SOME USEFUL THEOREMS FOR ASYMPTOTIC FORMULAS AND THEIR APPLICATIONS TO SKEW PLANE PARTITIONS AND CYLINDRIC PARTITIONS

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Abstract. Inspired by the works of Dewar, Murty and Kotěšovec, we establish some useful theorems for asymptotic formulas. As an application, we obtain asymptotic formulas for the numbers of skew plane partitions and cylindric partitions. We prove that the order of the asymptotic formula for the skew plane partitions of fixed width depends only on the width of the region, not on the profile (the skew zone) itself, while this is not true for cylindric partitions.

1. Introduction

Inspired by the works of Dewar, Murty and Kotěšovec [7, 12], we establish some useful theorems for asymptotic formulas. Define

$$\psi_n(v,r,b;p) := v\sqrt{\frac{p(1-p)}{2\pi}}\,\frac{r^{b+(1-p)/2}}{n^{b+1-p/2}}\exp(n^p r^{1-p}) \tag{1.1}$$

for $n \in \mathbb{N}$, $v, b \in \mathbb{R}$, $r > 0$, $0 < p < 1$.

Theorem 1.1. *Let t_1 and t_2 be two given positive integers with $gcd(t_1, t_2) = 1$. Suppose that*

$$F_1(q) = \sum_{n=0}^{\infty} a_{t_1 n} q^{t_1 n} \qquad \text{and} \qquad F_2(q) = \sum_{n=0}^{\infty} c_{t_2 n} q^{t_2 n}$$

are two power series such that their coefficients satisfy the asymptotic formulas

$$a_{t_1 n} \sim t_1 \psi_{t_1 n}(v_1, r_1, b_1; p), \tag{1.2}$$

$$c_{t_2 n} \sim t_2 \psi_{t_2 n}(v_2, r_2, b_2; p), \tag{1.3}$$

where $v_1, b_1, v_2, b_2 \in \mathbb{R}$, $r_1, r_2 > 0$, $0 < p < 1$. Then, the coefficients d_n in the product

$$F_1(q)F_2(q) = \sum_{n=0}^{\infty} d_n q^n$$

satisfy the following asymptotic formula

$$d_n \sim \psi_n(v_1 v_2, r_1 + r_2, b_1 + b_2; p). \tag{1.4}$$

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Some special cases of Theorem 1.1 have been established. In 2013, Dewar and Murty [7] proved the case of $p = 1/2$, $t_1 = t_2 = 1$. Later, Kotěšovec [12] obtained the case of $0 < p < 1$, $t_1 = t_2 = 1$. We add two more parameters t_1 and t_2 in order to calculate the asymptotic formulas for plane partitions, without them Theorem 1.2 would not be proven. Our further contribution is to reformulate the asymptotic formula in a much more simpler form (1.2), (1.3) and (1.4), so that the result of Theorem 1.1 can be easily iterated for handling a product of multiple power series $F_1(q)F_2(q)\cdots F_k(q)$ (see Theorem 2.3). We also obtain the following two theorems, which are useful to find asymptotic formulas for various plane partitions.

Theorem 1.2. *Let m be a positive integer. Suppose that x_i and y_i $(1 \le i \le m)$ are positive integers such that $\gcd(x_1, x_2, \ldots, x_m, y_1, y_2, \ldots, y_m) = 1$. Then, the coefficients d_n in the following infinite product*

$$\prod_{i=1}^{m}\prod_{k\ge 0}\frac{1}{1-q^{x_ik+y_i}} = \sum_{n=0}^{\infty} d_n q^n$$

have the following asymptotic formula

$$d_n \ \sim\ v\frac{1}{2\sqrt{2\pi}}\frac{r^{b+1/4}}{n^{b+3/4}}\exp(\sqrt{nr}), \tag{1.5}$$

where

$$v = \prod_{i=1}^{m}\frac{\Gamma(y_i/x_i)}{\sqrt{x_i\pi}}\left(\frac{x_i}{2}\right)^{y_i/x_i}, \qquad r = \sum_{i=1}^{m}\frac{2\pi^2}{3x_i}, \qquad b = \sum_{i=1}^{m}\left(\frac{y_i}{2x_i} - \frac{1}{4}\right).$$

Theorem 1.3. *Let $t_i \in \mathbb{N}$ for $1 \le i \le m$. Suppose that*

$$F(q) = \sum_{n=0}^{\infty} a_n q^n \qquad \text{and} \qquad F(q)\prod_{i=1}^{m}\frac{1}{1-q^{t_i}} = \sum_{n=0}^{\infty} d_n q^n$$

are two power series. If

$$a_n \sim n^{\alpha}\exp(\beta n^p)$$

where $0 < p < 1$, $\alpha \in \mathbb{R}, \beta > 0$, then we have

$$d_n \sim \frac{n^{\alpha+m(1-p)}}{\beta^m p^m \prod_{i=1}^{m} t_i}\exp(\beta n^p). \tag{1.6}$$

An *ordinary plane partition* (PP) is a filling $\omega = (\omega_{i,j})$ of the quarter plane $\Lambda = \{(i,j) \mid i,j \ge 1\}$ with nonnegative integers such that rows and columns decrease weakly, and the size $|\omega| = \sum \omega_{i,j}$ is finite. The generating function of ordinary plane partitions is known since MacMahon [16, 17]:

$$\sum_{\omega\in \mathrm{PP}} z^{|\omega|} = \prod_{i=1}^{\infty}\prod_{j=1}^{\infty}\frac{1}{1-z^{i+j-1}} = \prod_{i=1}^{\infty}(1-z^k)^{-k}. \tag{1.7}$$

The generating functions for various kinds of plane partitions can be found in [1, 2, 3, 4, 5, 6, 9, 13, 15, 18, 19, 20, 21, 22, 24].

For two partitions λ and μ, We write $\lambda \succ \mu$ or $\mu \prec \lambda$ if λ/μ is a horizontal strip (see [15, 25]). When reading an ordinary plane partition ω along the diagonals from left to right, we obtain a sequence of partitions $(\lambda^0, \lambda^1, \ldots, \lambda^h)$ such that $\lambda^{i-1} \prec \lambda^i$

or $\lambda^{i-1} \succ \lambda^i$ for $1 \le i \le h$. For simplicity, we identify the ordinary plane partition ω and the sequence of partitions by writing

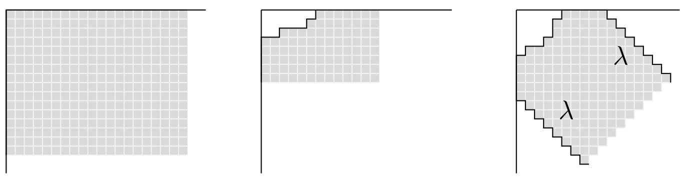
$$\omega = (\lambda^0, \lambda^1, \ldots, \lambda^h).$$

A ± 1-sequence δ is called a *profile*. Let $|\delta|_1$ (resp. $|\delta|_{-1}$) be the number of letters 1 (resp. -1) in δ. A *skew plane partition* (SkewPP) with profile $\delta = (\delta_1, \delta_2, \ldots, \delta_h)$ is a sequence of partitions $\omega = (\lambda^0, \lambda^1, \ldots, \lambda^h)$ such that $\lambda^0 = \lambda^h = \emptyset$, and $\lambda^{i-1} \prec \lambda^i$ (resp. $\lambda^{i-1} \succ \lambda^i$) if $\delta_i = 1$ (resp. $\delta_i = -1$). Its *size* is defined by $|\omega| = \sum_{i=0}^{h} |\lambda^i|$. For example, $\omega = (\emptyset, (2), (3, 2), (2), (3), (4, 3), (3, 2), (3), \emptyset)$ is a skew plane partition with profile $\delta = (1, 1, -1, 1, 1, -1, -1, -1)$ and size 27. This skew plane partition can also be visualized as the following:

$$\begin{array}{cccc} & 4 & 3 & 3 \\ & 3 & 3 & 2 \\ 3 & 2 & & \\ 2 & 2 & & \end{array}$$

The generating function for skew plane partitions with profile δ is (see [6, 21, 23])

$$\sum_{\omega \in \mathrm{SkewPP}_\delta} z^{|\omega|} = \prod_{\substack{i<j \\ \delta_i > \delta_j}} \frac{1}{1 - z^{j-i}}. \tag{1.8}$$



A: Ordinary PP B: Skew PP C: Cylindric PP

Fig. 1. Skew plane partitions and cylindric partitions.

Cylindric partitions (CP) were first introduced by Gessel and Krattenthaler [9], see also [6] for an equivalent definition. A cylindric partition with profile $\delta = (\delta_1, \delta_2, \ldots, \delta_h)$ is a sequence of partitions $\omega = (\lambda^0, \lambda^1, \ldots, \lambda^h)$ such that $\lambda^0 = \lambda^h$, and $\lambda^{i-1} \prec \lambda^i$ (resp. $\lambda^{i-1} \succ \lambda^i$) if $\delta_i = 1$ (resp. $\delta_i = -1$). Its *size* is defined by $|\omega| = \sum_{i=0}^{h-1} |\lambda^i|$ (notice that λ^h is not counted here, which is a little different from skew plane partitions). For example, $\omega = ((2, 1), (3, 1), (4, 1), (3), (4, 2), (2, 1))$ is a cylindric partition with profile $\delta = (1, 1, -1, 1, -1)$ and size 21. This cylindric partition can be visualized as the following:

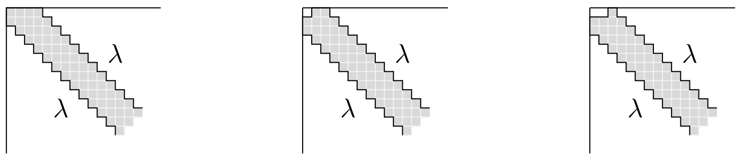
$$\begin{array}{cccc} & 4 & 2 & \\ 4 & 3 & 2 & 1 \\ 3 & 1 & & \\ 2 & 1 & & \\ & 1 & & \end{array}$$

Borodin obtained the generating function for cylindric partitions with profile $\delta = (\delta_i)_{1\le i\le h}$ (see [6, 14, 26]):

$$\sum_{\omega\in \mathrm{CP}_\delta} z^{|\omega|} = \prod_{k\ge 0}\left(\frac{1}{1-z^{hk+h}} \prod_{\substack{i<j\\ \delta_i>\delta_j}} \frac{1}{1-z^{hk+j-i}} \prod_{\substack{i<j\\ \delta_i<\delta_j}} \frac{1}{1-z^{hk+h+i-j}} \right). \tag{1.9}$$

By using Theorems 1.2 and 1.3 we obtain the asymptotic formulas for the numbers of skew plane partitions and cylindric partitions with size n for fixed widths in Sections 3 and 4 respectively. Let us reproduce the asymptotic formulas for some special cases below:

$\mathrm{PPa} \sim \frac{1.93}{n^3} e^{4.44\sqrt{n}}$ $\mathrm{PPb} \sim \frac{5.81}{n^3} e^{4.44\sqrt{n}}$ $\mathrm{PPc} \sim \frac{11.62}{n^3} e^{4.44\sqrt{n}}$



$\mathrm{CPa} \sim \frac{0.144}{n} e^{2.56\sqrt{n}}$ $\mathrm{CPb} \sim \frac{0.161}{n} e^{2.86\sqrt{n}}$ $\mathrm{CPc} \sim \frac{0.114}{n} e^{2.86\sqrt{n}}$

Fig. 2. Asymptotic formulas for skew PP and CP of fixed widths.

We see that the order of the asymptotic formula for skew plane partitions of fixed width depends only on the width, not on the profile (the skew zone) itself. We may think that this is natural by intuition. However, the case for cylindric partitions shows that this is not always true.

The rest of the paper is arranged in the following way. First, in Section 2 we prove our main theorems on asymptotic formulas. Later, we compute the asymptotic formulas for the numbers of skew plane partitions and cylindric partitions in Sections 3 and 4 respectively.

2. Proofs of main asymptotic formulas

In this section we prove the three main asymptotic formulas stated in Theorems 1.1, 1.2 and 1.3. The basic idea of the proofs comes from the work of Dewar and Murty [7]. First let us recall Laplace's method (see, for example, [8, p. 36]).

Lemma 2.1 (Laplace's method)**.** *Assume that $f(x)$ is a twice continuously differentiable function on $[a, b]$ with $x_0 \in (a, b)$ the unique point such that $f(x_0) = \max_{[a,b]} f(x)$. Assume additionally that $f''(x_0) < 0$. Then*

$$\int_a^b e^{nf(x)}dx \sim e^{nf(x_0)}\sqrt{\frac{2\pi}{-nf''(x_0)}}.$$

The sign $\sim$ means that the quotient of the left-hand side by the right-hand side tends to 1 as $n \to +\infty$.

We also need the following lemma.

Lemma 2.2. *Suppose that n is a positive integer. Let $f(x)$ be a non-negative Lebesgue integrable function on $[a,b]$ with $x_0 \in (a,b)$ the unique point such that $f(x_0) = \max_{[a,b]} f(x)$. Assume additionally that $f(x)$ increases on (a,x_0) and decreases on (x_0,b). Then,*

$$\int_a^b f(x)dx - \frac{f(x_0)}{n} \le \frac{1}{n}\sum_{i=\lceil na\rceil}^{\lfloor nb\rfloor} f(\frac{i}{n}) \le \int_a^b f(x)dx + \frac{f(x_0)}{n}.$$

Proof. Let $f(x) = 0$ for $x \notin [a,b]$. Since $f(x)$ increases on (a,x_0), we have

$$\frac{f(k)}{n} \le \int_k^{k+\frac{1}{n}} f(x)dx \le \frac{f(k+\frac{1}{n})}{n}$$

when $a \le k < k+\frac{1}{n} \le x_0$. Since $f(x)$ decreases on (x_0,b), we obtain

$$\frac{f(k+\frac{1}{n})}{n} \le \int_k^{k+\frac{1}{n}} f(x)dx \le \frac{f(k)}{n}$$

when $x_0 \le k < k+\frac{1}{n} \le b$.

Let k_0 be the integer such that $\frac{k_0}{n} \le x_0 < \frac{k_0+1}{n}$, we have

$$\int_{\frac{k_0}{n}}^{\frac{k_0+1}{n}} f(x)dx \le \frac{f(x_0)}{n}.$$

Therefore

$$\begin{aligned}\int_a^b f(x)dx &\le \int_{\frac{\lceil na\rceil-1}{n}}^{\frac{\lfloor nb\rfloor+1}{n}} f(x)dx \\ &= \int_{\frac{\lceil na\rceil-1}{n}}^{\frac{k_0}{n}} f(x)dx + \int_{\frac{k_0}{n}}^{\frac{k_0+1}{n}} f(x)dx + \int_{\frac{k_0+1}{n}}^{\frac{\lfloor nb\rfloor+1}{n}} f(x)dx \\ &\le \frac{1}{n}\sum_{i=\lceil na\rceil}^{k_0} f(\frac{i}{n}) + \frac{1}{n}f(x_0) + \frac{1}{n}\sum_{i=k_0+1}^{\lfloor nb\rfloor} f(\frac{i}{n}) \\ &\le \frac{1}{n}\sum_{i=\lceil na\rceil}^{\lfloor nb\rfloor} f(\frac{i}{n}) + \frac{1}{n}f(x_0).\end{aligned}$$

On the other hand,

$$\begin{aligned}\int_a^b f(x)dx &\ge \int_{\frac{\lceil na\rceil}{n}}^{\frac{\lfloor nb\rfloor}{n}} f(x)dx \\ &= \int_{\frac{\lceil na\rceil}{n}}^{\frac{k_0}{n}} f(x)dx + \int_{\frac{k_0}{n}}^{\frac{k_0+1}{n}} f(x)dx + \int_{\frac{k_0+1}{n}}^{\frac{\lfloor nb\rfloor}{n}} f(x)dx \\ &\ge \frac{1}{n}\sum_{i=\lceil na\rceil}^{k_0-1} f(\frac{i}{n}) + \frac{1}{n}\left(-f(x_0) + f(\frac{k_0}{n}) + f(\frac{k_0+1}{n})\right) + \frac{1}{n}\sum_{i=k_0+2}^{\lfloor nb\rfloor} f(\frac{i}{n}) \\ &\ge \frac{1}{n}\sum_{i=\lceil na\rceil}^{\lfloor nb\rfloor} f(\frac{i}{n}) - \frac{1}{n}f(x_0).\end{aligned}$$

□

Now we can give the proof of Theorem 1.1.

Proof of Theorem 1.1. Without loss of generality, we can assume that $v_1, v_2 > 0$. For $0 < x < 1$, let

$$f(x) = r_1^{1-p} x^p + r_2^{1-p} (1-x)^p$$

and

$$g(x) = x^{-b_1-1+\frac{p}{2}} (1-x)^{-b_2-1+\frac{p}{2}}.$$

Then

$$f'(x) = p r_1^{1-p} x^{p-1} - p r_2^{1-p} (1-x)^{p-1}$$

and

$$f''(x) = p(p-1) r_1^{1-p} x^{p-2} + p(p-1) r_2^{1-p} (1-x)^{p-2} < 0.$$

The function $f'(x)$ has only one zero point

$$x_0 = \frac{r_1}{r_1 + r_2}.$$

Therefore $f(x)$ is increasing on $(0, x_0)$, has a maximum of $(r_1 + r_2)^{1-p}$ at x_0, and is decreasing on $(x_0, 1)$.

Let $0 < \epsilon < 1$ be a given constant. By continuity, there exists $0 < \delta < \min\{\frac{1}{2}x_0, \frac{1}{2}(1-x_0)\}$ such that if $|x - x_0| < 2\delta$, then

$$(1-\epsilon) g(x_0) < g(x) < (1+\epsilon) g(x_0). \tag{2.1}$$

From (1.2) and (1.3), for large enough n we have

$$(1-\epsilon) t_1 \psi_{t_1 n}(v_1, r_1, b_1; p) < a_{t_1 n} < (1+\epsilon) t_1 \psi_{t_1 n}(v_1, r_1, b_1; p) \tag{2.2}$$

and

$$(1-\epsilon) t_2 \psi_{t_2 n}(v_2, r_2, b_2; p) < c_{t_2 n} < (1+\epsilon) t_2 \psi_{t_2 n}(v_2, r_2, b_2; p). \tag{2.3}$$

Suppose that $0 \leq i \leq t_1 t_2 - 1$ is a given integer. We just need to prove that (1.4) is true for $n = m t_1 t_2 + i$ where $m \in \mathbb{N}$. By Bézout's identity, there exists some $\alpha_i, \beta_i \in \mathbb{N}_{\geq 0}, 0 \leq \alpha_i \leq t_2 - 1$ such that

$$t_1 \alpha_i + t_2 \beta_i = i.$$

For large enough $n = m t_1 t_2 + i$, let

$$j_1(n) = \left\lceil \frac{(x_0 - \delta)n - \alpha_i t_1}{t_1 t_2} \right\rceil,$$
$$j_2(n) = \left\lfloor \frac{(x_0 + \delta)n - \alpha_i t_1}{t_1 t_2} \right\rfloor,$$
$$j_3(n) = \left\lfloor \frac{n - \alpha_i t_1}{t_1 t_2} \right\rfloor.$$

We have

$$d_n = H_1(n) + H_2(n) + H_3(n),$$

where

$$H_1(n) = \sum_{j=0}^{j_1(n)-1} a_{(\alpha_i + j t_2) t_1} c_{(m t_1 - j t_1 + \beta_i) t_2},$$

$$H_2(n) = \sum_{j=j_2(n)+1}^{j_3(n)} a_{(\alpha_i+jt_2)t_1} c_{(mt_1-jt_1+\beta_i)t_2},$$

$$H_3(n) = \sum_{j=j_1(n)}^{j_2(n)} a_{(\alpha_i+jt_2)t_1} c_{(mt_1-jt_1+\beta_i)t_2}.$$

For $H_1(n)$, we have

$$\begin{aligned} H_1(n) &= O\left(n^{|b_1+1-\frac{p}{2}|+|b_2+1-\frac{p}{2}|} \sum_{j=0}^{j_1(n)-1} \exp\left(n^p f\left(\frac{(\alpha_i+jt_2)t_1}{n}\right)\right)\right) \\ &= O\left(n^{|b_1+1-\frac{p}{2}|+|b_2+1-\frac{p}{2}|+1} \exp\left(n^p f(x_0-\delta)\right)\right) \\ &= o\left(n^{-b_1-b_2-1+\frac{p}{2}} \exp\left(n^p f(x_0)\right)\right). \end{aligned}$$

Similarly, we have

$$H_2(n) = o\left(n^{-b_1-b_2-1+\frac{p}{2}} \exp\left(n^p f(x_0)\right)\right).$$

Next we just need to estimate $H_3(n)$. For large enough n, we can assume that every $a_{(\alpha_i+jt_2)t_1}$ and $c_{(mt_1-jt_1+\beta_i)t_2}$ in $H_3(n)$ satisfy (2.2) and (2.3). Let

$$\begin{aligned} A_0 &= \frac{p(1-p)}{2\pi} t_1 t_2 v_1 v_2 r_1^{b_1+(1-p)/2} r_2^{b_2+(1-p)/2}, \\ A_1(n) &= A_0 n^{-b_1-b_2-2+p} \sum_{j=j_1(n)}^{j_2(n)} g\left(\frac{(\alpha_i+jt_2)t_1}{n}\right) \exp\left(n^p f\left(\frac{(\alpha_i+jt_2)t_1}{n}\right)\right), \\ A_2(n) &= g(x_0) A_0 n^{-b_1-b_2-2+p} \sum_{j=j_1(n)}^{j_2(n)} \exp\left(n^p f\left(\frac{(\alpha_i+jt_2)t_1}{n}\right)\right), \\ A_3(n) &= \int_{\frac{(x_0-\delta)n-\alpha_i t_1}{t_1 t_2 n}}^{\frac{(x_0+\delta)n-\alpha_i t_1}{t_1 t_2 n}} \exp\left(n^p f\left(\frac{(\alpha_i+nxt_2)t_1}{n}\right)\right) dx. \end{aligned}$$

Therefore

$$(1-\epsilon)^2 A_1(n) < H_3(n) < (1+\epsilon)^2 A_1(n).$$

Then by (2.1), we obtain

$$(1-\epsilon)^3 A_2(n) < H_3(n) < (1+\epsilon)^3 A_2(n). \tag{2.4}$$

Replace $f(x)$ by $\exp\left(n^p f\left(\frac{(\alpha_i+nxt_2)t_1}{n}\right)\right)$ in Lemma 2.2, we have

$$\begin{aligned} A_3(n) - \frac{\exp\left(n^p f(x_0)\right)}{n} &\leq \frac{1}{n} \sum_{j=j_1(n)}^{j_2(n)} \exp\left(n^p f\left(\frac{(\alpha_i+jt_2)t_1}{n}\right)\right) \\ &\leq A_3(n) + \frac{\exp\left(n^p f(x_0)\right)}{n}. \end{aligned} \tag{2.5}$$

Put (2.4) and (2.5) together, we obtain

$$(1-\epsilon)^3 \left(A_3(n) - \frac{\exp\left(n^p f(x_0)\right)}{n}\right) < \frac{H_3(n)}{g(x_0) A_0 n^{-b_1-b_2-1+p}} \tag{2.6}$$

$$< (1+\epsilon)^3 \left(A_3(n) + \frac{\exp\left(n^p f(x_0)\right)}{n} \right).$$

Notice that when n is large enough,

$$\frac{x_0 - \frac{3\delta}{2}}{t_1 t_2} < \frac{(x_0-\delta)n - \alpha_i t_1}{t_1 t_2 n} < \frac{x_0 - \frac{\delta}{2}}{t_1 t_2}, \tag{2.7}$$

$$\frac{x_0 + \frac{\delta}{2}}{t_1 t_2} < \frac{(x_0+\delta)n - \alpha_i t_1}{t_1 t_2 n} < \frac{x_0 + \frac{3\delta}{2}}{t_1 t_2}. \tag{2.8}$$

Also we have

$$\begin{aligned} n^p f\left(\frac{(\alpha_i + nxt_2)t_1}{n}\right) &= r_1^{1-p}\left((\alpha_i + nxt_2)t_1\right)^p + r_2^{1-p}\left(1 - (\alpha_i + nxt_2)t_1\right)^p \\ &= n^p f(xt_1t_2) + o(1). \end{aligned}$$

Then by (2.7), (2.8) and Lemma 2.1 (*Laplace's method*) we have

$$\begin{aligned} A_3(n) &\sim \int_{\frac{(x_0-\delta)n-\alpha_i t_1}{t_1 t_2 n}}^{\frac{(x_0+\delta)n-\alpha_i t_1}{t_1 t_2 n}} \exp\left(n^p f(xt_1t_2)\right) dx \\ &\sim \exp(n^p f(x_0)) \sqrt{\frac{2\pi}{-n^p t_1^2 t_2^2 f''(x_0)}}. \end{aligned} \tag{2.9}$$

This means that when n is large enough,

$$\begin{aligned} (1-\epsilon)\exp(n^p f(x_0)) \sqrt{\frac{2\pi}{-n^p t_1^2 t_2^2 f''(x_0)}} &< A_3(n) \\ &< (1+\epsilon)\exp(n^p f(x_0)) \sqrt{\frac{2\pi}{-n^p t_1^2 t_2^2 f''(x_0)}}. \end{aligned}$$

Therefore

$$\begin{aligned} &(1-\epsilon)^4 n^{-\frac{p}{2}} \exp(n^p f(x_0)) \sqrt{\frac{2\pi}{-t_1^2 t_2^2 f''(x_0)}} + o\left(n^{-\frac{p}{2}} \exp\left(n^p f(x_0)\right)\right) \\ &< \frac{H_3(n)}{g(x_0) A_0 n^{-b_1-b_2-1+p}} \\ &< (1+\epsilon)^4 n^{-\frac{p}{2}} \exp(n^p f(x_0)) \sqrt{\frac{2\pi}{-t_1^2 t_2^2 f''(x_0)}} + o\left(n^{-\frac{p}{2}} \exp\left(n^p f(x_0)\right)\right). \end{aligned}$$

Finally we obtain

$$\begin{aligned} d_n &= H_1(n) + H_2(n) + H_3(n) \sim H_3(n) \\ &\sim g(x_0) A_0 n^{-b_1-b_2-1+\frac{p}{2}} \exp(n^p f(x_0)) \sqrt{\frac{2\pi}{-t_1^2 t_2^2 f''(x_0)}}. \end{aligned}$$

But

$$\begin{aligned} g(x_0) &= \left(\frac{r_1}{r_1+r_2}\right)^{-b_1-1+\frac{p}{2}} \left(\frac{r_2}{r_1+r_2}\right)^{-b_2-1+\frac{p}{2}}, \\ f(x_0) &= (r_1+r_2)^{1-p}, \\ f''(x_0) &= \frac{p(p-1)(r_1+r_2)^{3-p}}{r_1 r_2}. \end{aligned}$$

Therefore

$$d_n \sim \psi_n(v_1 v_2, r_1 + r_2, b_1 + b_2; p). \qquad \square$$

Our asymptotic formula can be easily iterated for handling a product of multiple power series $F_1(q)F_2(q)\cdots F_k(q)$.

Theorem 2.3. *Suppose that $m > 0$, $1 \le i \le m$, $t = \gcd(z_1, z_2, \ldots, z_m)$. Let*

$$F_i(q) = \sum_{n=0}^{\infty} a_n^{(i)} q^{z_i n}$$

and

$$G(q) = \prod_{i=1}^{m} F_i(q) = \sum_{n=0}^{\infty} d_n q^{tn}$$

where

$$a_n^{(i)} \sim z_i \cdot \psi_{z_i n}(v_i, r_i, b_i; p). \tag{2.10}$$

Then

$$d_n \sim t \cdot \psi_{tn}(\prod_{i=1}^{m} v_i, \sum_{i=1}^{m} r_i, \sum_{i=1}^{m} b_i; p). \tag{2.11}$$

Proof. Without loss of generality, we can assume that $m = 2$. For $i = 1, 2$, we have

$$F_i(q) = \sum_{n=0}^{\infty} a_n^{(i)} (q^t)^{z_i n/t}$$

where

$$a_n^{(i)} \sim z_i \cdot \psi_{z_i n}(v_i, r_i, b_i; p) = \frac{z_i}{t} \psi_{z_i n/t}(v_i t^{-\frac{b_i}{1-p}}, r_i t^{\frac{p}{1-p}}, b_i; p).$$

Replace q by q^t, t_i by $\frac{z_i}{t}$ in Theorem 1.1 we have

$$\begin{aligned} d_n &\sim \psi_n(v_1 v_2 t^{-\frac{b_1+b_2}{1-p}}, (r_1 + r_2) t^{\frac{p}{1-p}}, b_1 + b_2; p) \\ &= t\psi_{tn}(v_1 v_2, r_1 + r_2, b_1 + b_2; p). \qquad \square \end{aligned}$$

Hardy-Ramanujan [10] have discovered the asymptotic formula for the number of integer partitions, which was extended by Ingham [11] in 1941.

Lemma 2.4 (Ingham)**.** *Let x and y be two positive integers with $\gcd(x, y) = 1$. Suppose that*

$$\prod_{k\ge 0} \frac{1}{1 - q^{xk+y}} = \sum_{n=0}^{\infty} a_n q^n.$$

Then

$$a_n \sim \psi_n(v, r, b; \frac{1}{2}),$$

where

$$v = \frac{\Gamma(y/x)}{\sqrt{x\pi}} (\frac{x}{2})^{y/x}, \qquad r = \frac{2\pi^2}{3x}, \qquad b = \frac{y}{2x} - \frac{1}{4}.$$

Ingham's result can be further generalized as follows, which will be useful for finding the asymptotic formula for skew plane partitions.

Theorem 2.5. *Suppose that* $m > 0$, $x_i > 0, y_i > 0$, $z_i = \gcd(x_i, y_i)$ *for* $1 \le i \le m$, $t = \gcd(z_1, z_2, \ldots, z_m)$ *and*

$$v_i = \frac{\Gamma(y_i/x_i)}{\sqrt{x_i \pi}} (\frac{x_i}{2})^{y_i/x_i}, \qquad r_i = \frac{2\pi^2}{3x_i}, \qquad b_i = \frac{y_i}{2x_i} - \frac{1}{4}.$$

Let

$$\prod_{i=1}^{m} \prod_{k \ge 0} \frac{1}{1 - q^{x_i k + y_i}} = \sum_{n=0}^{\infty} d_n q^{tn}.$$

Then

$$d_n \sim t \cdot \psi_{tn}(\prod_{i=1}^{m} v_i, \sum_{i=1}^{m} r_i, \sum_{i=1}^{m} b_i; \frac{1}{2}). \tag{2.12}$$

Proof. Let

$$\prod_{k \ge 0} \frac{1}{1 - q^{x_i k + y_i}} = \sum_{n=0}^{\infty} a_n^{(i)} q^{z_i n}.$$

It is easy to check that

$$\psi_n(v_i z_i^{\frac{1}{2} - \frac{y_i}{x_i}}, r_i z_i, b_i; \frac{1}{2}) = z_i \cdot \psi_{z_i n}(v_i, r_i, b_i; \frac{1}{2}).$$

Replace q by q^{z_i}, x by x_i/z_i, y by y_i/z_i in Lemma 2.4 we obtain

$$a_n^{(i)} \sim \psi_n(v_i z_i^{\frac{1}{2} - \frac{y_i}{x_i}}, r_i z_i, b_i; \frac{1}{2}) \sim z_i \cdot \psi_{z_i n}(v_i, r_i, b_i; \frac{1}{2}).$$

Thus (2.12) follows from Theorem 2.3. □

The above result implies Theorem 1.2 by letting $t = 1$. Next we give the proof of Theorem 1.3.

Proof of Theorem 1.3. By induction, it is easy to see that we just need to prove the case $m = 1$, $t_1 = t$. Notice that $(x^\alpha \exp(\beta x^p))' = (\beta p x^p + \alpha) x^{\alpha - 1} \exp(\beta x^p)$. Let $0 < \epsilon < 1$. Then there exists some $N > 0$ such that for any $x \ge N$, we have $(x^\alpha \exp(\beta x^p))' > 0$; and for any $n \ge N$, we have

$$(1 - \epsilon) n^\alpha \exp(\beta n^p) < a_n < (1 + \epsilon) n^\alpha \exp(\beta n^p). \tag{2.13}$$

But

$$d_n = \sum_{j=0}^{\lfloor \frac{n}{t} \rfloor} a_{n-tj} = \sum_{j=0}^{\lfloor \frac{n-N}{t} \rfloor} a_{n-tj} + \sum_{j=\lfloor \frac{n-N}{t} \rfloor + 1}^{\lfloor \frac{n}{t} \rfloor} a_{n-tj}.$$

First we have

$$\sum_{j=\lfloor \frac{n-N}{t} \rfloor + 1}^{\lfloor \frac{n}{t} \rfloor} a_{n-tj} = O(1).$$

On the other hand, we have

$$(1 - \epsilon) \sum_{j=0}^{\lfloor \frac{n-N}{t} \rfloor} (n - tj)^\alpha \exp(\beta (n - tj)^p) < \sum_{j=0}^{\lfloor \frac{n-N}{t} \rfloor} a_{n-tj} \tag{2.14}$$

$$< (1+\epsilon) \sum_{j=0}^{\lfloor \frac{n-N}{t} \rfloor} (n-tj)^{\alpha} \exp(\beta(n-tj)^p). \tag{2.15}$$

Since $x^{\alpha}\exp(\beta x^p)$ increases for $x \geq N$, we have

$$\begin{aligned}
&\frac{1}{t}\int_{n-t\lfloor \frac{n-N-t}{t} \rfloor}^{n} x^{\alpha}\exp(\beta x^p)dx \leq \sum_{j=0}^{\lfloor \frac{n-N}{t} \rfloor} (n-tj)^{\alpha}\exp(\beta(n-tj)^p) \\
&\leq \frac{1}{t}\int_{n-t\lfloor \frac{n-N}{t} \rfloor}^{n+t} x^{\alpha}\exp(\beta x^p)dx.
\end{aligned}$$

But when n is large enough, we have

$$\begin{aligned}
\int_{n-t\lfloor \frac{n-N}{t} \rfloor}^{n+t} x^{\alpha}\exp(\beta x^p)dx &\sim \int_{n-t\lfloor \frac{n-N}{t} \rfloor}^{n+t} \Big(x^{\alpha} + \frac{\alpha+1-p}{\beta p}x^{\alpha-p}\Big)\exp(\beta x^p)dx \\
&= \int_{n-t\lfloor \frac{n-N}{t} \rfloor}^{n+t} \big(\frac{x^{\alpha+1-p}}{\beta p}\exp(\beta x^p)\big)' dx \\
&\sim \frac{(n+t)^{\alpha+1-p}}{\beta p}\exp(\beta(n+t)^p) \\
&\sim \frac{n^{\alpha+1-p}}{\beta p}\exp(\beta n^p),
\end{aligned}$$

where the last $\sim$ is guaranteed by the condition $0 < p < 1$.

Similarly, we have

$$\int_{n-t\lfloor \frac{n-N-t}{t} \rfloor}^{n} x^{\alpha}\exp(\beta x^p)dx \sim \frac{n^{\alpha+1-p}}{\beta p}\exp(\beta n^p).$$

Therefore when n is large enough, we have

$$(1-\epsilon)^2\frac{n^{\alpha+1-p}}{\beta pt}\exp(\beta n^p) < \sum_{j=0}^{\lfloor \frac{n-N}{t} \rfloor} a_{n-tj} < (1+\epsilon)^2\frac{n^{\alpha+1-p}}{\beta pt}\exp(\beta n^p).$$

Finally we obtain

$$d_n \sim \frac{n^{\alpha+1-p}}{\beta pt}\exp(\beta n^p). \qquad \square$$

3. Asymptotic formulas for skew plane partitions

Various plane partitions have been widely studied since MacMahon [16, 17]. In particular, the generating function for skew plane partitions with profile δ has been derived (see [6, 21, 23]). In this section, first we obtain the asymptotic formula for ordinary plane partitions of fixed width. We say that a (skew) plane partition ω has a width m if $\omega_{i,j} = 0$ for $i > m$.

Theorem 3.1. *Let $\mathrm{PP}_m(n)$ be the number of plane partitions ω of width m and size n. Then,*

$$\mathrm{PP}_m(n) \ \sim\ 2^{-\frac{m^2+2m+5}{4}}(\frac{m}{3})^{\frac{m^2+1}{4}}\pi^{\frac{m^2-m}{2}}\prod_{i=1}^{m-1} i! \ \times n^{-\frac{m^2+3}{4}}\exp(\pi\sqrt{2mn/3}). \tag{3.1}$$

Proof. Let $\delta = 1^m(-1)^\infty$ in (1.8) we have

$$\sum_{\omega\in \mathrm{PP}_m} z^{|\omega|} = \prod_{k\ge 0}\prod_{i=1}^{m} \frac{1}{(1-z^{k+i})}.$$

Therefore by Theorem 1.2 the number of plane partitions with profile δ and size n is asymptotic to

$$\psi_n(\prod_{i=1}^{m} \frac{(i-1)!}{\sqrt{\pi}}(\frac{1}{2})^i,\ \frac{2m\pi^2}{3},\ \frac{m^2}{4};\ \frac{1}{2}),$$

which is equal to the right-hand side of (3.1). $\square$

When $m=1$, the above theorem gives the Hardy-Ramanujan asymptotic formula for the number of integer partitions

$$\mathrm{PP}_1(n) \ \sim\ \frac{1}{4\sqrt{3}n}\exp(\pi\sqrt{2n/3}). \tag{3.2}$$

When $m=3$, this is the example (PPa) illustrated in Fig. 2. Actually we have

$$\sum_{\omega\in \mathrm{PPa}} z^{|\omega|} = \prod_{k\ge 0} \frac{1}{(1-z^{k+1})(1-z^{k+2})(1-z^{k+3})}.$$

Therefore the number of plane partitions of width 3 and size n is asymptotic to

$$2^{-4}\pi^3 n^{-3}\exp(\pi\sqrt{2n}).$$

More generally, we can derive the asymptotic formula for the number of skew plane partitions with fixed width.

Theorem 3.2. *Let $\delta = (\delta', (-1)^\infty) = (\delta'_1, \delta'_2, \ldots, \delta'_{m-1}, (-1)^\infty)$ be a profile, and $\mathrm{SkewPP}_\delta(n)$ be the number of skew plane partitions with profile δ and size n. Then*

$$\mathrm{SkewPP}_\delta(n) \ \sim\ 2^{-\frac{\ell^2+2\ell+5}{4}}(\frac{\ell}{3})^{\frac{\ell^2+1}{4}}\pi^{\frac{\ell^2-\ell}{2}} \prod_{\substack{i<j\\ \delta'_i>\delta'_j}} \frac{1}{j-i}\prod_{\delta'_i=1}(m-i-1)!$$

$$\times\, n^{-\frac{\ell^2+3}{4}}\exp(\pi\sqrt{2\ell n/3}), \tag{3.3}$$

where $\ell := |\delta'|_1$.

Proof. By (1.8) we have

$$\sum_{\omega\in \mathrm{SkewPP}_\delta} z^{|\omega|} = \prod_{\substack{i<j\\ \delta'_i>\delta'_j}} \frac{1}{1-z^{j-i}} \times \prod_{k\ge 0}\prod_{\delta'_i=1}\frac{1}{1-z^{k+m-i}}.$$

By Theorem 1.2 the coefficient of z^n in

$$\prod_{k\ge 0}\prod_{\delta'_i=1}\frac{1}{1-z^{k+m-i}}$$

is asymptotic to

$$\psi_n(\prod_{\delta'_i=1}\frac{(m-i-1)!}{\sqrt{\pi}}(\frac{1}{2})^{m-i},\ \frac{2\ell\pi^2}{3},\ \frac{1}{2}\sum_{\delta'_i=1}(m-i)-\frac{\ell}{4};\ \frac{1}{2}).$$

Denote by $\mathrm{inv}(\delta')$ the number of pairs (i,j) such that $i<j, \delta'_i>\delta'_j$. Notice that $\sum_{\delta'_i=1}(m-i)-\mathrm{inv}(\delta')=\binom{\ell+1}{2}$. Then by Theorem 1.3 the number $\mathrm{SkewPP}_\delta(n)$ is asymptotic to

$$2^{-3/2+\mathrm{inv}(\delta')}\pi^{-\frac{1}{2}}(\frac{2\ell\pi^2}{3})^{\frac{\ell^2+1}{4}}\prod_{\substack{i<j\\ \delta'_i>\delta'_j}}\frac{1}{j-i}\prod_{\delta'_i=1}\frac{(m-i-1)!}{\sqrt{\pi}}(\frac{1}{2})^{m-i}$$
$$\times\, n^{-\frac{\ell^2+3}{4}}\exp(\pi\sqrt{2\ell n/3}),$$

which is equal to the right-hand side of (3.3). □

The two examples (PPb) and (PPc) for $\ell=|\delta'|_1=3$ illustrated in Fig. 2 correspond to the following special cases of Theorem 3.2.

(PPb) Let $\delta=(1,1,-1,1,(-1)^\infty)$ we have

$$\sum_{\omega\in\mathrm{PPb}}z^{|\omega|}=\frac{1}{1-z}\frac{1}{1-z^2}\prod_{k\geq 0}\frac{1}{(1-z^{k+1})(1-z^{k+3})(1-z^{k+4})}.$$

Therefore by Theorems 1.2 and 1.3 the number of skew plane partitions with profile δ and size n is asymptotic to

$$3\cdot 2^{-4}\pi^3 n^{-3}\exp(\pi\sqrt{2n}).$$

(PPc) Let $\delta=(1,1,-1,-1,1,(-1)^\infty)$ we have

$$\sum_{\omega\in\mathrm{PPc}}z^{|\omega|}=\frac{1}{1-z}\frac{1}{1-z^2}\frac{1}{1-z^2}\frac{1}{1-z^3}\prod_{k\geq 0}\frac{1}{(1-z^{k+1})(1-z^{k+4})(1-z^{k+5})}.$$

Therefore the number of skew plane partitions with profile δ and size n is asymptotic to

$$3\cdot 2^{-3}\pi^3 n^{-3}\exp(\pi\sqrt{2n}).$$

4. Asymptotic formula for cylindric partitions

First we recall Borodin's formula (1.9) written in the following form.

Lemma 4.1 (Borodin [6])**.** *Let $\delta=(\delta_i)_{1\leq i\leq h}$ be a profile. Then the generating function for the cylindric partitions with profile δ is*

$$\sum_{\omega\in\mathrm{CP}_\delta}z^{|\omega|}=\prod_{k\geq 0}\prod_{t\in W_\delta}\frac{1}{1-z^{hk+t}},$$

where

$$W_\delta=\{h\}\cup\{j-i: i<j,\ \delta_i>\delta_j\}\cup\{h+i-j: i<j,\ \delta_i<\delta_j\}.$$

In this section we derive the asymptotic formula for the number of cylindric partitions.

Theorem 4.2. *Let $\delta=(\delta_j)_{1\leq j\leq h}$ be a profile. When $1\leq|\delta|_1\leq h-1$, the number of cylindric partitions with profile δ is asymptotic to*

$$\prod_{\substack{i<j\\ \delta_i>\delta_j}}\Gamma(\frac{j-i}{h})\prod_{\substack{i<j\\ \delta_i<\delta_j}}\Gamma(\frac{h+i-j}{h})\,\frac{\sqrt{1+2K}}{4\sqrt{3}\cdot(2\pi)^K}\times\frac{1}{n}\exp\left(\pi\sqrt{\frac{2(1+2K)n}{3h}}\right),$$

where $K=|\delta|_1|\delta|_{-1}/2$.

Proof. We have

$$\begin{aligned}\sum_{t\in W_\delta}\frac{t}{h} &= \frac{h}{h} + \sum_{i<j,\ \delta_i>\delta_j}\frac{j-i}{h} + \sum_{i<j,\ \delta_i<\delta_j}\frac{h+i-j}{h}\\ &= 1+\sum_{\substack{i<j\\ \delta_i<\delta_j}} 1 + \frac{|\delta|_1}{h}\sum_{\delta_i=-1} i - \frac{|\delta|_{-1}}{h}\sum_{\delta_j=1} j. \qquad (4.1)\end{aligned}$$

If we exchange any two adjacent letters in δ, the right-hand side of (4.1) doesn't change. Therefore, the above summation is independent of δ when h and $|\delta|_1$ are given. Hence we have

$$\sum_{t\in W_\delta}\frac{t}{h} = 1+K.$$

On the other hand, $\#W_\delta = 1+2K$. Then we obtain

$$\sum_{t\in W_\delta}(\frac{t}{2h}-\frac14) = \frac14. \qquad (4.2)$$

By Lemma 4.1 , Theorem 1.2 and Identity (4.2) the number of cylindric partitions with profile δ and size n is asymptotic to

$$\psi_n(v,r,b;\frac12),$$

where

$$\begin{aligned} v &= \prod_{t\in W_\delta}\left(\frac{\Gamma(t/h)}{\sqrt{h\pi}}(\frac{h}{2})^{t/h}\right) = \prod_{t\in W_\delta}\Gamma(\frac{t}{h})\cdot 2^{-1-K}\sqrt{h}\pi^{-\frac12-K},\\ r &= \sum_{t\in W_\delta}\frac{2\pi^2}{3h} = \frac{2\pi^2}{3h}(1+2K),\\ b &= \sum_{t\in W_\delta}(\frac{t}{2h}-\frac14) = \frac14.\end{aligned}$$

The proof is achieved by the definition (1.1) of ψ. □

Notice that in the above theorem, the profile δ contains both steps "1" and "-1". In fact, when $\delta=(-1)^h$ or $\delta=(1^h)$, the number of cylindric partitions with profile δ is 0 if n is not a multiple of h. If $n=hn_1$, it is equal to the number of integer partitions of size n_1.

The three examples (CPa)-(CPc) for $h=4$ (here we say that these cylindric partitions have width h=4) illustrated in Fig. 2 correspond to the following special cases of Theorem 4.2.

(CPa) Let $\delta=(1-1,-1,-1)$. By Lemma 4.1 we have

$$\sum_{\omega\in\mathrm{CPa}} z^{|\omega|} = \prod_{k\ge0}\frac{1}{(1-z^{4k+1})(1-z^{4k+2})(1-z^{4k+3})(1-z^{4k+4})} = \prod_{k\ge0}\frac{1}{1-z^{k+1}}.$$

Therefore the number of such cylindric partitions with size n is asymptotic to

$$\frac{\sqrt3}{12}\times\frac1n\exp\Bigl(\pi\sqrt{\frac{2n}{3}}\Bigr).$$

(CPb) Let $\delta = (1-1, 1, -1)$. By Lemma 4.1 we have

$$\sum_{\omega \in \mathrm{CPb}} z^{|\omega|} = \prod_{k \geq 0} \frac{1}{(1-z^{4k+1})^2(1-z^{4k+3})^2(1-z^{4k+4})}$$

$$= \prod_{k \geq 0} \frac{1}{(1-z^{2k+1})^2(1-z^{4k+4})}.$$

Therefore the number of such cylindric partitions with size n is asymptotic to

$$\frac{1}{8}\sqrt{\frac{5}{3}} \times \frac{1}{n} \exp\Big(\pi\sqrt{\frac{5n}{6}}\Big).$$

(CPc) Let $\delta = (1-1, -1, 1)$. By Lemma 4.1 we have

$$\sum_{\omega \in \mathrm{CPc}} z^{|\omega|} = \prod_{k \geq 0} \frac{1}{(1-z^{4k+1})(1-z^{4k+2})^2(1-z^{4k+3})^2(1-z^{4k+4})}$$

$$= \prod_{k \geq 0} \frac{1}{(1-z^{k+1})(1-z^{4k+2})}.$$

Therefore the number of such cylindric partitions with size n is asymptotic to

$$\frac{1}{8}\sqrt{\frac{5}{6}} \times \frac{1}{n} \exp\Big(\pi\sqrt{\frac{5n}{6}}\Big).$$

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