( I^{n-2} = (-1)^{l+1}I \) and \( (-1)^{(n-1)/2} = (-1)^l \). Thus, \( I^{n-2}c_n(m, k) = -Ib_{n+1}(k + 1, m + 1) \). The imaginary part of identity
(7.12) becomes
\[
\sum_{1 \leq m+1 \leq k \leq n-1} b_{n+1}(k+1, m+1) \frac{x^{n-k-1}}{(n-k-1)!} \frac{y^{k-m-1}}{(k-m-1)!} \frac{z^m}{m!}
\]
\[
= -\frac{\sin x \cos(x+y)}{\cos(x+y+z)} + \frac{\cos x \cos(x+y) \sin(x+y+z)}{\cos^2(x+y+z)}
\]
\[
= \frac{\cos(x+y) \sin(y+z)}{\cos^2(x+y+z)}.
\]

With the change of variables \( n \leftarrow 2n-1, m+1 \leftarrow k, k+1 \leftarrow m \), we get (1.20) from Theorem 1.5.

If \( n = 2l \), then \( I^{n-2} = (-1)^{l-1} \) and \( (-1)^{(n-2)/2} = (-1)^{l-1} \), so that
\[
I^{n-2} c_n(m, k) = a_{n+1}(k+1, m+1).
\]
As for the real part, we have
\[
\sum_{1 \leq m+1 \leq k \leq n-1} a_{n+1}(k+1, m+1) \frac{x^{n-k-1}}{(n-k-1)!} \frac{y^{k-m-1}}{(k-m-1)!} \frac{z^m}{m!}
\]
\[
= \frac{\cos x \cos(x+y)}{\cos(x+y+z)} + \frac{\sin x \cos(x+y) \sin(x+y+z)}{\cos^2(x+y+z)}
\]
\[
= \frac{\cos(x+y) \cos(y+z)}{\cos^2(x+y+z)}.
\]

With the change of variables \( n \leftarrow 2n-1, k+1 \leftarrow m, m+1 \leftarrow k \), we get (1.18) from Theorem 1.4.

9. The formal Laplace transform

The purpose of this section is to show that, when the Entringer numbers \( E_n(k) \) are defined by relations (1.1), without any reference to their combinatorial interpretations, they can be proved to be a refinement of the tangent/secant numbers:
\[
\sum_k E_n(k) = E_n \quad (n \geq 1).
\]
In the same manner, when the twin Seidel matrix sequence \((A_n), (B_n)\) is analytically defined, as it was stated in Subsection 1.5, also without reference to any combinatorial interpretation, their entries \((a_n(m, k)), (b_n(m, k))\) make up a refinement of the Entringer numbers, by row and by column, and
\[
\sum_{m, k} a_n(m, k) = \sum_{m, k} b_n(m, k) = E_n.
\]

The proofs of these results make use of the closed expressions found for the generating functions in the preceding section, and of a well-adapted formal Laplace transform technique.
Theorem 9.1. (1) Let \((E_n(k))\) be the sequence of the Entringer numbers, defined by
\[
E_1(1) := 1; \quad E_n(n) := 0 \text{ for all } n \geq 2; \\
\Delta E_n(m) + E_{n-1}(n-m) = 0 \quad (n \geq 2; m = n - 1, \ldots, 2, 1);
\]
Then,
\[
\sum_{1 \leq k \leq 2n-1} E_{2n-1}(k) = E_{2n-1}; \\
\sum_{1 \leq k \leq 2n} E_{2n}(k) = E_{2n}; \quad (n \geq 1).
\]

(2) Let \((a_n(m, k)), (b_n(m, k))\) be the entries of the twin Seidel matrix sequence \((A_n), (B_n)\), as they are defined in Subsection 1.5. Then,
\[
a_n(m, \cdot) = E_n(m), \quad b_n(m, \cdot) = E_n(n + 1 - m), \quad (1 \leq m \leq n); \\
a_n(\cdot, k) = b_n(\cdot, k) = E_n(n - k) \quad (1 \leq k \leq n).
\]

The proof of (9.1) is fully given. Next, we reproduce the proof of \(a_{2n}(m, \cdot) = E_{2n}(m)\), based on Theorem 1.3. The other identities in (9.2) and (9.3) can also be derived following the same method by using Theorems 1.4, 1.5, 1.6. Their proofs are omitted.

The formal Laplace transform, already used in our previous paper [FH14], maps a function \(f(x)\) to a function \(L(f(x), x, s)\) defined by
\[
L(f(x), x, s) := \int_0^\infty f(x)e^{-xs}dx.
\]
In particular, \(L(\cdot, x, s)\) maps \(x^k/k!\) to \(1/s^{k+1}\):
\[
L\left(\frac{x^k}{k!}, x, s\right) = \frac{1}{s^{k+1}}.
\]

For proving (9.1), start with identity (7.7) involving the generating function for the numbers \(E_{2n+1}(k)\) and apply the Laplace transform twice with respect to \((x, s), (y, t)\), respectively. We get
\[
\sum_{1 \leq k \leq 2n+1} \frac{1}{s^{2n-k+2}} \frac{1}{t^k} E_{2n+1}(k) = \int_0^\infty \int_0^\infty \frac{\cos y}{\cos(x + y)} e^{-xs-ys} dx dy,
\]
which becomes, with \(t \leftarrow s\) and \(r = x + y,\)
\[
\sum_{1 \leq k \leq 2n+1} \frac{1}{s^{2n+2}} E_{2n+1}(k) = \int_0^\infty \int_0^\infty \frac{\cos y}{\cos(x + y)} e^{-xs-ys} dx dy.
\]
50 DOMINIQUE FOATA AND GUO-NIU HAN

\[
\begin{align*}
&= \int_0^\infty \int_0^r \frac{\cos y}{\cos r} e^{-rs} dy \, dr \\
&= \int_0^\infty \frac{\sin r}{\cos r} e^{-rs} dr \\
&= \int_0^\infty (\tan r) e^{-rs} dr \\
&= \sum_{n \geq 1} \frac{1}{e^{2n}} E_{2n-1}.
\end{align*}
\]

Hence,

\[
\sum_{1 \leq k \leq 2n-1} E_{2n-1}(k) = E_{2n-1}.
\]

In the same manner, apply the Laplace transform to identity (7.8) twice with respect to \((x,s), (y,t)\), respectively. We get

\[
\sum_{1 \leq k \leq 2n} \frac{1}{e^{k}} \frac{1}{e^{2n-k+1}} E_{2n}(k) = \int_0^\infty \int_0^\infty \frac{\sin y}{\cos(x+y)} e^{-(x-s-y)t} dx \, dy,
\]

which becomes, with \(s \leftarrow t\) and \(r = x + y\),

\[
\sum_{1 \leq k \leq 2n} \frac{1}{e^{2n+1}} E_{2n}(k) = \int_0^\infty \int_0^\infty \frac{\sin y}{\cos(x+y)} e^{-xt-yt} dx \, dy
\]

\[
= \int_0^\infty \int_0^r \frac{\sin y}{\cos r} e^{-rt} dy \, dr \\
= \int_0^\infty \frac{1 - \cos r}{\cos r} e^{-rt} dr \\
= \int_0^\infty (\sec r - 1) e^{-rt} dr \\
= \sum_{n \geq 1} \frac{1}{e^{2n+1}} E_{2n}.
\]

Hence,

\[
\sum_{1 \leq k \leq 2n} E_{2n}(k) = E_{2n}.
\]

Next, to prove \(a_{2n}(m, \bullet) = E_{2n}(m)\) start with identity (1.15) of Theorem 1.5 and apply the Laplace transform to its left-hand side three times with respect to \((x, s), (y, t), (z, u)\), respectively. We get

\[
\sum_{2 \leq m+1 \leq k \leq 2n-1} \frac{1}{e^m} \frac{1}{e^{k-m}} \frac{1}{e^{u^2-n-k}} a_{2n}(m, k),
\]

which becomes
\begin{align}
\sum_{2 \leq m+1 \leq k \leq 2n-1} \frac{1}{s^m} \frac{1}{u^{2n}} a_{2n}(m,k),
\end{align}

when \( t \leftarrow u \) and \( s \leftarrow su \). Apply the Laplace transform to the right-hand side of \( (1.15) \) three times with respect to \((x, s), (y, t), (z, u)\), respectively, and let \( t \leftarrow u, s \leftarrow su \). With \( r = y + z \), we get

\begin{align}
\int_0^\infty \int_0^\infty \int_0^\infty \cos x \cos z \sin(x + y + z) e^{-xu - yu - zu} dx dy dz \\
= \int_0^\infty \int_0^\infty \int_0^r \cos x \cos z \sin(x + r) e^{-xu - ru} dz dr dx \\
= \int_0^\infty \int_0^\infty \cos x \sin r \sin(x + r) e^{-xu - ru} dr dx.
\end{align}

With identity \((1.16)\) apply the Laplace transform to its left-hand side three times with respect to \((x, u), (y, s), (z, t)\), respectively. We get

\begin{align}
\sum_{2 \leq k+1 \leq m \leq 2n-1} \frac{1}{u^{2n-m}} \frac{1}{s^{m-k}} \frac{1}{t^k} a_{2n}(m,k),
\end{align}

which becomes

\begin{align}
\sum_{2 \leq k+1 \leq m \leq 2n-1} \frac{1}{u^{2n-m}} \frac{1}{s^{m-k}} \frac{1}{t^k} a_{2n}(m,k),
\end{align}

when \( s \leftarrow su \) and \( t \leftarrow su \). Apply the Laplace transform to the right-hand side of \((1.16)\) three times with respect to \((x, u), (y, s), (z, t)\), respectively, and let \( s \leftarrow su, t \leftarrow su \). With \( r = y + z \) we get

\begin{align}
\int_0^\infty \int_0^\infty \int_0^\infty \left( \frac{\cos x \sin z}{\cos(x + y + z)} + \frac{\sin x \cos(x + y)}{\cos^2(x + y + z)} \right) e^{-xu - yu - zu} dx dy dz \\
= \int_0^\infty \int_0^\infty \int_0^r \left( \frac{\cos x \sin z}{\cos(x + r)} + \frac{\sin x \cos(x + r - z)}{\cos^2(x + r)} \right) e^{-xu - ru} dz dr dx \\
= \int_0^\infty \int_0^\infty \left( \frac{\cos x (1 - \cos r)}{\cos(x + r)} + \frac{\sin x (\sin(x + r) - \sin x)}{\cos^2(x + r)} \right) e^{-xu - ru} dr dx \\
= \int_0^\infty \int_0^\infty \left( \frac{\cos r (1 - \cos x)}{\cos(x + r)} + \frac{\sin r (\sin(x + r) - \sin r)}{\cos^2(x + r)} \right) e^{-ru - xsu} dr dx.
\end{align}

By \((9.4)-(9.7)\), we have

\begin{align}
\sum_{1 \leq k, m \leq 2n-1; k \neq m} \frac{1}{s^m} \frac{1}{u^{2n}} a_{2n}(m,k) = \int_0^\infty \int_0^\infty F(x, r) e^{-xu - ru} dr dx,
\end{align}
where
\[ F(x, r) = \frac{\cos x \sin r \sin(x + r)}{\cos^2(x + r)} + \frac{\cos r (1 - \cos x)}{\cos(x + r)} + \frac{\sin r (\sin(x + r) - \sin r)}{\cos^2(x + r)} \]
\[ = \frac{\cos x}{\cos^2(x + r)} - 1. \]

However, from (7.8), we conclude
\[ \sum_{1 \leq m \leq 2n} E_{2n}(m) \frac{x^{m-1} r^{2n-m-1}}{(m-1)! (2n-m-1)!} = \frac{\partial}{\partial r} \frac{\sin r}{\cos(x + r)} \]
(9.9)
\[ = \frac{\cos x}{\cos^2(x + r)}. \]

Apply the Laplace transform to (9.9) twice with respect to \((x, s), (y, u)\), respectively. We get
\[ \sum_{1 \leq m \leq 2n} \frac{1}{s^m u^{2n-m}} E_{2n}(m) = \int_0^\infty \int_0^\infty \frac{\cos x}{\cos^2(x + r)} e^{-xs-ru} dx dr, \]
or still
(9.10)
\[ \sum_{1 \leq m \leq 2n} \frac{1}{s^m u^{2n}} E_{2n}(m) = \int_0^\infty \int_0^\infty \frac{\cos x}{\cos^2(x + r)} e^{-xsu-ru} dx dr. \]

By (9.8) and (9.10), we obtain
\[ \sum_{1 \leq k, m \leq 2n-1; k \neq m} \frac{1}{s^m u^{2n}} a_{2n}(m, k) = \sum_{1 \leq m \leq 2n} \frac{1}{s^m u^{2n}} E_{2n}(m) - \frac{1}{su^2}, \]
and then
\[ \sum_{1 \leq k, m \leq 2n-1} \frac{1}{s^m u^{2n}} a_{2n}(m, k) = \sum_{1 \leq m \leq 2n} \frac{1}{s^m u^{2n}} E_{2n}(m). \]

Hence,
\[ \sum_{1 \leq k \leq 2n-1} a_{2n}(m, k) = E_{2n}(m). \Box \]

**Acknowledgement.** The authors would like to thank Volker Strehl very highly for having read the paper so carefully and having singled out its spirit, and also Christian Krattenthaler for his magnificent editorial work.
References

 http://www.mat.univie.ac.at/~slc/.
 http://www.mat.univie.ac.at/~slc/.


[Vi88] Xavier G. Viennot. *Séries génératrices énumératives*, chap. 3, 160 pp., 1988; notes of lectures given at the École Normale Supérieure Ulm (Paris), the UQAM (Montréal, Québec) and the University of Wuhan (China) [http://www.xavierviennot.org/xavier/cours.html](http://www.xavierviennot.org/xavier/cours.html).

Dominique Foata
Institut Lothaire
1, rue Murner
F-67000 Strasbourg, France
foata@unistra.fr

Guo-Niu Han
I.R.M.A. UMR 7501
Université de Strasbourg et CNRS
7, rue René-Descartes
F-67084 Strasbourg, France
guoniu.han@unistra.fr