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Hankel continued fraction and its applications

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Abstract. The Hankel determinants of a given power series f can be evaluated by using the Jacobi continued fraction expansion of f. However the existence of the Jacobi continued fraction needs that all Hankel determinants of f are nonzero. We introduce *Hankel continued fraction*, whose existence and uniqueness are guaranteed without any condition for the power series f. The Hankel determinants can also be evaluated by using the Hankel continued fraction.

It is well known that the continued fraction expansion of a quadratic irrational number is ultimately periodic. We prove a similar result for power series. If a power series f over a finite field satisfies a quadratic equation, then the Hankel continued fraction is ultimately periodic. As an application, we derive the Hankel determinants of several automatic sequences, in particular, the regular paperfolding sequence. Thus we provide an automatic proof of a result obtained by Guo, Wu and Wen, which was conjectured by Coons-Vrbik.

1. Introduction

Let $\mathbb{F}$ be a field and x be a indeterminate. We identify a sequence $\mathbf{a} = (a_0, a_1, a_2, \ldots)$ over $\mathbb{F}$ and its generating function $f = f(x) = a_0 + a_1x + a_2x^2 + \cdots \in \mathbb{F}[[x]]$. Usually, $a_0 = 1$. For each $n \geq 1$ and $k \geq 0$ the Hankel determinant of the series f (or of the sequence $\mathbf{a}$) is defined by

$$H_n^{(k)}(f) := \begin{vmatrix} a_k & a_{k+1} & \cdots & a_{k+n-1} \\ a_{k+1} & a_{k+2} & \cdots & a_{k+n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{k+n-1} & a_{k+n} & \cdots & a_{k+2n-2} \end{vmatrix} \in \mathbb{F}. \tag{1.1}$$

Let $H_n(f) := H_n^{(0)}(f)$, for short; the *sequence of the Hankel determinants* of f is defined to be:

$$H(f) := (H_0(f) = 1, H_1(f), H_2(f), H_3(f), \ldots).$$

The Hankel determinants play an important role in the study of the irrationality exponent of automatic numbers. In 1998, Allouche, Peyrière,

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Wen and Wen proved that all Hankel determinants of the Thue-Morse sequence are nonzero [APWW]. Bugeaud [Bu11] was able to prove that the irrationality exponent of the Thue-Morse-Mahler number is equal to 2 by using APWW's result. Using Bugeaud's method, several authors obtained the following results: first, Coons [Co13] who proved that the irrationality exponent of the sum of the reciprocals of the Fermat numbers is 2; then, Guo, Wu and Wen who showed that the irrationality exponents of the regular paperfolding numbers are exactly 2 [GWW]. However, the evaluations of the Hankel determinants still rely on the method developed by Allouche, Peyrière, Wen and Wen, which consists of proving sixteen recurrence relations between determinants (see [APWW, Co13, GWW]). A combinatorial proof of the results by APWW and Coons about the Hankel determinants is derived by Bugeaud and the author [BH14]. In our previous paper [Ha15] short proofs of those results are presented by using Jacobi continued fractions.

The Hankel determinants of a given power series f can be evaluated by using the Jacobi continued fraction expansion of f (see, e.g., [Kr98, Kr05, Fl80, Wa48, Vi83, Ha15]). However the existence of the Jacobi continued fraction needs that all Hankel determinants of f are nonzero. In Section 2 we introduce *Hankel continued fraction*, whose existence and uniqueness are guaranteed without any condition for the power series. The Hankel determinants can also be evaluated by using the Hankel continued fraction (see Theorem 2.1). Let p be a prime number and $\mathbb{F}_p = \mathbb{Z}/p\mathbb{Z}$ be the finite field of size p. In Section 3 we prove the following result.

Theorem 1.1. *Let p be a prime number and $F(x) \in \mathbb{F}_p[[x]]$ be a power series satisfying the following quadratic equation*

$$A(x) + B(x)F(x) + C(x)F(x)^2 = 0, \tag{1.2}$$

where $A(x), B(x), C(x) \in \mathbb{F}_p[x]$ are three polynomials with one of the following conditions

(i) $B(0) = 1,\ C(0) = 0,\ C(x) \neq 0$;

(ii) $B(0) = 1,\ C(x) = 0$;

(iii) $B(0) = 1,\ C(0) \neq 0,\ A(0) = 0$;

(iv) $B(x) = 0,\ C(0) = 1,\ A(x) = -(a_k x^k)^2 + O(x^{2k+1})$ for some $k \in \mathbb{N}$ and $a_k \neq 0$ when $p \neq 2$.

Then, the Hankel continued fraction expansion of $F(x)$ exists and is ultimately periodic. Also, the Hankel determinant sequence $H(F)$ is ultimately periodic.

It is well known that the simple continued fraction for a real number r is infinite and ultimately periodic if and only if r is a quadratic irrational number [Hal13, Theorem 2.2.2 (Euler-Lagrange)]. The first part of Theorem 1.1 can be viewed as a power series analog of Lagrange's theorem for real number. The converse, stated in Theorem 3.6, is an analog of Euler's theorem. We insist that there is no similar result with traditional Jacobi continued fraction because of that its existence is not guaranteed. The idea of introducing the concept of the Hankel continued fractions plays a crucial role.

Notice that the Hankel continued fraction and the Hankel determinant sequence in Theorem 1.1 can be *entirely* calculated by Algorithm 3.3. By using Theorem 1.1 we derive the Hankel determinants of several automatic sequences.

Theorem 1.2. *For each pair of positive integers a, b, let*

$$G_{a,b}(x) = \frac{1}{x^{2^a}} \sum_{n=0}^{\infty} \frac{x^{2^{n+a}}}{1 - x^{2^{n+b}}} \in \mathbb{F}_2[[x]]. \tag{1.3}$$

Then $H(G_{a,b})$ is ultimately periodic.

A list of Hankel determinants for the special cases of Theorem 1.2 obtained by Algorithm 3.3 is given in Corollary 4.1. When $a = b = 0$, we then reprove Coons's Theorem [Co13]. The cases, where $(a, b) = (2, 1), (2, 0), (1, 1)$, are obtained in [Ha15] by using the Jacobi continued fraction expansion. The case, where $a = 0$ and $b = 2$ was conjectured by Coons and Vrbik [CV12] and recently proved by Guo, Wu and Wen [GWW] by using APWW's method. The sequence $G_{0,2}$ is usually called *regular paperfolding sequence* [Al87].

An ultimately periodic sequence is written in contracted form by using the star sign. For instance, the sequence $\mathbf{a} = (1, (3, 0)^*)$ represents $(1, 3, 0, 3, 0, 3, 0, \ldots)$, that is, $a_0 = 1$ and $a_{2k+1} = 3, a_{2k+2} = 0$ for each positive integer k. Recall that the *Rudin-Shapiro sequence* (u_n) is defined by

$$\begin{cases} u_0 = 0, \\ u_{2n} = u_n, \quad u_{4n+1} = u_n, \quad u_{4n+3} = 1 - u_{2n+1}. \quad (n \geq 0) \end{cases} \tag{1.4}$$

Proposition 1.3. *Let (u_n) be the Rudin-Shapiro sequence and*

$$f(x) = \sum_{n \geq 0} u_n x^n; \qquad f_1(x) = \sum_{n \geq 0} u_{n+1} x^n;$$

$$f_2(x) = \sum_{n \geq 0} u_{n+2} x^n; \quad f_3(x) = \sum_{n \geq 0} u_{n+3} x^n.$$

Then,

$$H(f) \equiv (1, 0, 0, 0, 1, 0)^* \pmod 2;$$

$$H(f_1) \equiv (1, 0, 0, 1, 0, 0, 1, 0, 0, 1, 1, 1, 0, 0, 0, 0, 1, 1)^* \pmod 2;$$

$$H(f_2) \equiv (1,0,1,1,0,1,1,0,1,1,1,1,0,0,0,1,1,1)^* \pmod 2;$$
$$H(f_3) \equiv (1,1,0,1,1,1,1,1,1,0,1,1,0,0,1,0,1,0)^* \pmod 2.$$

Recall that Stern's sequence $(a_n)_{n=0,1,\dots}$ is defined by (see [St58])

$$\begin{cases} a_0 = 0, \quad a_1 = 1 \\ a_{2n} = a_n, \quad a_{2n+1} = a_n + a_{n+1}. \quad (n \geq 1) \end{cases}$$

The twisted version of Stern's sequence (b_n) is defined by (see [BV13, Ba10, Al12])

$$\begin{cases} b_0 = 0, \quad b_1 = 1, \\ b_{2n} = -b_n, \quad b_{2n+1} = -(b_n + b_{n+1}). \quad (n \geq 1) \end{cases}$$

Let

$$S(x) = \sum_{n=0}^{\infty} a_{n+1} x^n \quad \text{and} \quad B(x) = \sum_{n \geq 0} b_{n+1} x^n$$

be the generating function for Stern's sequence and twisted Stern's sequence.

Proposition 1.4. *The Hankel determinants of the Stern's sequence and the twisted Stern's sequence verify the following relations*

$$H_n(S)/2^{n-2} \equiv H_n(B)/2^{n-2} \equiv (0,0,1,1)^* \pmod 2.$$

The proofs of Theorem 1.2 and Propositions 1.3-4 are given in Section 4. The results obtained in the paper about Hankel determinants can be used for studying irrationality exponents [BHWY].

2. Hankel continued fractions

Let $\mathbf{u} = (u_1, u_2, \ldots)$ and $\mathbf{v} = (v_0, v_1, v_2, \ldots)$ be two sequences. Recall that the *Jacobi continued fraction* attached to $(\mathbf{u}, \mathbf{v})$, or *J-fraction*, for short, is a continued fraction of the form

$$f(x) = \cfrac{v_0}{1 + u_1 x - \cfrac{v_1 x^2}{1 + u_2 x - \cfrac{v_2 x^2}{1 + u_3 x - \cfrac{v_3 x^2}{\ddots}}}}.$$

The basic properties on J-fractions, we now recall, can be found in [Kr98, Kr05, Fl80, Wa48, Vi83, Ha15]. The J-fraction of a given power series f

exists if and only if all the Hankel determinants $H_n(f)$ are nonzero. The first values of the coefficients u_n and v_n in the J-fraction expansion can be calculated by the *Stieltjes Algorithm*. Also, Hankel determinants can be calculated from the J-fraction by means of the following fundamental relation:

$$H_n(f) = v_0^n v_1^{n-1} v_2^{n-2} \cdots v_{n-2}^2 v_{n-1}.$$

The Hankel determinants of a power series f can be calculated by the above fundamental relation if the J-fraction exists, which is equivalent to the fact that all Hankel determinants of f are nonzero. In this section we define the so-called *Hankel continued fraction expansion* (*Hankel fraction* or *H-fraction*, for short) whose existence and uniqueness are guaranteed without any condition for the power series. The Hankel determinants can also be evaluated by using the Hankel continued fraction.

The relation between continued fractions and Hankel determinants are widely studied. See [Kr05, Vi83, Fl80] for S- and J-fractions; [Bu10] and [Ci13] for C-fraction. The following table shows that the Hankel continued fraction has some advantage over any other type of continued fractions.

Fraction type	Parameters	Fraction existence	Fraction uniqueness	Hankel det. formula
S, J-fraction	$\delta = 1, 2; k_j = 0$	No	Yes	Yes
C-fraction	$\delta = 1, u_j(x) = 0$	Yes	Yes	No
H-fraction	$\delta = 2$	Yes	Yes	Yes

Definition 2.1. For each positive integer δ, a *super continued fraction* associated with δ, called *super δ-fraction* for short, is defined to be a continued fraction of the following form

$$F(x) = \cfrac{v_0 x^{k_0}}{1 + u_1(x)x - \cfrac{v_1 x^{k_0+k_1+\delta}}{1 + u_2(x)x - \cfrac{v_2 x^{k_1+k_2+\delta}}{1 + u_3(x)x - \ddots}}} \tag{2.1}$$

where $v_j \neq 0$ are constants, k_j are nonnegative integers and $u_j(x)$ are polynomials of degree less than or equal to $k_{j-1} + \delta - 2$. By convention, 0 is of degree -1.

When $\delta = 1$ (resp. $\delta = 2$) and all $k_j = 0$, the super δ-fraction (2.1) is the traditional S-fraction (resp. J-fraction). A super 2-fraction is called *Hankel continued fraction.* When $\delta = 1$ and $u_j(x) = 0$, the super 1-fraction is a special C-fraction (set $b_j = k_0 + k_1 + \cdots k_{j-1} + \lfloor j/2 \rfloor$ in

[Ci13]). Notice that every power series has a unique C-fraction expansion, but not all C-fractions have Hankel determinant formula, and only those who are also super 1-fractions have.

Theorem 2.1. *(i) Let δ be a positive integer. Each super δ-fraction defines a power series, and conversely, for each power series $F(x)$, the super δ-fraction expansion of $F(x)$ exists and is unique.*

(ii) Let $F(x)$ be a power series such that its H-fraction is given by (2.1) with $\delta = 2$. Then, all non-vanishing Hankel determinants of $F(x)$ are given by

$$H_{s_j}(F(x)) = (-1)^{\epsilon} v_0^{s_j} v_1^{s_j - s_1} v_2^{s_j - s_2} \cdots v_{j-1}^{s_j - s_{j-1}}, \tag{2.2}$$

where $\epsilon = \sum_{i=0}^{j-1} k_i(k_i+1)/2$ and $s_j = k_0 + k_1 + \cdots + k_{j-1} + j$ for every $j \geq 0$.

The first part of Theorem 2.1 is a consequence of Definition 2.1 and can be proved by using an algorithm. In fact, if $F(x) = v_0 x^{k_0} + O(x^{k_0+1})$ with $v_0 \neq 0$, then, $F(x)/(v_0 x^{k_0}) = 1 + O(x)$. The polynomial $u_1(x)$ can be calculated by

$$\frac{v_0 x^{k_0}}{F(x)} = 1 + u_1(x)x - x^{k_0+\delta} F_1(x).$$

We repeat the same operation for $F_1(x)$ and get $v_1, k_1, u_2(x)$, etc. The second part of Theorem 2.1 follows from the next lemma.

Lemma 2.2. *Let k be a nonnegative integer and let $F(x), G(x)$ be two power series satisfying*

$$F(x) = \frac{x^k}{1 + u(x)x - x^{k+2}G(x)}, \tag{2.3}$$

where $u(x)$ is a polynomial of degree less than or equal to k. Then,

$$H_n(F) = (-1)^{k(k+1)/2} H_{n-k-1}(G). \tag{2.4}$$

Proof. Let $F(x) = \sum_j f_j x^j$. We have $f_j = 0$ for $j \leq k-1$ and $f_k = 1$. Let $x^k/F(x) = \sum_j b_j x^j$ and $G(x) = \sum_j g_j x^j$. We have $g_j = -b_{j+k+2}$ for $j \geq 0$. Let $b_j = f_j = 0$ when $j < 0$. We define four matrices by

$$\begin{aligned}
\mathbf{F}_1 &= (f_{i-j+k})_{0 \leq i,j \leq n-1}, \\
\mathbf{G} &= \mathrm{Diag}\big((b_{i+j-k})_{0 \leq i,j \leq k},\ (g_{i+j})_{0 \leq i,j \leq n-k-1}\big), \\
\mathbf{F} &= (f_{i+j})_{0 \leq i,j \leq n-1}, \\
\mathbf{B} &= (b_{j-i})_{0 \leq i,j \leq n-1},
\end{aligned}$$

and show that

$$\mathbf{F}_1 \times \mathbf{G} = \mathbf{F} \times \mathbf{B}. \tag{2.5}$$

For example, when $k = 3$, $n = 7$, the four matrices and (2.5) are reproduced as follows.

$$\begin{pmatrix} 1 & . & . & . & . & . & . \\ f_4 & 1 & . & . & . & . & . \\ f_5 & f_4 & 1 & . & . & . & . \\ f_6 & f_5 & f_4 & 1 & . & . & . \\ f_7 & f_6 & f_5 & f_4 & 1 & . & . \\ f_8 & f_7 & f_6 & f_5 & f_4 & 1 & . \\ f_9 & f_8 & f_7 & f_6 & f_5 & f_4 & 1 \end{pmatrix} \begin{pmatrix} . & . & . & 1 & . & . & . \\ . & . & 1 & b_1 & . & . & . \\ . & 1 & b_1 & b_2 & . & . & . \\ 1 & b_1 & b_2 & b_3 & . & . & . \\ . & . & . & . & g_0 & g_1 & g_2 \\ . & . & . & . & g_1 & g_2 & g_3 \\ . & . & . & . & g_2 & g_3 & g_4 \end{pmatrix}$$
$$= \begin{pmatrix} . & . & . & 1 & f_4 & f_5 & f_6 \\ . & . & 1 & f_4 & f_5 & f_6 & f_7 \\ . & 1 & f_4 & f_5 & f_6 & f_7 & f_8 \\ 1 & f_4 & f_5 & f_6 & f_7 & f_8 & f_9 \\ f_4 & f_5 & f_6 & f_7 & f_8 & f_9 & f_{10} \\ f_5 & f_6 & f_7 & f_8 & f_9 & f_{10} & f_{11} \\ f_6 & f_7 & f_8 & f_9 & f_{10} & f_{11} & f_{12} \end{pmatrix} \begin{pmatrix} 1 & b_1 & b_2 & b_3 & b_4 & b_5 & b_6 \\ . & 1 & b_1 & b_2 & b_3 & b_4 & b_5 \\ . & . & 1 & b_1 & b_2 & b_3 & b_4 \\ . & . & . & 1 & b_1 & b_2 & b_3 \\ . & . & . & . & 1 & b_1 & b_2 \\ . & . & . & . & . & 1 & b_1 \\ . & . & . & . & . & . & 1 \end{pmatrix}$$

Relations (2.5) are trivial for the entry (i, j) when $0 \le j \le k$ and when $j \ge k + 1, i \le k$. For $i, j \ge k + 1$, the entries of the two sides of (2.5) are respectively

$$\begin{aligned} \text{LHS} &= f_{i-1}g_{j-k-1} + f_{i-2}g_{j-k} + \cdots + f_{i-n+k-1}g_{j+n-2k-1} \\ &= -(f_{i-1}b_{j+1} + f_{i-2}b_{j+2} + \cdots + f_{i-n+k-1}b_{j+n-k+1}); \\ \text{RHS} &= f_i b_j + f_{i+1}b_{j-1} \cdots + f_{i+n-1}b_{j-n+1}. \end{aligned}$$

Since $F(x)\sum b_j x_j = x^k$, we have RHS − LHS = 0. Moreover, $\det \mathbf{F}_1 = 1$, $\det \mathbf{G} = (-1)^{k(k+1)/2} H_{n-k-1}(G)$, $\det \mathbf{F} = H_n(F)$, $\det \mathbf{B} = 1$. This completes the proof of (2.4). □

Example 2.1. Let

$$f(x) = \frac{1 - \sqrt{1 - \frac{4x^4}{1+x}}}{2x^4} \in \mathbb{Q}[[x]]. \tag{2.6}$$

Then the H-fraction (i.e. the super 2-fraction) of $f(x)$ is equal to

$$f(x) = \cfrac{1}{1 + x - \cfrac{x^4}{1 - \cfrac{x^4}{1 + x - \cfrac{x^4}{1 - \cfrac{x^4}{\ddots}}}}}. \tag{2.7}$$

In view of (2.1) we have $v_i = 1$, $k_{2i} = 0$, $k_{2i+1} = 2$ for all i and $(s_j)_{j=0,1,\ldots} = (0,1,4,5,8,9,12,13,\ldots)$ where s_j is defined in Theorem 2.1. By Theorem 2.1 the Hankel determinant sequence is (see also [Ha15, Proposition 3.7]) $H(f) = (1,1,0,0,-1,-1,0,0)^*$.

Example 2.2. Let $g(x)$ be the generating function for the number of distinct partitions

$$\begin{aligned} g(x) &= \prod_{k\geq 1}(1+x^k) \in \mathbb{Q}[[x]] \\ &= 1 + x + x^2 + 2x^3 + 2x^4 + 3x^5 + 4x^6 + 5x^7 + 6x^8 + 8x^9 + \cdots \end{aligned}$$

Then the H-fraction of $g(x)$ is equal to

$$g(x) = \cfrac{1}{1 - x - \cfrac{x^3}{1 + x + \cfrac{x^5}{1 - x + x^2 - x^3 - \cfrac{x^5}{1 + x + x^2 + \cfrac{x^3}{1 - x + \cfrac{x^3}{\ddots}}}}}}.$$

We have

$$\begin{aligned} (k_j)_{j=0,1,\ldots} &= (0,1,2,1,0,1,0,1,0,0,0,0,0,1,1,0,0,0,2,\ldots), \\ (v_j)_{j=0,1,\ldots} &= (1,1,-1,1,-1,-1,-1,1,-4,-\tfrac{1}{4},\tfrac{1}{4},-8,\ldots), \\ (s_j)_{j=0,1,\ldots} &= (0,1,3,6,8,9,11,12,14,15,16,17,18,19,21,\ldots), \\ (H_j(g))_{j=0,1,\ldots} &= (1,1,0,-1,0,0,-1,0,1,1,0,-1,-1,0,1,-4,\ldots). \end{aligned}$$

Example 2.3. Let $h(x) = (1-x)^{1/3} \in \mathbb{F}_2[[x]]$. Then the H-fraction of $h(x)$ is equal to

$$h(x) = \cfrac{1}{1 + x + \cfrac{x^4}{1 + x + x^2 + x^3 + \cfrac{x^4}{1 + x + \cfrac{x^8}{1 + x + x^2 + x^3 + \cfrac{x^{16}}{\ddots}}}}}.$$

We have

$$\begin{aligned} (k_j)_{j=0,1,\ldots} &= (0,2,0,6,8,22,40,\ldots), \\ (v_j)_{j=0,1,\ldots} &= (1,-1,-1,-1,-1,-1,-1,\ldots), \\ (s_j)_{j=0,1,\ldots} &= (0,1,4,5,12,21,44,85,\ldots), \\ (H_j(h))_{j=0,1,\ldots} &= (1,1,0,0,1,1,0,0,0,0,0,0,1,0,0,0,0,\ldots). \end{aligned}$$

Example 2.4. Let $f(x) \in \mathbb{Q}[[x]]$ be the power series defined by (2.6) in Example 2.1. The super 2-fraction of $f(x)$ is periodic, as shown in (2.7) with $(k_j)_{j=0,1,\ldots} = (0,2)^*$. Note that the super 3-fraction of $f(x)$ is also periodic and given by (2.7) with $(k_j)_{j=0,1,\ldots} = (0,1)^*$. Finally, the super 5-fraction (resp. 7-fraction) is ultimately periodic and equal to

$$f(x) = \cfrac{1}{1 + x - x^4 - \cfrac{x^8}{1 + x - 2x^4 - \cfrac{x^8}{1 + x - 2x^4 - \cfrac{x^8}{\ddots}}}}$$

with $(k_j)_{j=0,1,\ldots} = (0,3)^*$ (resp. $(k_j)_{j=0,1,\ldots} = (0,1)^*$).

3. The periodicity

In this section we prove Theorem 1.1. Let $\delta \in \mathbb{N}^+$ and $\mathbb{F}$ be a field. With condition (i) in Theorem 1.1, the quadratic equation (1.2) has the unique solution

$$\begin{aligned} F^{(1)}(x) &= \frac{-B + \sqrt{B^2 - 4AC}}{2C} \\ &= -\frac{A}{B} \sum_{k=0}^{\infty} C_k \Big(\frac{CA}{B^2}\Big)^k = -\frac{A}{B} - \frac{CA^2}{B^3} - \frac{2\,C^2A^3}{B^5} - \frac{5\,C^3A^4}{B^7} - \cdots , \end{aligned} \tag{3.1}$$

where $C_k = \frac{(2k)!}{k!(k+1)!}$ is the k-th Catalan number. The above power series shows that $F^{(1)}$ is well-defined even the characteristic of $\mathbb{F}$ is equal to 2. Under condition (iii), there are two solutions, $F^{(1)}(x)$ and

$$\begin{aligned} F^{(2)}(x) &= \frac{-B - \sqrt{B^2 - 4AC}}{2C} \\ &= -\frac{B}{C} + \frac{A}{B} \sum_{k=0}^{\infty} C_k \Big(\frac{CA}{B^2}\Big)^k = -\frac{B}{C} + \frac{A}{B} + \frac{CA^2}{B^3} + \frac{2\,C^2A^3}{B^5} + \cdots . \end{aligned}$$

The power series $F(x)$ satisfying (1.2) can be uniquely determined by the constant term $F(0)$.

Algorithm 3.1 [NextABC].

Prototype: $(A^*, B^*, C^*; k, A_k, D) = \texttt{NextABC}(A, B, C; \delta)$

Input: $A(x), B(x), C(x) \in \mathbb{F}[x]$ *three polynomials such that* $B(0) = 1$, $A(0)C(0) = 0$, $C(x) \neq 0, A(x) \neq 0$;

Output: $A^*(x), B^*(x), C^*(x) \in \mathbb{F}[x]$, $k \in \mathbb{N}^+$, $A_k \neq 0 \in \mathbb{F}$, $D(x) \in \mathbb{F}[x]$ *a polynomial of degree less than or equal to* $k+\delta-1$ *such that* $D(0)=1$.

Step 1 [Define k, A_k*]. Since* $A(x) \neq 0$, *let* $A(x) = A_k x^k + O(x^{k+1})$ *with* $A_k \neq 0$.

Step 2. Let

$$F(x) = F^{(1)}(x) = \frac{-A(x)}{B(x)+C(x)F(x)}. \tag{3.2}$$

Using (3.1) or (3.2) to get the first terms of $F(x)$, $F(x)/(-A_k x^k)$ *and of* $-A_k x^k/F(x)$*:*

$$\begin{aligned} F(x) &= -A_k x^k + \cdots + O(x^{2k+\delta}); \\ \frac{F(x)}{-A_k x^k} &= 1 + \cdots + O(x^{k+\delta}); \\ \frac{-A_k x^k}{F(x)} &= 1 + \cdots + O(x^{k+\delta}). \end{aligned} \tag{3.3}$$

Step 3 [Define D]. Define $D(x), G(x)$ *by*

$$\frac{-A_k x^k}{F(x)} = D(x) - x^{k+\delta} G(x) \tag{3.4}$$

where $D(x)$ *is a polynomial of degree less than or equal to* $k+\delta-1$ *such that* $D(0)=1$ *and* $G(x)$ *is a power series. The value of* $D(x)$ *is obtained by (3.3).*

Step 4 [Define A^*, B^*, C^**]. Let*

$$\begin{aligned} A^*(x) &= \big(-D^2 A/A_k + BDx^k - CA_k x^{2k}\big)/x^{2k+\delta}; \\ B^*(x) &= 2AD/(A_k x^k) - B; \\ C^*(x) &= -Ax^\delta/A_k. \end{aligned} \tag{3.5}$$

We will prove in Lemma 3.2 that A^*, B^*, *and* C^* *are polynomials.*

Notice that in Step 2 we always take the solution $F^{(1)}(x)$. The case of condition (iii) and the solution $F^{(2)}(x)$ is discussed in the proof of Theorem 1.1 (see (3.12-13)).

Lemma 3.2. *Let* $A(x), B(x), C(x) \in \mathbb{F}[x]$ *be three polynomials such that* $B(0)=1, C(0)=0, C(x) \neq 0, A(x) \neq 0$ *and*

$$(A^*, B^*, C^*; k, A_k, D) = \texttt{NextABC}(A, B, C; \delta)$$

obtained by Algorithm 3.1. If $F(x)$ is the power series defined by (1.2), then, $F(x)$ can be written as

$$F(x) = \frac{-A_k x^k}{D(x) - x^{k+\delta} G(x)} \tag{3.6}$$

where $G(x)$ is a power series satisfying

$$A^*(x) + B^*(x)G(x) + C^*(x)G(x)^2 = 0. \tag{3.7}$$

Furthermore, $A^*(x), B^*(x), C^*(x)$ are three polynomials in $\mathbb{F}[x]$ such that $B^*(0) = 1, C^*(0) = 0, C^*(x) \neq 0$ and

$$\deg(A^*) \leq d;\ \deg(B^*) \leq d+1;\ \deg(C^*) \leq d+\delta, \tag{3.8}$$

where

$d := d(A, B, C) = \max(\deg(A), \deg(A) + \delta - 2, \deg(B) - 1, \deg(C) - \delta).$

Proof. From (1.2) and (3.6), we have

$$A(D - x^{k+\delta}G)^2 + B(-A_k x^k)(D - x^{k+\delta}G) + C\left(-A_k x^k\right)^2 = 0.$$

Thus, $G(x)$ satisfies

$$\bar{A}(x) + \bar{B}(x)G(x) + \bar{C}(x)G(x)^2 = 0 \tag{3.9}$$

where

$$\begin{aligned}
\bar{A} &= AD^2 - BA_k x^k D + CA_k^2 x^{2k};\\
\bar{B} &= -2ADx^{k+\delta} + BA_k x^{2k+\delta};\\
\bar{C} &= Ax^{2k+2\delta}.
\end{aligned}$$

By (1.2) and (3.6) the polynomial A is divisible by x^k. Hence, $\bar{C}$ and $\bar{B}$ are divisible by $x^{2k+\delta}$. Thanks to (3.9), $\bar{A}$ is also divisible by $x^{2k+\delta}$. Consequently, (3.5) defines three polynomials A^*, B^*, C^*. Since $A(x) = A_k x^k + O(x^{k+1})$, we have $A(x)/(A_k x^k)|_{x=0} = 1$. By (3.3) and (3.4), $D(0) = 1$. Hence

$$\begin{aligned}
B^*(0) &= \big(2AD/(A_k x^k) - B\big)|_{x=0} = 2 \cdot 1 \cdot D(0) - B(0) = 2 - 1 = 1;\\
C^*(x) &= -Ax^\delta/A_k = -x^{k+\delta} + O(x^{k+\delta+1}) \neq 0;\\
C(0) &= 0.
\end{aligned}$$

Moreover,

$$\begin{aligned}
\deg(A^*) &\leq \max(\deg(A) + \delta - 2, \deg(B) - 1, \deg(C) - \delta);\\
\deg(B^*) &\leq \max(\deg(A) + \delta - 1, \deg(B));\\
\deg(C^*) &= \deg(A) + \delta.
\end{aligned}$$

Let $d_A = \deg(A),\ d_B = \deg(B) - 1,\ d_C = \deg(C) - \delta$ and $d_A^* = \deg(A^*)$, $d_B^* = \deg(B^*) - 1,\ d_C^* = \deg(C^*) - \delta$. The above inequalities become

$$\begin{aligned} d_A^* &\le \max(d_A + \delta - 2, d_B, d_C); \\ d_B^* &\le \max(d_A + \delta - 2, d_B); \\ d_C^* &= d_A. \end{aligned}$$

So that $d_A^*, d_B^*, d_C^* \le \max(d_A, d_A + \delta - 2, d_B, d_C)$. $\square$

Keep the same notation as in Lemma 3.2 and let

$$\begin{aligned} d^* &:= d(A^*, B^*, C^*) \\ &= \max(\deg(A^*), \deg(A^*) + \delta - 2, \deg(B^*) - 1, \deg(C^*) - \delta). \end{aligned}$$

By Lemma 3.2, we have

$$d^* \le \max(d, d + \delta - 2, d, d) = \max(d, d + \delta - 2).$$

Hence,

$$d(A^*, B^*, C^*) \le d(A, B, C) \quad \text{for} \quad \delta = 1, 2. \tag{3.10}$$

Algorithm 3.3 [HFrac].

Prototype: $(a_k, d_k, D_k)_{k=0,1,\ldots} = \texttt{HFrac}(A, B, C; p, \delta)$

Input: $\delta = 1$ *or* 2*;*

p a prime number;

$A(x), B(x), C(x) \in \mathbb{F}_p[x]$ three polynomials such that $B(0) = 1$, $C(0) = 0$ and $C(x) \neq 0$;

Output: a finite or infinite sequence $(a_k, d_k, D_k)_{k=0,1,\ldots}$

Step 1. $j := 0$, $A^{(j)} := A$, $B^{(j)} := B$, $C^{(j)} := C$.

Step 2. If $A^{(j)} = 0$, then return the finite sequence $(a_k, d_k, D_k)_{k=0,1,\ldots,j-1}$. The algorithm terminates.

Step 3. If $A^{(j)} \neq 0$, then let

$$(A^{(j+1)}, B^{(j+1)}, C^{(j+1)}; d_j, a_j, D_j) := \texttt{NextABC}(A^{(j)}, B^{(j)}, C^{(j)}; \delta).$$

Let $j := j + 1$.

Step 4. If there exists i with $0 \le i < j$ such that

$$(A^{(i)}, B^{(i)}, C^{(i)}) = (A^{(j)}, B^{(j)}, C^{(j)}),$$

then return the infinite sequence

$$((a_k, d_k, D_k)_{k=0,1,\ldots,i-1}, (a_k, d_k, D_k)^*_{k=i,i+1,\ldots,j-1}).$$

The algorithm terminates. Else, go to Step 2.

Remarks. (i) In Step 3 the conditions

$$B^{(j)}(0) = 1,\ C^{(j)}(0) = 0,\ C^{(j)}(x) \neq 0$$

are guaranteed by Lemma 3.2. Algorithm 3.1 can be applied repeatedly. (ii) The loop Steps 2-4 will be broken at Step 2 or Step 4, since the degrees of the polynomials $A^{(i)}, B^{(i)}, C^{(i)}$ are bounded (see (3.10)), and the coefficients are taken from $\mathbb{F}_p$. The number of different triplets $(A^{(i)}, B^{(i)}, C^{(i)})$ is finite.

Proof of Theorem 1.1. There are several cases to be considered.
(i) If $B(0) = 1, C(0) = 0, C(x) \neq 0$, let

$$(a_k, d_k, D_k)_{k=0,1,\ldots} = \texttt{HFrac}(A, B, C; p, 2).$$

By Lemma 3.2,

$$F(x) = \cfrac{-a_0 x^{d_0}}{D_0(x) + \cfrac{a_1 x^{d_0+d_1+2}}{D_1(x) + \cfrac{a_2 x^{d_1+d_2+2}}{D_2(x) + \cfrac{a_3 x^{d_2+d_3+2}}{\ddots}}}} \tag{3.11}$$

and the above H-fraction is ultimately periodic (see Steps 2 and 4 in Algorithm 3.3). Note that if $A(x) = 0$, then the output sequence $((a_k, d_k, D_k))$ $(k = 0, 1, 2, \ldots)$ is the empty sequence. In this case $F(x) = 0$.

(ii) If $B(0) = 1, C(x) = 0$, then $F(x) = -A(x)/B(x)$ is rational.

(iii) If $B(0) = 1, C(0) \neq 0$ and $A(x) = 0$, then $F(x)$ is rational. If $B(0) = 1, C(0) \neq 0$ and $A(x) = A_k x^k + O(x^{k+1})$ with $k \geq 1$ and $A_k \neq 0$, then equation (1.2) has two solutions, namely, $F^{(1)}(x)$ and $F^{(2)}(x)$. Note that $F_{(1)}(x) = -A_k x^k + O(x^{k+1})$ and $F^{(2)}(x) = -1/C(0) + O(x)$.

(iii.1) In the case of $F^{(1)}(x)$, let

$$F^{(1)}(x) = \frac{-A_k x^k}{D(x) - x^{k+2} G(x)}.$$

Then, $G(x)$ satisfies (3.7) with polynomials A^*, B^*, C^* defined by (3.5) (see the proof of Lemma 3.2). Since $B^*(0) = 1, C^*(0) = 0, C^*(x) \neq 0$, the H-fraction expansion of $G(x)$ exists and is ultimately periodic by case (i), the same property holds for the H-fraction expansion of $F^{(1)}(x)$.

(iii.2) In the case of $F^{(2)}(x)$, let

$$F^{(2)}(x) = \frac{-1/C(0)}{D(x) - x^2 G(x)}. \tag{3.12}$$

Then, $G(x)$ satisfies (3.7) with polynomials A^*, B^*, C^* defined (same proof as Lemma 3.2):

$$\begin{aligned} A^*(x) &= \big(D^2 AC(0) - BD + C/C(0)\big)/x^2; \\ B^*(x) &= -2ADC(0) + B; \\ C^*(x) &= C(0)Ax^2. \end{aligned} \tag{3.13}$$

Since $B^*(0) = 1, C^*(0) = 0, C^*(x) \neq 0$, the H-fraction expansion of $G(x)$ exists and is ultimately periodic by case (i), the same property holds for the H-fraction expansion of $F^{(2)}(x)$.

(iv) If $B(x) = 0$, $C(0) = 1$ (or $C(0) \neq 0$) and $A(x) = -(a_k x^k)^2 + O(x^{2k+1})$ for some $k \in \mathbb{N}$ and $a_k \neq 0$, then $F(x)$ exists

$$F(x) = \sqrt{\frac{-A(x)}{C(x)}} = \sqrt{\frac{(a_k x^k)^2 + \cdots}{C(x)}} = a_k x^k \sqrt{\frac{1 + \cdots}{C(x)}}$$

Let

$$F(x) = \frac{a_k x^k}{D(x) - x^{k+2} G(x)}.$$

Then, $G(x)$ satisfies (3.7) with A^*, B^*, C^* defined (same proof as Lemma 3.2):

$$\begin{aligned} A^*(x) &= (D^2 A + C a_k^2 x^{2k})/x^{3k+2}; \\ B^*(x) &= -2ADx^{k+2}/x^{3k+2}; \\ C^*(x) &= Ax^{2k+4}/x^{3k+2}. \end{aligned} \tag{3.14}$$

If $p \neq 2$, then A^*, B^*, C^* are polynomials such that $B^*(0) \neq 0, C^*(0) = 0, C^*(x) \neq 0$. The H-fraction expansion of $G(x)$ exists and is ultimately periodic by case (i), the same property holds for the H-fraction expansion of $F(x)$.

The periodicity of the Hankel determinant sequence $H(F)$ is a consequence of Lemma 3.4 stated below. $\square$

Lemma 3.4. *Let p be a prime number. If the H-fraction expansion of a power series $F(x) \in \mathbb{F}_p[[x]]$ is ultimately periodic, then the Hankel determinant sequence $H(F)$ is ultimately periodic.*

Proof. Using the notation of Theorem 2.1 the two sequences (v_i) and (k_i) can be written as

$$\begin{aligned} (v_i) &= (v_0, v_1, \ldots, v_{m-1}, (v_m, v_{m+1}, \ldots, v_{m+t-1})^*); \\ (k_i) &= (k_0, k_1, \ldots, k_{m-1}, (k_m, k_{m+1}, \ldots, k_{m+t-1})^*). \end{aligned}$$

Let

$$\gamma_1 = \prod_{i=m}^{m+t-1} (-1)^{k_i(k_i+1)/2}, \quad \gamma_2 = \prod_{i=m}^{m+t-1} v_i^{s_m - s_i}, \quad \gamma_3 = \prod_{i=0}^{m-1} v_i,$$

$$\beta = \prod_{i=m}^{m+t-1} v_i, \quad r = s_{m+t} - s_m,$$

$$\gamma = \gamma_1 \gamma_3^r \gamma_2 \beta^{r - s_m}, \quad \eta = \lceil \frac{\ell - m}{t} \rceil, \quad \rho = \ell - m - \eta t.$$

For each $\ell \geq m$ we have

$$H_{s_\ell}(F) = \prod_{i=0}^{\ell-1} (-1)^{k_i(k_i+1)/2} v_i^{s_\ell - s_i}$$

and

$$H_{s_\ell + r}(F) = \prod_{i=0}^{\ell+t-1} (-1)^{k_i(k_i+1)/2} v_i^{s_\ell + r - s_i}. \tag{3.15}$$

If $p = 2$, then $H_{s_\ell}(F) = H_{s_\ell + r}(F) = 1$. Hence $H(F)$ is ultimately periodic. For general p we need to evaluate (3.15).

$$\begin{aligned}
H_{s_\ell + r}(F) &= \prod_{i=0}^{\ell-1} (-1)^{k_i(k_i+1)/2} v_i^{s_\ell + r - s_i} \times \prod_{i=\ell}^{\ell+t-1} (-1)^{k_i(k_i+1)/2} v_i^{s_\ell + r - s_i} \\
&= H_{s_\ell}(F) \prod_{i=0}^{\ell-1} v_i^r \times \gamma_1 \prod_{i=\ell}^{\ell+t-\rho-1} v_i^{s_\ell + r - s_i} \prod_{i=\ell+t-\rho}^{\ell+t-1} v_i^{s_\ell + r - s_i} \\
&= H_{s_\ell}(F) \prod_{i=0}^{\ell-1} v_i^r \times \gamma_1 \prod_{i=\ell}^{\ell+t-\rho-1} v_i^r \prod_{i=\ell}^{\ell+t-\rho-1} v_i^{s_\ell - s_i} \prod_{i=\ell-\rho}^{\ell-1} v_i^{s_\ell - s_i} \\
&= H_{s_\ell}(F) \gamma_1 \prod_{i=0}^{\ell+t-\rho-1} v_i^r \prod_{i=\ell}^{\ell+t-\rho-1} v_i^{s_\ell - s_i} \prod_{i=\ell-\rho}^{\ell-1} v_i^{s_\ell - s_i} \\
&= H_{s_\ell}(F) \gamma_1 \gamma_3^r \prod_{i=m}^{\ell+t-\rho-1} v_i^r \prod_{i=\ell-\rho}^{\ell+t-\rho-1} v_i^{s_\ell - s_i} \\
&= H_{s_\ell}(F) \gamma_1 \gamma_3^r \prod_{i=m}^{m+\eta t+t-1} v_i^r \prod_{i=m}^{m+t-1} v_i^{s_{m+\rho} - s_i} \\
&= H_{s_\ell}(F) \gamma_1 \gamma_3^r \prod_{i=m}^{m+\eta t+t-1} v_i^r \prod_{i=m}^{m+t-1} v_i^{s_{m+\rho} - s_m} \times \gamma_2 \\
&= H_{s_\ell}(F) \gamma_1 \gamma_3^r \gamma_2 \beta^{(\eta+1)r} \beta^{s_{m+\rho} - s_m} \\
&= H_{s_\ell}(F) \beta^{s_\ell} \gamma. \qquad (3.16)
\end{aligned}$$

We apply (3.16) recursively and get

$$H_{s_\ell+2\theta r}(F) = H_{s_\ell}(F)\beta^{2\theta s_\ell}\beta^{\theta(2\theta-1)}\gamma^{2\theta}.$$

Since $\beta \neq 0$ and $\gamma \neq 0$, there exists θ such that $\beta^\theta = 1 \in \mathbb{F}_p$ and $\gamma^{2\theta} = 1 \in \mathbb{F}_p$. Hence,

$$H_{s_\ell+2\theta r}(F) = H_{s_\ell}(F).$$

So that $H(F)$ is an ultimately periodic sequence. Moreover, $2\theta r$ is a period with offset s_m. If $p = 2$, then r is a period with offset s_m. The *least period* is a divisor of $2\theta r$ (or r when $p = 2$) and the *least offset* is less than or equal to s_m. $\square$

For convenience, the following notation is used for continued fraction

$$\frac{v_0}{u_1} + \frac{v_1}{u_2} + \frac{v_2}{u_3} + \frac{v_3}{u_4} + \cdots = \cfrac{v_0}{u_1 + \cfrac{v_1}{u_2 + \cfrac{v_2}{u_3 + \cfrac{v_3}{u_4 + \ddots}}}}.$$

Example (i.1). Let $p = 5$ and

$$F = \frac{1 - \sqrt{1 - \frac{4x}{1-x^4}}}{2x} \in \mathbb{F}_5[[x]]$$

or

$$-1 + (1 - x^4)F + (-x + x^5)F^2 = 0.$$

By Algorithm 3.3 [HFrac], we successively get

k	$A^{(k)}$	$B^{(k)}$	$C^{(k)}$	d_k	α_k
0	4	$1 + 4x^4$	$4x + x^5$	0	4
1	$4 + 4x^2 + 2x^3$	$1 + 3x + x^4$	$4x^2$	0	4
2	$3 + x + 4x^2 + x^3$	$1 + 3x + 2x^2 + 2x^3 + 2x^4$	$4x^2 + 4x^4 + 2x^5$	0	3
3	$4x + 4x^3$	$1 + 3x + 3x^2 + 3x^3 + 2x^4$	$4x^2 + 3x^3 + 2x^4 + 3x^5$	1	4
4	$4 + 3x + 2x^2 + 3x^3$	$1 + 3x + 3x^2 + 3x^3 + 2x^4$	$4x^3 + 4x^5$	0	4
5	$3 + 3x^2 + 4x^3$	$1 + 3x + 2x^2 + 2x^3 + 2x^4$	$4x^2 + 3x^3 + 2x^4 + 3x^5$	0	3
6	4	$1 + 3x + x^4$	$4x^2 + 4x^4 + 2x^5$	0	4
7	4	$1 + 3x + 4x^4$	$4x^2$	0	4
8	$4 + 4x^2 + 2x^3$	$1 + 3x + x^4$	$4x^2$	0	4

We see $(A^{(1)}, B^{(1)}, C^{(1)}) = (A^{(8)}, B^{(8)}, C^{(8)})$. The output of Algorithm 3.3 is reproduced next:

$$\big((4, 0, 4x + 1)\big), \big((4, 0, 3x + 1), (3, 0, x + 1), (4, 1, 2x^2 + 3x + 1),$$
$$(4, 0, x + 1), (3, 0, 3x + 1), (4, 0, 3x + 1), (4, 0, 3x + 1)\big)^*$$

In view of (3.11), the H-fraction expansion of the power series F is equal to

$$F = \frac{1}{1+4x} + \Big(\frac{4x^2}{1+3x} + \frac{3x^2}{1+x} + \frac{4x^3}{1+3x+2x^2} + \frac{4x^3}{1+x} + \frac{3x^2}{1+3x} + \frac{4x^2}{1+3x} + \frac{4x^2}{1+3x} + \Big)^*.$$

Hence, $m = 1, t = 7$, $(k_i)_{i\geq 0} = (0, (0, 0, 1, 0, 0, 0, 0)^*)$,

$$(s_i)_{i\geq 0} = (0, 1, 2, 3, 5, 6, 7, 8, 9, 10, \ldots),$$

$r = s_{m+t} - s_m = 8$, and $\beta = 4, \gamma_1 = -1, \gamma_2 = 4, \gamma_3 = 1, \gamma = 4, \theta = 2$. So that $2\theta r = 32$ is a period with offset $s_m = 1$. Checking the first $32+1 = 33$ terms, we observe that the least period is equal to 16 with the least offset 0. Hence

$$H(g) = (1, 1, 1, 2, 0, 0, 2, 4, 1, 4, 1, 4, 2, 0, 2, 1, 1)^*.$$

Example (i.2). Let $p = 2$ and

$$F = \frac{1 - \sqrt{1 - \frac{4x}{1-x^4}}}{2x} \in \mathbb{F}_2[[x]] \tag{3.17}$$

or

$$-1 + (1 - x^4)F + (-x + x^5)F^2 = 0. \tag{3.18}$$

By Algorithm 3.3 [HFrac], we get the following H-fraction expansion

$$F = \frac{1}{1+x} + \Big(\frac{x^2}{1} + \frac{x^4}{1} + \frac{x^6}{1} + \frac{x^4}{1} + \frac{x^2}{1} + \frac{x^2}{1} + \Big)^*.$$

Hence, $m = 1, t = 6$, $(k_i)_{i\geq 0} = (0, (0, 2, 2, 0, 0, 0)^*)$,

$$(s_i)_{i\geq 0} = (0, 1, 2, 5, 8, 9, 10, 11, \ldots),$$

$r = s_7 - s_1 = 10$. Since $p = 2$, 10 is a period with offset $s_m = 1$. Checking the first $10+1 = 11$ terms in $H(f)$, which are $(1, 1, 1, 0, 0, 1, 0, 0, 1, 1, 1, \ldots)$, we see that the least period is equal to 10 with offset 0. Finally,

$$H(f) = (1, 1, 1, 0, 0, 1, 0, 0, 1, 1)^*.$$

Example (iii.1). Let $p = 2$ and $G = xF$ where F is defined in Example (i.2) by (3.17) or (3.18). We have

$$G = \frac{1 - \sqrt{1 - \frac{4x}{1-x^4}}}{2} \in \mathbb{F}_2[[x]]$$

and

$$(3.19)\qquad -x+(1-x^4)G+(-1+x^4)G^2=0 \quad \text{with} \quad G(0)=0.$$

Since the coefficient of G^2 has constant term, we cannot apply Algorithm 3.3 directly. Let

$$G=\frac{x}{1+x+x^2+x^3G_1}.$$

Equation (3.19) becomes

$$x^3+(1+x^4)G_1+x^3G_1^2=0.$$

By Algorithm 3.3 [HFrac], we get the following H-fraction expansion

$$G_1=\frac{x^3}{1+x^4}\;+\;\left(\frac{x^6}{1}\;+\;\frac{x^4}{1+x^2}\;+\;\frac{x^4}{1}\;+\;\frac{x^6}{1+x^4}\;+\;\right)^*.$$

Hence

$$G=\frac{x}{1+x+x^2}\;+\;\left(\frac{x^6}{1+x^4}\;+\;\frac{x^6}{1}\;+\;\frac{x^4}{1+x^2}\;+\;\frac{x^4}{1}\;+\;\right)^*.$$

Example (iii.2). Let $p=2$ and

$$G=\frac{1+\sqrt{1-\frac{4x}{1-x^4}}}{2}\in\mathbb{F}_2[[x]].$$

We have

$$(3.20)\qquad -x+(1-x^4)G+(-1+x^4)G^2=0 \quad \text{with} \quad G(0)=1.$$

Since the coefficient of G^2 has constant term, we cannot apply Algorithm 3.3 directly. Let

$$G=\frac{1}{1+x+x^2G_1}.$$

Equation (3.20) becomes

$$(3.21)\qquad (x+x^3)+(1+x^4)G_1+x^3G_1^2=0.$$

By Algorithm 3.3 [HFrac], we get the following H-fraction expansion

$$(3.22)\qquad G_1=\frac{x}{1+x^2}\;+\;\left(\frac{x^4}{1}\;+\;\frac{x^6}{1+x^4}\;+\;\frac{x^6}{1}\;+\;\frac{x^4}{1+x^2}\;+\;\right)^*.$$

Hence

$$G=\frac{1}{1+x}\;+\;\frac{x^3}{1+x^2}\;+\;\left(\frac{x^4}{1}\;+\;\frac{x^6}{1+x^4}\;+\;\frac{x^6}{1}\;+\;\frac{x^4}{1+x^2}\;+\;\right)^*.$$

Example (iv). Let $p = 3$ and

$$F = \sqrt{\frac{x^2 - x^3}{1 + x^3}} \in \mathbb{F}_3[[x]]$$

or

$$(-x^2 + x^3) + (1 + x^3)F^2 = 0. \tag{3.23}$$

Since the coefficient of F is zero, we cannot apply Algorithm 3.3 directly. Let

$$F = \frac{x}{1 + 2x + x^3 F_1}.$$

Equation (3.23) becomes

$$2 + (1 + x + x^2)F_1 + (2x^3 + x^4)F_1^2 = 0.$$

By Algorithm 3.3 [HFrac], we get the following H-fraction expansion

$$F_1 = \frac{1}{1+x} \underset{+}{} \left(\frac{x^2}{1+x} \underset{+}{} \frac{2x^2}{1+x} \underset{+}{} \frac{x^2}{1+x} \underset{+}{} \frac{2x^3}{1+2x} \underset{+}{} \frac{2x^3}{1+x} \underset{+}{} \right)^*.$$

Hence, the H-fraction expansion of F is

$$\frac{x}{1+2x} \underset{+}{} \frac{x^3}{1+x} \underset{+}{} \left(\frac{x^2}{1+x} \underset{+}{} \frac{2x^2}{1+x} \underset{+}{} \frac{x^2}{1+x} \underset{+}{} \frac{2x^3}{1+2x} \underset{+}{} \frac{2x^3}{1+x} \underset{+}{} \right)^*.$$

Algorithm 3.3 and the proof of Theorem 1.1 are also valid for super 1-fractions with $\delta = 1$. Consequently, the following theorem holds.

Theorem 3.5. *Let p be a prime number and let $F(x) \in \mathbb{F}_p[[x]]$ be a power series satisfying the following quadratic equation*

$$A(x) + B(x)F(x) + C(x)F(x)^2 = 0,$$

where $A(x), B(x), C(x) \in \mathbb{F}_p[x]$ are three polynomials with one of the following conditions

(i) $B(0) = 1,\ C(0) = 0,\ C(x) \neq 0$;

(ii) $B(0) = 1,\ C(x) = 0$;

(iii) $B(0) = 1,\ C(0) \neq 0,\ A(0) = 0$;

(iv) $B(x) = 0,\ C(0) = 1,\ A(x) = -(a_k x^k)^2 + O(x^{2k+1})$ for some $k \in \mathbb{N}$ and $a_k \neq 0$ when $p \neq 2$.

Then, the super 1*-fraction expansion of* $F(x)$ *exists and is ultimately periodic.*

For example, take the power series $G_1(x) \in \mathbb{F}_2[[x]]$ defined by (3.21), we have

$$\begin{aligned} G_1 &= \frac{x}{1+x^2} + \left(\frac{x^4}{1} + \frac{x^6}{1+x^4} + \frac{x^6}{1} + \frac{x^4}{1+x^2} + \right)^* \quad [H\text{-fraction}] \\ &= \frac{x}{1} + \left(\frac{x^2}{1} + \frac{x^2}{1} + \frac{x^2}{1} + \frac{x^4}{1} + \frac{x^6}{1} + \frac{x^4}{1} + \right)^*. \quad [\text{Super 1-frac.}] \end{aligned}$$

The converse of Theorem 3.5 and of the first part of Theorem 1.1 is stated next, which can be viewed as a power series analog of Euler's theorem for quadratic irrational number.

Theorem 3.6. *Let* δ *be a nonnegative integer and* $F(x)$ *a power series. If the super* δ*-continued fraction expansion of* $F(x)$ *is ultimately periodic, then* $F(x)$ *satisfies the quadratic equation (1.2).*

Proof. Without lost of generality, we suppose that the super δ-continued fraction of $F(x)$ is ultimately periodic of periodic 3 with offset 3. So that

$$F(x) = \cfrac{v_0 x^{k_0}}{1 + u_1(x)x - \cfrac{v_1 x^{k_0+k_1+\delta}}{1 + u_2(x)x - \cfrac{v_2 x^{k_1+k_2+\delta}}{1 + u_3(x)x - x^{k_2+\delta}G(x)}}}, \tag{3.24}$$

where

$$G(x) = \cfrac{v_3 x^{k_3}}{1 + u_4(x)x - \cfrac{v_4 x^{k_3+k_4+\delta}}{1 + u_5(x)x - \cfrac{v_5 x^{k_4+k_5+\delta}}{1 + u_6(x)x - x^{k_5+\delta}G(x)}}}. \tag{3.25}$$

The right-hand side of the above two equalities is always of form

$$\frac{P(x) + Q(x)\,G(x)}{R(x) + S(x)\,G(x)},$$

where P, Q, R, S are four polynomials. From (3.25) and (3.24), $G(x)$ and $F(x)$ satisfy the quadratic equation (1.2). ∎

Note that the converse of the second part of Theorem 1.1 is not true. For example, the Hankel determinant sequence $H(f)$, where

$$f(x) = \frac{1}{1-x} - \frac{x^2}{1-2x} - \frac{x^2}{1-3x} - \frac{x^2}{1-4x} - \cdots \tag{3.26}$$

is periodic of period 1, but the J-fraction expansion of f is not ultimately periodic.

4. Application to automatic sequences

Proof of Theorem 1.2. Let $f(x) = G_{a,b}(x) \in \mathbb{F}_2[[x]]$. Then

$$\begin{aligned}
x^{2^a} f(x) &= \sum_{n=0}^{\infty} \frac{x^{2^{n+a}}}{1 - x^{2^{n+b}}};\\
x^{2^{a+1}} f(x^2) &= \sum_{n=1}^{\infty} \frac{x^{2^{n+a}}}{1 - x^{2^{n+b}}};\\
x^{2^a} f(x^2) &= f(x) - \frac{1}{1 - x^{2^b}};\\
1 + (1 + x^{2^b}) f(x) + x(1 + x^{2^b}) &x^{2^a-1} f(x)^2 = 0.
\end{aligned}$$

The above equation is of type (1.2). By Theorem 1.1 the Hankel determinant sequence $H(f)$ is ultimately periodic. $\square$

The following corollary is obtained by Algorithm 3.3. The case for the regular paperfolding sequence, i.e., $a = 0, b = 2$, is verified in Section 3, Example (i.2).

Corollary 4.1. *Let $G_{a,b}(x)$ be the power series in $\mathbb{F}_2[[x]]$ defined by (1.3). Over the field $\mathbb{F}_2$ we have*

$$\begin{aligned}
&H(G_{0,0}) = (1)^*; && \text{[APWW 1998; Coons 2013]}\\
&H(G_{0,1}) = 1, 1, (0)^*;\\
&H(G_{1,0}) = (1)^*;\\
&H(G_{0,2}) = (1, 1, 1, 0, 0, 1, 0, 0, 1, 1)^*; && \text{[Guo-Wu-Wen 2013]}\\
&H(G_{1,1}) = (1, 1, 0, 0, 1, 1)^*;\\
&H(G_{2,0}) = (1, 1, 0, 0)^*;\\
&H(G_{0,3}) = (1^5 0^2 1^1 0^6 1^3 0^2 1^2 0^2 1^2 0^4 1^1 0^4 1^1 0^2 1^1 0^2 1^1\\
&\qquad\qquad 0^4 1^1 0^4 1^2 0^2 1^2 0^2 1^3 0^6 1^1 0^2 1^4)^*; && \text{[least period is 74]}\\
&H(G_{1,2}) = 1, 1, 1, (0)^*;\\
&H(G_{2,1}) = (1, 1, 1, 1, 1, 1, 0, 0)^*;\\
&H(G_{3,0}) = (1, 1, 0, 0, 0, 0, 0, 0)^*;\\
&H(G_{0,4}) = (1^9 0^2 1^1 0^2 \cdots 1^1 0^2 1^8)^*; && \text{[least period is 1078]}\\
&H(G_{1,3}) = (1, 1, 1, 1, 1, 0, 0, 0, 0, 1, 1, 0, 0, 0, 0, 1, 1, 1, 1, 1)^*;\\
&H(G_{2,2}) = (1, 1, 0, 0, 0, 0, 0, 0, 1, 1, 0, 0)^*;\\
&H(G_{3,1}) = (1, 1, 1, 0, 0, 0, 0, 1, 1, 1, 0, 0, 0, 0, 0, 0)^*;\\
&H(G_{4,0}) = (1, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0)^*.
\end{aligned}$$

Proof of Proposition 1.3. Let $f(x)=\sum_{n\geq 0} u_n x^n \in \mathbb{F}_2[[x]]$ where (u_n) is the Rudin-Shapiro sequence defined by (1.4). Then

$$x^3+(1+x)^4 f(x)+(1+x)^5 f^2(x)=0. \tag{4.1}$$

Since $u_0=u_1=u_2=0$, the following divisions yield power series:

$$f_1(x)=f(x)/x;\quad f_2(x)=f(x)/x^2;\quad f_3(x)=f(x)/x^3.$$

From (4.1) we derive

$$\begin{aligned} x^2+(1+x)^4 f_1(x)+(1+x)^5 x f_1^2(x)=0;\\ x+(1+x)^4 f_2(x)+(1+x)^5 x^2 f_2^2(x)=0;\\ 1+(1+x)^4 f_3(x)+(1+x)^5 x^3 f_3^2(x)=0. \end{aligned}$$

From Algorithm 3.3 we obtain the following H-fractions for f_1, f_2, f_3:

$$\begin{aligned} f_1 &= \frac{x^2}{1+x^3} + \Big(\frac{x^6}{1+x^2} + \frac{x^6}{1} + \frac{x^4}{1} + \frac{x^2}{1} + \frac{x^6}{1+x^4} \\ &\qquad + \frac{x^6}{1} + \frac{x^2}{1} + \frac{x^4}{1} + \Big)^*;\\ f_2 &= \frac{x}{1} + \Big(\frac{x^3}{1} + \frac{x^3}{1+x^2} + \frac{x^3}{1} + \frac{x^3}{1} + \frac{x^3}{1+x} + \frac{x^2}{1} \\ &\qquad + \frac{x^2}{1+x} + \frac{x^5}{1+x^4} + \frac{x^5}{1+x} + \frac{x^2}{1} + \frac{x^2}{1+x} + \frac{x^3}{1} + \Big)^*;\\ f_3 &= \frac{1}{1} + \Big(\frac{x^3}{1} + \frac{x^3}{1} + \frac{x^2}{1+x} + \frac{x^2}{1+x} + \frac{x^2}{1} + \frac{x^2}{1+x} + \frac{x^3}{1} \\ &\qquad + \frac{x^3}{1+x} + \frac{x^4}{1} + \frac{x^5}{1+x+x^2} + \frac{x^4}{1+x+x^2} + \frac{x^3}{1} + \Big)^*. \end{aligned}$$

The Hankel determinants $H(f_1), H(f_2), H(f_3)$ given in Proposition 1.3 follow from the above H-fractions. For the unshifted Rudin-Shapiro sequence f, we have

$$A=x^3,\ B=(1+x)^4,\ C=(1+x)^5.$$

Since $B(0)=C(0)=1$, we are in the case (iii) in the proof of Theorem 1.1. Let

$$f=\frac{x^3}{1+x^3+x^5 g}.$$

Equation (4.1) becomes

$$(x+x^3)+(1+x^4)g+x^5g^2=0.$$

By Algorithm 3.3 [HFrac], we get the following H-fraction expansion for g

$$g = \frac{x}{1+x^2} + \left(\frac{x^6}{1+x^4} + \frac{x^6}{1+x^2} + \right)^*.$$

Hence,

$$f = \frac{x^3}{1+x^3} + \left(\frac{x^6}{1+x^2} + \frac{x^6}{1+x^4} + \right)^*.$$

The Hankel determinants $H(f)$ given in Proposition 1.3 follow from the above H-fraction of f, or Lemma 2.2. $\square$

Let k be a positive integer and $f_k(x)$ the generating function of the k-shifted Rudin-Shapiro sequence. Then, the Hankel determinant sequence of $f_k(x)$ is periodic of least period $r(k)$. The first values of $r(k)$ are listed next.

k	:	0	1	2	3	4	5	6	7	8	9	10	$\cdots$
$r(k)$	:	6	18	18	18	18	36	36	36	36	72	72	$\cdots$

Conjecture 4.2. *Let $k \geq 3$. The least period of the Hankel determinant sequence of $f_k(x)$ is equal to $r(k) = 9 \times 2^m$, where m is the positive integer satisfying $2^m < k \leq 2^{m+1}$.*

Conjecture 4.2 has been checked by computer for k up to 129.

Proof of Proposition 1.4. It is well known that [BV13]

$$S(x) = (1+x+x^2)S(x^2) \in \mathbb{Q}[[x]]. \tag{4.2}$$

Since $S(x) \pmod 2$ is rational, there exists a positive integer N such that $H_k(S) \equiv 0 \pmod 2$ for all $k \geq N$. We will use the grafting technique, introduced in [Ha15, Section 2]. First, the H-fraction of $S(x)$ is

$$S(x) = \frac{1}{1-x} - \frac{x^2}{1+2x} + \frac{2x^2}{1} - \frac{2x^3}{1-\frac{3}{2}x+\frac{11}{4}x^2} + \cdots.$$

The even number 2 occurs in the sequence (v_j), in particular at position v_2 in view of (2.1). Define $G(x)$ by

$$S(x) = \cfrac{1}{1 - x - \cfrac{x^2}{1+2x+2x^2G(x)}}.$$

From (4.2) the power series $G(x)$ satisfies the following relation

$$(1+x+x^2) + (1+x+x^2)G(x) + x^4G(x^2) \equiv 0 \pmod 2.$$

By Algorithm 3.3 we get $H(G) \equiv (1,1,0,0)^* \pmod 2$. By Lemma 2.2, $H_n(S) = (-1)^n 2^{n-2} H_{n-2}(G)$. Hence

$$H_n(S)/2^{n-2} \equiv (0,0,1,1)^* \pmod 2.$$

In the same manner, $B(x)$ is a rational function modulo 2. We use the grafting technique. Since [BV13]

$$B(x) = 2 - (1+x+x^2)B(x^2) \tag{4.3}$$

and

$$B(x) = \frac{1}{1+x} \; + \; \frac{x^2}{1} \; + \; \frac{2x^2}{1-2x} \; + \; \frac{x^4}{\ddots}$$

we define $U(x)$ by

$$B(x) = \cfrac{1}{1+x+\cfrac{x^2}{1+2x^2U(x)}}.$$

From (4.3) the power series $U(x)$ satisfies the following functional equation

$$(1+x+x^2) + (1+x+x^2)U(x) + x^4U(x^2) \equiv 0 \pmod 2.$$

By Algorithm 3.3 we get $H(U) = (1,1,0,0)^* \pmod 2$. On the other hand, the Hankel determinant $H_n(B) = -2^{n-2}H_{n-2}(U)$ by Lemma 2.2. Hence, $H_n(B)/2^{n-2} \equiv (0,0,1,1)^* \pmod 2$. □

5. Some conjectures about Hankel determinant sequences

The Hankel determinants associated with a given power series $f(x)$ are always nonzero if and only if the Jacobi fraction of $f(x)$ exists. This non-vanishing property can be used for studying irrationality exponents in Number Theory. Even some progress is made in the present paper together with two previous papers [Ha15, FH16], the problem is still very hard to solve in the general case. We reproduce several conjectures on this topic, comparing with known results. Let

$$P_1 = 3\prod_{n=1}^{\infty}(1-x^{3^n}) - \frac{2}{1-x}, \qquad C_2 = 3\prod_{n=1}^{\infty}(1+x^{3^n}) - \frac{2}{1-x},$$

$$P_3 = \frac{1}{x}\sum_{n=0}^{\infty}\frac{x^{2^n}}{1-x^{2^n}}, \qquad C_4 = 1 + \sum_{n=0}^{\infty}\frac{x^{2^n}}{1-x^{2^n}},$$

$$F_5 = \frac{1}{x}\sum_{n=0}^{\infty}\frac{x^{2^n}}{1-x^{2^{n+2}}},$$

$$P_7 = \prod_{n\geq 0}(1 - x^{5^n} - x^{2\cdot 5^n} - x^{3\cdot 5^n} + x^{4\cdot 5^n}),$$
$$C_8 = \prod_{n\geq 0}(1 - x^{6^n} - x^{2\cdot 6^n} - x^{3\cdot 6^n} + x^{4\cdot 6^n} - x^{5\cdot 6^n}),$$
$$C_9 = \prod_{n\geq 0}(1 - x^n)^3 = \sum_{n=0}^{\infty}(-1)^n(2n+1)x^{n(n+1)/2},$$
$$C_{10} = \sum_{n=0}^{\infty}(-1)^n x^{n(n+1)/2}.$$

Proposition 5.1 [Ha15, Proposition 2.5]**.** $H_k(P_1) \neq 0$ *for all* k*.*

Conjecture 5.2. $H_k(C_2) \neq 0$ *for all* k*.*

Proposition 5.3 [Corollary 4.1; Co13, APWW]**.** $H_k(P_3) \neq 0$ *for all* k*.*

Conjecture 5.4. $H_k(C_4) \neq 0$ *for all* k*.*

Proposition 5.5 [Corollary 4.1; GWW]**.** $H(F_5) \pmod 2$ *is periodic.*

Conjecture 5.6. $H(F_5) \pmod p$ *is not periodic for prime integer* $p \geq 3$*.*

Proposition 5.7 [FH16, Theorem 1.2]**.** $H_k(P_7) \neq 0$ *for all* k*.*

Conjecture 5.8. $H_k(C_8) \neq 0$ *for all* k*.*

Conjecture 5.9. $H_k(C_9) \neq 0$ *for all* k*.*

Conjecture 5.10. $H_k(C_{10}) \neq 0$ *for all* k*.*

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References

[Al87] Allouche, Jean-Paul. — Automates finis en théorie des nombres, *Expo. Math.*, **5** (1987), pp. 239–266.

[Al12] Allouche, Jean-Paul. — On the Stern sequence and its twisted version, *Integers*, **12** (2012), A58.

[APWW] Allouche, Jean-Paul; Peyrière, Jacques; Wen, Zhi-Xiong; Wen, Zhi-Ying. — Hankel determinants of the Thue-Morse sequence, *Ann. Inst. Fourier, Grenoble*, **48** (1998), pp. 1–27.

[Ba10] Bacher, Roland. — Twisting the Stern sequence, arxiv.org/abs/1005.5627, 2010, 19 pages.
[BH14] Bugeaud, Yann; Han, Guo-Niu. — A combinatorial proof of the non-vanishing of Hankel determinants of the Thue–Morse sequence, *Electronic J. Combinatorics*, **21(3)** (2014), pp. 16 pages.
[BHWY] Bugeaud, Yann; Han, Guo-Niu; Wen, Zhi-Ying; Yao, Jia-Yan. — Hankel Determinants, Padé Approximations, and Irrationality Exponents, *International Mathematics Research Notices (IMRN)*, **5** (2016), pp. 1467–1496.
[Bu10] Buslaev, V.I. — On Hankel determinants of functions given by their expansions in P-fractions, *Ukrainian Math. J.*, **62** (2010), pp. 358–372.
[Bu11] Bugeaud, Yann. — On the rational approximation to the Thue-Morse-Mahler numbers, *Ann. Inst. Fourier, Grenoble*, **61** (2011), pp. 2065–2076.
[BV13] Bundschuh, Peter; Vaananen, Keijo. — Algebraic independence of the generating functions of Stern's sequence and of its twist, *J. Théorie des Nombres de Bordeaux*, **25** (2013), pp. 43–57.
[Ci13] Cigler, Johann. — A special class of Hankel determinants, arXiv:1302.4235 [math.CO], 2013, 26 pages.
[CV12] Coons, Michael; Vrbik, Paul. — An irrationality measure for regular paperfolding numbers, *Journal of Integer Sequences*, **15** (2012), pp. Article 12.1.6.
[Co13] Coons, Michael. — On the rational approximation of the sum of the reciprocals of the Fermat numbers, *The Ramanujan Journal*, **30** (2013), pp. 39–65.
[Fl80] Flajolet, Philippe. — Combinatorial aspects of continued fractions, *Discrete Math.*, **32** (1980), pp. 125–161.
[FH16] Fu, Hao; Han, Guo-Niu. — Computer assisted proof for Apwenian sequences, *ISSAC 2016 Conference, Waterloo, Ontario, Canada*, 2016, 8×2 pages.
[GWW] Guo, Ying-Jun; Wu, Wen; Wen, Zhi-Xiong. — On the irrationality exponent of the regular paperfolding numbers, *Linear Algebra Appl.*, **446** (2014), pp. 237–264.
[Ha15] Han, Guo-Niu. — Hankel Determinant Calculus for the Thue-Morse and related sequences, *J. Number Theory*, **147** (2015), pp. 374-395.
[Hal13] Halter-Koch, Franz. — *Quadratic Irrationals: An Introduction to Classical Number Theory*, CRC Press, 2013.
[Kr98] Krattenthaler, Christian. — Advanced determinant calculus, *Sém. Lothar. Combin.*, **B42q** (1998), 67pp.
[Kr05] Krattenthaler, Christian. — Advanced determinant calculus: A complement, *Linear Algebra and its Applications*, **411** (2005), pp. 68–166.
[St58] Stern, M.A.. — Über eine zahlentheoretische Funktion, *J. Reine Angew. Math*, **55** (1858), pp. 193–220.
[Vi83] Viennot, Xavier. — Une théorie combinatoire des polynômes orthogonaux généraux, *UQAM, Montreal, Quebec*, 1983.
[Wa48] Wall, H. S. — *Analytic theory of continued fractions*, Chelsea publishing company, Bronx, N.Y., 1948.

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