NEW HOOK-CONTENT FORMULAS FOR STRICT PARTITIONS

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Abstract. We introduce the difference operator for functions defined on strict partitions and prove a polynomiality property for a summation involving the hook length and content statistics. As an application, several new hook-content formulas for strict partitions are derived.

1. Introduction

The basic knowledge on partitions, Young tableaux and symmetric functions could be found in [17]. In this paper, we focus on strict partitions. A *strict partition* is a finite strict decreasing sequence of positive integers $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_\ell)$. The integer $|\lambda| = \sum_{1\le i\le \ell} \lambda_i$ is called the *size* of the partition λ and $\ell(\lambda) = \ell$ is called the *length* of λ. For convenience, let $\lambda_i = 0$ for $i > l(\lambda)$. A strict partition λ can be identified with its shifted Young diagram, which means that the i-th row of the usual Young diagram is shifted to the right by i boxes. Therefore the leftmost box in the i-th row has coordinate $(i, i+1)$. For the (i, j)-box in the shifted Young diagram of the strict partition λ, we can associate its *hook length*, denoted by $h_{(i,j)}$, which is the number of boxes exactly to the right, or exactly above, or the box itself, plus λ_j. For example, consider the box $\square = (i, j) = (1, 3)$ in the shifted Young diagram of the strict partition $(7, 5, 4, 1)$. There are 1 and 5 boxes above and to the right of the box $\square$ respectively. Since $\lambda_3 = 4$, the hook length of $\square$ is equal to $1 + 5 + 1 + 4 = 11$, as illustrated in Figure 1. The *content* of $\square = (i, j)$ is defined to be $c_\square = j - i$, so that the leftmost box in each row has content 1. Also, let $\mathcal{H}(\lambda)$ be the multi-set of hook lengths of boxes and H_λ be the product of all hook lengths of boxes in λ.

Our goal is to find some hook-content formulas for *strict partitions*, by analogy with that for *ordinary partitions*. For the ordinary partition ν, it is well known that (see [3, 9, 17])

$$f_\nu = \frac{|\nu|!}{H_\nu} \qquad \text{and} \qquad \frac{1}{n!}\sum_{|\nu|=n} f_\nu^2 = 1, \tag{1.1}$$

where H_ν denotes the product of all hook lengths of boxes in ν and f_ν denotes the number of standard Young tableaux of shape ν. The first author conjectured [4] that

$$P(n) = \frac{1}{n!}\sum_{|\nu|=n} f_\nu^2 \sum_{\square\in\nu} h_\square^{2k}$$

2010 *Mathematics Subject Classification.* 05A15, 05A17, 05A19, 05E05, 05E10, 11P81.

Key words and phrases. strict partition, hook length, content, shifted Young tableau, difference operator.

An extended abstract for this paper appeared in the Proceedings of the FPSAC'16 conference.

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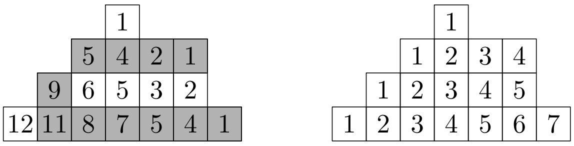

FIGURE 1. The shifted Young diagram of the strict partition $(7, 5, 4, 1)$ with its hook lengths and contents.

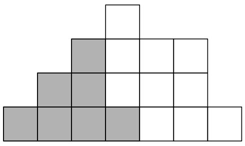

FIGURE 2. The skew shifted Young diagram of the skew strict partition $(7, 5, 4, 1)/(4, 2, 1)$.

is always a polynomial in n for all $k \in \mathbb{N}$, which was generalized and proved by Stanley [16], and later generalized in [7] (see also [2, 5, 6, 8, 10, 11, 12, 13]).

For two strict partitions λ and μ, we write $\lambda \supseteq \mu$ if $\lambda_i \geq \mu_i$ for all $i \geq 1$. In this case, the skew strict partition λ/μ can be identified with its skew shifted Young diagram. For example, the skew strict partition $(7, 5, 4, 1)/(4, 2, 1)$ is represented by the white boxes in Figure 2. Let f_λ (resp. $f_{\lambda/\mu}$) be the number of standard shifted Young tableaux of shape λ (resp. λ/μ). The following are well-known formulas (see [1, 15, 18]) analogous to (1.1):

$$f_\lambda = \frac{|\lambda|!}{H_\lambda} \qquad \text{and} \qquad \frac{1}{n!} \sum_{|\lambda|=n} 2^{n-\ell(\lambda)} f_\lambda^2 = 1. \tag{1.2}$$

In this paper, we generalize the latter equality of (1.2) by means of the following results.

Theorem 1.1. *Suppose that Q is a given symmetric function, and μ is a given strict partition. Then*

$$P(n) = \sum_{|\lambda/\mu|=n} \frac{2^{|\lambda|-|\mu|-\ell(\lambda)+\ell(\mu)} f_{\lambda/\mu}}{H_\lambda} Q\left(\binom{c_\square}{2} : \square \in \lambda\right)$$

is a polynomial in n, where $Q(\binom{c_\square}{2} : \square \in \lambda)$ means that $|\lambda|$ of the variables are substituted by $\binom{c_\square}{2}$ for $\square \in \lambda$, and all other variables by 0.

Theorem 1.2. *Suppose that k is a given nonnegative integer. Then*

$$\sum_{|\lambda|=n} \frac{2^{|\lambda|-\ell(\lambda)} f_\lambda}{H_\lambda} \sum_{\square \in \lambda} \binom{c_\square + k - 1}{2k} = \frac{2^k}{(k+1)!} \binom{n}{k+1}.$$

When $k = 0$ we derive the latter identity of (1.2). When $k = 1$, Theorem 1.2 becomes

$$\sum_{|\lambda|=n} \frac{2^{|\lambda|-\ell(\lambda)} f_\lambda}{H_\lambda} \sum_{\square \in \lambda} \binom{c_\square}{2} = \binom{n}{2},$$

which could also be obtained by setting $\mu = \emptyset$ in the next theorem.

Theorem 1.3. *Let μ be a strict partition. Then*

$$\sum_{|\lambda/\mu|=n} \frac{2^{|\lambda|-\ell(\lambda)-|\mu|+\ell(\mu)} f_{\lambda/\mu} H_\mu}{H_\lambda} \left(\sum_{\square\in\lambda} \binom{c_\square}{2} - \sum_{\square\in\mu} \binom{c_\square}{2} \right) = \binom{n}{2} + n|\mu|. \tag{1.3}$$

The proofs of those theorems are given in Section 4, by using the difference operator technique.

2. Difference operators

For each strict partition λ, the symbol λ^+ (resp. λ^-) always represents a strict partition obtained by adding (resp. removing) a box to (resp. from) λ. In other words, $|\lambda^+/\lambda| = 1$ and $|\lambda/\lambda^-| = 1$. By analogy with the difference operator for ordinary partitions introduced in [7], we define *the difference operator for strict partitions* by

$$D\left(g(\lambda)\right) := \sum_{\ell(\lambda^+)>\ell(\lambda)} g(\lambda^+) + 2 \sum_{\ell(\lambda^+)=\ell(\lambda)} g(\lambda^+) \ - \ g(\lambda),$$

where λ and λ^+ are strict partitions and g is a function on strict partitions. Notice that $\#\{\lambda^+ : \ell(\lambda^+) > \ell(\lambda)\} = 0$ or 1.

For each skew strict partition λ/μ, let ${f'}_{\lambda/\mu} := 2^{|\lambda|-|\mu|-\ell(\lambda)+\ell(\mu)} f_{\lambda/\mu}$.

Lemma 2.1. *For two different strict partitions $\lambda \supseteq \mu$ we have*

$${f'}_{\lambda/\mu} = \sum_{\substack{\lambda^- :\, \lambda\supseteq\lambda^-\supseteq\mu \\ \ell(\lambda^-)<\ell(\lambda)}} {f'}_{\lambda^-/\mu} + 2 \sum_{\substack{\lambda^- :\, \lambda\supseteq\lambda^-\supseteq\mu \\ \ell(\lambda^-)=\ell(\lambda)}} {f'}_{\lambda^-/\mu}.$$

Proof. By the construction of standard shifted Young tableaux we have

$$f_{\lambda/\mu} = \sum_{\lambda\supseteq\lambda^-\supseteq\mu} f_{\lambda^-/\mu}$$

and therefore

$$2^{|\lambda|-\ell(\lambda)} f_{\lambda/\mu} = \sum_{\substack{\lambda^- :\, \lambda\supseteq\lambda^-\supseteq\mu \\ \ell(\lambda^-)<\ell(\lambda)}} 2^{|\lambda^-|-\ell(\lambda^-)} f_{\lambda^-/\mu} + 2 \sum_{\substack{\lambda^- :\, \lambda\supseteq\lambda^-\supseteq\mu \\ \ell(\lambda^-)=\ell(\lambda)}} 2^{|\lambda^-|-\ell(\lambda^-)} f_{\lambda^-/\mu}.$$

Then by the definition of ${f'}_{\lambda/\mu}$ we prove the claim. $\square$

Lemma 2.2. *For each strict partition μ and each function g of strict partitions, let*

$$A(n) := \sum_{|\lambda/\mu|=n} {f'}_{\lambda/\mu} g(\lambda)$$

and

$$B(n) := \sum_{|\lambda/\mu|=n} {f'}_{\lambda/\mu} Dg(\lambda).$$

Then

$$A(n) = A(0) + \sum_{k=0}^{n-1} B(k).$$

Proof. We have

$$\begin{aligned}
A(n+1)-A(n) &= \sum_{|\gamma/\mu|=n+1} {f'}_{\gamma/\mu} g(\gamma) - \sum_{|\lambda/\mu|=n} {f'}_{\lambda/\mu} g(\lambda) \\
&= \sum_{|\gamma/\mu|=n+1} \left(\sum_{\substack{\gamma^-:\ \gamma\supseteq\gamma^-\supseteq\mu \\ \ell(\gamma^-)<\ell(\gamma)}} {f'}_{\gamma^-/\mu} + 2 \sum_{\substack{\gamma^-:\ \gamma\supseteq\gamma^-\supseteq\mu \\ \ell(\gamma^-)=\ell(\gamma)}} {f'}_{\gamma^-/\mu} \right) g(\gamma) \\
&\quad - \sum_{|\lambda/\mu|=n} {f'}_{\lambda/\mu} g(\lambda) \\
&= \sum_{|\lambda/\mu|=n} {f'}_{\lambda/\mu} \left(\sum_{\ell(\lambda^+)>\ell(\lambda)} g(\lambda^+) + 2 \sum_{\ell(\lambda^+)=\ell(\lambda)} g(\lambda^+) - g(\lambda) \right) \\
&= \sum_{|\lambda/\mu|=n} {f'}_{\lambda/\mu} Dg(\lambda) = B(n).
\end{aligned}$$

Thus

$$\begin{aligned}
A(n+1) &= A(n) + B(n) \\
&= A(n-1) + B(n-1) + B(n) \\
&= \cdots \\
&= A(0) + \sum_{k=0}^{n} B(k).
\end{aligned}$$

□

Theorem 2.3. *Let g be a function on strict partitions and μ be a given strict partition. Then we have*

$$\sum_{|\lambda/\mu|=n} 2^{|\lambda|-|\mu|-\ell(\lambda)+\ell(\mu)} f_{\lambda/\mu} g(\lambda) = \sum_{k=0}^{n} \binom{n}{k} D^k g(\mu) \tag{2.1}$$

and

$$D^n g(\mu) = \sum_{k=0}^{n} (-1)^{n+k} \binom{n}{k} \sum_{|\lambda/\mu|=k} 2^{|\lambda|-|\mu|-\ell(\lambda)+\ell(\mu)} f_{\lambda/\mu} g(\lambda). \tag{2.2}$$

In particular, if there exists some positive integer r such that $D^r g(\lambda) = 0$ for every strict partition λ, then the left-hand side of (2.1) *is a polynomial in n with degree at most $r-1$.*

Proof. We will prove (2.1) by induction. The case $n=0$ is trivial. Assume that (2.1) is true for some nonnegative integer n. Then by Lemma 2.2 we have

$$\begin{aligned}
\sum_{|\lambda/\mu|=n+1} {f'}_{\lambda/\mu} g(\lambda) &= \sum_{|\lambda/\mu|=n} {f'}_{\lambda/\mu} g(\lambda) + \sum_{|\lambda/\mu|=n} {f'}_{\lambda/\mu} Dg(\lambda) \\
&= \sum_{k=0}^{n} \binom{n}{k} D^k g(\mu) + \sum_{k=0}^{n} \binom{n}{k} D^{k+1} g(\mu) \\
&= \sum_{k=0}^{n+1} \binom{n+1}{k} D^k g(\mu).
\end{aligned}$$

Identity (2.2) follows from the Möbius inversion formula [14]. $\square$

Example. Let $g(\lambda) = 1/H_\lambda$. Then $Dg(\lambda) = 0$ by Theorem 3.3. The two quantities defined in Lemma 2.2 are:

$$A(n) = \sum_{|\lambda/\mu|=n} \frac{f'_{\lambda/\mu}}{H_\lambda} \qquad \text{and} \qquad B(n) = 0.$$

Consequently,

$$\sum_{|\lambda/\mu|=n} \frac{2^{|\lambda|-|\mu|-\ell(\lambda)+\ell(\mu)} f_{\lambda/\mu}}{H_\lambda} = \frac{1}{H_\mu}. \tag{2.3}$$

In particular, $\mu = \emptyset$ implies

$$\sum_{|\lambda|=n} \frac{2^{|\lambda|-\ell(\lambda)} f_\lambda}{H_\lambda} = 1, \tag{2.4}$$

or equivalently,

$$\sum_{|\lambda|=n} 2^{|\lambda|-\ell(\lambda)} f_\lambda^2 = n!. \tag{2.5}$$

3. Corners of strict partitions

For a strict partition λ, the *outer corners* are the boxes which can be removed to get a new strict partition λ^-. Let $(\alpha_1, \beta_1), \ldots, (\alpha_m, \beta_m)$ be the coordinates of outer corners such that $\alpha_1 > \alpha_2 > \cdots > \alpha_m$. Let $y_j = \beta_j - \alpha_j$ be the contents of outer corners for $1 \le j \le m$. We set $\alpha_{m+1} = 0$, $\beta_0 = \ell(\lambda) + 1$ and call $(\alpha_1, \beta_0), (\alpha_2, \beta_1), \ldots, (\alpha_{m+1}, \beta_m)$ the *inner corners* of λ. Let $x_i = \beta_i - \alpha_{i+1}$ be the contents of inner corners for $0 \le i \le m$ (see Figure 3). The following relations of x_i and y_j are obvious.

$$x_0 = 1 \le y_1 < x_1 < y_2 < x_2 < \cdots < y_m < x_m. \tag{3.1}$$

Notice that $y_1 = 1$ iff $\lambda_{\ell(\lambda)} = 1$.

We define

$$q_k(\lambda) := \sum_{i=0}^{m} \binom{x_i}{2}^k - \sum_{i=1}^{m} \binom{y_i}{2}^k \tag{3.2}$$

for each $k \ge 0$. For each partition $\nu = (\nu_1, \nu_2, \ldots, \nu_\ell)$ we define the function $q_\nu(\lambda)$ of strict partitions by

$$q_\nu(\lambda) := q_{\nu_1}(\lambda) q_{\nu_2}(\lambda) \cdots q_{\nu_\ell}(\lambda). \tag{3.3}$$

First we consider the difference between the hook length sets of λ and $\lambda^+ = \lambda \cup \{\square\}$ for some box $\square$.

Theorem 3.1. *Suppose that $\lambda^+ = \lambda \cup \{\square\}$ such that $c_\square = x_i$. If $i = 0$, then*

$$\frac{H_\lambda}{H_{\lambda^+}} = \frac{\prod\limits_{1 \le j \le m} \left(\binom{x_0}{2} - \binom{y_j}{2} \right)}{\prod\limits_{1 \le j \le m} \left(\binom{x_0}{2} - \binom{x_j}{2} \right)}.$$

If $1 \le i \le m$, *then*

$$\frac{H_\lambda}{H_{\lambda^+}} = \frac{1}{2} \cdot \frac{\prod\limits_{1\le j\le m} \left(\binom{x_i}{2} - \binom{y_j}{2}\right)}{\prod\limits_{\substack{0\le j\le m\\ j\neq i}} \left(\binom{x_i}{2} - \binom{x_j}{2}\right)}.$$

Proof. First we consider the case $i = 0$, which means that $y_1 \ge 2$. In this case we add the box $\square = (\ell+1, \ell+2)$ to λ with $c_\square = x_0 = 1$ where ℓ is the length of λ. By the definition, it is easy to see that the hook lengths of boxes which are in the $(\ell+1)$-th column and $(\ell+2)$-th column of λ increase by 1, and the hook lengths of boxes in the other columns don't change. Since the boxes which are in the $\ell+1$-column and $\ell+2$-column of λ have hook lengths

$$\bigcup_{j=1}^{m} \left(\{h : y_j - 1 \le h \le x_j - 2\} \cup \{h : y_j \le h \le x_j - 1\}\right),$$

then the boxes which are in the $\ell+1$-column and $\ell+2$-column of λ^+ have hook lengths

$$\{1\} \cup \left(\bigcup_{j=1}^{m} \left(\{h : y_j \le h \le x_j - 1\} \cup \{h : y_j + 1 \le h \le x_j\}\right) \right).$$

Therefore

$$\begin{aligned}\mathcal{H}(\lambda) \setminus \mathcal{H}(\lambda^+) &= \{y_1, y_1 - 1, y_2, y_2 - 1, \cdots, y_m, y_m - 1\}\\ &\quad \setminus \{1, x_1, x_1 - 1, x_2, x_2 - 1, \cdots, x_m, x_m - 1\},\end{aligned}$$

which means that

$$\frac{H_\lambda}{H_{\lambda^+}} = \frac{\prod\limits_{1\le j\le m} y_j(y_j-1)}{\prod\limits_{1\le j\le m} x_j(x_j-1)} = \frac{\prod\limits_{1\le j\le m} \left(\binom{x_0}{2} - \binom{y_j}{2}\right)}{\prod\limits_{1\le j\le m} \left(\binom{x_0}{2} - \binom{x_j}{2}\right)}$$

since $x_0 = 1$.

Similarly, for the case $1 \le i \le m$, we add the box $\square = (\alpha_{i+1} + 1, \beta_i + 1)$ to λ with $c_\square = x_i$. By the definition, it is easy to see that the hook lengths of boxes which are in the $(\alpha_{i+1}+1)$-th column, (β_i+1)-th column and $(\alpha_{i+1}+1)$-th row of λ increase by 1, and the hook lengths of other boxes don't change. Since the boxes

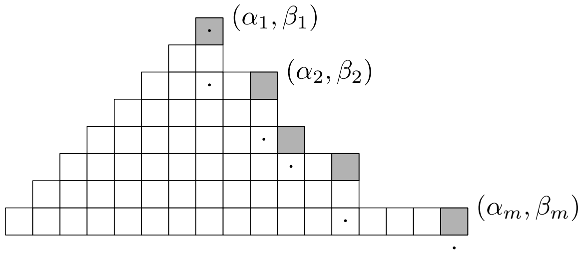

FIGURE 3. A strict partition and its corners. The outer corners are labelled with (α_i, β_i) $(i = 1, 2, \ldots, m)$. The inner corners are indicated by the dot symbol "$\cdot$".

which are in the $(\alpha_{i+1}+1)$-th column, (β_i+1)-th column and $(\alpha_{i+1}+1)$-th row of λ have hook lengths

$$\bigcup_{j=i+1}^{m}\{h : x_i+y_j-1\le h\le x_i+x_j-2\},$$

$$\bigcup_{j=i+1}^{m}\{h : y_j-x_i\le h\le x_j-x_i-1\},$$

and

$$\begin{aligned}&\{h : x_i-y_1\le h\le x_i-1\}\cup\{h : x_i+y_i-1\le h\le 2x_i-3\}\\ &\cup\bigcup_{j=1}^{i-1}\left(\{h : x_i-y_{j+1}\le h\le x_i-x_j-1\}\cup\{h : x_i+y_j-1\le h\le x_i+x_j-2\}\right)\end{aligned}$$

respectively, then the same boxes which are in the $(\alpha_{i+1}+1)$-th column, (β_i+1)-th column and $(\alpha_{i+1}+1)$-th row of λ^+ have hook lengths

$$\bigcup_{j=i+1}^{m}\{h : x_i+y_j\le h\le x_i+x_j-1\},$$

$$\bigcup_{j=i+1}^{m}\{h : y_j-x_i+1\le h\le x_j-x_i\},$$

and

$$\begin{aligned}&\{h : x_i-y_1+1\le h\le x_i\}\cup\{h : x_i+y_i\le h\le 2x_i-2\}\\ &\cup\bigcup_{j=1}^{i-1}\left(\{h : x_i-y_{j+1}+1\le h\le x_i-x_j\}\cup\{h : x_i+y_j\le h\le x_i+x_j-1\}\right)\end{aligned}$$

respectively. Notice that the box $\square=(\alpha_{i+1}+1,\beta_i+1)$ in λ^+ has hook length 1, then we have

$$\begin{aligned}\mathcal{H}(\lambda)\setminus\mathcal{H}(\lambda^+)=&\left(\{|x_i-y_j| : 1\le j\le m\}\cup\{x_i+y_j-1 : 1\le j\le m\}\right)\\ &\setminus\big(\{1,x_i,2x_i-2\}\cup\{|x_i-x_j| : 1\le j\le m, j\ne i\}\\ &\cup\{x_i+x_j-1 : 1\le j\le m, j\ne i\}\big),\end{aligned}$$

which means that

$$\frac{H_\lambda}{H_{\lambda^+}}=\frac{1}{x_i(2x_i-2)}\cdot\frac{\prod\limits_{1\le j\le m}(x_i-y_j)(x_i+y_j-1)}{\prod\limits_{\substack{1\le j\le m\\ j\ne i}}(x_i-x_j)(x_i+x_j-1)}=\frac{1}{2}\cdot\frac{\prod\limits_{1\le j\le m}\left(\binom{x_i}{2}-\binom{y_j}{2}\right)}{\prod\limits_{\substack{0\le j\le m\\ j\ne i}}\left(\binom{x_i}{2}-\binom{x_j}{2}\right)}.\qquad\square$$

Suppose that $a_0<a_1<\cdots<a_m$ and $b_1<\cdots<b_m$ are real numbers. Let

$$q_k(\{a_i\},\{b_i\}) := \sum_{i=0}^{m}{a_i}^k-\sum_{i=1}^{m}{b_i}^k \tag{3.4}$$

for each $k \geq 0$ and

$$q_\nu(\{a_i\},\{b_i\}) := \prod_{j=1}^{\ell} q_{\nu_j}(\{a_i\},\{b_i\}) \tag{3.5}$$

for the usual partition $\nu = (\nu_1, \nu_2, \ldots, \nu_\ell)$. Notice that

$$q_k(\lambda) = q_k\left(\{\binom{x_i}{2}\}_{0\leq i\leq m}, \{\binom{y_i}{2}\}_{1\leq i\leq m}\right).$$

Theorem 3.2. *Let k be a nonnegative integer. Then there exist some $\xi_\nu \in \mathbb{Q}$ such that*

$$\sum_{0\leq i\leq m} \frac{\prod\limits_{1\leq j\leq m}(a_i - b_j)}{\prod\limits_{\substack{0\leq j\leq m\\ j\neq i}}(a_i - a_j)} a_i^k = \sum_{|\nu|\leq k} \xi_\nu q_\nu(\{a_i\},\{b_i\})$$

for all real numbers $a_0 < a_1 < \cdots < a_m$ and $b_1 < b_2 < \cdots < b_m$.

Proof. First notice we just need to prove the case that $a_i \neq 0$ for all i. Because if we multiply by $\prod_{0\leq i<j\leq n}(a_i - a_j)$ on both sides of the above formula, then both sides become polynomials in $a_0, a_1, \ldots, a_m$ and $b_1, b_2, \ldots, b_m$, which means they are continuous functions on such variables. Therefore if the above formula is true for all nonzero a_i, then it is also true for the case $a_i = 0$ for some i. Let

$$g(z) = \prod_{1\leq j\leq m}(1 - b_j z) - \sum_{0\leq i\leq m} \frac{\prod\limits_{1\leq j\leq m}(a_i - b_j)}{\prod\limits_{\substack{0\leq j\leq m\\ j\neq i}}(a_i - a_j)} \prod_{\substack{0\leq j\leq m\\ j\neq i}}(1 - a_j z).$$

Then for $0 \leq i \leq m$ we obtain

$$g\left(\frac{1}{a_i}\right) = \prod_{1\leq j\leq m}\left(1 - \frac{b_j}{a_i}\right) - \frac{\prod\limits_{1\leq j\leq m}(a_i - b_j)}{\prod\limits_{\substack{0\leq j\leq m\\ j\neq i}}(a_i - a_j)} \prod_{\substack{0\leq j\leq m\\ j\neq i}}\left(1 - \frac{a_j}{a_i}\right) = 0.$$

This means that $g(z)$ has at least $m+1$ roots, so that $g(z) = 0$ since $g(z)$ is a polynomial in z with degree at most m. Therefore

$$\sum_{0\leq i\leq m} \frac{\prod\limits_{1\leq j\leq m}(a_i - b_j)}{\prod\limits_{\substack{0\leq j\leq m\\ j\neq i}}(a_i - a_j)} \cdot \frac{1}{1 - a_i z} = \frac{\prod_{1\leq j\leq m}(1 - b_j z)}{\prod_{0\leq j\leq m}(1 - a_j z)},$$

which means that

$$\begin{aligned} &\sum_{0\leq i\leq m} \frac{\prod\limits_{1\leq j\leq m}(a_i - b_j)}{\prod\limits_{\substack{0\leq j\leq m\\ j\neq i}}(a_i - a_j)} \left(\sum_{k\geq 0}(a_i z)^k\right) \\ &= \exp\left(\sum_{1\leq j\leq m} \ln(1 - b_j z) - \sum_{0\leq i\leq m} \ln(1 - a_i z)\right) \end{aligned} \tag{3.6}$$

$$= \exp\left(\sum_{k\geq 1}\frac{q_k(\{a_i\},\{b_i\})}{k}z^k\right).$$

Comparing the coefficients of z^k on both sides, we obtain there exist some $\xi_\nu \in \mathbb{Q}$ such that

$$\sum_{0\leq i\leq m}\frac{\prod\limits_{1\leq j\leq m}(a_i-b_j)}{\prod\limits_{\substack{0\leq j\leq m\\ j\neq i}}(a_i-a_j)}a_i^k = \sum_{|\nu|\leq k}\xi_\nu q_\nu(\{a_i\},\{b_i\})$$

for all real numbers $a_0 < a_1 < \cdots < a_m$ and $b_1 < b_2 < \cdots < b_m$. □

By (3.6), when $k = 0, 1, 2$, we obtain

$$\sum_{0\leq i\leq m}\frac{\prod\limits_{1\leq j\leq m}(a_i-b_j)}{\prod\limits_{\substack{0\leq j\leq m\\ j\neq i}}(a_i-a_j)} = 1, \tag{3.7}$$

$$\sum_{0\leq i\leq m}\frac{\prod\limits_{1\leq j\leq m}(a_i-b_j)}{\prod\limits_{\substack{0\leq j\leq m\\ j\neq i}}(a_i-a_j)}a_i = q_1(\{a_i\},\{b_i\}), \tag{3.8}$$

$$\sum_{0\leq i\leq m}\frac{\prod\limits_{1\leq j\leq m}(a_i-b_j)}{\prod\limits_{\substack{0\leq j\leq m\\ j\neq i}}(a_i-a_j)}a_i^2 = \frac{q_1^2(\{a_i\},\{b_i\})+q_2(\{a_i\},\{b_i\})}{2}. \tag{3.9}$$

Let $\lambda^{i+} = \lambda\cup\{\square_i\}$ such that $c_{\square_i} = x_i$ for $1\leq i\leq m$. If $y_1 > 1$, let $\lambda^{0+} = \lambda\cup\{\square_0\}$ such that $c_{\square_0} = x_0 = 1$.

Theorem 3.3. *Suppose that λ is a given strict partition. Then*

$$D\left(\frac{1}{H_\lambda}\right) = 0.$$

Proof. Notice that when $y_1 = 1$, we have $\{\lambda^+ : \ell(\lambda^+) > \ell(\lambda)\} = \emptyset$, therefore

$$\sum_{\ell(\lambda^+)>\ell(\lambda)}\frac{H_\lambda}{H_{\lambda^+}} = 0 = \frac{\prod\limits_{1\leq j\leq m}\left(\binom{x_0}{2}-\binom{y_j}{2}\right)}{\prod\limits_{1\leq j\leq m}\left(\binom{x_0}{2}-\binom{x_j}{2}\right)}.$$

When $y_1 > 1$, we have $\{\lambda^+ : \ell(\lambda^+) > \ell(\lambda)\} = \{\lambda^{0+}\}$, therefore by Theorem 3.1 we also obtain

$$\sum_{\ell(\lambda^+)>\ell(\lambda)}\frac{H_\lambda}{H_{\lambda^+}} = \frac{\prod\limits_{1\leq j\leq m}\left(\binom{x_0}{2}-\binom{y_j}{2}\right)}{\prod\limits_{1\leq j\leq m}\left(\binom{x_0}{2}-\binom{x_j}{2}\right)}$$

and

$$\sum_{\ell(\lambda^+)>\ell(\lambda)}\frac{H_\lambda}{H_{\lambda^+}} + 2\sum_{\ell(\lambda^+)=\ell(\lambda)}\frac{H_\lambda}{H_{\lambda^+}} = \sum_{0\leq i\leq m}\frac{\prod\limits_{1\leq j\leq m}\left(\binom{x_i}{2}-\binom{y_j}{2}\right)}{\prod\limits_{\substack{0\leq j\leq m\\ j\neq i}}\left(\binom{x_i}{2}-\binom{x_j}{2}\right)}.$$

Let $a_i = \binom{x_i}{2}$ and $b_i = \binom{y_i}{2}$ in (3.7), we obtain

$$D\left(\frac{1}{H_\lambda}\right) = \frac{1}{H_\lambda}\left(\sum_{\ell(\lambda^+)>\ell(\lambda)} \frac{H_\lambda}{H_{\lambda^+}} + 2\sum_{\ell(\lambda^+)=\ell(\lambda)} \frac{H_\lambda}{H_{\lambda^+}} - 1\right) = 0. \qquad \square$$

Corollary 3.4. *Suppose that g is a function on strict partitions. Then*

$$D\left(\frac{g(\lambda)}{H_\lambda}\right) = \sum_{\ell(\lambda^+)>\ell(\lambda)} \frac{g(\lambda^+)-g(\lambda)}{H_{\lambda^+}} + 2\sum_{\ell(\lambda^+)=\ell(\lambda)} \frac{g(\lambda^+)-g(\lambda)}{H_{\lambda^+}}$$

for every strict partition λ.

Proof. The corollary follows directly from the definition of the operator D and the last identity in the proof of Theorem 3.3. $\square$

Theorem 3.5. *Let k be a given nonnegative integer and λ be a strict partition. Then there exist some $\xi_j \in \mathbb{Q}$ such that*

$$q_k(\lambda^{i+}) - q_k(\lambda) = \sum_{j=0}^{k-1} \xi_j \binom{x_i}{2}^j$$

for every strict partition λ and every i.

Proof. Denote by $X = \{x_0, x_1, \ldots, x_m\}$ and $Y = \{y_1, y_2, \ldots, y_m\}$. For $1 \le i \le m$, four cases are to be considered.

(i) If $\beta_i + 1 < \beta_{i+1}$ and $\alpha_{i+1} + 1 < \alpha_i$. Then it is easy to see that the contents of inner corners and outer corners of λ^{i+} are $X \cup \{x_i - 1, x_i + 1\} \setminus \{x_i\}$ and $Y \cup \{x_i\}$ respectively.

(ii) If $\beta_i + 1 = \beta_{i+1}$ and $\alpha_{i+1} + 1 < \alpha_i$, so that $y_{i+1} = x_i + 1$. Hence the contents of inner corners and outer corners of λ^{i+} are $X \cup \{x_i - 1\} \setminus \{x_i\}$ and $Y \cup \{x_i\} \setminus \{x_i + 1\}$ respectively.

(iii) If $\beta_i + 1 < \beta_{i+1}$ and $\alpha_{i+1} + 1 = \alpha_i$, so that $y_i = x_i - 1$. Then the contents of inner corners and outer corners of λ^{i+} are $X \cup \{x_i + 1\} \setminus \{x_i\}$ and $Y \cup \{x_i\} \setminus \{x_i - 1\}$ respectively.

(iv) If $\beta_i + 1 = \beta_{i+1}$ and $\alpha_{i+1} + 1 = \alpha_i$. Then $y_i + 1 = x_i = y_{i+1} - 1$. The contents of inner corners and outer corners of λ^{i+} are $X \setminus \{x_i\}$ and $Y \cup \{x_i\} \setminus \{x_i - 1, x_i + 1\}$ respectively.

For $i = 0$, two cases are to be considered.

(v) If $y_1 = 2$, the contents of inner corners and outer corners of λ^{0+} are X and $Y \cup \{1\} \setminus \{2\}$ respectively.

(vi) If $y_1 > 2$, the contents of inner corners and outer corners of λ^{i+} are $X \cup \{2\}$ and $Y \cup \{1\}$ respectively.

In each of the six cases, we always have

$$q_k(\lambda^{i+}) - q_k(\lambda) = \binom{x_i+1}{2}^k + \binom{x_i-1}{2}^k - 2\binom{x_i}{2}^k. \tag{3.10}$$

Next we have for all $z \in \mathbb{R}$,

$$(z+2)^{2k} + (z-2)^{2k} - 2z^{2k} = 2\sum_{1\le j\le k} \binom{2k}{2j} 2^{2j} z^{2k-2j}.$$

Replace z by $2z-1$, we obtain

$$(2z+1)^{2k}+(2z-3)^{2k}-2(2z-1)^{2k}=2\sum_{1\le j\le k}\binom{2k}{2j}2^{2j}(2z-1)^{2k-2j},$$

or

$$\begin{aligned}&\left(8\binom{z+1}{2}+1\right)^k+\left(8\binom{z-1}{2}+1\right)^k-2\left(8\binom{z}{2}+1\right)^k\\ =&2\sum_{1\le j\le k}\binom{2k}{2j}2^{2j}\left(8\binom{z}{2}+1\right)^{k-j}.\end{aligned}$$

Then by induction on k we have

$$\binom{x_i+1}{2}^k+\binom{x_i-1}{2}^k-2\binom{x_i}{2}^k=\sum_{j=0}^{k-1}\xi_j\binom{x_i}{2}^j$$

for some constants $\xi_j\in\mathbb{Q}$. □

Theorem 3.6. *Let $\nu=(\nu_1,\nu_2,\dots,\nu_\ell)$ be a partition. Then there exist some $\xi_\delta\in\mathbb{Q}$ such that*

$$D\left(\frac{q_\nu(\lambda)}{H_\lambda}\right)=\sum_{|\delta|\le|\nu|-1}\xi_\delta\frac{q_\delta(\lambda)}{H_\lambda} \tag{3.11}$$

for every strict partition λ.

Proof. For $0\le i\le m$, we have

$$\prod_{k=1}^{\ell}q_{\nu_k}(\lambda^{i+})-\prod_{k=1}^{\ell}q_{\nu_k}(\lambda)=\sum_{(*)}\prod_{k\in U}q_{\nu_k}(\lambda)\prod_{k'\in V}\big(q_{\nu_{k'}}(\lambda^{i+})-q_{\nu_{k'}}(\lambda)\big), \tag{3.12}$$

where the sum $(*)$ ranges over all pairs (U,V) of positive integer sets such that $U\cup V=\{1,2,\dots,\ell\}$, $U\cap V=\emptyset$ and $V\neq\emptyset$. Actually the Identity (3.12) follows by the inclusion-exclusion principle. By Corollary 3.4 and Theorem 3.1 we have

$$\begin{aligned}H_\lambda D(\frac{q_\nu(\lambda)}{H_\lambda})=&\sum_{\ell(\lambda^+)>\ell(\lambda)}\frac{H_\lambda\left(q_\nu(\lambda^+)-q_\nu(\lambda)\right)}{H_{\lambda^+}}+2\sum_{\ell(\lambda^+)=\ell(\lambda)}\frac{H_\lambda\left(q_\nu(\lambda^+)-q_\nu(\lambda)\right)}{H_{\lambda^+}}\\ =&\sum_{0\le i\le m}\frac{\prod\limits_{1\le j\le m}\left(\binom{x_i}{2}-\binom{y_j}{2}\right)}{\prod\limits_{\substack{0\le j\le m\\ j\neq i}}\left(\binom{x_i}{2}-\binom{x_j}{2}\right)}\left(\prod_{k=1}^{\ell}q_{\nu_k}(\lambda^{i+})-\prod_{k=1}^{\ell}q_{\nu_k}(\lambda)\right)\\ =&\sum_{(*)}\prod_{k\in U}q_{\nu_k}(\lambda)\sum_{0\le i\le m}\frac{\prod\limits_{1\le j\le m}\left(\binom{x_i}{2}-\binom{y_j}{2}\right)}{\prod\limits_{\substack{0\le j\le m\\ j\neq i}}\left(\binom{x_i}{2}-\binom{x_j}{2}\right)}\prod_{k'\in V}\big(q_{\nu_{k'}}(\lambda^{i+})-q_{\nu_{k'}}(\lambda)\big).\end{aligned}$$

Then the claim follows from Theorems 3.5 and 3.2. □

4. Proofs of Theorems

Instead of proving Theorem 1.1, we prove the following more general result, which implies Theorem 1.1 when $\nu = \emptyset$.

Theorem 4.1. *Suppose that $\nu = (\nu_1, \nu_2, \ldots, \nu_\ell)$ is a given partition, μ is a given strict partition and Q is a symmetric function. Then there exists some $r \in \mathbb{N}$ such that*

$$D^r\left(\frac{Q\left(\binom{c_\square}{2} : \square \in \lambda\right) q_\nu(\lambda)}{H_\lambda}\right) = 0$$

for every strict partition λ. Consequently,

$$P(n) = \sum_{|\lambda/\mu|=n} \frac{2^{|\lambda|-|\mu|-\ell(\lambda)+\ell(\mu)} f_{\lambda/\mu}}{H_\lambda} Q\left(\binom{c_\square}{2} : \square \in \lambda\right) q_\nu(\lambda)$$

is a polynomial in n.

Proof. By linearity we can assume that

$$Q\left(\binom{c_\square}{2} : \square \in \lambda\right) = \prod_{t=1}^{s} \sum_{\square\in\lambda} \binom{c_\square}{2}^{r_t}$$

for some tuple $(r_1, r_2, \ldots, r_s)$. Let

$$\begin{aligned}
A &= q_\nu(\lambda),\\
\Delta_i A &= q_\nu(\lambda^{i+}) - q_\nu(\lambda),\\
B &= \prod_{t=1}^{s} \sum_{\square\in\lambda} \binom{c_\square}{2}^{r_t},\\
\Delta_i B &= \prod_{t=1}^{s} \sum_{\square\in\lambda^{i+}} \binom{c_\square}{2}^{r_t} - \prod_{t=1}^{s} \sum_{\square\in\lambda} \binom{c_\square}{2}^{r_t}.
\end{aligned}$$

We have

$$\begin{aligned}
\Delta_i A &= \sum_{(*)} \prod_{k\in U} q_{\nu_k}(\lambda) \prod_{k'\in V} \left(q_{\nu_{k'}}(\lambda^{i+}) - q_{\nu_{k'}}(\lambda)\right),\\
\Delta_i B &= \sum_{(**)} \prod_{t\in U} \sum_{\square\in\lambda} \binom{c_\square}{2}^{r_t} \prod_{t'\in V} \left(\sum_{\square\in\lambda^{i+}} \binom{c_\square}{2}^{r_{t'}} - \sum_{\square\in\lambda} \binom{c_\square}{2}^{r_{t'}} \right)\\
&= \sum_{(**)} \prod_{t\in U} \sum_{\square\in\lambda} \binom{c_\square}{2}^{r_t} \prod_{t'\in V} \binom{x_i}{2}^{r_{t'}},
\end{aligned}$$

where the sum $(*)$ (resp. $(**)$) ranges over all pairs (U, V) of positive integer sets such that $U \cup V = \{1, 2, \ldots, \ell\}$ (resp. $U \cup V = \{1, 2, \ldots, s\}$), $U \cap V = \emptyset$ and $V \neq \emptyset$. It follows from Corollary 3.4 and Theorem 3.1 that

$$\begin{aligned}
&H_\lambda D\left(\frac{q_\nu(\lambda) \prod_{t=1}^{s} \sum_{\square\in\lambda} \binom{c_\square}{2}^{r_t}}{H_\lambda}\right)\\
&= \sum_{\ell(\lambda^+)>\ell(\lambda)} \frac{H_\lambda}{H_{\lambda^+}} \left(q_\nu(\lambda^+) \prod_{t=1}^{s} \sum_{\square\in\lambda^+} \binom{c_\square}{2}^{r_t} - q_\nu(\lambda) \prod_{t=1}^{s} \sum_{\square\in\lambda} \binom{c_\square}{2}^{r_t} \right)
\end{aligned}$$

$$
\begin{aligned}
&+2\sum_{\ell(\lambda^+)=\ell(\lambda)}\frac{H_\lambda}{H_{\lambda^+}}\left(q_\nu(\lambda^+)\prod_{t=1}^{s}\sum_{\square\in\lambda^+}\binom{c_\square}{2}^{r_t}-q_\nu(\lambda)\prod_{t=1}^{s}\sum_{\square\in\lambda}\binom{c_\square}{2}^{r_t}\right)\\
&=\sum_{i=0}^{m}\frac{\prod\limits_{1\le j\le m}\left(\binom{x_i}{2}-\binom{y_j}{2}\right)}{\prod\limits_{\substack{0\le j\le m\\ j\ne i}}\left(\binom{x_i}{2}-\binom{x_j}{2}\right)}\left(q_\nu(\lambda^{i+})\prod_{t=1}^{s}\sum_{\square\in\lambda^{i+}}\binom{c_\square}{2}^{r_t}-q_\nu(\lambda)\prod_{t=1}^{s}\sum_{\square\in\lambda}\binom{c_\square}{2}^{r_t}\right)\\
&=\sum_{i=0}^{m}\frac{\prod\limits_{1\le j\le m}\left(\binom{x_i}{2}-\binom{y_j}{2}\right)}{\prod\limits_{\substack{0\le j\le m\\ j\ne i}}\left(\binom{x_i}{2}-\binom{x_j}{2}\right)}\left(A\cdot\Delta_iB+B\cdot\Delta_iA+\Delta_iA\cdot\Delta_iB\right).
\end{aligned}
$$

Then by Theorems 3.2, 3.5, and 3.6 each of the above three terms could be written as a linear combination of some $\prod_{\bar t=1}^{\bar s}\sum_{\square\in\lambda}\binom{c_\square}{2}^{\bar r_{\bar t}}q_{\bar\nu}(\lambda)$ satisfying one of the following two conditions:

(1) $\bar s<s$;

(2) $\bar s=s$ and $|\bar\nu|\le|\nu|-1$.

Therefore the theorem follows by induction on s and $|\nu|$. □

Proof of Theorem 1.2. The special case in the proof of Theorem 4.1 with $\nu=\emptyset$ and $s=1$ yields

$$
\begin{aligned}
H_\lambda D\left(\frac{\sum_{\square\in\lambda}\binom{c_\square}{2}^{r_1}}{H_\lambda}\right)&=\sum_{0\le i\le m}\frac{\prod\limits_{1\le j\le m}\left(\binom{x_i}{2}-\binom{y_j}{2}\right)}{\prod\limits_{\substack{0\le j\le m\\ j\ne i}}\left(\binom{x_i}{2}-\binom{x_j}{2}\right)}\binom{x_i}{2}^{r_1}\\
&=\sum_{|\nu|\le r_1}\xi_\nu q_\nu(\lambda),
\end{aligned}
$$

where ξ_ν are some constants. The last equality is due to Theorem 3.2. Notice that

$$
(2k)!\binom{z+k-1}{2k}=2^k\prod_{i=1}^{k}\left(\binom{z}{2}-\binom{i}{2}\right).
$$

Then by Theorems 3.6 and 2.3 we know that

$$
P(n)=\sum_{|\lambda|=n}\frac{f'_\lambda}{H_\lambda}\sum_{\square\in\lambda}\binom{c_\square+k-1}{2k}
$$

is a polynomial in n with degree at most $k+1$.

On the other hand,

$$
P(k+1)=\frac{f'_{(k+1)}}{H_{(k+1)}}\binom{2k}{2k}=\frac{2^k}{(k+1)!}
$$

since $\lambda=(k+1)$ is the only strict partition with size $k+1$ who has contents greater than k. Moreover, it is obvious that $P(0)=P(1)=\cdots=P(k)=0$. Since the polynomial $P(n)$ is uniquely determined by those values, we obtain

$$
P(n)=\frac{2^k}{(k+1)!}\binom{n}{k+1}.
$$

□

By Theorem 4.1, the left-hand side of (1.3) is a polynomial in n. To evaluate this polynomial explicitly, we need the following lemma.

Lemma 4.2. *Let λ be a strict partition. Then*

$$q_1(\lambda) = |\lambda|.$$

Proof. By the definition of the size of λ, we have

$$|\lambda| = \sum_{i=1}^{m} \sum_{j=y_i}^{x_i-1} j = \sum_{i=1}^{m} \left(\binom{x_i}{2} - \binom{y_i}{2} \right) = q_1(\lambda).$$

□

Proof of Theorem 1.3. It is easy to check that both sides of (1.3) are equal for $n = 0, 1, 2$. By Corollary 3.4, Theorem 3.1 and Identity (3.8) it is easy to see that

$$\begin{aligned} H_\lambda D\left(\frac{\sum_{\square\in\lambda}\binom{c_\square}{2}}{H_\lambda}\right) &= \sum_{\ell(\lambda^+)>\ell(\lambda)} \frac{H_\lambda}{H_{\lambda^+}} \left(\sum_{\square\in\lambda^+} \binom{c_\square}{2} - \sum_{\square\in\lambda} \binom{c_\square}{2} \right) \\ &\quad + 2 \sum_{\ell(\lambda^+)=\ell(\lambda)} \frac{H_\lambda}{H_{\lambda^+}} \left(\sum_{\square\in\lambda^+} \binom{c_\square}{2} - \sum_{\square\in\lambda} \binom{c_\square}{2} \right) \\ &= \sum_{0\le i\le m} \frac{\prod\limits_{1\le j\le m} \left(\binom{x_i}{2} - \binom{y_j}{2}\right)}{\prod\limits_{\substack{0\le j\le m \\ j\neq i}} \left(\binom{x_i}{2} - \binom{x_j}{2}\right)} \binom{x_i}{2} \\ &= q_1(\lambda) \\ &= |\lambda|. \end{aligned}$$

Therefore we have

$$H_\lambda D^2\left(\frac{\sum_{\square\in\lambda}\binom{c_\square}{2}}{H_\lambda}\right) = 1,$$

$$H_\lambda D^3\left(\frac{\sum_{\square\in\lambda}\binom{c_\square}{2}}{H_\lambda}\right) = 0.$$

Then our claim follows from Theorem 2.3. □

Similarly, by (3.7), (3.8) and (3.9) we have

$$H_\lambda D\left(\frac{\sum_{\square\in\lambda}\binom{c_\square+1}{4}}{H_\lambda}\right) = \frac{1}{12}\left(q_2(\lambda) + |\lambda|^2 - 2|\lambda|\right),$$

$$H_\lambda D^2\left(\frac{\sum_{\square\in\lambda}\binom{c_\square+1}{4}}{H_\lambda}\right) = \frac{2}{3}|\lambda|,$$

$$H_\lambda D^3\left(\frac{\sum_{\square\in\lambda}\binom{c_\square+1}{4}}{H_\lambda}\right) = \frac{2}{3},$$

$$H_\lambda D^4\left(\frac{\sum_{\square\in\lambda}\binom{c_\square+1}{4}}{H_\lambda}\right) = 0.$$

Thus by Theorem 2.3 we obtain the following result.

Theorem 4.3. *Let μ be a strict partition. Then*

$$\sum_{|\lambda/\mu|=n} \frac{2^{|\lambda|-\ell(\lambda)-|\mu|+\ell(\mu)} f_{\lambda/\mu} H_\mu}{H_\lambda} \left(\sum_{\square\in\lambda} \binom{c_\square+1}{4} - \sum_{\square\in\mu} \binom{c_\square+1}{4} \right)$$
$$= \frac{2}{3}\binom{n}{3} + \frac{2}{3}|\mu|\binom{n}{2} + \frac{1}{12}\left(q_2(\mu)+|\mu|^2-2|\mu|\right)n.$$

5. Acknowledgments

The authors thank the referees for carefully reading the paper and for giving helpful comments. The second author thanks Prof. P.-O. Dehaye for the encouragements and helpful suggestions. The second author is supported by Grant [PP00P2_138906] of the Swiss National Science Foundation and Forschungskredit [FK-14-093] of the University of Zurich.

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