

New hook-content formulas for strict partitions (extended abstract)

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Abstract. We introduce the difference operator for functions defined on strict partitions and prove a polynomiality property for a summation involving the bar length (hook length) and content statistics. As an application, several new hook-content formulas for strict partitions are derived.

Résumé. Nous introduisons l'opérateur de différence pour les fonctions définies sur les partitions strictes et démontrons qu'une somme en fonction des équerres et des contenus est un polynôme. En particulier, nous obtenons plusieurs nouvelles formules des équerres et des contenus pour les partitions strictes.

Keywords: strict partition, hook length, bar length, content, shifted Young tableau, difference operator.

1 Introduction

The basic knowledge on partitions, Young tableaux and symmetric functions could be found in [20]. For the usual partitions, it is well known (cf. [9]) that complex irreducible characters of the alternating group A_n and the symmetric group S_n are determined by partitions of size n . Also, a famous result in representation of finite groups, Nakayama Conjecture, says that two irreducible characters of S_n lie in the same p -block if and only if their corresponding partitions have the same p -core. Strict partitions are also closely related to the representation of finite groups. For example, the irreducible spin characters of the covering groups of the alternating group A_n and the symmetric group S_n are determined by strict partitions with size n (see [17]).

In this paper, we focus on strict partitions. A *strict partition* is a finite strict decreasing sequence of positive integers $\bar{\lambda} = (\bar{\lambda}_1, \bar{\lambda}_2, \dots, \bar{\lambda}_\ell)$. The integer $|\bar{\lambda}| = \sum_{1 \leq i \leq \ell} \bar{\lambda}_i$ is called the *size* of the partition $\bar{\lambda}$ and $\ell(\bar{\lambda}) = \ell$ is called the *length* of $\bar{\lambda}$. For convenience, let $\bar{\lambda}_i = 0$ for $i > \ell(\bar{\lambda})$. A strict partition $\bar{\lambda}$ could be identical with its shifted Young diagram, which means that the i -th row of the usual Young diagram is shifted to the right by i boxes. Therefore the leftmost box in the i -th row has coordinate $(i, i + 1)$. For the (i, j) -box in the shifted Young diagram of the

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strict partition $\bar{\lambda}$, we can associate its *bar length* (in some other papers, it is also called *hook length*), denoted by $\bar{h}_{(i,j)}$, which is the number of boxes exactly to the right, or exactly above, or the box itself, plus $\bar{\lambda}_j$. For example, consider the box $\square = (1, 3)$ in the shifted Young diagram of the strict partition $(7, 5, 4, 1)$. There are 1 and 5 boxes above and to the right of the box \square respectively. Since $\bar{\lambda}_3 = 4$, the bar length of \square is equal to $1 + 5 + 1 + 4 = 11$, as illustrated in Figure 1. The *content* of $\square = (i, j)$ in the shifted Young diagram is defined to be $\bar{c}_\square = j - i$, so that the leftmost box in each row has content 1. Also, let $\bar{\mathcal{H}}(\bar{\lambda})$ be the multiset of bar lengths of boxes and $\bar{H}_{\bar{\lambda}}$ be the product of all bar lengths of boxes in $\bar{\lambda}$.

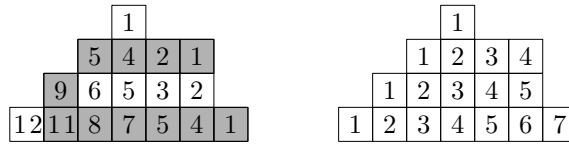


Fig. 1: The shifted Young diagram, the bar lengths and the contents of the strict partition $(7, 5, 4, 1)$.

Our goal is to find some formulas involving bar lengths and contents for strict partitions, also called *hook-content formulas for strict partitions*, by analogy with that for usual partitions. For the usual partition ν , it is well known that (see [3, 10, 20])

$$f_\nu = \frac{|\nu|!}{H_\nu} \quad \text{and} \quad \frac{1}{n!} \sum_{|\nu|=n} f_\nu^2 = 1, \quad (1)$$

where H_ν denotes the product of all hook lengths of boxes in ν and f_ν denotes the number of standard Young tableaux of shape ν . The first author conjectured [4] that

$$P(n) = \frac{1}{n!} \sum_{|\nu|=n} f_\nu^2 \sum_{\square \in \nu} h_\square^{2k}$$

is always a polynomial in n for any $k \in \mathbf{N}$, which was generalized and proved by Stanley [18], and later generalized in [7] (see also [2, 5, 8, 14, 15]).

For two strict partitions $\bar{\lambda}$ and $\bar{\mu}$, we write $\bar{\lambda} \supseteq \bar{\mu}$ if $\bar{\lambda}_i \geq \bar{\mu}_i$ for any $i \geq 1$. In this case, the skew strict partition $\bar{\lambda}/\bar{\mu}$ is identical with its skew shifted Young diagram. For example, the skew strict partition $(7, 5, 4, 1)/(4, 2, 1)$ is represented by the white boxes in Figure 2. Let $\bar{f}_{\bar{\lambda}}$ (resp. $\bar{f}_{\bar{\lambda}/\bar{\mu}}$) be the number of standard shifted Young tableaux of shape $\bar{\lambda}$ (resp. $\bar{\lambda}/\bar{\mu}$). The following are well-known formulas (see [1, 17, 21]) analogous to (1):

$$\bar{f}_{\bar{\lambda}} = \frac{|\bar{\lambda}|!}{\bar{H}_{\bar{\lambda}}} \quad \text{and} \quad \frac{1}{n!} \sum_{|\bar{\lambda}|=n} 2^{n-\ell(\bar{\lambda})} \bar{f}_{\bar{\lambda}}^2 = 1. \quad (2)$$

In this paper, we generalize the latter equality of (2) by means of the following results.

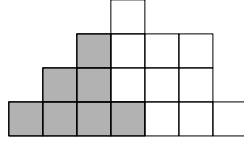


Fig. 2: The skew shifted Young diagram of the skew strict partition $(7, 5, 4, 1)/(4, 2, 1)$.

Theorem 1.1 *Suppose that Q is a given symmetric function and $\bar{\mu}$ is a given strict partition. Then*

$$P(n) = \sum_{|\bar{\lambda}/\bar{\mu}|=n} \frac{2^{n-\ell(\bar{\lambda})} \bar{f}_{\bar{\lambda}/\bar{\mu}}}{\bar{H}_{\bar{\lambda}}} Q\left(\binom{\bar{c}_{\square}}{2} : \square \in \bar{\lambda}\right)$$

is a polynomial in n , where $Q\left(\binom{\bar{c}_{\square}}{2} : \square \in \bar{\lambda}\right)$ means that $|\bar{\lambda}|$ of the variables are substituted by $\binom{\bar{c}_{\square}}{2}$ for $\square \in \bar{\lambda}$, and all other variables by 0.

Theorem 1.2 *Suppose that k is a given nonnegative integer. Then we have*

$$\sum_{|\bar{\lambda}|=n} \frac{2^{n-\ell(\bar{\lambda})} \bar{f}_{\bar{\lambda}}}{\bar{H}_{\bar{\lambda}}} \sum_{\square \in \bar{\lambda}} \binom{\bar{c}_{\square} + k - 1}{2k} = \frac{2^k}{(k+1)!} \binom{n}{k+1}.$$

When $k = 0$ we derive the latter identity of (2). When $k = 1$, Theorem 1.2 becomes

$$\sum_{|\bar{\lambda}|=n} \frac{2^{n-\ell(\bar{\lambda})} \bar{f}_{\bar{\lambda}}}{\bar{H}_{\bar{\lambda}}} \sum_{\square \in \bar{\lambda}} \binom{\bar{c}_{\square}}{2} = \binom{n}{2},$$

which could also be obtained by setting $\bar{\mu} = \emptyset$ in the next theorem.

Theorem 1.3 *Let $\bar{\mu}$ be a strict partition. Then we have*

$$\sum_{|\bar{\lambda}/\bar{\mu}|=n} \frac{2^{n-\ell(\bar{\lambda})+\ell(\bar{\mu})} \bar{f}_{\bar{\lambda}/\bar{\mu}} \bar{H}_{\bar{\mu}}}{\bar{H}_{\bar{\lambda}}} \left(\sum_{\square \in \bar{\lambda}} \binom{\bar{c}_{\square}}{2} - \sum_{\square \in \bar{\mu}} \binom{\bar{c}_{\square}}{2} \right) = \binom{n}{2} + n|\bar{\mu}|. \quad (3)$$

The proofs of those theorems are given in Section 4, by using the difference operator technique.

2 The difference operator for strict partitions

For each strict partition $\bar{\lambda}$, the symbol $\bar{\lambda}^+$ (resp. $\bar{\lambda}^-$) always represents a strict partition obtained by adding (resp. removing) a box to (resp. from) $\bar{\lambda}$. In other words, $|\bar{\lambda}^+/\bar{\lambda}| = 1$ and $|\bar{\lambda}/\bar{\lambda}^-| = 1$. By analogy with the difference operator for usual partitions introduced in [7], we define *the difference operator for strict partitions* by

$$D(g(\bar{\lambda})) := \sum_{\ell(\bar{\lambda}^+) > \ell(\bar{\lambda})} g(\bar{\lambda}^+) + 2 \sum_{\ell(\bar{\lambda}^+) = \ell(\bar{\lambda})} g(\bar{\lambda}^+) - g(\bar{\lambda}),$$

where $\bar{\lambda}$ is a strict partition and g is a function of strict partitions. Notice that $\#\{\bar{\lambda}^+ : \ell(\bar{\lambda}^+) > \ell(\bar{\lambda})\} = 0$ or 1 .

For each skew strict partition $\bar{\lambda}/\bar{\mu}$, let $f'_{\bar{\lambda}/\bar{\mu}} := 2^{|\bar{\lambda}|-|\bar{\mu}|-\ell(\bar{\lambda})+\ell(\bar{\mu})} \bar{f}_{\bar{\lambda}/\bar{\mu}}$. We obtain the following two lemmas.

Lemma 2.1 *For two different strict partitions $\bar{\lambda} \supseteq \bar{\mu}$ we have*

$$f'_{\bar{\lambda}/\bar{\mu}} = \sum_{\bar{\lambda}^-: \substack{\bar{\lambda} \supseteq \bar{\lambda}^- \supseteq \bar{\mu} \\ \ell(\bar{\lambda}^-) < \ell(\bar{\lambda})}} f'_{\bar{\lambda}^-/\bar{\mu}} + 2 \sum_{\bar{\lambda}^-: \substack{\bar{\lambda} \supseteq \bar{\lambda}^- \supseteq \bar{\mu} \\ \ell(\bar{\lambda}^-) = \ell(\bar{\lambda})}} f'_{\bar{\lambda}^-/\bar{\mu}}.$$

Lemma 2.2 *For each strict partition $\bar{\mu}$ and each function g of strict partitions, let*

$$A(n) := \sum_{|\bar{\lambda}/\bar{\mu}|=n} f'_{\bar{\lambda}/\bar{\mu}} g(\bar{\lambda})$$

and

$$B(n) := \sum_{|\bar{\lambda}/\bar{\mu}|=n} f'_{\bar{\lambda}/\bar{\mu}} Dg(\bar{\lambda}).$$

Then

$$A(n) = A(0) + \sum_{k=0}^{n-1} B(k).$$

By induction and Lemma 2.2 we obtain the following result.

Theorem 2.3 *Let g be a function of strict partitions and $\bar{\mu}$ be a given strict partition. Then we have*

$$\sum_{|\bar{\lambda}/\bar{\mu}|=n} f'_{\bar{\lambda}/\bar{\mu}} g(\bar{\lambda}) = \sum_{k=0}^n \binom{n}{k} D^k g(\bar{\mu}) \quad (4)$$

and

$$D^n g(\bar{\mu}) = \sum_{k=0}^n (-1)^{n+k} \binom{n}{k} \sum_{|\bar{\lambda}/\bar{\mu}|=k} f'_{\bar{\lambda}/\bar{\mu}} g(\bar{\lambda}). \quad (5)$$

In particular, if there exists some positive integer r such that $D^r g(\bar{\lambda}) = 0$ for every strict partition $\bar{\lambda}$, then the left-hand side of (4) is a polynomial in n with degree at most $r - 1$.

Example. Let $g(\bar{\lambda}) = 1/\bar{H}_{\bar{\lambda}}$. Then $Dg(\bar{\lambda}) = 0$ by Theorem 3.3. The two quantities defined in Lemma 2.2 are:

$$A(n) = \sum_{|\bar{\lambda}/\bar{\mu}|=n} \frac{f'_{\bar{\lambda}/\bar{\mu}}}{\bar{H}_{\bar{\lambda}}} \quad \text{and} \quad B(n) = 0.$$

Consequently,

$$\sum_{|\bar{\lambda}/\bar{\mu}|=n} \frac{2^{n-\ell(\bar{\lambda})+\ell(\bar{\mu})} f'_{\bar{\lambda}/\bar{\mu}}}{\bar{H}_{\bar{\lambda}}} = \frac{1}{\bar{H}_{\bar{\mu}}}. \quad (6)$$

In particular, $\bar{\mu} = \emptyset$ implies

$$\sum_{|\bar{\lambda}|=n} 2^{n-\ell(\bar{\lambda})} \bar{f}_{\bar{\lambda}}^2 = n!. \quad (7)$$

3 Corners of strict partitions

For a strict partition $\bar{\lambda}$, the *outer corners* are the boxes which can be removed to get a new strict partition $\bar{\lambda}^-$. Let $(\alpha_1, \beta_1), \dots, (\alpha_m, \beta_m)$ be the coordinates of outer corners such that $\alpha_1 > \alpha_2 > \dots > \alpha_m$. Let $y_j = \beta_j - \alpha_j$ be the contents of outer corners for $1 \leq j \leq m$. We set $\alpha_{m+1} = 0$, $\beta_0 = \ell(\bar{\lambda}) + 1$ and call $(\alpha_1, \beta_0), (\alpha_2, \beta_1), \dots, (\alpha_{m+1}, \beta_m)$ the *inner corners* of $\bar{\lambda}$. Let $x_i = \beta_i - \alpha_{i+1}$ be the contents of inner corners for $0 \leq i \leq m$ (see Figure 3). The following relation of x_i and y_j are obvious.

$$x_0 = 1 \leq y_1 < x_1 < y_2 < x_2 < \dots < y_m < x_m. \quad (8)$$

Notice that $x_0 = y_1 = 1$ if and only if $\bar{\lambda}_{\ell(\bar{\lambda})} = 1$.

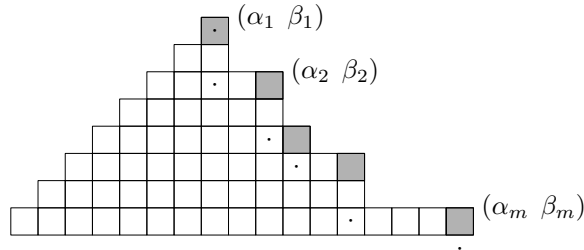


Fig. 3: A strict partition and its corners. The outer corners are labelled with (α_i, β_i) ($i = 1, 2, \dots, m$). The inner corners are indicated by the dot symbol “.”.

For arbitrary two finite alphabets A and B , the power sum of the alphabet $A - B$ is defined by [11, p.5]

$$\Psi^k(A, B) := \sum_{a \in A} a^k - \sum_{b \in B} b^k \quad (9)$$

for each integer $k \geq 0$. And for each partition $\nu = (\nu_1, \nu_2, \dots, \nu_{\ell(\nu)})$, we define

$$\Psi^\nu(A, B) := \prod_{j=1}^{\ell(\nu)} \Psi^{\nu_j}(A, B). \quad (10)$$

Let $\bar{\lambda}$ be a strict partition. Motivated by the work of Olshanski [13] we define

$$\Phi^\nu(\bar{\lambda}) := \Psi^\nu\left(\left\{\binom{x_i}{2}\right\}, \left\{\binom{y_i}{2}\right\}\right). \quad (11)$$

First we consider the difference between the bar length sets of $\bar{\lambda}$ and $\bar{\lambda}^+ = \bar{\lambda} \cup \{\square\}$ for some box \square .

Theorem 3.1 Suppose that $\bar{\lambda}^+ = \bar{\lambda} \cup \{\square\}$ such that $\bar{c}_\square = x_i$. If $i = 0$, then

$$\frac{\bar{H}_{\bar{\lambda}}}{\bar{H}_{\bar{\lambda}^+}} = \frac{\prod_{1 \leq j \leq m} \left(\binom{x_0}{2} - \binom{y_j}{2} \right)}{\prod_{1 \leq j \leq m} \left(\binom{x_0}{2} - \binom{x_j}{2} \right)}.$$

If $1 \leq i \leq m$, then

$$\frac{\bar{H}_{\bar{\lambda}}}{\bar{H}_{\bar{\lambda}^+}} = \frac{1}{2} \cdot \frac{\prod_{1 \leq j \leq m} \left(\binom{x_i}{2} - \binom{y_j}{2} \right)}{\prod_{\substack{0 \leq j \leq m \\ j \neq i}} \left(\binom{x_i}{2} - \binom{x_j}{2} \right)}.$$

Suppose that $a_0 < a_1 < \dots < a_m$ and $b_1 < \dots < b_m$ are real numbers.

Theorem 3.2 Let k be a nonnegative integer. Then there exist some $\xi_\nu \in \mathbb{Q}$ such that

$$\sum_{0 \leq i \leq m} \frac{\prod_{1 \leq j \leq m} (a_i - b_j)}{\prod_{\substack{0 \leq j \leq m \\ j \neq i}} (a_i - a_j)} a_i^k = \sum_{|\nu| \leq k} \xi_\nu \Psi^\nu(\{a_i\}, \{b_i\})$$

for arbitrary real numbers $a_0 < a_1 < \dots < a_m$ and $b_1 < b_2 < \dots < b_m$.

By Theorem 3.2, when $k = 0, 1, 2$, we obtain

$$\sum_{0 \leq i \leq m} \frac{\prod_{1 \leq j \leq m} (a_i - b_j)}{\prod_{\substack{0 \leq j \leq m \\ j \neq i}} (a_i - a_j)} = 1, \quad (12)$$

$$\sum_{0 \leq i \leq m} \frac{\prod_{1 \leq j \leq m} (a_i - b_j)}{\prod_{\substack{0 \leq j \leq m \\ j \neq i}} (a_i - a_j)} a_i = \Psi^1(\{a_i\}, \{b_i\}), \quad (13)$$

$$\sum_{0 \leq i \leq m} \frac{\prod_{1 \leq j \leq m} (a_i - b_j)}{\prod_{\substack{0 \leq j \leq m \\ j \neq i}} (a_i - a_j)} a_i^2 = \frac{\Psi^{(1,1)}(\{a_i\}, \{b_i\}) + \Psi^2(\{a_i\}, \{b_i\})}{2}. \quad (14)$$

By Theorems 3.1 and 3.2 we find a formula to compute $D\left(\frac{g(\bar{\lambda})}{\bar{H}_{\bar{\lambda}}}\right)$.

Theorem 3.3 Suppose that g is a function of strict partitions. Then

$$D\left(\frac{g(\bar{\lambda})}{\bar{H}_{\bar{\lambda}}}\right) = \sum_{\ell(\bar{\lambda}^+) > \ell(\bar{\lambda})} \frac{g(\bar{\lambda}^+) - g(\bar{\lambda})}{\bar{H}_{\bar{\lambda}^+}} + 2 \sum_{\ell(\bar{\lambda}^+) = \ell(\bar{\lambda})} \frac{g(\bar{\lambda}^+) - g(\bar{\lambda})}{\bar{H}_{\bar{\lambda}^+}}$$

for every strict partition $\bar{\lambda}$.

Let $\bar{\lambda}^{i+} = \bar{\lambda} \cup \{\square_i\}$ such that $c_{\square_i} = x_i$ for $1 \leq i \leq m$. If $y_1 > 1$, let $\bar{\lambda}^{0+} = \bar{\lambda} \cup \{\square_0\}$ such that $c_{\square_0} = x_0 = 1$. We obtain the following result for $\Phi^k(\bar{\lambda}^{i+})$ and $\Phi^k(\bar{\lambda})$.

Theorem 3.4 *Let k be a given nonnegative integer and $\bar{\lambda}$ be a strict partition. Then there exist some $\xi_j \in \mathbb{Q}$ such that*

$$\Phi^k(\bar{\lambda}^{i+}) - \Phi^k(\bar{\lambda}) = \sum_{j=0}^{k-1} \xi_j \binom{x_i}{2}^j$$

for every strict partition $\bar{\lambda}$.

By Theorem 3.4 we can show that $D\left(\frac{\Phi^\nu(\bar{\lambda})}{H_{\bar{\lambda}}}\right)$ could be written as a linear combination of some $\frac{\Phi^\delta(\bar{\lambda})}{H_{\bar{\lambda}}}$ for some partitions δ .

Theorem 3.5 *Let $\nu = (\nu_1, \nu_2, \dots, \nu_\ell)$ be a partition. Then there exist some $\xi_\delta \in \mathbb{Q}$ such that*

$$D\left(\frac{\Phi^\nu(\bar{\lambda})}{H_{\bar{\lambda}}}\right) = \sum_{|\delta| \leq |\nu| - 1} \xi_\delta \frac{\Phi^\delta(\bar{\lambda})}{H_{\bar{\lambda}}} \quad (15)$$

for every strict partition $\bar{\lambda}$.

4 Proofs of Theorems

Instead of proving Theorem 1.1, we prove the following more general result, which implies Theorem 1.1 when $\nu = \emptyset$.

Theorem 4.1 *Suppose that $\nu = (\nu_1, \nu_2, \dots, \nu_\ell)$ is a given partition, $\bar{\mu}$ is a given strict partition and Q is a symmetric function. Then there exists some $r \in \mathbb{N}$ such that*

$$D^r\left(\frac{Q\left(\binom{\bar{c}_\square}{2} : \square \in \bar{\lambda}\right) \Phi^\nu(\bar{\lambda})}{H_{\bar{\lambda}}}\right) = 0$$

for every strict partition $\bar{\lambda}$. Consequently,

$$P(n) = \sum_{|\bar{\lambda}/\bar{\mu}|=n} \frac{2^{n-\ell(\bar{\lambda})+\ell(\bar{\mu})} f_{\bar{\lambda}/\bar{\mu}}}{H_{\bar{\lambda}}} Q\left(\binom{\bar{c}_\square}{2} : \square \in \bar{\lambda}\right) \Phi^\nu(\bar{\lambda})$$

is a polynomial in n .

Proof (sketch): By linearity we assume that

$$Q\left(\binom{\bar{c}_\square}{2} : \square \in \bar{\lambda}\right) = \prod_{t=1}^s \sum_{\square \in \bar{\lambda}} \binom{\bar{c}_\square}{2}^{r_t}$$

for some tuple (r_1, r_2, \dots, r_s) . Then by Theorems 3.2, 3.4 and 3.5 we can show that

$$\bar{H}_{\bar{\lambda}} D \left(\frac{\Phi^\nu(\bar{\lambda}) \prod_{t=1}^s \sum_{\square \in \bar{\lambda}} \binom{\bar{c}_\square}{2}^{r_t}}{\bar{H}_{\bar{\lambda}}} \right)$$

could be written as a linear combination of some $\Phi^{\nu'}(\bar{\lambda}) \prod_{t'=1}^{s'} \sum_{\square \in \bar{\lambda}} \binom{\bar{c}_\square}{2}^{r_{t'}}$ satisfying one of the following two conditions:

- (1) $s' < s$;
- (2) $s' = s$ and $|\nu'| \leq |\nu| - 1$.

Then the claim follows from induction on s and $|\nu|$. \square

Proof of Theorem 1.2: The special case of the proof of Theorem 4.1 with $\nu = \emptyset$ and $s = 1$ yields

$$\begin{aligned} \bar{H}_{\bar{\lambda}} D \left(\frac{\sum_{\square \in \bar{\lambda}} \binom{\bar{c}_\square}{2}^{r_1}}{\bar{H}_{\bar{\lambda}}} \right) &= \sum_{0 \leq i \leq m} \frac{\prod_{1 \leq j \leq m} \binom{x_i}{2} - \binom{y_j}{2}}{\prod_{\substack{0 \leq j \leq m \\ j \neq i}} \binom{x_i}{2} - \binom{x_j}{2}} \binom{x_i}{2}^{r_1} \\ &= \sum_{|\nu| \leq r_1} \xi_\nu \Phi^\nu(\bar{\lambda}), \end{aligned}$$

where ξ_ν are some constants. The last equality is due to Theorem 3.2. Notice that

$$(2k)! \binom{z+k-1}{2k} = 2^k \prod_{i=1}^k \left(\binom{z}{2} - \binom{i}{2} \right).$$

Then by Theorems 3.5 and 2.3 we know that

$$P(n) = \sum_{|\bar{\lambda}|=n} \frac{f'_{\bar{\lambda}}}{\bar{H}_{\bar{\lambda}}} \sum_{\square \in \bar{\lambda}} \binom{\bar{c}_\square + k - 1}{2k}$$

is a polynomial in n with degree at most $k+1$.

On the other hand,

$$P(k+1) = \frac{f'_{(k+1)}}{\bar{H}_{(k+1)}} \binom{2k}{2k} = \frac{2^k}{(k+1)!}$$

since $(k+1)$ is the only strict partition with size $k+1$ who has contents greater than k . Moreover, It is obvious that $P(0) = P(1) = \dots = P(k) = 0$. Since the polynomial $P(n)$ is uniquely determined by those values, we obtain $P(n) = \frac{2^k}{(k+1)!} \binom{n}{k+1}$. \square

From Theorem 4.1, the left-hand side of (3) in Theorem 1.3 is a polynomial in n . To evaluate this polynomial explicitly, we need the following lemma.

Lemma 4.2 *Let $\bar{\lambda}$ be a strict partition. Then $\Phi^1(\bar{\lambda}) = |\bar{\lambda}|$.*

Proof of Theorem 1.3: By Corollary 3.3, Theorem 3.1 and Identity (13) it is easy to see that

$$\begin{aligned}
\bar{H}_{\bar{\lambda}} D \left(\frac{\sum_{\square \in \bar{\lambda}} \binom{\bar{c}_{\square}}{2}}{\bar{H}_{\bar{\lambda}}} \right) &= \sum_{\ell(\bar{\lambda}^+) > \ell(\bar{\lambda})} \frac{\bar{H}_{\bar{\lambda}}}{\bar{H}_{\bar{\lambda}^+}} \left(\sum_{\square \in \bar{\lambda}^+} \binom{\bar{c}_{\square}}{2} - \sum_{\square \in \bar{\lambda}} \binom{\bar{c}_{\square}}{2} \right) \\
&+ 2 \sum_{\ell(\bar{\lambda}^+) = \ell(\bar{\lambda})} \frac{\bar{H}_{\bar{\lambda}}}{\bar{H}_{\bar{\lambda}^+}} \left(\sum_{\square \in \bar{\lambda}^+} \binom{\bar{c}_{\square}}{2} - \sum_{\square \in \bar{\lambda}} \binom{\bar{c}_{\square}}{2} \right) \\
&= \sum_{0 \leq i \leq m} \frac{\prod_{1 \leq j \leq m} \binom{x_i}{2} - \binom{y_j}{2}}{\prod_{\substack{0 \leq j \leq m \\ j \neq i}} \binom{x_i}{2} - \binom{x_j}{2}} \binom{x_i}{2} \\
&= \Phi^1(\bar{\lambda}) \\
&= |\bar{\lambda}|.
\end{aligned}$$

Therefore we have

$$\begin{aligned}
\bar{H}_{\bar{\lambda}} D^2 \left(\frac{\sum_{\square \in \bar{\lambda}} \binom{\bar{c}_{\square}}{2}}{\bar{H}_{\bar{\lambda}}} \right) &= 1, \\
\bar{H}_{\bar{\lambda}} D^3 \left(\frac{\sum_{\square \in \bar{\lambda}} \binom{\bar{c}_{\square}}{2}}{\bar{H}_{\bar{\lambda}}} \right) &= 0.
\end{aligned}$$

Then our claim follows from Theorem 2.3. \square

Similarly, by (12), (13) and (14) we have

$$\begin{aligned}
\bar{H}_{\bar{\lambda}} D \left(\frac{\sum_{\square \in \bar{\lambda}} \binom{\bar{c}_{\square}+1}{4}}{\bar{H}_{\bar{\lambda}}} \right) &= \frac{1}{12} (\Phi^2(\bar{\lambda}) + |\bar{\lambda}|^2 - 2|\bar{\lambda}|), \\
\bar{H}_{\bar{\lambda}} D^2 \left(\frac{\sum_{\square \in \bar{\lambda}} \binom{\bar{c}_{\square}+1}{4}}{\bar{H}_{\bar{\lambda}}} \right) &= \frac{2}{3} |\bar{\lambda}|, \\
\bar{H}_{\bar{\lambda}} D^3 \left(\frac{\sum_{\square \in \bar{\lambda}} \binom{\bar{c}_{\square}+1}{4}}{\bar{H}_{\bar{\lambda}}} \right) &= \frac{2}{3}, \\
\bar{H}_{\bar{\lambda}} D^4 \left(\frac{\sum_{\square \in \bar{\lambda}} \binom{\bar{c}_{\square}+1}{4}}{\bar{H}_{\bar{\lambda}}} \right) &= 0.
\end{aligned}$$

Thus by Theorem 2.3 we obtain the following result.

Theorem 4.3 *Let $\bar{\mu}$ be a strict partition. Then*

$$\begin{aligned}
&\sum_{|\bar{\lambda}/\bar{\mu}|=n} \frac{2^{n-\ell(\bar{\lambda})+\ell(\bar{\mu})} f_{\bar{\lambda}/\bar{\mu}} \bar{H}_{\bar{\mu}}}{\bar{H}_{\bar{\lambda}}} \left(\sum_{\square \in \bar{\lambda}} \binom{\bar{c}_{\square}+1}{4} - \sum_{\square \in \bar{\mu}} \binom{\bar{c}_{\square}+1}{4} \right) \\
&= \frac{2}{3} \binom{n}{3} + \frac{2}{3} |\bar{\mu}| \binom{n}{2} + \frac{1}{12} (\Phi^2(\bar{\mu}) + |\bar{\mu}|^2 - 2|\bar{\mu}|) n.
\end{aligned}$$

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