# THE ENTRINGER-POUPARD MATRIX SEQUENCE

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Abstract. The so-called Entringer-Poupard matrices naturally occur when the distribution of the statistical pair ("last letter", "greater neighbor of maximum") is under study on the set of alternating permutations. They also provide a matrix refinement of the tangent/secant numbers. Moreover, their generating function can be explicitly derived.

## 1. Introduction

The aim of this paper is to construct a well-defined sequence of matrices $(A_n = (a_n(k,\ell))_{(1\le k,\ell\le n)})$ $(n\ge 1)$ with integral entries, called the *Entringer-Poupard matrix sequence*, which provides a *matrix refinement* $\sum_{k,\ell} a_n(k,\ell) = E_n$ of the *tangent and secant numbers*, in such a way that the row and column sums $\sum_\ell a_n(k,\ell)$ and $\sum_k a_n(k,\ell)$ are themselves *Entringer* and *Poupard numbers*, respectively. The sequence $(A_n)$ is defined by a system of *partial finite difference equations* and, moreover, the generating function for the entries $a_n(k,\ell)$ of the matrices $A_n$ can be explicitly evaluated.

This characterization of the Entringer-Poupard matrix sequence completes the program initiated in our previous papers, where matrix refinements of the tangent and secant numbers have been found having the property that *both* row and column sums were equal to Poupard numbers as in [FH14] and [FH14b], and to Entringer numbers as done in [FH14c] and [FH15]. There remains to say something relevant when both Entringer and Poupard numbers are involved.

1.1. **Tangent and Secant numbers; Entringer and Poupard numbers.** The classical Euler numbers $(E_n)_{\ge 0}$ are the (integer) coefficients in exponential series expansion of the tangent resp. the secant function, viz.

$$\tan u = \sum_{n\ge 0} \frac{u^{2n+1}}{(2n+1)!} E_{2n+1} = \frac{u}{1!}1 + \frac{u^3}{3!}2 + \frac{u^5}{5!}16 + \frac{u^7}{7!}272 + \frac{u^9}{9!}7936 + \cdots,$$

$$\sec u = \sum_{n\ge 0} \frac{u^{2n}}{(2n)!} E_{2n} = 1 + \frac{u^2}{2!}1 + \frac{u^4}{4!}5 + \frac{u^6}{6!}61 + \frac{u^8}{8!}1385 + \frac{u^{10}}{10!}50521 + \cdots,$$

see e.g. [Nie23] (p. 177–178) or [Com74] (p. 258–259) for the expansions, sequences A000182 and A000364 of the Sloane's Encyclopedia [OF15] for tables and more information.

The *Entringer numbers* $E_n(k)$ $(1 \le k \le n)$ are traditionally defined by a *first*-order difference equation system. See, e.g., Sloane's Encyclopedia of integers

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[OF15], where they are registered as the A008282 sequence. The *Poupard numbers* $P_n(k)$ $(1 \le k \le n-1)$ are registered as the A236934 and A125053 sequences, respectively, in that Encyclopedia. A full study of those latter two sequences was made in our previous paper [FH13]. With $\Delta$ standing for the classical finite difference operator $\Delta E_n(k) := E_n(k+1) - E_n(k)$, their definitions can be stated as follows:

$$\Delta^2 E_n(k) + E_{n-2}(k) = 0 \quad (1 \le k \le n-2)$$

with the initial conditions:

$$E_1(1) = 1; \quad E_n(1) = E_n(2) = \sum_k E_{n-1}(k) \quad (n \text{ odd} \ge 3);$$

$$E_n(1) = 0 \quad n \text{ (even)}; \quad E_2(2) = 1; \quad E_n(2) = \sum_k E_{n-2}(k) \quad (n \text{ even} \ge 4);$$

and then by

$$\Delta^2 P_n(k) + 4\, P_{n-2}(k) = 0 \quad (1 \le k \le n-3)$$

with the initial conditions:

$$P_3(1) = 0; \qquad P_3(2) = 2; \quad P_n(1) = 0, \quad P_n(2) = 2\sum_k P_{n-2}(k) \quad (n \text{ odd} \ge 5);$$

$$P_2(1) = 1; \quad P_n(1) = \sum_k P_{n-2}(k), \quad P_n(2) = 3\sum_k P_{n-2}(k) \quad (n \text{ even} \ge 4).$$

These numbers provide *linear refinements* of tangent and secant numbers: $E_{2n+1} = \sum_k E_{2n+1}(k) = \sum_k P_{2n+1}(k)$ and $E_{2n} = \sum_k E_{2n}(k) = \sum_k P_{2n}(k)$, as can be seen in Table 1 where the row sums have been written between triangles. Those linear refinements have been proved *combinatorially* by Entringer himself [Ent66], for the Entringer numbers, and by Christiane Poupard [Pou82] for the Poupard numbers, only for $E_{2n+1}$, the even case being completed in our previous paper [FH13]. The numbers $E_{2n+1}(k)$ ($E_{2n}(k)$, resp.) will be referred to as *Entringer tangent numbers* (*Entringer secant numbers*, resp.). Likewise, $P_{2n+1}(k)$ ($P_{2n}(k)$, resp.) will be called *Poupard tangent numbers* (*Poupard secant numbers*). The first values of these numbers are reproduced in Table 1.

| $n$ | $E_n(k)$ | 1 | 2 | 3 | 4 | 5 | 6 | $E_n$ | $P_n(k)$ | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 2 | | 0 | 1 | | | | | 1 | | 1 | | | | |
| 3 | | 1 | 1 | 0 | | | | 2 | | 0 | 2 | | | |
| 4 | | 0 | 1 | 2 | 2 | | | 5 | | 1 | 3 | 1 | | |
| 5 | | 5 | 5 | 4 | 2 | 0 | | 16 | | 0 | 4 | 8 | 4 | |
| 6 | | 0 | 5 | 10 | 14 | 16 | 16 | 61 | | 5 | 15 | 21 | 15 | 5 |

Table 1. Entringer and Poupard numbers

In Sec. 4 the rows of these arrays will be met again as anti-diagonals in particular infinite skew Seidel matrices (to be defined later).

1.2. **Alternating permutations.** To generate a sequence $(A_n = (a_n(k,\ell))$ of matrices, with nonnegative entries, whose row sums (resp. column sums) are equal to Entringer (resp. Poupard) numbers, it suffices to get

(1) a specific sequence $(\mathfrak{E}_n)$ $(n \ge 1)$ of finite sets such that $\#\mathfrak{E}_n = E_n$;

(2) two statistics $\mathrm{stat}_1$, $\mathrm{stat}_2$, defined on each set $\mathfrak{E}_n$ having the property that $\#\{w \in \mathfrak{E}_n : \mathrm{stat}_1\, w = k\} = E_n(k)$ and $\#\{w \in \mathfrak{E}_n : \mathrm{stat}_2\, w = \ell\} = P_n(\ell)$ for all $k, \ell$.

(3) then define $a_n(k,\ell) := \#\{w \in \mathfrak{E}_n : \mathrm{stat}_1(w) = k, \mathrm{stat}_2(w) = \ell\}$ and make sure that the generating function for the matrices $A_n = (a_n(k,\ell))$ thereby constructed has an interesting *closed form*.

This program is achieved by taking the old Désiré André's model $\mathrm{Alt}_n$ of *alternating permutations* [And79], [And81] as $\mathfrak{E}_n$, together with the two statistics $\mathbf{L}$ ("last letter") and $\mathbf{grn}$ ("**gr**eater **n**eighbor of maximum"). Recall that a permutation $w = w_1w_2\dots w_n$ of the sequence $12\dots n$ is said to be *alternating* if $w_1 < w_2 > w_3 < w_4 > \dots > w_{2k-1} < w_{2k} > w_{2k+1} < \dots w_n$, the set of these alternating permutations of $12\dots n$ being denoted by $\mathrm{Alt}_n$. What Désiré André proved is that $\#\mathrm{Alt}_n = E_n$ for all $n$. This old model is still being studied from various perspectives, see [Arn91], [GHZ11], [KPP94], [MSY96], [Sta10].

Next, the *last letter* $\mathbf{L}(w)$ of a permutation $w = w_1w_2\dots w_n$ of $12\dots n$ is simply its rightmost letter $w_n$. Finally, $\mathbf{grn}(w)$ is defined as follows: let $j$ be the position of $n$ in $w$, i.e., $w_j = n$, then $\mathbf{grn}(w) := \max\{w_{j-1}, w_{j+1}\}$, with the convention that $w_0 = 0 = w_{n+1}$.

In the sequel, we use the fact that $\mathbf{L}$ (resp. $\mathbf{grn}$) has the *Entringer* (resp. *Poupard*) *distribution* on each $\mathrm{Alt}_n$, i.e.,

$$E_n(k) = \#\{w \in \mathrm{Alt}_n\,;\, \mathbf{L}(w) = k\};$$
$$P_n(k) = \#\{w \in \mathrm{Alt}_n\,;\, \mathbf{grn}(w) = k\}.$$

The first identity is due to Entringer himself [Ent66], the second one was proved in [FH13]. The purpose of the work presented here is then the study of the *joint distribution* of $(\mathbf{L}, \mathbf{grn})$ on $\mathrm{Alt}_n$. For any fixed $n \geq 2$, the numerical information about this distribution will consequently be stored in a matrix $A_n$ of size $n \times n$, which we call *Entringer-Poupard* matrix.

As explained in the next section, our purpose will be to compute the generating function for these matrices in an appropriate manner. Similar computations have been made in our previous papers, first using the algebra of the so-called *Poupard matrices* in [FH14b], [FH14c], then that of *Seidel triangles* in [FH14], [FH15]. The core of the present paper lies in a new approach to the Seidel matrices, together with their skew variants, that naturally leads to an easy calculation of the generating function for sequences of the matrices $A_n$.

1.3. **The Entringer-Poupard Matrix Sequence.** We then define a sequence $(A_n)_{n\geq 2}$ of matrices with non-negative integer entries, $A_n = \big(a_n(k,\ell)\big)_{1\leq k,\ell\leq n}$, with coefficients given by

$$a_n(k,\ell) = \sharp\{w \in \mathrm{Alt}_n\,;\, \mathbf{grn}(w) = k, \mathbf{L}(w) = \ell\} \qquad (1 \leq k,\ell \leq n).$$

As an example, we display $A_6$, together with row sums (the $\mathbf{grn}$ statistics) and column sums (the $\mathbf{L}$ statistics) in Table 2. The Entringer-Poupard matrices $A_n$ for $2 \leq n \leq 9$ are reproduced, together with row and column sums, in the Appendix.

Our goal is to globally encode the numerical information contained in the *Entringer-Poupard Matrix Sequence* $\big(A_n\big)_{n\geq 2}$ in generating functions that, among others, allow for a rapid computation of the values $a_n(k,\ell)$ of the joint distribution. For that purpose we partition each matrix $A_n$ into three parts:

- The *upper triangular part* (or *upper triangle*, for short) $\mathrm{Up}_n$, consisting of the positions $(k,\ell)$ above the main diagonal, but excluding the last column, i.e., $1 \leq k < \ell < n$.

$$A_6 = \begin{array}{c|cccccc|c} & 1 & 2 & 3 & 4 & 5 & 6 & \\ \hline 1 & 0 & 0 & 0 & 0 & 0 & 5 & 5 \\ 2 & 0 & 0 & 2 & 4 & 4 & 5 & 15 \\ 3 & 0 & 1 & 0 & 8 & 8 & 4 & 21 \\ 4 & 0 & 3 & 6 & 0 & 4 & 2 & 15 \\ 5 & 0 & 1 & 2 & 2 & 0 & 0 & 5 \\ 6 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ & & & & & & & \\ & 0 & 5 & 10 & 14 & 16 & 16 & \end{array}$$

Table 2. Entringer-Poupard matrix $A_6$

- The *lower triangular part* (or *lower triangle*, for short) $\mathrm{Low}_n$, consisting of the positions $(k, \ell)$ below the main diagonal, but excluding the last row, i.e., $1 \le \ell < k < n$.
- The *diagonal part* (or *diagonals*, for short) $\mathrm{Diag}_n$, consisting of the positions $(k, \ell)$ on the main diagonal, plus the last row and the last column, i.e. $1 \le k = \ell \le n$ or $1 \le \ell \le k = n$ or $1 \le k \le \ell = n$.

Note that $\mathrm{Up}_2$ and $\mathrm{Low}_2$ are empty. Furthermore, the last row of $A_n$ is always a zero row, so that only the elements of the main diagonal and of the last column matter. Furthermore: for $n$ even the main diagonal has only zero entries, for $n$ odd the last column has only zero entries.

This segmentation is indicated by the coloring the entries of the examples in the Appendix: *red* for $\mathrm{Up}_n$, *blue* for $\mathrm{Low}_n$, *black* for $\mathrm{Diag}_n$.

To each of the three sequences $\big(\mathrm{Up}_n\big)_{n\ge2}$, $\big(\mathrm{Low}_n\big)_{n\ge2}$, $\big(\mathrm{Diag}_n\big)_{n\ge2}$ will be defined a generating function:

- For the upper triangles:

$$\Upsilon(x,y,z) = \sum_{n\ge3} \sum_{(k,\ell)\in\mathrm{Up}_n} a_n(k,\ell)\, M(n-\ell-1, \ell-k-1, k-1). \tag{1.1}$$

- For the lower triangles:

$$\Lambda(x,y,z) = \sum_{n\ge3} \sum_{(k,\ell)\in\mathrm{Low}_n} a_n(k,\ell)\, M(\ell-1, k-\ell-1, n-k-1). \tag{1.2}$$

- For the diagonals:

$$\Delta(x,y,z) = \sum_{n\ge2} \sum_{(k,\ell)\in\mathrm{Diag}_n} a_n(k,\ell)\, M(k-1, n-k-1, 0), \tag{1.3}$$

  where the $M(a,b,c)$ are the egf monomials

$$M(a,b,c) = \frac{x^a}{a!}\,\frac{y^b}{b!}\,\frac{z^c}{c!}.$$

  The rather weird looking choice of the monomials will find a natural explanation later on (Subsection 7.1), but note for the moment that by the case distinction the actually occurring exponents are always non-negative integers.

This allows as to spell out our main result.

**Theorem 1.4.** *The three generating functions attached to the Entringer-Poupard Matrix Sequence are*

$$\Upsilon(x,y,z) = \frac{(\sin x + \cos x)\sin(2z)}{\cos^2(x+y+z)}, \tag{1.5}$$

$$\Lambda(x,y,z) = \frac{(\sin x + \cos x)\cos(x+y-z)}{\cos^2(x+y+z)}, \tag{1.6}$$

$$\Delta(x,y) = \frac{\sin x + \cos x}{\cos(x+y)}. \tag{1.7}$$

1.4. **Outline.** The article is organized as follows. In Sec. 2 we fix some notation for matrices and generating functions, and then show how to compute exponential anti-diagonal generating functions. The important (and very classical) concept of *Seidel matrices* is introduced, together with a particular *skew* variant, in Sec 3. Also generating functions for infinite sequences of Seidel matrices are discussed. Seidel matrices containing the Entringer and Poupard statistics along their even anti-diagonals are presented in Sec. 4. Particular properties (special values and difference schemes) of the Entringer-Poupard matrices are obtained from their combinatorial definition in Sec. 5. From the segmentation of these matrices into upper and lower triangles and diagonals we get two sequences of skew Seidel matrices (for "lower" and "upper") in Sec. 6, which then give rise in Sec. 7 to the two generating functions (1.5) and (1.6) by using the results from Sec. 3.2, whereas the "diagonal" generating function (1.7) can be directly computed from the information given in Sec 5. This concludes the proof of our main result. Finally, in Sec. 8 we show how to recover the Poupard statistics (row sums of the Entringer-Poupard matrices) from the three generating functions of Thm. 1.4 without referring to the combinatorial definition. In the Appendix we display the Entringer-Poupard matrices $A_n$ for $2 \le n \le 9$, together with Entringer numbers (column sums) and Poupard number (row sums).

## 2. Notation

We will identify any infinite matrix $M = (m_{i,j})_{i,j\ge 0}$ (of complex numbers, say) with its corresponding exponential generating function (*egf*)

$$M(x,y) = \sum_{i,j\ge 0} m_{i,j} \frac{x^j}{i!}\frac{y^j}{j!}.$$

Furthermore, if $\mathcal{M} = \left(M^{(n)}\right)_{n\ge 0}$ is an infinite sequence of matrices, then its egf is denoted by

$$\mathcal{M}(x,y,z) = \sum_{n\ge 0} M^{(n)}(x,y)\frac{z^n}{n!} = \sum_{n,i,j\ge 0} m_{i,j}^{(n)} \frac{x^i}{i!}\frac{y^j}{j!}\frac{z^n}{n!}.$$

The *even* resp. *odd* positions of a matrix correspond to the terms of even resp. odd total degree in the corresponding generating function. The first row (resp. first column) of a matrix $M$ can then identified with $M(0,y)$ (resp. $M(x,0)$).

The $n$-th *anti-diagonal sum* of $M$, viz. $\widehat{M}_n = \sum_{i+j=n} m_{i,j}$, sums the coefficients of the terms of total degree $n$ in $M(x,y)$. The sequence of anti-diagonal sums of $M$ is $\widehat{M} = (\widehat{M}_n)_{n\ge 0}$, with exponential generating function

$$\widehat{M}(z) = \sum_{n\ge 0} \widehat{M}_n \frac{z^n}{n!} = \sum_{n\ge 0}\sum_{i+j=n} m_{i,j}\frac{z^n}{n!}.$$

$\widehat{M}(z)$ is not obtained by a simple specialization of $M(x, y)$, as it would be the case for *ordinary* generating functions (*ogf*). One may circumvent this obstacle by passing between egf's and ogf's via (formal) Laplace transform. A simpler method, which is convenient for the current work, is as follows.

**Lemma 2.1.** *For any matrix $M = (m_{i,j})_{i,j\geq 0}$ with egf $M(x, y)$,*

$$\widehat{M}(z) = \frac{d}{dz}\mu(z), \qquad \textit{where} \qquad \mu(z) = \int_0^z M(z-y, y)\, dy,$$

*or, equivalently,*

$$\widehat{M}(z) = M(0, z) + \int_0^z (\partial_x M)(z-y, y)\, dy.$$

*Proof.* This could be done by using (formal) Laplace transform. It is even easier to invoke the classical $\beta$-integral. By linearity, it suffices to consider (for fixed $i, j \geq 0$) the term $\delta_{i,j}(x, y) = x^i y^j$. Then

$$\begin{aligned}\int_0^z \delta_{i,j}(z-y, y)\, dz = \int_0^z (z-y)^i y^j\, dy = \int_0^1 (z-zy)^i (zy)^j z\, dy \\ = B(i+1, j+1)\, z^{i+j+1} = \frac{i!\, j!}{(i+j+1)!} z^{i+j+1}.\end{aligned}$$

□

A mild, but useful extension is

**Lemma 2.2.** *If $M(x, y) = g(x+y)\cdot k(x, y)$, then, with $\kappa(z) = \int_0^z k(z-y, y)\, dy$,*

$$\widehat{M}(z) = \frac{d}{dz}\Big(g(z)\cdot\kappa(z)\Big).$$

This is of interest below, because factors like $g(x+y)$, which behave like *scalars* w.r.t. the differential operator $\partial_x - \partial_y$ and its powers, play an important role for Seidel matrices.

## 3. Seidel matrices and sequences of Seidel matrices

3.1. **Seidel matrices and skew Seidel matrices.** Seidel matrices are a classical tool in enumerative combinatorics in particular, and for certain transformations of generating functions in general, where they go back to Euler and Leibniz, although their nomination refers to Seidel [Sei77], see Dumont's survey [Dum82] for many examples and references, or the Wikipedia entry [Wik16] for their importance for computing with Bernoulli numbers.

**Lemma 3.1.** *(and Definition)*
*For a matrix $M = (m_{i,j})_{i,j\geq 0}$ the following assertions are (easily seen to be) equivalent:*

(1) $m_{i+1,j} - m_{i,j+1} = m_{i,j}$ $(i, j \geq 0)$
(2) $(\partial_x - \partial_y)M(x, y) = M(x, y)$
(3) $M(x, y) = e^x M(0, x+y)$
(4) $M(x, y) = e^{-y} M(x+y, 0)$

*A matrix satisfying these conditions is called a* Seidel matrix.

For the present work we continue to relate matrices to bivariate *exponential* generating functions. But this is only for convenience. One could use *ordinary* generating functions as well. Indeed, Euler already states that if $a(t) = \sum_k m_{0,\ell} t^\ell$ is the ogf of the first row of a Seidel matrix $M$, then $\overline{a}(t) = a(t/(1-t))/(1-t) = \sum_k m_{k,0} t^k$ is the ogf of the first column. In this way Seidel matrices were popular for transformations of series expansions. In Seidel [Sei77], see also [Dum82] and [Wik16], one finds the classical example of the array

$$\begin{array}{rrrrrrrl}
1 & -1 & 0 & 2 & 0 & -16 & 0 & \dots \\
0 & -1 & 2 & 2 & -16 & -16 & \ddots & \\
-1 & 1 & 4 & -14 & -32 & \ddots & & \\
0 & 5 & -10 & -64 & \ddots & & & \\
5 & -5 & -56 & \ddots & & & & \\
0 & -61 & \ddots & & & & & \\
-61 & \ddots & & & & & & \\
\vdots & & & & & & &
\end{array}$$

which shows how to obtain the secant coefficients from the tangent coefficients (and vice versa) using a simple difference scheme. Particular *skew* Seidel matrices (see the following definition) are given in subsequent sections. They could be turned into Seidel matrices by Remark 3.3. Indeed, the example just given is equivalent to Ex. 4.3.

In the sequel we will use a variant of Seidel matrices, not really more general, but handy enough to merit a separate naming.

**Lemma 3.2.** *(and Definition)*
*For a matrix $M = (m_{i,j})_{i,j\geq 0}$ the following assertions are equivalent:*

(1) $m_{i+1,j} - m_{i,j+1} = (-1)^{i+j} m_{i,j}$ $(i, j \geq 0)$
(2) $(\partial_x - \partial_y)M(x, y) = M(-x, -y)$
(3) $M(x, y) = \cos(x)M(0, x + y) + \sin(x)M(0, -x - y)$
(4) $M(x, y) = \cos(y)M(x + y, 0) - \sin(y)M(-x - y, 0)$

*A matrix satisfying these conditions is called a* skew Seidel matrix

*Proof.* Implications (1) ⇔ (2), (3) ⇒ (2) and (4) ⇒ (2) of the Lemma are obvious. For (2) ⇒ (3) use the fact that the general solution of (2) iterated, viz.

$$(\partial_x - \partial_y)^2 M(x, y) = -M(x, y),$$

is given by

$$M(x, y) = e^{ix} g(x + y) + e^{-ix} h(x + y),$$

where $g(\,.\,)$ and $h(\,.\,)$ are arbitrary. Substituting this form back into (2) gives (3) with $M(0, y) = g(y) + h(y)$.
As for (3) ⇒ (4), note that the first column of $M$ is

$$M(x, 0) = \cos(x)\, M(0, x) + \sin(x)\, M(0, -x)$$

Hence by expanding, regrouping and using standard trig formulas:

$$\begin{aligned}
&\cos(y)\, M(x+y,0) - \sin(y)\, M(-x-y,0)\\
&\quad = \cos(y)\ [\cos(x+y)\, M(0,x+y) + \sin(x+y)\, M(0,-x-y)\,]\\
&\qquad - \sin(y)\ [\cos(-x-y)\, M(0,-x-y) + \sin(-x-y)\, M(0,x+y)\,]\\
&\qquad = [\,\cos(y)\ \cos(x+y) + \sin(y)\ \sin(x+y)\,]\ M(0,x+y)\\
&\qquad\quad + [\,\cos(y)\ \sin(x+y) - \sin(y)\ \cos(x+y)\,]\ M(0,-x-y)\\
&\qquad\qquad = \cos(x)\, M(0,x+y) + \sin(x)\, M(0,-x-y) = M(x,y)
\end{aligned}$$

□

*Remark* 3.3. If $M=(m_{i,j})_{i,j\ge 0}$ is a skew Seidel matrix, then $\overline{M}=(\overline{m}_{i,j})_{i,j\ge 0}$ with $\overline{m}_{i,j}=(-1)^{\binom{i+j+a}{2}} m_{i,j}$ is a Seidel matrix (resp. transpose of a Seidel matrix) if $a$ is even (res. is odd), and vice versa.

*Remark* 3.4. By exchanging the roles of the coordinates in Lemmas 3.1 and 3.2:

– If a matrix $M$ is the *transpose* of a Seidel matrix, then

$$M(x,y) = e^{-x} M(0,x+y).$$

– If a matrix $M$ is the *transpose* of a skew Seidel matrix, then

$$M(x,y) = \cos(x)\, M(0,x+y) - \sin(x)\, M(0,-x-y).$$

The other characterizations translate in a similar way.

3.2. **Sequences of Seidel matrices.** We turn to infinite sequences of (skew) Seidel matrices. The next proposition is mentioned for completeness, it will not be used in the sequel – but the following ones will.

**Proposition 3.5.** *If* $\mathcal{S}=\big(S^{(n)}\big)_{n\ge 0}$ *is a sequence of Seidel matrices* $S^{(n)}=\big(S^{(n)}_{i,j}\big)_{i,j\ge 0}$, *then the egf of* $\mathcal{S}$ *is*

$$\mathcal{S}(x,y,z) = e^{x} H_{\mathcal{S}}(x+y,z),$$

*where* $H_{\mathcal{S}}=\big(S^{(n)}_{0,k}\big)_{k,n\ge 0}$ *(with* $k$ *indexing rows,* $n$ *indexing columns).*

This is immediate from Lemma 3.1 by comparing the coefficients of $z^n/n!$ on both sides.

*Remark* 3.6. For $n\ge 0$, the matrix $H_{\mathcal{S}}$ contains the first *row* of the matrix $S^{(n)}$ as its $n$-th *column*. Since Seidel matrices are completely specified by their first row, the matrix $H_{\mathcal{S}}$ contains the complete information about $\mathcal{S}$.

**Proposition 3.7.** *If* $\mathcal{S}=\big(S^{(n)}\big)_{n\ge 0}$ *is a sequence of* skew *Seidel matrices, then, with* $H_{\mathcal{S}}$ *as before,*

$$\mathcal{S}(x,y,z) = \cos(x) H_{\mathcal{S}}(x+y,z) + \sin(x) H_{\mathcal{S}}(-x-y,z).$$

This is immediate from Lemma 3.2 by comparing the coefficients of $z^n/n!$ on both sides.

**Proposition 3.8.** *If* $\mathcal{S}=\big(S^{(n)}\big)_{n\ge 0}$ *is a sequence of matrices such that for* $n$ *even (resp.* $n$ *odd)* $S^{(n)}$ *is a skew Seidel matrix (resp. the* transpose *of a skew Seidel matrix), then, with* $H_{\mathcal{S}}$ *as before,*

$$\mathcal{S}(x,y,z) = \cos(x) H_{\mathcal{S}}(x+y,z) + \sin(x) H_{\mathcal{S}}(-x-y,-z).$$

This is immediate from Lemma 3.2 and Remark 3.4 by comparing the coefficients of $z^n/n!$ on both sides.

*Remark* 3.9. If the role of skew and transposed skew Seidel matrices is inverted, then the result reads $\ldots - \sin(x)\ldots$.

## 4. Seidel matrices for Entringer's and Poupard's statistics

In the following skew Seidel matrices $G^{\ell,a}$ (for $a \in \{0,1\}, \ell \geq 0$) are defined by giving the generating function $G^{\ell,a}(x,y)$, from which the marginal series $G^{\ell,a}(0,y)$ (top row) and $G^{\ell,a}(x,0)$ (leftmost column) are obvious. One could go the other way round: by Lemma 3.2 on gets $G^{\ell,a}(x,y)$ by specifying $G^{\ell,a}(0,y)$, say, and requiring that the matrix has to be skew Seidel.

**Definition 4.1.** For $\ell \geq 0$ and $a \in \{0,1\}$[1] let $G^{\ell,a}$ be the matrix belonging to

$$G^{\ell,a}(x,y) = \sec^{\ell}(x+y)(\cos(ax-y) - \sin(ax-y)).$$

Let us list some properties of these matrices:

**Properties of the $G^{\ell,0}$:** As seen from

$$(\partial_x - \partial_y)G^{\ell,0}(x,y) = -G^{\ell,0}(-x,-y),$$

$G^{\ell,0}$ is a transposed skew Seidel matrix with row and column egf's

$$\begin{aligned} G^{\ell,0}(0,y) &= \sec^{\ell}(y)(\cos(y)+\sin(y)), \\ G^{\ell,0}(x,0) &= \sec^{\ell}(x). \end{aligned}$$

The anti-diagonal sums of $G^{\ell,0}$, from Lemma 2.2, are given by

$$\begin{aligned} \widehat{G^{\ell,0}}(z) &= \frac{d}{dz}\,(p(z)\cdot q(z)) \ \text{ with } p(z) = \sec^{\ell}(z) \text{ and} \\ q(z) &= \int_0^z (\cos(y)+\sin(y))\,dy = 1-\cos(z)+\sin(z). \end{aligned}$$

The *odd* part of $p(z)\cdot q(z)$ is $\sec^{\ell}(z)\cdot\sin(z) = \sec^{\ell-1}(z)\cdot\tan(z)$.
The *even* part of $G^{\ell,0}$ is obviously given by $\sec^{\ell}(x+y)\cdot\cos(y)$.

**Properties of the $G^{\ell,1}$:** As seen from

$$(\partial_x - \partial_y)G^{\ell,1}(x,y) = 2\,G^{\ell,1}(-x,-y),$$

$G^{\ell,1}$ is, apart from the multiplier 2, a skew Seidel matrix. Constant multipliers can easily be accommodated in the (skew) Seidel matrix setup, so we don't worry about this. The row and column egf's are

$$\begin{aligned} G^{\ell,1}(0,y) &= \sec^{\ell}(y)(\cos(y)-\sin(y)), \\ G^{\ell,1}(x,0) &= \sec^{\ell}(x)(\cos(x)+\sin(x)). \end{aligned}$$

As for the anti-diagonal sums of $G^{\ell,1}$, they are given by

$$\begin{aligned} \widehat{G^{\ell,1}}(z) &= \frac{d}{dz}\,(p(z)\cdot q(z)) \ \text{ with } p(z) = \sec^{\ell}(z) \text{ and} \\ q(z) &= \int_0^z (\cos(z-2y)+\sin(z-2y))\,dy = \sin(z). \end{aligned}$$

[1]One could study the situation for other values of $a$, but that is not our concern here.

Again, the *odd* part of $p(z) \cdot q(z)$ is $\sec^{\ell}(z) \cdot \sin(z) = \sec^{\ell-1}(z) \cdot \tan(z)$.
The even part of $G^{\ell,1}$ is obviously given by $\sec^{\ell}(x+y) \cdot \cos(x-y)$.

*Remark* 4.2. The $G$-matrices are skew Seidel matrices, which means that their entries satisfy first order difference equations for the entries, or equivalently, first order differential equations for the exponential generating functions. If one is only interested in the numbers in *even* and wants to forget about the entries in *odd* position, then this essentially means considering the differential operator $(\partial_x - \partial_y)^2$ instead of $\partial_x - \partial_y$ for the generating functions (and similarly for the difference operators acting on the entries).

We will now look at the examples for $\ell = 0$ and $\ell = 1$. For higher values of $\ell$ one encounters the so-called *$\ell$-tangent numbers*, which are not relevant for what follows. It will turn out that the Entringer and Poupard numbers occur in the *even* positions of these matrices, displayed in **boldface** in the examples that follow.

*Example* 4.3. The skew Seidel matrix $G^{1,0}$ contains the *Entringer-tangent numbers* in the even positions (in boldface) and the *Entringer-secant numbers* in the odd positions. The first few entries of $G^{1,0}$ are

$$G^{1,0} = \begin{matrix} \mathbf{1} & 1 & \mathbf{0} & 2 & \mathbf{0} & 16 & \mathbf{0} & \cdots \\ 0 & \mathbf{1} & 2 & \mathbf{2} & 16 & \mathbf{16} & 272 & \cdots \\ \mathbf{1} & 1 & \mathbf{4} & 14 & \mathbf{32} & 256 & \mathbf{544} & \cdots \\ 0 & \mathbf{5} & 10 & \mathbf{46} & 224 & \mathbf{800} & 7120 & \cdots \\ \mathbf{5} & 5 & \mathbf{56} & 178 & \mathbf{1024} & 6320 & \mathbf{30656} & \cdots \\ 0 & \mathbf{61} & 122 & \mathbf{1202} & 5296 & \mathbf{36976} & 275792 & \cdots \\ \mathbf{61} & 61 & \mathbf{1324} & 4094 & \mathbf{42272} & 238816 & \mathbf{1965664} & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \ddots \end{matrix}$$

with the tangent numbers in the first row ($G^{1,0}(0,y) = 1 + \tan(y)$) and the secant numbers in the first column ($G^{1,0}(x,0) = \sec(x)$). The exponential generating function for the skew diagonal sums of the $G^{1,0}$ matrix,

$$(\widehat{G_n^{1,0}})_{n\geq 0} = 1, 1, 2, 5, 16, 61, 272, 1385, 7936, 50521, 353792, \ldots,$$

shows tangent and secant numbers simultaneously, indeed

$$\widehat{G^{1,0}}(z) = \frac{d}{dz}\big(\sec(z)\cdot(1-\cos(z)+\sin(z)\big) = \sec(z)(\sec(z)+\tan(z)) = \frac{d}{dz}(\sec(z)+\tan(z)).$$

Note that the $n$-th anti-diagonal of $G^{1,0}$ (starting with $n = 0$) is equal to the vector of column sums of $A_{n+1}$, i.e., Entringer tangent and secant numbers.

*Example* 4.4. The skew Seidel matrix $G^{2,0}$ contains the *Entringer-secant numbers* in the even positions (in boldface). The first few entries of $G^{2,0}$ are

$$G^{2,0} = \begin{matrix} \mathbf{1} & 1 & \mathbf{1} & 5 & \mathbf{5} & 61 & \mathbf{61} & \cdots \\ 0 & \mathbf{2} & 4 & \mathbf{10} & 56 & \mathbf{122} & 1324 & \cdots \\ \mathbf{2} & 2 & \mathbf{14} & 46 & \mathbf{178} & 1202 & \mathbf{4094} & \cdots \\ 0 & \mathbf{16} & 32 & \mathbf{224} & 1024 & \mathbf{5296} & 42272 & \cdots \\ \mathbf{16} & 16 & \mathbf{256} & 800 & \mathbf{6320} & 36976 & \mathbf{238816} & \cdots \\ 0 & \mathbf{272} & 544 & \mathbf{7120} & 30656 & \mathbf{275792} & 1965664 & \cdots \\ \mathbf{272} & 272 & \mathbf{7664} & 23536 & \mathbf{306448} & 1689872 & \mathbf{17180144} & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \ddots \end{matrix}$$

The first row contains secant numbers with repetition ($G^{2,0}(0,y) = \sec^2(y)(\cos(y)+\sin(y)) = \sec(y)+(d/dy)\sec(y)$). The first column gives the tangent numbers again ($G^{2,0}(x,0) = \sec^2(x) = (d/dx)\tan(x)$).
The egf for the skew diagonal sums of the $G^{2,0}$ matrix,

$$(G_n^{2,0})_{n\geq 0} = 1, 1, 5, 11, 61, 211, 1385, 6551, 50521, 303271, 2702765, \ldots,$$

is obtained as

$$\widehat{G^{2,0}}(z) = \frac{d}{dz}\big(\sec^2(z)\cdot(1-\cos(z)+\sin(z))\big) = \sec(z)(1+(2\sec(z)-1)\tan(z)+2\tan^2(z)).$$

Extracting the even part gives the secant numbers again:

$$\sec(z)(1+2\tan^2(z)) = \frac{d^2}{dz^2}\sec(z).$$

The odd part can be written as

$$(2-\cos(z))\sec^2(z)\tan(z) \text{ or } \sec(s)\tan(z)(2\sec(z)-1).$$

Note that the $2n$-th anti-diagonal of $G^{1,1}$ (starting with $n = 0$) is equal to the vector of column sums of $A_{2n+2}$ in reverse order, i.e., Entringer secant numbers.

*Example* 4.5. The skew Seidel matrix $G^{1,1}$ contains the *Poupard-tangent numbers* in the even positions (in boldface). The first few entries of $G^{1,1}$ are

$$G^{1,1} = \begin{array}{cccccccc}
\mathbf{1} & -1 & \mathbf{0} & -2 & \mathbf{0} & -16 & \mathbf{0} & \ldots \\
1 & \mathbf{2} & -2 & \mathbf{4} & -16 & \mathbf{32} & -272 & \ldots \\
\mathbf{0} & 2 & \mathbf{8} & -8 & \mathbf{64} & -208 & \mathbf{1088} & \ldots \\
2 & \mathbf{4} & 8 & \mathbf{80} & -80 & \mathbf{1504} & -4672 & \ldots \\
\mathbf{0} & 16 & \mathbf{64} & 80 & \mathbf{1664} & -1664 & \mathbf{54784} & \ldots \\
16 & \mathbf{32} & 208 & \mathbf{1504} & 1664 & \mathbf{58112} & -58112 & \ldots \\
\mathbf{0} & 272 & \mathbf{1088} & 4672 & \mathbf{54784} & 58112 & \mathbf{3027968} & \ldots \\
\ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ddots
\end{array}$$

The first row contains the negatives of the tangent numbers ($G^{1,1}(0,y) = \sec(y)(\cos(y)-\sin(y)) = 1-\tan(y)$), the first column shows the tangent numbers as usual ($G^{1,1}(x,0) = \sec(x)(\cos(x)+\sin(x)) = 1+\tan(y)$).
The skew-diagonal sums of the $G^{1,1}$-array,

$$(\widehat{G_n^{1,1}})_{n\geq 0} = (1, 0, 2, 0, 16, 0, 272, 0, 7936, 0, 353792, \ldots),$$

are visibly tangent numbers, their egf is obtained from

$$\widehat{G^{1,1}}(z) = \frac{d}{dz}(\sec(z)\cdot\sin(z)) = \frac{d}{dz}\tan(z) = \sec(z)^2,$$

which is an even function, so that the odd part vanishes.

Note that the $2n$-th anti-diagonal of $G^{1,1}$ (starting with $n = 0$) is equal to the vector of row sums of $A_{2n+1}$, i.e., Poupard tangent statistics.

*Example* 4.6. The skew Seidel matrix $G^{2,1}$ contains the *Poupard-secant numbers* in the even positions (in boldface). The first few entries of $G^{2,1}$ are

$$\begin{array}{cccccccc}
\mathbf{1} & -1 & \mathbf{1} & -5 & \mathbf{5} & -61 & \mathbf{61} & \ldots \\
1 & \mathbf{3} & -3 & \mathbf{15} & -51 & \mathbf{183} & -1263 & \ldots \\
\mathbf{1} & 3 & \mathbf{21} & -21 & \mathbf{285} & -897 & \mathbf{6681} & \ldots \\
5 & \mathbf{15} & 21 & \mathbf{327} & -327 & \mathbf{8475} & -26079 & \ldots \\
\mathbf{5} & 51 & \mathbf{285} & 327 & \mathbf{9129} & -9129 & \mathbf{378105} & \ldots \\
61 & \mathbf{183} & 897 & \mathbf{8475} & 9129 & \mathbf{396363} & -396363 & \ldots \\
\mathbf{61} & 1263 & \mathbf{6681} & 26079 & \mathbf{378105} & 396363 & \mathbf{24615741} & \ldots \\
\ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ddots
\end{array}$$

The first row features secant numbers repeated with alternating signs ($G^{2,1}(0,y) = \sec^2(y)(\cos(y) - \sin(y)) = \sec(y) - (d/dy)\sec(y)$), the first column does the same without alternating signs ($G^{2,1}(x,0) = \sec^2(x)(\cos(x)+\sin(x)) = \sec(y)+(d/dy)\sec(y)$). The exponential generating function for the skew diagonal sums of the $G^{2,1}$-array,

$$(\widehat{G_n^{2,1}})_{n\geq 0} = (1, 0, 5, 0, 61, 0, 1385, 0, 50521, 0, 2702765, \ldots),$$

featuring the tangent numbers, can be obtained as before:

$$\widehat{G^{2,1}}(z) = \frac{d}{dz}\big(\sec^2(z)\cdot\sin(z)\big) = \sec(z)(\sec(z)^2 + \tan(z)^2) = \frac{d^2}{dz^2}\sec(z),$$

which is an even function, so that the odd part vanishes.

Note that the $2n$-th anti-diagonal of $G^{1,1}$ (starting with $n = 0$) is equal to the vector of row sums of $A_{2n+2}$, i.e., Poupard secant numbers.

## 5. Properties of the Entringer-Poupard matrices

From the *combinatorial* definition of entries the $a_n(k,\ell)$ of the Entringer-Poupard matrices $A_n$ we are going to derive a number of properties that will be instrumental in the sequel. We emphasize that here we only make use of combinatorial arguments.

**Proposition 5.1.** *(Special values)*
*For the matrices $A_n = \big(a_n(k,\ell)\big)_{1\leq k,\ell\leq n}$ the following properties hold:*

(1) *Entries along the main diagonal of $A_n$:*

$$a_n(k,k) = \begin{cases} 0 & \text{if } n \text{ is even,} \\ E_{n-1}(k) & \text{if } n \text{ is odd.} \end{cases} \quad (1 \leq k < n)$$

(2) *Entries along the rightmost column af $A_n$:*

$$a_n(k,n) = \begin{cases} 0 & \text{if } n \text{ is odd,} \\ E_{n-1}(k) & \text{if } n \text{ is even.} \end{cases} \quad (1 \leq k < n)$$

(3) *Entries along the rightmost column of* $\mathrm{Up}_n$:

$$a_n(k,n-1) = \begin{cases} 0 & \text{if } n \text{ is odd,} \\ P_{n-1}(k) & \text{if } n \text{ is even.} \end{cases} \quad (1 \leq k < n-1)$$

(4) *Entries along the leftmost column of* $\mathrm{Low}_n$:

$$a_n(k,1) = \begin{cases} 0 & \text{if } n \text{ is even,} \\ P_{n-1}(k-1) & \text{if } n \text{ is odd.} \end{cases} \quad (1 < k < n)$$

(5) *For the bottom row of* $A_n$*:* $a_n(n,k) = 0$ *for* $1 \le k \le n$.

*Proof.* Write $w = w_1 w_2 \dots w_n$ for an arbitrary element of $\mathrm{Alt}_n$.

(1) If $w_n = k$ and $n$ is even, then $w_{n-1} < k$, i.e., $w_{n-1} \ne n$, the maximum value, hence $a_n(k,k) = 0$.
If $w_n = k$ and $n$ is odd, then $w_{n-1} = n$ is possible. But if also $\mathbf{grn}(w) = k$, then $w_{n-2} < w_n$, and eliminating $w_{n-1} = n$ from $w$ gives a permutation $w' \in \mathrm{Alt}_{n-1}$ with $\mathbf{L}(w') = k$ – and vice versa. Hence $a_n(k,k) = E_{n-1}(k)$.

(2) If $n > 1$ is odd, then $\mathbf{L}(w) = n$ is impossible, thus $a_n(k,n) = 0$ for any $k$.
If $n$ is even and $\mathbf{L}(w) = n$, then $\mathbf{grn}(w) = w_{n-1} = k$, say. Eliminating $w_n = n$ from $w$ gives $w' \in \mathrm{Alt}_{n-1}$ with $\mathbf{L}(w') = k$ – and vice versa. Hence $a_n(k,n) = E_{n-1}(k)$.

(3) If $n$ is odd and $\mathbf{L}(w) = n-1$, then $w_{n-1} = n$ and hence $\mathbf{grn}(w) = n-1$, so $a_n(k,n-1) = 0$ for $k < n-1$.
If $n$ is even and $\mathbf{L}(w) = n-1$, then $\mathbf{grn}(w) = k < n-1$. Eliminating $w_n$ from $w$ and replacing the maximum $n$ by $n-1$ gives $w' \in \mathrm{Alt}_{n-1}$, where $\mathbf{grn}(w') = k = \mathbf{grn}(w) < n-1$ – and vice versa. Hence $a_n(k,n-1) = P_{n-1}(k)$.

(4) $\mathrm{L}(w) = 1$ is impossible for even $n$, hence $a_n(k,1) = 0$ for all $k$. If $n$ is odd and $\mathrm{L}(w) = 1$, then removing $w_n = 1$ from $w$ and decreasing all letters of $w$ by one gives a $w' \in \mathrm{Alt}_{n-1}$ with $\mathbf{grn}(w') = k-1$ if before $\mathbf{grn}(w) = k$ – and vice versa. Hence $a_n(k,1) = P_{n-1}(k-1)$.

(5) By definition, $\mathbf{grn}(w)$ is always $< n$.

□

**Proposition 5.2.** *(Difference scheme)*
*For the matrices* $A_n = \big(a_n(k,\ell)\big)_{1\le k,\ell\le n}$, $n \ge 3$, *the following properties hold:*

(1) *For the upper triangles, i.e.,* $1 \le k < \ell \le n-2$,

$$a_n(k,\ell+1) - a_n(k,\ell) = (-1)^n a_{n-1}(k,\ell); \tag{5.3}$$

(2) *for the lower triangles, i.e., for* $3 \le \ell + 2 \le k \le n-2$,

$$a_n(k,\ell+1) - a_n(k,\ell) = (-1)^n a_{n-1}(k-1,\ell). \tag{5.4}$$

*Proof.* We prove (1), case (2) is similar.
Generally, if $w = w_1 w_2 \dots w_n \in \mathrm{Alt}_n$ and if $1 \le \ell < n$ is a letter which is not maximum, then the positions of the letters $\ell$ and $\ell+1$ in $w$ may be interchanged and the resulting permutation $w'$ still belongs to $\mathrm{Alt}_n$, unless $\ell$ and $\ell+1$ occupied neighboring positions in $w$. Furthermore, if $\mathbf{grn}(w) = k < \ell < n-1$, then the interchange would not affect the neighbors of $n$, nor $n$ itself, i.e. $\mathbf{grn}(w') = k$ as well. Hence, if $1 \le k < \ell < n-1$, then, by an obvious involution argument, permutations in which $\ell$ and $\ell+1$ are not neighbors do not contribute to the left-hand side of (5.3). It remains to consider the contribution coming from permutations in which either $w_{n-1} = \ell, w_n = \ell+1$ (so $n$ ist even), or $w_{n-1} = \ell, w_n = \ell+1$ (so $n$ is odd).
If $n$ is even, then in all permutations with $w_{n-1} = \ell, w_n = \ell+1$ the last letter can be eliminated and all letters $> \ell+1$ decremented by one. This results in an element $w'$ of $\mathrm{Alt}_{n-1}$ with $\mathbf{grn}(w') = k$ and $\mathbf{L}(w') = \ell$ – and vice versa.
If $n$ is odd, a similar argument applies. □

*Remark* 5.5. It should be noted that the two preceding propositions together with the trivial initial case $A_2 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ completely specify the sequence of Entringer-Poupard matrices. Prop. 5.1 serves to initialize enough "border" or "diagonal" values of $A_n$ so that the difference scheme from Prop. 5.2 can be used inductively to fill the remaining positions of $A_n$, given complete knowledge of $A_{n-1}$.

## 6. From Entringer-Poupard matrices to Seidel matrices

From the sequence $\big(A_n\big)_{n\geq 3}$ of Entringer-Poupard matrices, referring to their segmentation into upper triangles $\mathrm{Up}_n$, lower triangles $\mathrm{Low}_n$, and diagonals $\mathrm{Diag}_n$, we now construct the following two sequences of matrices $\Upsilon = (U_{\nu+3})_{\nu\geq 3}$ and $\Lambda = (L_{\nu+3})_{\nu\geq 3}$. The matrices $U_n^{\blacktriangle}$ and $L_n^{\blacktriangle}$ that show up in the following are only intermediate steps that help to visualize the construction.

(1) Upper triangles: For each $n \geq 3$ construct the matrix $U_n^{\blacktriangle}$ by selecting rows of increasing length from $\mathrm{Up}_m$ for $m \geq n$, so that each row is the difference of the following row (with alternating signs), see (5.3), precisely:

$$U_n^{\blacktriangle} = (a_{n+k}(n-2, n-1+\ell))_{k,\ell\geq 0} =$$
$$\begin{bmatrix} a_n(n-2,n-1) & 0 & 0 & 0 & \dots \\ a_{n+1}(n-2,n-1) & a_{n+1}(n-2,n) & 0 & 0 & \dots \\ a_{n+2}(n-2,n-1) & a_{n+2}(n-2,n) & a_{n+2}(n-2,n+1) & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

To obtain $U_n$, push the elements in the $\ell$-th column of $U_n^{\blacktriangle}$ $\ell$ positions upward.

$$U_n = (a_{n+k+\ell}(n-2, n-1+\ell))_{k,\ell\geq 0} =$$
$$\begin{bmatrix} a_n(n-2,n-1) & a_{n+1}(n-2,n) & a_{n+2}(n-2,n+1) & \dots \\ a_{n+1}(n-2,n-1) & a_{n+2}(n-2,n) & a_{n+3}(n-2,n+1) & \dots \\ a_{n+2}(n-2,n-1) & a_{n+3}(n-2,n) & a_{n+4}(n-2,n+1) & \dots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix},$$

and it will be seen that, by construction, this is a skew Seidel matrix (or transposed skew Seidel matrix, depending on the parity of $n$). The first row of $U_n$ equals the diagonal of $U_n^{\blacktriangle}$, and these numbers are Poupard numbers by item (3) of Proposition 5.1.

(2) Lower triangles: For each $n \geq 3$ construct the matrix $L_n^{\blacktriangle}$ by selecting rows of increasing length from $\mathrm{Low}_m$ for $m \geq n$, so that each row is the difference of the following row (with alternating signs), see (5.4), precisely:

$$L_n^{\blacktriangle} = (a_{n+k}(k+2, \ell+1))_{k,\ell\geq 0} =$$
$$\begin{bmatrix} a_n(2,1) & 0 & 0 & 0 & \dots \\ a_{n+1}(3,1) & a_{n+1}(3,2) & 0 & 0 & \dots \\ a_{n+2}(4,1) & a_{n+2}(4,2) & a_{n+2}(4,3) & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

To obtain $L_n$, push the elements in the $\ell$-th column of $L_n^{\blacktriangle}$ $\ell$ positions upward.

$$L_n = (a_{n+k+\ell}(k+\ell+2,\ell+1))_{k,\ell\geq 0} = \begin{bmatrix} a_n(2,1) & a_{n+1}(3,2) & a_{n+2}(4,3) & \dots \\ a_{n+1}(3,1) & a_{n+2}(4,2) & a_{n+3}(5,3) & \dots \\ a_{n+2}(4,1) & a_{n+3}(5,2) & a_{n+4}(6,3) & \dots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix},$$

and it will be seen that, by construction, this is a skew Seidel matrix (or transposed skew Seidel, depending on the parity of $n$). The first columns of $L_n^{\blacktriangle}$ and $L_n$ agree, and these numbers are Poupard numbers by item (4) of Proposition 5.1.

**Lemma 6.1.** *(Checking the skew Seidel property)*
*The matrices $U_n$ and $L_n$ are skew Seidel matrices (or transposed skew Seidel matrices, depending on the parity of $n$).*

*Proof.* (a) For the $U_n$: Recall that from (5.3) we have

$$a_n(k,\ell) - a_n(k,\ell+1) = (-1)^{n-1}a_{n-1}(k,\ell).$$

From the definition of $U_n$

$$U_n(k,\ell) = a_{n+k+\ell}(n-2,n+\ell-1) = a_\nu(\kappa,\lambda),$$

with $\nu = n+k+\ell$, $\kappa = n-2$, $\lambda = n-\ell-1$. Then

$$\begin{aligned} U_n(k+1,\ell) - U_n(k,\ell+1) &= a_{\nu+1}(\kappa,\lambda) - a_{\nu+1}(\kappa,\lambda+1) \\ &= (-1)^\nu a_\nu(\kappa,\lambda) \\ &= (-1)^{n+k+\ell}a_{n+k+\ell}(n-2,n+\ell) \\ &= (-1)^n\cdot(-1)^{k+\ell}U_n(k,\ell). \end{aligned}$$

(b) For the $L_n$: Recall that from (5.4) we have

$$a_n(k,\ell) - a_n(k,\ell+1) = (-1)^{n-1}a_{n-1}(k-1,\ell).$$

From the definition of $L_n$

$$L_n(k,\ell) = a_{n+k+\ell}(k+\ell+2,\ell+1) = a_\nu(\kappa,\lambda)$$

with $\nu = n+k+\ell$, $\kappa = k+\ell+2$, $\lambda = \ell+1$. Then

$$\begin{aligned} L_n(k+1,\ell) - L_n(k,\ell+1) &= a_{\nu+1}(\kappa+1,\lambda) - a_{\nu+1}(\kappa+1,\lambda+1) \\ &= (-1)^\nu a_\nu(\kappa,\lambda) \\ &= (-1)^{n+k+\ell}a_{n+k+\ell}(k+\ell+2,\ell+1) \\ &= (-1)^n\cdot(-1)^{k+\ell}L_n(k,\ell). \end{aligned}$$

□

## 7. The generating functions for Entringer-Poupard matrices

We are now ready for completing the proof of Thm. 1.4, our main result. Recall from the Introduction the definition of the generating functions $\Upsilon(x,y,z)$, resp. $\Lambda(x,y,z)$, that we want to identify.

7.1. **Relabelling.**

**Proposition 7.1.** *The generating functions $\Upsilon(x,y,z)$ resp. $\Lambda(x,y,z)$ are indeed the generating functions for the skew Seidel matrix sequences $\Upsilon = \big(U_{n+3}\big)_{n\ge 0}$ resp. $\Lambda = \big(L_{n+3}\big)_{n\ge 0}$, i.e.,*

$$\Upsilon(x,y,z) = \sum_{\nu\ge 0} U_{\nu+3}(x,y)\frac{z^\nu}{\nu!}, \qquad \Lambda(x,y,z) = \sum_{\nu\ge 0} L_{\nu+3}(x,y)\frac{z^\nu}{\nu!}.$$

*Proof.* This first step is only a matter of relabelling.

(1) For the upper triangles: In

$$\begin{aligned}\Upsilon(x,y,z) &= \sum_{n\ge 3}\sum_{(k,\ell)\in \mathrm{Up}_n} a_n(k,\ell) M(n-\ell-1,\ell-k-1,k-1)\\ &= \sum_{1\le k<\ell<n} a_n(k,\ell)\frac{x^{n-\ell-1}}{(n-\ell-1)!}\frac{y^{\ell-k-1}}{(\ell-k-1)!}\frac{z^{k-1}}{(k-1)!}\end{aligned}$$

the linear change of indices

$$\left\{\begin{array}{rcl} n-\ell-1 &=& \kappa\\ \ell-k-1 &=& \lambda\\ k-1 &=& \nu \end{array}\right\} \Longleftrightarrow \left\{\begin{array}{rcl} k &=& \nu+1\\ \ell &=& \lambda+\nu+2\\ n &=& \kappa+\lambda+\nu+3 \end{array}\right\}$$

turns the triple sum into

$$\sum_{\kappa,\lambda,\nu\ge 0} a_{\kappa+\lambda+\nu+3}(\nu+1,\lambda+\nu+2)\frac{x^\kappa}{\kappa!}\frac{y^\lambda}{\lambda!}\frac{z^\nu}{\nu!} = \sum_{\nu\ge 0} U_{\nu+3}(x,y)\frac{z^\nu}{\nu!}.$$

For later use: By construction, the $\rho$-th anti-diagonal of $U_{\nu+3}$ is

$$[\,a_{\nu+\rho+3}(\nu+1,\lambda+\nu+2)\,]_{\rho\le\lambda\le\rho+\nu+2}$$

which is the row with index $(\nu+1)$ of $\mathrm{Up}_{\nu+\rho+3}$, so the sum over these elements is conveniently denoted by $up_{\nu+\rho+3}(\nu+1,\bullet)$.

(2) For the lower triangles: In

$$\begin{aligned}\Lambda(x,y,z) &= \sum_{n\ge 3}\sum_{(k,\ell)\in \mathrm{Low}_n} a_n(k,\ell) M(\ell-1,k-\ell-1,n-k-1)\\ &= \sum_{1\le \ell<k<n} a_n(k,\ell)\frac{x^{\ell-1}}{(\ell-1)!}\frac{y^{k-\ell-1}}{(k-\ell-1)!}\frac{z^{n-k-1}}{(n-k-1)!}\end{aligned}$$

the linear change of indices

$$\left\{\begin{array}{rcl} \ell-1 &=& \lambda\\ k-\ell-1 &=& \kappa\\ n-k-1 &=& \nu \end{array}\right\} \Longleftrightarrow \left\{\begin{array}{rcl} \ell &=& \lambda+1\\ k &=& \kappa+\lambda+2\\ n &=& \kappa+\lambda+\nu+3 \end{array}\right\}$$

turns the triple sum into

$$\sum_{\kappa,\lambda,\nu\ge 0} a_{\kappa+\lambda+\nu+3}(\kappa+\lambda+2,\lambda+1)\frac{x^\kappa}{\kappa!}\frac{y^\lambda}{\lambda!}\frac{z^\nu}{\nu!} = \sum_{\nu\ge 0} L_{\nu+3}(x,y)\frac{z^\nu}{\nu!}.$$

For later use: By construction, the $\rho$-th anti-diagonal of $L_{\nu+3}$ is

$$[\,a_{\nu+\rho+3}(\rho+2,\lambda+1)\,]_{0\le\lambda\le\rho}\,,$$

which is the row with index $\rho+2$ of $\mathrm{Low}_{\nu+\rho+3}$, so the sum over these elements is conveniently denoted by $low_{\nu+\rho+3}(\nu+1,\bullet)$.

$\square$

7.2. **The $H$-matrices and the main result.** Now that we have identified the generating functions $\Upsilon(x,y,z)$ and $\Lambda(x,y,z)$ as generating functions sequences of skew Seidel matrices, we can proceed and make use of Prop. 3.8. We have to determine what the respective matrices $H_\Upsilon$ and $H_\Lambda$ are.

(1) From the Seidel matrix sequence for upper triangles $\Upsilon = (U_{\nu+3})_{\nu\geq 3}$ we get

$$\begin{aligned} H_\Upsilon &= \left[a_{3+k+\ell}(k+\ell+2,k+1)\right]_{k,\ell\geq 0} \\ &= \begin{bmatrix} a_3(1,2) & a_4(2,3) & a_5(3,4) & \dots \\ a_4(1,3) & a_5(2,4) & a_6(3,5) & \dots \\ a_5(1,4) & a_6(2,5) & a_7(3,6) & \dots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix} = \begin{bmatrix} 0 & 2 & 0 & 4 & 0 & \dots \\ 0 & 0 & 8 & 0 & 64 & \dots \\ 0 & 4 & 0 & 80 & 0 & \dots \\ 0 & 0 & 64 & 0 & 1664 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}. \end{aligned}$$

This is essentially the even part of $G^{1,1}$ from Example 4.5 (omitting the first row of the even part of $G^{1,1}$), hence

$$H_\Upsilon(x,y) = \partial_x(\sec(x+y)\cdot\cos(x-y)) = \sec(x+y)^2\cdot\sin(2y).$$

For $\ell\geq 0$, column $\ell$ of $H_\Upsilon$ is the first (i.e., $k=0$) row of the skew (or transposed skew) Seidel matrix $U_{3+\ell}$. Hence by Prop. 3.8, taking the Remark 3.9 into account, the "upper" generating function is

$$\begin{aligned} \Upsilon(x,y,z) &= \cos(x)H_\Upsilon(x+y,z) - \sin(x)H_\Upsilon(-x-y,-z) \\ &= \sec(x+y+z)^2(\cos(x)+\sin(x))\sin(2z). \end{aligned}$$

(2) From the Seidel matrix sequence for lower triangles $\Lambda = (L_{\nu+3})_{\nu\geq 3}$:

$$\begin{aligned} H_\Lambda(x,y) &= \left[a_{3+k+\ell}(k+3,1)\right]_{k,\ell\geq 0} \\ &= \begin{bmatrix} a_3(2,1) & a_4(2,1) & a_5(2,1) & \dots \\ a_4(3,1) & a_5(3,1) & a_6(3,1) & \dots \\ a_5(4,1) & a_6(4,1) & a_7(4,1) & \dots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 & 0 & \dots \\ 0 & 3 & 0 & 15 & \dots \\ 1 & 0 & 21 & 0 & \dots \\ 0 & 15 & 0 & 327 & \dots \\ \vdots & \vdots & \vdots & \ddots & \end{bmatrix} \end{aligned}$$

This is the even part of $G^{2,1}$, see Example 4.6, hence

$$H_\Lambda(x,y) = \sec(x+y)^2\cdot\cos(x-y).$$

For $\ell\geq 0$,column $\ell$ of $H_\Lambda$ is the first column of the skew (or transposed skew) Seidel matrix $L_{3+n}$ (and hence the first row of the transposed skew (or skew) Seidel matrix $L^t_{3+n}$. Hence, by Prop. 3.8, the "lower" generating function is

$$\begin{aligned} \Lambda(x,y,z) &= \cos(x)H_\Lambda(x+y,z) + \sin(x)H_\Lambda(-x-y,-z) \\ &= \sec(x+y+z)^2(\cos(x)+\sin(x))\cos(x+y-z). \end{aligned}$$

(3) The generating function for the diagonals of the $A_n$ is simply

$$\sum_{i,j\geq 0}(a_{i+j+2}(i+1,i+1)+a_{i+j+2}(i+1,k+2))\,\frac{x^i}{i!}\frac{y^j}{j!}=\frac{\sin(x)+\cos(x)}{\cos(x+y)}.$$

Note that from Prop. 5.1 we know that the entries of the diagonal resp. of the last column of $A_n$ are the Entringer numbers $E_{n-1}(k), 1\leq k<n$, hence

$$\begin{aligned}\Delta(x,y)&=\sum_{1\leq k<n}\big(a_n(k,n)+a_n(k,k)\big)\frac{x^{k-1}}{(k-1)!}\frac{y^{n-k-1}}{(n-k-1)!}\\&=\sum_{1\leq k<n}E_{n-1}(k)\frac{x^{k-1}}{(k-1)!}\frac{y^{n-k-1}}{(n-k-1)!}\\&=\sum_{i,j\geq 0}E_{i+j+1}(i+1)\frac{x^i}{i!}\frac{y^j}{j!}\end{aligned}$$

and this is precisely the generating function attached to the transpose of the skew Seidel matrix $G^{1,0}$, hence

$$\Delta(x,y)=\frac{\sin(x)+\cos(x)}{\cos(x+y)}.$$

This concludes the proof of Thm. 1.4. □

## 8. Recovering the row sums of the Entringer-Poupard matrices

The Entringer resp. Poupard numbers and statistics occur as column sums and row sums of the Entringer-Poupard matrices by definition. In order to show that the three generating function obtained in the previous section contain the full information about these matrices, one may ask how to recover the Entringer resp. Poupard statistics from the knowledge of these generating functions alone. In this section we will sketch how this can be done for the row sums using the idea techniques of anti-diagonal sums (see Lemma 2.2). We leave it to the reader to obtain the column sums in a similar way.

Denoting by $a_n(k,\bullet)$ the row sum over the $k$-the row of $A_n$, we can set up the matrix of row sums which displays the Poupard statistics in the anti-diagonals.

$$\begin{aligned}RS&=\big[a_{i+j+2}(i+1,\bullet)\big]_{i,j\geq 0}\\&=\begin{bmatrix}a_2(1,\bullet)&a_3(1,\bullet)&a_4(1,\bullet)&\dots\\a_3(2,\bullet)&a_4(2,\bullet)&a_5(2,\bullet)&\dots\\a_4(3,\bullet)&a_5(3,\bullet)&a_6(3,\bullet)&\dots\\\vdots&\vdots&\vdots&\ddots\end{bmatrix}=\begin{bmatrix}1&0&1&0&5&\dots\\2&3&4&15&\ddots&\\1&8&21&\ddots&&\\4&15&\ddots&&&\\5&\ddots&&&&\\\vdots&&&&&\end{bmatrix}\end{aligned}$$

The generating functions is that of Poupard:

$$\begin{aligned} RS(x,y) &= \sum_{i,j\geq 0} a_{i+j+2}(i+1,\bullet)\frac{x^i}{i!}\frac{y^j}{j!} \\ &= \partial_y\big(\sec(x+y)\cdot\cos(x-y)\big)+\sec(x+y)^2\cdot\cos(x-y) \\ &= \sec(x+y)^2\cdot(\cos(x-y)+\sin(2y)). \end{aligned}$$

For $n \geq 2$ and $1 \leq k < n$, one has the decomposition of the row sums according to the segmentation of the $A_n$ into upper and lower triangles, and diagonals (see Subsec. 7.1 for the notation)

$$a_n(k,\bullet) = low_n(k,\bullet) + up_n(k,\bullet) + diag_n(k),$$

where $diag_n(k) = a_n(k,k) + a_n(k,n)$.

Now recall Prop. 7.1 and the remarks made during the proof:

- the $\rho$-th anti-diagonal of $U_{\nu+3}$ is is the row with index $\nu+1$ of $\mathrm{Up}_{\nu+\rho+3}$;
- the $\rho$-th anti-diagonal of $L_{\nu+3}$ is the row with index $\rho+2$ of $\mathrm{Low}_{\nu+\rho+3}$.

Thus we get row sums for the Up-parts resp. Low-parts of the $A_n$ as anti-diagonals of the $U_\nu$ resp. $L_\nu$ by applying the technique of Lemma 2.2 to $\Upsilon = (U_{\nu+3})_{\nu\geq 0}$ resp. $\Lambda = (L_{\nu+3})_{\nu\geq 0}$ w.r.t. the first two variables of $\Upsilon(x,y,z)$ resp. $\Lambda(x,y,z)$, i.e.,

$$\begin{aligned} \widehat{\Upsilon}(x,y) &= (d/dx)\left[\sec(x+y)^2\cdot\sin(2y)\cdot(1-\cos(x)+\sin(x))\right] \\ \widehat{\Lambda}(x,y) &= (d/dx)\left[\sec(x+y)^2\cdot\cos(x-y)\cdot(1-\cos(x)+\sin(x))\right] \end{aligned}$$

(1) The upper triangle:
Set $\nu+\rho+3 = i+j+2$ and $\nu = i$, so that $\rho+1 = j$. As noted, $up_{i+j+2}(i+1,\bullet)$ is the $\rho$-th anti-diagonal sum in $U_{\nu+3}$, thus it appears
   - as coefficient of $(x^\nu/\nu!)(y^{\rho+1}/(\rho+1)!)$ in $RS(x,y)$
   - as coefficient of $(x^\rho/\rho!)(y^\nu/\nu!)$ in $\widehat{\Upsilon}(x,y)$

(2) The lower triangle:
Set $\nu+\rho+3 = i+j+2$ and $\rho+2 = i+1$, so that $\nu = j$. As noted, $up_{i+j+2}(i+1,\bullet)$ is the $\rho$-th anti-diagonal sum in $L_{\nu+3}$, thus it appears
   - as coefficient of $(x^{\rho+1}/(\rho+1)!)(y^\nu/(\nu)!)$ in $RS(x,y)$
   - as coefficient of $(x^\rho/\rho!)(y^\nu/\nu!)$ in $\widehat{\Upsilon}(x,y)$

(3) The diagonal is simply:

$$\sum_{i,j\geq 0}(a_{i+j+2}(i+1,i+1)+a_{i+j+2}(i+1,k+2))\,\frac{x^i}{i!}\frac{y^j}{j!} = \frac{\sin(x)+\cos(x)}{\cos(x+y)}.$$

Putting all contributions for the row sums together we should get

$$RS(x,y) = \int_y \widehat{\Upsilon}(y,x) + \int_x \widehat{\Lambda}(x,y) + \frac{\sin(x)+\cos(x)}{\cos(x+y)},$$

(note the switch of the variables for the upper triangles!) where the discrepancy of the exponents is eliminated by indefinite integration, which has to be taken such that the "constant term" vanishes.

The last identity is an identity for trigonometric functions which is readily verified by using computer algebra. This shows that indeed the Poupard statistics can be obtained from the three generating functions.

## 9. Appendix

Entringer-Poupard matrices $A_n$ for $2 \le n \le 9$, with upper triangles $\mathrm{Up}_n$ (red), lower triangles $\mathrm{Low}_n$ (blue), diagonals $\mathrm{Diag}_n$ (black) indicated. Each dot $\cdot$ (of any color) represents a zero entry.

$$A_2 = \begin{array}{c|cc|c} & 1 & 2 & \\ \hline 1 & \cdot & 1 & 1 \\ 2 & \cdot & \cdot & 0 \\ \hline & 0 & 1 & 1 = E_2 \end{array}; \quad A_3 = \begin{array}{c|ccc|c} & 1 & 2 & 3 & \\ \hline 1 & \cdot & \cdot & \cdot & 0 \\ 2 & 1 & 1 & \cdot & 2 \\ 3 & \cdot & \cdot & \cdot & 0 \\ \hline & 1 & 1 & 0 & 2 = E_3 \end{array}; \quad A_4 = \begin{array}{c|cccc|c} & 1 & 2 & 3 & 4 & \\ \hline 1 & \cdot & \cdot & \cdot & 1 & 1 \\ 2 & \cdot & \cdot & 2 & 1 & 3 \\ 3 & \cdot & 1 & \cdot & \cdot & 1 \\ 4 & \cdot & \cdot & \cdot & \cdot & 0 \\ \hline & 0 & 1 & 2 & 2 & 5 = E_4 \end{array};$$

$$A_5 = \begin{array}{c|ccccc|c} & 1 & 2 & 3 & 4 & 5 & \\ \hline 1 & \cdot & \cdot & \cdot & \cdot & \cdot & 0 \\ 2 & 1 & 1 & 2 & \cdot & \cdot & 4 \\ 3 & 3 & 3 & 2 & \cdot & \cdot & 8 \\ 4 & 1 & 1 & \cdot & 2 & \cdot & 4 \\ 5 & \cdot & \cdot & \cdot & \cdot & \cdot & 0 \\ \hline & 5 & 5 & 4 & 2 & 0 & 16 = E_5 \end{array}; \quad A_6 = \begin{array}{c|cccccc|c} & 1 & 2 & 3 & 4 & 5 & 6 & \\ \hline 1 & \cdot & \cdot & \cdot & \cdot & \cdot & 5 & 5 \\ 2 & \cdot & \cdot & 2 & 4 & 4 & 5 & 15 \\ 3 & \cdot & 1 & \cdot & 8 & 8 & 4 & 21 \\ 4 & \cdot & 3 & 6 & \cdot & 4 & 2 & 15 \\ 5 & \cdot & 1 & 2 & 2 & \cdot & \cdot & 5 \\ 6 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 0 \\ \hline & 0 & 5 & 10 & 14 & 16 & 16 & 61 = E_6 \end{array}$$

$$A_7 = \begin{array}{c|ccccccc|c} & 1 & 2 & 3 & 4 & 5 & 6 & 7 & \\ \hline 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 0 \\ 2 & 5 & 5 & 10 & 8 & 4 & \cdot & \cdot & 32 \\ 3 & 15 & 15 & 10 & 16 & 8 & \cdot & \cdot & 64 \\ 4 & 21 & 21 & 20 & 14 & 4 & \cdot & \cdot & 80 \\ 5 & 15 & 15 & 12 & 6 & 16 & \cdot & \cdot & 64 \\ 6 & 5 & 5 & 4 & 2 & \cdot & 16 & \cdot & 32 \\ 7 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 0 \\ \hline & 61 & 61 & 56 & 46 & 32 & 16 & 0 & 272 = E_7 \end{array};$$

$$A_8 = \begin{array}{c|cccccccc|c} & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & \\ \hline 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 61 & 61 \\ 2 & \cdot & \cdot & 10 & 20 & 28 & 32 & 32 & 61 & 183 \\ 3 & \cdot & 5 & \cdot & 40 & 56 & 64 & 64 & 56 & 285 \\ 4 & \cdot & 15 & 30 & \cdot & 76 & 80 & 80 & 46 & 327 \\ 5 & \cdot & 21 & 42 & 62 & \cdot & 64 & 64 & 32 & 285 \\ 6 & \cdot & 15 & 30 & 42 & 48 & \cdot & 32 & 16 & 183 \\ 7 & \cdot & 5 & 10 & 14 & 16 & 16 & \cdot & \cdot & 61 \\ 8 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 0 \\ \hline & 0 & 61 & 122 & 178 & 224 & 256 & 272 & 272 & 1385 = E_8 \end{array};$$

$$A_9 = \begin{array}{c|ccccccccc|c} & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & \\ \hline 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \\ 2 & 61 & 61 & 122 & 112 & 92 & 64 & 32 & \cdot & \cdot & 544 \\ 3 & 183 & 183 & 122 & 224 & 184 & 128 & 64 & \cdot & \cdot & 1088 \\ 4 & 285 & 285 & 280 & 178 & 236 & 160 & 80 & \cdot & \cdot & 1504 \\ 5 & 327 & 327 & 312 & 282 & 224 & 128 & 64 & \cdot & \cdot & 1664 \\ 6 & 285 & 285 & 264 & 222 & 160 & 256 & 32 & \cdot & \cdot & 1504 \\ 7 & 183 & 183 & 168 & 138 & 96 & 48 & 272 & \cdot & \cdot & 1088 \\ 8 & 61 & 61 & 56 & 46 & 32 & 16 & \cdot & 272 & \cdot & 544 \\ 9 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 0 \\ \hline & 1385 & 1385 & 1324 & 1202 & 1024 & 800 & 544 & 272 & 0 & 7936 = E_9 \end{array}$$

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