The Nekrasov-Okounkov hook length formula: refinement, elementary proof, extension and applications

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ABSTRACT. — The paper is devoted to the derivation of the expansion formula for the powers of the Euler Product in terms of partition hook lengths, discovered by Nekrasov and Okounkov in their study of the Seiberg-Witten Theory. We provide a refinement based on a new property of t-cores, and give an elementary proof by using the Macdonald identities. We also obtain an extension by adding two more parameters, which appears to be a discrete interpolation between the Macdonald identities and the generating function for t-cores. Several applications are derived, including the "marked hook formula".

La formule de longueur d'équerre de Nekrasov-Okounkov: raffinement, démonstration élémentaire, extension et applications

RÉSUMÉ. — Nekrasov et Okounkov ont obtenu une nouvelle formule pour le développement des puissances du produit d'Euler, à l'aide des longueurs d'équerre des partitions d'entiers, dans leur étude de la théorie de Seiberg-Witten. Nous proposons un raffinement de cette formule reposant sur une propriété nouvelle des t-cores, qui permet de donner une démonstration élémentaire en faisant usage des identités de Macdonald. Nous obtenons aussi une extension, en ajoutant deux paramètres supplémentaires, qui peut être considérée comme une interpolation discrète entre les identités de Macdonald et la fonction génératrice des t-cores. Plusieurs applications en sont déduites, y compris la "formule d'équerre pointée".

1. Introduction

An explicit expansion formula for the powers of the Euler Product in terms of partition hook lengths was discovered by Nekrasov and Okounkov in their study of the Seiberg-Witten Theory [NO06] (see also [CO08], where a Jack polynomial analogue was derived) and re-discovered by the author recently [Ha08a] using an appropriate hook length expansion technique [Ha08b] *. In the present paper we again take up the study of the Nekrasov-Okounkov formula and obtain several results in the following four directions:

(1) We establish new properties on t-cores, which can be seen as a refinement of the Nekrasov-Okounkov formula. The proof involves a bijection between t-cores and integer vectors constructed by Garvan, Kim and Stanton [GKS90].

(2) We provide an elementary proof of the Nekrasov-Okounkov formula by using the Macdonald identities for $A_\ell^{(a)}$ [Ma72] and the properties on t-cores mentioned in (1).

(3) We obtain an extension by adding two more parameters t and y, so that the resulting formula appears to be a discrete interpolation between the Macdonald identities and the generating function for t-cores (see Corollary 5.3). Our extension opens the way to richer specializations, including the generating function for partitions, the Jacobi triple product identity, the Macdonald identity for $A_\ell^{(a)}$, the classical hook length formula, the marked hook formula [Ha08a], the generating function for t-cores, and the t-core analogues of the hook formula and of the marked hook formula. We also prove another extension of the generating functions for t-cores.

(4) As applications, we derive some new formulas about hook lengths, including the "marked hook formula". We also improve a result due to Kostant [Ko04]. A hook length expression of integer value is obtained by using the Lagrange inversion formula.

The basic notions needed here can be found in [Ma95, p.1; St99, p.287; La01, p.1; Kn98, p.59; An76, p.1]. A *partition* λ is a sequence of positive integers $\lambda = (\lambda_1, \lambda_2, \cdots, \lambda_\ell)$ such that $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_\ell > 0$. The

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* The author has indeed deposited a paper on arXiv [Ha08a; April, 2008] that contained an explicit expansion formula for the powers of the Euler Product in terms of partition hook lengths. A few days later he received an email from Andrei Okounkov who kindly pointed out that the expansion formula already appeared in his joint paper, which was deposited on arXiv in Section "High Energy Physics - Theory" [NO06; June, 2003; 90 pages]. Although the ultimate formula is the same in both papers, the methods of proof belong to different cultures. The author's original paper has remained on arXiv. The present one contains parts of it, plus several new results.

integers $(\lambda_i)_{i=1,2,\ldots,\ell}$ are called the *parts* of λ, the number ℓ of parts being the *length* of λ denoted by $\ell(\lambda)$. The sum of its parts $\lambda_1 + \lambda_2 + \cdots + \lambda_\ell$ is denoted by $|\lambda|$. Let n be an integer, a partition λ is said to be a partition of n if $|\lambda| = n$. We write $\lambda \vdash n$. The set of all partitions of n is denoted by $\mathcal{P}(n)$. The set of all partitions is denoted by $\mathcal{P}$, so that

$$\mathcal{P} = \bigcup_{n \geq 0} \mathcal{P}(n).$$

Each partition can be represented by its Ferrers diagram. For example, $\lambda = (6, 3, 3, 2)$ is a partition and its Ferrers diagram is reproduced in Fig. 1.1.
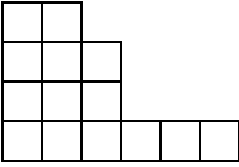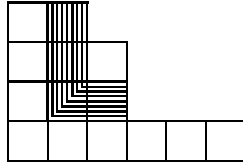

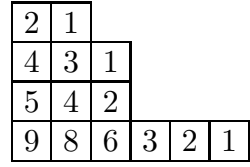Fig. 1.1. Partition   Fig. 1.2. Hook length   Fig. 1.3. Hook lengths

For each box v in the Ferrers diagram of a partition λ, or for each box v in λ, for short, define the *hook length* of v, denoted by $h_v(\lambda)$ or h_v, to be the number of boxes u such that $u = v$, or u lies in the same column as v and above v, or in the same row as v and to the right of v (see Fig. 1.2). The *hook length multi-set* of λ, denoted by $\mathcal{H}(\lambda)$, is the multi-set of all hook lengths of λ. Let t be a positive integer. We write

$$\mathcal{H}_t(\lambda) = \{h \mid h \in \mathcal{H}(\lambda), h \equiv 0 \,(\mathrm{mod}\, t)\}.$$

In Fig. 1.3 the hook lengths of all boxes for the partition $\lambda = (6, 3, 3, 2)$ have been written in each box. We have $\mathcal{H}(\lambda) = \{2, 1, 4, 3, 1, 5, 4, 2, 9, 8, 6, 3, 2, 1\}$ and $\mathcal{H}_2(\lambda) = \{2, 4, 4, 2, 8, 6, 2\}$.

Recall that a partition λ is a t-core if the *hook length multi-set* of λ does not contain the integer t. It is known that the hook length multi-set of each t-core does not contain any *multiple* of t [Kn98. p.69, p.612; St99, p.468; JK81, p.75]. In other words, a partition λ is a t-core if and only if $\mathcal{H}_t(\lambda) = \emptyset$.

Definition 1.1. Let $t = 2t'+1$ be an odd positive integer. Each vector of integers $(v_0, v_1, \ldots, v_{t-1}) \in \mathbb{Z}^t$ is called *V-coding* if the following conditions hold: (i) $v_i \equiv i \,(\mathrm{mod}\, t)$ for $0 \leq i \leq t-1$; (ii) $v_0 + v_1 + \cdots + v_{t-1} = 0$.

The V-coding is implicitly introduced in [Ma72]. It can be identified with the *set* $\{v_0, v_1, \ldots, v_{t-1}\}$ thanks to condition (i).

Our first result is the following property on t-cores, which can be seen as a refinement of the Nekrasov-Okounkov formula. The proof of this property involves a bijection between t-cores and integer vectors constructed by Garvan, Kim and Stanton [GKS90].

Theorem 1.1. *Let $t = 2t' + 1$ be an odd positive integer. There is a bijection $\phi_V : \lambda \mapsto (v_0, v_1, \ldots, v_{t-1})$ which maps each t-core onto a V-coding such that*

$$(1.1) \qquad |\lambda| = \frac{1}{2t}(v_0^2 + v_1^2 + \cdots + v_{t-1}^2) - \frac{t^2 - 1}{24}$$

and

$$(1.2) \qquad \prod_{v \in \lambda}\left(1 - \frac{t^2}{h_v^2}\right) = \frac{(-1)^{t'}}{1! \cdot 2! \cdot 3! \cdots (t-1)!} \prod_{0 \leq i < j \leq t-1} (v_i - v_j).$$

We will describe the bijection ϕ_V and prove the two equalities (1.1) and (1.2) in Section 2. An example is given after the construction of the bijection ϕ_V.

Next we provide an elementary proof of the following hook length formula, discovered by Nekrasov and Okounkov in their study of the Seiberg-Witten Theory [NO06, formula (6.12)]. Our proof is based on the Macdonald identities for $A_\ell^{(a)}$ [Ma72] and Theorem 1.1.

Theorem 1.2 [Nekrasov-Okounkov]. *For any complex number z we have*

$$(1.3) \qquad \sum_{\lambda \in \mathcal{P}} x^{|\lambda|} \prod_{h \in \mathcal{H}(\lambda)}\left(1 - \frac{z}{h^2}\right) = \prod_{k \geq 1}(1 - x^k)^{z-1}.$$

Then we prove the following (t, y)-extension of Theorem 1.2. When $y = t = 1$ in (1.4) we recover the Nekrasov-Okounkov formula. This extension unifies the Macdonald identities and the generating function for t-cores.

Theorem 1.3. *Let t be a positive integer. For any complex numbers y and z we have*

$$(1.4) \qquad \sum_{\lambda \in \mathcal{P}} x^{|\lambda|} \prod_{h \in \mathcal{H}_t(\lambda)}\left(y - \frac{tyz}{h^2}\right) = \prod_{k \geq 1} \frac{(1 - x^{tk})^t}{(1 - (yx^t)^k)^{t-z}(1 - x^k)}.$$

The proof of Theorem 1.3, given in Section 4, is based on the Nekrasov-Okounkov formula (1.3) and on the properties of a classical bijection which maps each partition to its t-core and t-quotient [Ma95, p.12; St99, p.468; JK81, p.75; GSK90]. The following result has a similar proof.

Theorem 1.4. *For any complex number y we have*

(1.5) $$\sum_{\lambda \in \mathcal{P}} x^{|\lambda|} y^{\#\{h \in \mathcal{H}(\lambda), h=t\}} = \prod_{k \geq 1} \frac{(1 + (y-1)x^{tk})^t}{1 - x^k}.$$

Last, we derive several applications of Theorems 1.2 and 1.3. Let us single out some of them in this introduction. See [Ha08a] (resp. Section 5) for other applications of Theorem 1.2 (resp. Theorem 1.3).

Theorem 1.5 [marked hook formula]. *We have*

(1.6) $$\sum_{\lambda \vdash n} f_\lambda^2 \sum_{h \in \mathcal{H}(\lambda)} h^2 = \frac{n(3n-1)}{2} n!,$$

where f_λ is the number of standard Young tableaux of shape λ.

Theorem 1.5 will be proved in Section 5. The two sides of (1.6) can be combinatorially interpreted (see [Ha08a]). However, a natural bijection between those two sides remains to be constructed. Theorem 1.5 is to be compared with the following well-known formula, which is also a consequence of the Robinson-Schensted-Knuth correspondence (see, for example, [Kn98, p.49-59; St99, p.324]).

(1.7) $$\sum_{\lambda \vdash n} f_\lambda^2 = n!$$

The following theorem, proved in Section 6, improves a result due to Kostant [Ko04].

Theorem 1.6. *Let k be a positive integer and s be a real number such that $s \geq k^2 - 1$. Then $(-1)^k f_k(s) > 0$, where $f_k(s)$ is defined by*

$$\prod_{n \geq 1} (1 - x^n)^s = \sum_{k \geq 0} f_k(s) x^k.$$

In section 7 we study the reversion of the Euler Product and obtain, in particular, the following result.

Theorem 1.7. *For any positive integers n and k the following two expressions*

(1.8) $$\sum_{\lambda \vdash n} \prod_{v \in \lambda} \left(1 + \frac{k}{h_v^2}\right)$$

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and

$$(1.9) \qquad \frac{1}{n+1} \sum_{\lambda \vdash n} \prod_{v \in \lambda} \left(1 + \frac{n}{h_v^2}\right)$$

are integers.

The following specializations have similar forms, namely, Corollaries 1.8, 1.9 and 1.10 on the one hand, Corollaries 1.11 and 1.12 on the other hand. In fact, our motivation for Theorem 1.3 was to look for a formula that could interpolate the following two formulas (1.10) and (1.11).

Corollary 1.8 $[y = t = 1, z = t^2$ in Theorem 1.3$]$. *We have*

$$(1.10) \qquad \sum_{\lambda} x^{|\lambda|} \prod_{h \in \mathcal{H}(\lambda)} \left(1 - \frac{t^2}{h^2}\right) = \prod_{k \geq 1} \frac{(1 - x^k)^{t^2}}{1 - x^k},$$

where the sum ranges over all t-cores.

Corollary 1.9 $[z = t$ or $y = 0$ in Theorem 1.3$]$. *We have*

$$(1.11) \qquad \sum_{\lambda} x^{|\lambda|} = \prod_{k \geq 1} \frac{(1 - x^{tk})^t}{1 - x^k},$$

where the sum ranges over all t-cores.

Note that identity (1.11) is the well-known generating function for t-cores [Ma95, p.12; St99, p.468; GSK90]. It is also the special case $y = 0$ of Theorem 1.4. The following identity is similar to the above two identities. It is also a consequence of Theorem 1.4.

Corollary 1.10 $[y = 2$ in Theorem 1.4$]$. *We have*

$$(1.12) \qquad \sum_{\lambda \in \mathcal{P}} x^{|\lambda|} 2^{\#\{h \in \mathcal{H}(\lambda), h = t\}} = \prod_{k \geq 1} \frac{(1 + x^{tk})^t}{1 - x^k}.$$

Corollary 1.11 $[y = t = 1, z = 2$ in Theorem 1.3$]$. *We have*

$$(1.13) \qquad \sum_{\lambda \in \mathcal{P}} x^{|\lambda|} \prod_{h \in \mathcal{H}(\lambda)} \left(1 - \frac{2}{h^2}\right) = \prod_{k \geq 1} (1 - x^k).$$

Corollary 1.12 $[t = 2, y = z = 1$ in Theorem 1.3$]$. *We have*

$$(1.14) \qquad \sum_{\lambda \in \mathcal{P}} x^{|\lambda|} \prod_{h \in \mathcal{H}_2(\lambda)} \left(1 - \frac{2}{h^2}\right) = \prod_{k \geq 1} (1 + x^k).$$

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We end the introduction with some remarks. The right-hand side of (1.13) can be expanded by using the Euler pentagonal theorem [Eu83; An76, p.11]

$$(1.15) \qquad \prod_{k \geq 1}(1 - x^k) = \sum_{m=-\infty}^{\infty} (-1)^m x^{m(3m+1)/2},$$

so that Corollary 1.11 says that

$$(1.16) \qquad \sum_{\lambda \vdash n} \prod_{h \in \mathcal{H}(\lambda)} \left(1 - \frac{2}{h^2}\right)$$

is equal to $-1, 0, 1$ depending on the numerical value of n.

The right-hand side of (1.14) is the generating function for partitions with *distinct* parts, so that Corollary 1.12 says that

$$(1.17) \qquad \sum_{\lambda \vdash n} \prod_{h \in \mathcal{H}_2(\lambda)} \left(1 - \frac{2}{h^2}\right)$$

is equal to the number of partitions of n with distinct parts.

For example, there are five partitions of $n = 4$ and two of them have distinct parts.
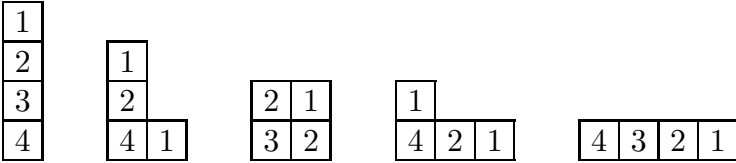

Fig. 1.4. The multi-set of hook lengths for $\mathcal{P}(4)$

We have

$$2\left(1 - \frac{2}{1^2}\right)\left(1 - \frac{2}{2^2}\right)\left(1 - \frac{2}{3^2}\right)\left(1 - \frac{2}{4^2}\right)$$
$$+2\left(1 - \frac{2}{1^2}\right)\left(1 - \frac{2}{1^2}\right)\left(1 - \frac{2}{2^2}\right)\left(1 - \frac{2}{4^2}\right)$$
$$+\left(1 - \frac{2}{1^2}\right)\left(1 - \frac{2}{2^2}\right)\left(1 - \frac{2}{2^2}\right)\left(1 - \frac{2}{3^2}\right) = 0$$

and

$$2\left(1 - \frac{2}{2^2}\right)\left(1 - \frac{2}{4^2}\right) + 2\left(1 - \frac{2}{2^2}\right)\left(1 - \frac{2}{4^2}\right) + \left(1 - \frac{2}{2^2}\right)\left(1 - \frac{2}{2^2}\right) = 2.$$

It would be interesting to explain directly why (1.16) and (1.17) are integers.

2. New properties of t-cores

In this section we first describe the bijection ϕ_V required in Theorem 1.1 and then prove equalities (1.1) and (1.2). Let $t = 2t' + 1$ be an odd positive integer. Each finite set of integers $A = \{a_1, a_2, \ldots, a_n\}$ is said to be t-*compact* if the following conditions hold:

(i) $-1, -2, \ldots, -t \in A$;

(ii) for each $a \in A$ such that $a \neq -1, -2, \ldots, -t$, we have $a \geq 1$ and $a \not\equiv 0 \bmod t$;

(iii) let $b > a \geq 1$ be two integers such that $a \equiv b \bmod t$. If $b \in A$, then $a \in A$.

Let A be a t-compact set. An element $a \in A$ is said to be t-*maximal* if $b \notin A$ for every $b > a$ such that $a \equiv b \bmod t$. The set of t-maximal elements of A is denoted by $\max_t(A)$. Let λ be a t-core. The H-*set* of the t-core λ is defined to be

$$H(\lambda) = \{h_v \mid v \text{ is a box in the leftmost column of } \lambda\} \cup \{-1, -2, \ldots - t\}.$$

The notion of H-set is a variation of the β-*numbers* introduced by James and Kerber, who also introduced the runners-beads-abacus model [JK81, p.75] in the study of t-cores. In this section we prefer to work directly on the H-sets, as our goal is to prove identities (1.1) and (1.2).

Lemma 2.1. *For each t-core λ its H-set $H(\lambda)$ is a t-compact set.*

Proof. Let $c = tk + r$ ($k \geq 1, 0 \leq r \leq t - 1$) be an element in $H(\lambda)$ and a be the maximal element in $H(\lambda)$ such that $a < t(k - 1) + r$. We must show that $t(k - 1) + r$ is also in $H(\lambda)$. If it were not the case, let $z > t(k - 1) + r, y_1, y_2, \ldots, y_d$ be the hook lengths as shown in Fig. 2.1, where only the relevant horizontal section of the partition diagram has been represented. We have $y_1 = c - a - 1 \geq tk + r - t(k - 1) - r = t$ and $y_d = c - z + 1 \leq tk + r - t(k - 1) - r = t$; so that there is one hook $y_i = t$. This is a contradiction since λ is supposed to be a t-core. $\square$
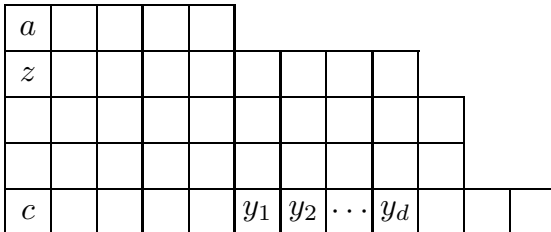

Fig. 2.1. Hook length and t-compact set

Construction of ϕ_V. Let λ be a t-core and $H(\lambda)$ be its H-set. The U-coding of λ is defined to be the set $U := \max_t(H(\lambda))$, which can be identified with the vector $(u_0, u_1, \ldots, u_{t-1})$ such that $u_0 = -t$, $u_i > -t$ and $u_i \equiv i \bmod t$ for $1 \leq i \leq t-1$. Let

(2.1) $$S := u_0 + u_1 + \cdots + u_{t-1}.$$

The integer S is a multiple of t because

(2.2) $$S = \sum u_i = \sum (tk_i + i) = t \sum k_i + t(t-1)/2$$

(remember that $t = 2t' + 1$ is an odd integer). The V-coding $\phi_V(\lambda)$ is the set V obtained from U by the following *normalization*:

(2.3) $$\phi_V(\lambda) = V := \{u - S/t \; : \; u \in U\}.$$

In fact, we can prove that $S/t = \ell(\lambda) - t' - 1$ (see (2.8)). The set V can be identified with a vector V-coding because

$$\sum v_i = \sum (u_i - S/t) = \sum u_i - S = 0.$$

Example 2.1. Consider the 5-core

$$\lambda = (14, 10, 6, 6, 4, 4, 4, 2, 2, 2).$$

The H-set of λ (see Fig. 2.2)

$$H(\lambda) = \{23, 18, 13, 12, 9, 8, 7, 4, 3, 2, -1, -2, -3, -4, -5\}$$

is 5-compact. The U-coding of λ is $U = \max_5(H(\lambda)) = \{23, 12, 9, -4, -5\}$, or in vector form

$$(u_0, u_1, u_2, u_3, u_4) = (-5, -4, 12, 23, 9).$$

As $S = \sum u_i = 35$, the V-coding is given by

$$V = \{-5 - 7, -4 - 7, 12 - 7, 23 - 7, 9 - 7\} = \{-12, -11, 5, 16, 2\},$$

or in vector form

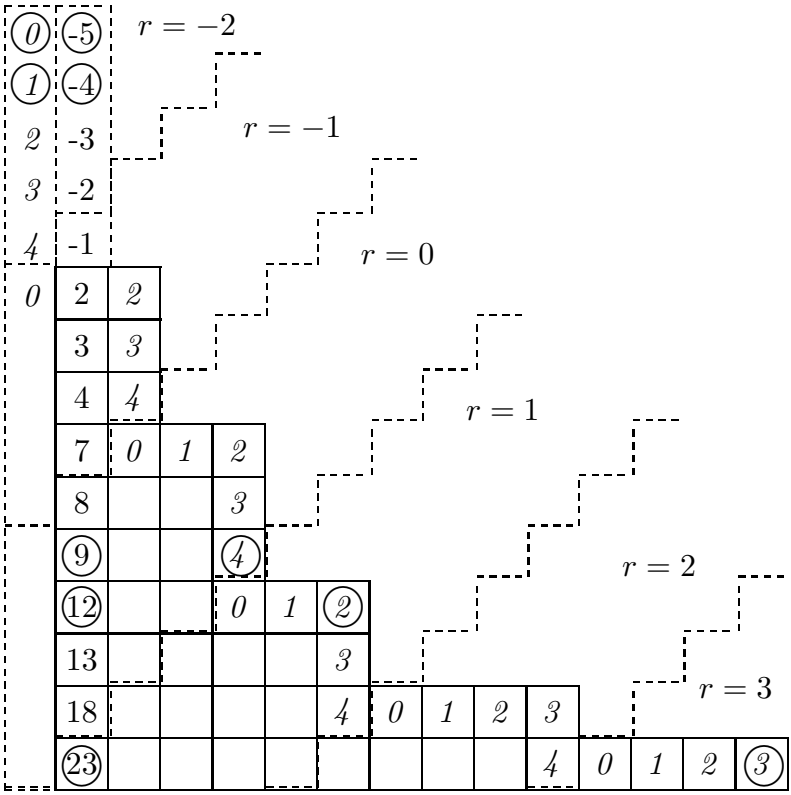$$\phi_V(\lambda) = (v_0, v_1, v_2, v_3, v_4) = (5, 16, 2, -12, -11).$$

Fig. 2.2. U-coding and N-coding of t-core

We have

$$|\lambda| = \frac{1}{2t}(v_0^2 + v_1^2 + \cdots + v_{t-1}^2) - \frac{t^2-1}{24}$$
$$= \frac{1}{2 \cdot 5}(5^2 + 16^2 + 2^2 + (-12)^2 + (-11)^2) - \frac{5^2-1}{24} = 54.$$

and

$$\prod_{v \in \lambda}\left(1 - \frac{5^2}{h_v^2}\right) = \frac{1}{1! \cdot 2! \cdot 3! \cdots (t-1)!} \prod_{0 \le i < j \le t-1}(v_i - v_j)$$
$$= (-11)(3)(17)(16) \cdot (14)(28)(27) \cdot (14)(13) \cdot (-1)/288$$
$$= 60035976.$$

Notice that, as expected, the above two numbers are positive integers.

A vector of integers $(n_0, n_1, \ldots, n_{t-1}) \in \mathbb{Z}^t$ is said to be an N-coding if $n_0 + n_1 + \cdots + n_{t-1} = 0$. Garvan, Kim and Stanton have defined a bijection ϕ_N between N-codings and t-cores. We now recall its definition using their own words [GKS90,p.3] (see also [BG06]).

Let λ be a t-core. Define the vector $(n_0, \ldots, n_{t-1}) = \phi_N(\lambda)$ in the following way. Label the box in the i-th row and j-column of λ by $j - i \bmod t$. We also label the boxes in column 0 (in dotted lines in Fig. 2.2) in the same way, and call the resulting diagram the *extended t-residue diagram*. A box is called *exposed* if it is at the end of a row of the extended t-residue diagram. The set of boxes (i, j) satisfying $t(r-1) \leq j-i < tr$ of the extended t-residue diagram of λ is called *region* and numbered r. In Fig. 2.2 the regions have been bordered by dotted lines. We now define n_i to be the maximum region r which contains an exposed box labeled i.

In Fig. 2.2 the labels of all boxes lying on the maximal border strip (but the leftmost one) have been written in italic. This includes all the exposed boxes: 3,3,3,2,4,3,2,4,3,2,4,3,2,1,0, when reading from bottom to top. We have $(n_0, n_1, n_2, n_3, n_4) = (-2, -2, 1, 3, 0)$.

Theorem 2.2 [Garvan-Kim-Stanton]. *The bijection*

$$\phi_N : \lambda \mapsto (n_0, n_1, \ldots, n_{t-1})$$

has the following property:

$$(2.4) \qquad |\lambda| = \frac{t}{2} \sum_{i=0}^{t-1} n_i^2 + \sum_{i=0}^{t-1} i n_i.$$

Let $t' = (t-1)/2$ and let

$$\phi_V^N : (n_0, n_1, \ldots, n_{t-1}) \mapsto (v_0, v_1, \ldots, v_{t-1})$$

be the bijection that maps each N-coding onto the V-coding defined by

$$(2.5) \qquad v_i = \begin{cases} t n_{i+t'} + i & \text{if } 0 \leq i \leq t'; \\ t n_{i-t'-1} + i - t & \text{if } t'+1 \leq i \leq t-1 \end{cases}$$

or in set form

$$(2.6) \qquad \{v_i \mid 0 \leq i \leq t-1\} = \{t n_i + i - t' \mid 0 \leq i \leq t-1\}.$$

The bijective property of the map ϕ_V^N is easy to verify. More essentially, the bijection ϕ_V is the composition product of the two previous bijections as is now shown.

Lemma 2.3. *We have $\phi_V = \phi_V^N \circ \phi_N$.*

Proof. Let $(v_0, \ldots, v_{t-1}) = \phi_V(\lambda)$, $(n_0, \ldots, n_{t-1}) = \phi_N(\lambda)$ and

$$(v_0', \ldots, v_{t-1}') = \phi_V^N(n_0, \ldots, n_{t-1}).$$

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We need prove that $v_i = v_i'$. The number n_i in the N-coding is defined to be the maximum region r which contains an exposed box labelled i. This exposed box is called *critical italic box*. In Fig. 2.2 a circle is drawn around the label of each critical italic box. On the other hand, the U-coding is defined to be the set $\max_t(H(\lambda))$, where $H(\lambda)$ is the H-set of λ. A box in the leftmost column whose hook length is an element of the U-coding is called *critical roman box*. In Fig. 2.2, a circle is drawn around the hook length number of each critical roman box. Let us write the labels of all the exposed boxes (the vector $L = (L_i)$) with its region numbers (the vector $R = (R_i)$) and the H-set of λ (the vector $H = (H_i) = H(\lambda)$), read from bottom to top.

$$
\begin{array}{rccccccccccccccc}
L = & ③ & 3 & 3 & ② & ④ & 3 & 2 & 4 & 3 & 2 & 4 & 3 & 2 & ① & ⓪ \\
R = & ③ & 2 & 1 & ① & ⓪ & 0 & 0 & -1 & -1 & -1 & -2 & -2 & -2 & ⑵ & ⑵ \\
H = & ㉓ & 18 & 13 & ⑫ & ⑨ & 8 & 7 & 4 & 3 & 2 & -1 & -2 & -3 & ⑷ & ⑸
\end{array}
$$

It is easy to see that $L_i \equiv (H_i - \ell(\lambda)) \bmod t$ and $R_i = \lfloor (H_i - \ell(\lambda))/t \rfloor + 1$. This means that L_i has a circle symbol if and only if H_i has a circle symbol. Define $i(k) := \max\{i \mid L_i = k\}$. Then

$$
\begin{aligned}
u_k &= H_{i(k)}, \\
n_k &= R_{i(k)} = 1 + \lfloor (u_i - \ell(\lambda))/t \rfloor, \\
k &= L_{i(k)} \equiv (H_i - \ell(\lambda)) \bmod t.
\end{aligned}
$$

We then have a natural bijection

$$(2.7) \qquad f : u_i \mapsto \lfloor (u_i - \ell(\lambda))/t \rfloor + 1 = n_{(u_i - \ell) \bmod t}$$

between the set $\{u_0, \ldots, u_{t-1}\}$ and $\{n_0, \ldots, n_{t-1}\}$. By (2.6) and (2.7) we have

$$
\begin{aligned}
\{v_i'\} &= \{tn_i + i - t'\} \\
&= \{tn_{(u_i - \ell) \bmod t} + (u_i - \ell) \bmod t - t'\} \\
&= \{t(\lfloor (u_i - \ell)/t \rfloor + 1) + (u_i - \ell) \bmod t - t'\} \\
&= \{u_i - \ell + t' + 1\}.
\end{aligned}
$$

On the other hand, (v_i') is a V-coding, because $v_i' \equiv i \bmod t$ and $\sum v_i' = t \sum n_i + \sum i - t(t-1)/2 = 0$; so that

$$(2.8) \qquad \left(\sum_i u_i \right)/t = \ell - t' - 1.$$

Hence

$$\{v_i'\} = \{u_i - \ell + t' + 1\} = \{u_i - (\sum_i u_i)/t\} = \{v_i\}. \quad \square$$

Take again the same partition as in Example 2.1; the N-coding is

$$(n_0, n_1, n_2, n_3, n_4) = (-2, -2, 1, 3, 0).$$

We verify that

$$
\begin{aligned}
(v_0', v_1', &v_2', v_3', v_4') \\
&= (1 \times 5 + 0, \ 3 \times 5 + 1, \ 0 \times 5 + 2, \ -2 \times 5 - 2, \ -2 \times 5 - 1). \\
&= (5, 16, 2, -12, -11) = (v_0, v_1, v_2, v_3, v_4).
\end{aligned}
$$

Proof of (1.1) in Theorem 1.1. From (2.6) we have

$$
\begin{aligned}
\sum v_i^2 &= \sum (tn_i + i - t')^2 \\
&= \sum \left((tn_i)^2 + 2tin_i - 2tt'n_i + i^2 + t'^2 - 2it' \right) \\
&= t^2 \sum n_i^2 + 2t \sum in_i + \frac{(t-1)t(2t-1)}{6} + tt'^2 - t't(t-1) \\
&= t^2 \sum n_i^2 + 2t \sum in_i + \frac{t(t^2-1)}{12}.
\end{aligned}
$$

Hence

$$\frac{1}{2t} \sum v_i^2 = \frac{t}{2} \sum n_i^2 + \sum in_i + \frac{t^2-1}{24} = |\lambda| + \frac{t^2-1}{24}. \quad \square$$

For proving (1.2) in Theorem 1.1, we first etablish the following two lemmas.

Lemma 2.4. *For any t-compact set A we have*

(2.9)
$$\prod_{a \in A, a > 0} \left(1 - \frac{t^2}{a^2} \right) = \prod_{a \in \max_t(A), a \neq -t} \frac{a+t}{a}.$$

Example 2.2. Take $t = 5$. Then the set

$$A = \{-5, -4, -3, -2, -1, 2, 3, 4, 7, 8, 9, 12, 13, 18, 23\}$$

is 5-compact. We have $\max_t(A) = \{-5, -4, 9, 12, 23\}$. Hence

$$(2.10) \qquad \prod_{a \in A, a > 0} \left(1 - \frac{25}{a^2}\right) = \frac{1 \cdot 14 \cdot 17 \cdot 28}{(-4) \cdot 9 \cdot 12 \cdot 23}.$$

Proof. Write

$$\prod_{a \in A, a > 0} \left(1 - \frac{t^2}{a^2}\right) = \prod_{a \in A, a > 0} \frac{(a - t) \cdot (a + t)}{a \cdot a},$$

then delete the common factors in numerator and denominator, as illustrated by means of Example 2.2.

$$\frac{1}{-4} \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (a \equiv 1 \bmod 5)$$

$$\frac{2}{-3} \quad \frac{-3}{2} \times \frac{7}{2} \quad \frac{2}{7} \times \frac{12}{7} \quad \frac{7}{12} \times \frac{17}{12} \qquad\qquad\qquad (a \equiv 2 \bmod 5)$$

$$\frac{3}{-2} \quad \frac{-2}{3} \times \frac{8}{3} \quad \frac{3}{8} \times \frac{13}{8} \quad \frac{8}{13} \times \frac{18}{13} \quad \frac{13}{18} \times \frac{23}{18} \quad \frac{18}{23} \times \frac{28}{23} \quad (a \equiv 3 \bmod 5)$$

$$\frac{4}{-1} \quad \frac{-1}{4} \times \frac{9}{4} \quad \frac{4}{9} \times \frac{14}{9} \qquad\qquad\qquad\qquad\qquad (a \equiv 4 \bmod 5)$$

The product $(a - 5)(a + 5)/a^2$ for $a > 0$ is reproduced in the row determined by $a \bmod 5$ in the above table, except for the leftmost column. But the product of the factors in the leftmost column is equal to 1 because t is an odd integer; so that the left-hand side of (2.10) is the product of the factors in the above table. After deleting the common factors, it remains the rightmost fraction in each row. □

Lemma 2.5. *Let λ be a t-core and $(u_0, u_1, \ldots, u_{t-1})$ be its U-coding (defined in the body of the construction of ϕ_V). Let λ' be the t-core obtained from λ by erasing the leftmost column of λ and $(u'_0, u'_1, \ldots, u'_{t-1})$ be its U-coding. Then*

$$\prod_{0 \le i < j \le t-1} \frac{u_i - u_j}{u'_i - u'_j} = \prod_{j=1}^{t-1} \frac{u_j + t}{u_j}.$$

Example 2.3. Take the 5-core λ given in Example 2.1. The U-coding of λ is $(u_0, u_1, u_2, u_3, u_4) = (-5, -4, 12, 23, 9)$. We have

$$\lambda' = (13, 9, 5, 5, 3, 3, 3, 1, 1, 1).$$

The U-coding of λ' is $(u'_0, u'_1, u'_2, u'_3, u'_4) = (-5, 11, 22, 8, -1)$. Now, consider the cyclic rearrangement

$$(u''_0, u''_1, u''_2, u''_3, u''_4) = (-1, -5, 11, 22, 8)$$

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of $(u'_0, u'_1, u'_2, u'_3, u'_4)$. We have $\prod(u'_i - u'_j) = \prod(u''_i - u''_j)$ because t is an odd integer. Moreover $u''_i = u_i - 1$ for all $1 \leq i \leq 4$. Hence

$$\prod_{0 \leq i < j \leq t-1} \frac{u_i - u_j}{u''_i - u''_j} = \prod_{j=1}^{t-1} \frac{u_0 - u_j}{u''_0 - u''_j}$$

$$= \frac{(-5+4)(-5-12)(-5-23)(-5-9)}{(-1+5)(-1-11)(-1-22)(-1-8)}$$

$$= \frac{(-4+5)(12+5)(23+5)(9+5)}{(-4)(12)(23)(9)}.$$

Proof. We suppose that λ contains δ parts equal to 1 $(0 \leq \delta \leq t - 1)$. Its H-set $H(\lambda)$ (viewed as a vector in decreasing order if necessary) can be split into six segments $H(\lambda) = A_1 A_2 A_3 A_4 A_5 A_6$ defined by (see Fig. 2.3)
   (i) $a \geq \delta + 2$ for each $a \in A_1$;
   (ii) $A_2 = (\delta, \delta - 1, \ldots, 3, 2, 1)$;
   (iii) $A_3 = (-1, -2, -3, \ldots, \delta + 2 - t)$;
   (iv) $A_4 = (\delta + 1 - t)$;
   (v) $A_5 = (\delta - t, \delta - 1 - t, \ldots, 1 - t)$;
   (vi) $A_6 = (-t)$.

On the other hand the H-set $H(\lambda')$ of λ' is split into five segments $H(\lambda') = A'_1 A'_2 A'_3 A'_4 A'_5$ defined by
   (i') $A'_1 = \{a - \delta - 1 : a \in A_1\}$;
   (ii') $A'_2 = \{a - \delta - 1 : a \in A_2\} = (-1, -2, \ldots, -\delta)$;
   (iii') $A'_3 = (-\delta - 1)$;
   (iv') $A'_4 = \{a - \delta - 1 : a \in A_3\} = (-\delta - 2, -\delta - 3, \ldots, -t + 1)$;
   (v') $A'_5 = (-t)$.
Notice that some segments A_i and A'_i may be empty. More precisely,

$$\begin{cases} A_2 = A_5 = A'_2 = \emptyset, & \text{if } \delta = 0; \\ A_3 = A'_4 = \emptyset, & \text{if } \delta = t - 2; \\ A_3 = A_4 = A'_3 = A'_4 = \emptyset, & \text{if } \delta = t - 1. \end{cases}$$

The basic facts are:
   (i) $a \notin \max_t(H(\lambda))$ for every $a \in A_5$; because $\{a \bmod t : a \in A_5\} = \{a \bmod t : a \in A_3\}$. In other words the set A_5 is *masked* by A_3.
   (ii) $\delta + 1 - t \in \max_t(H(\lambda))$; because $a \not\equiv 0 \bmod t$ for every $a \in A'_1$ so that $a \not\equiv \delta + 1 \bmod t$ for every $a \in A_1$. It is easy to see that $a \not\equiv \delta + 1 \bmod t$ for every $a \in A_2 \cup A_3$.
   (iii) $-\delta - 1 \in \max_t(H(\lambda'))$; because $a \not\equiv 0 \bmod t$ for every $a \in A_1 \cup A_2$ so that $a \not\equiv -\delta - 1 \bmod t$ for every $a \in A'_1 \cup A'_2$.
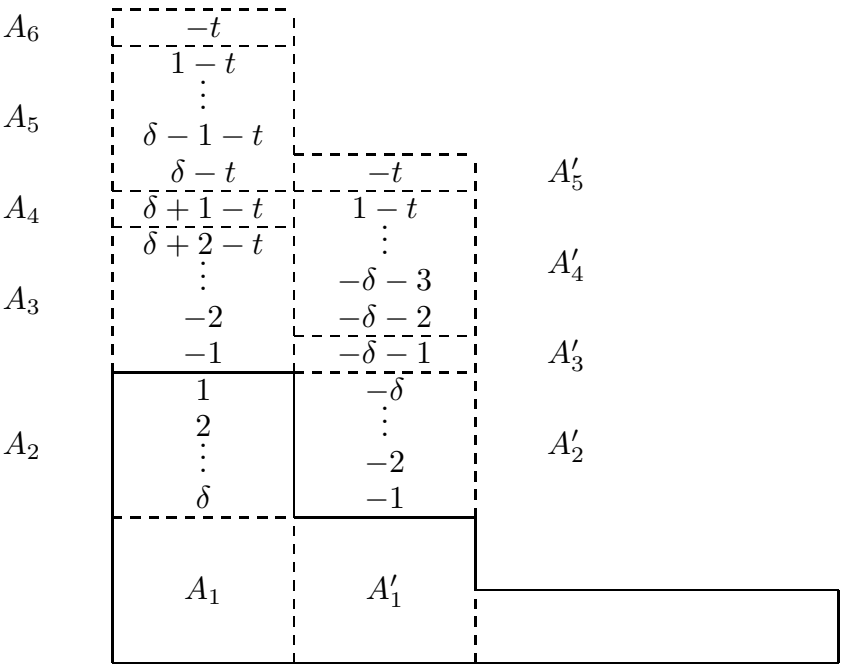
Fig. 2.3. Comparison of the hook lengths of λ and λ'

(iv) Since that $a \mapsto a - \delta - 1$ is a bijection between $A_1 \cup A_2 \cup A_3$ and $A'_1 \cup A'_2 \cup A'_4$, it is also a bijection between $\max_t(H(\lambda)) \setminus \{-t, \delta - t + 1\}$ and $\max_t(H(\lambda')) \setminus \{-t, -\delta - 1\}$.

The above facts enable us to derive the U-coding of λ' from the U-coding of λ as follows. Let

$$(u_i) = (u_0 = -t, u_1, u_2, \ldots, u_{k-1}, \delta + 1 - t, u_{k+1}, u_{k+2}, \ldots, u_{t-1})$$

be the U-coding of λ and define

$$(u''_i) = (u''_0 = -\delta - 1, u''_1, u''_2, \ldots, u''_{k-1}, -t, u''_{k+1}, u''_{k+2}, \ldots, u''_{t-1})$$

where $u''_i = u_i - \delta - 1$ for $i \geq 1$. Then, the U-coding of λ' is simply

$$(u'_i) = (u'_0 = -t, u''_{k+1}, u''_{k+2}, \ldots, u''_{t-1}, -\delta - 1, u''_1, u''_2, \ldots, u''_{k-1}).$$

We have $\prod(u'_i - u'_j) = \prod(u''_i - u''_j)$ because t is an odd integer. On the other hand, $u''_i - u''_j = u_i - u_j$ for all $1 \leq i < j \leq t - 1$. Hence

$$\prod_{0 \leq i < j \leq t-1} \frac{u_i - u_j}{u'_i - u'_j} = \prod_{0 \leq i < j \leq t-1} \frac{u_i - u_j}{u''_i - u''_j} = \prod_{j=1}^{t-1} \frac{u_0 - u_j}{u''_0 - u''_j}$$

$$= \prod_{j=1}^{t-1} \frac{-t - u_j}{-\delta - 1 - u''_j} = \prod_{j=1}^{t-1} \frac{u_j + t}{u_j}. \quad \square$$

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Proof of (1.2) in Theorem 1.1. Because the U-coding and V-coding of λ only differ by the normalization given in (2.3) and t is an odd integer, we have $\prod(v_i - v_j) = \prod(u_i - u_j)$. By Lemmas 2.5 and 2.4 we have

$$\prod_{0 \le i < j \le t-1} (u_i - u_j) = \prod_{j=1}^{t-1} \frac{u_j + t}{u_j} \times \prod_{0 \le i < j \le t-1} (u'_i - u'_j)$$

$$= \prod_{a \in H(\lambda), a > 0} \left(1 - \frac{t^2}{a^2}\right) \times \prod_{0 \le i < j \le t-1} (u'_i - u'_j)$$

$$= \cdots = K \times \prod_{v \in \lambda} \left(1 - \frac{t^2}{h_v^2}\right).$$

Taking λ as the empty t-core, the U-coding of λ is $(-t, -t + 1, -t + 2, \ldots, -3, -2, -1)$. We then obtain $K = (-1)^{t'} 1! \cdot 2! \cdot 3! \cdots (t-1)!$ ∎

3. Expansion formula for the powers of the Euler Product

The powers of the Euler Product and the hook lengths of partitions are two mathematical objects widely studied in the Theory of Partitions, in Algebraic Combinatorics and Group Representation Theory. In this section we give an elementary proof of Theorem 1.2, which establishes a new connection by giving an explicit expansion formula for all the powers s of the Euler Product in terms of partition hook lengths, where the exponent s is any complex number. Recall that the *Euler Product* is the infinite product $\prod_{m \ge 0}(1 - x^m)$. The following two formulas [Eu83; An76, p.11, p.21] go back to Euler (the pentagonal theorem)

$$(3.1) \qquad \prod_{m \ge 1} (1 - x^m) = \sum_{k=-\infty}^{\infty} (-1)^k x^{k(3k+1)/2}$$

and Jacobi (triple product identity, see [An76, p.21; Kn98, p.20; JS89; FH99; FK99])

$$(3.2) \qquad \prod_{m \ge 1} (1 - x^m)^3 = \sum_{m \ge 0} (-1)^m (2m + 1) x^{m(m+1)/2}.$$

Further explicit formulas for the powers of the Euler Product

$$(3.3) \qquad \prod_{m \ge 1} (1 - x^m)^s = \sum_{k \ge 0} f_k(s) x^k$$

have been derived for certain integers

$$(3.4) \qquad s = 1, 3, 8, 10, 14, 15, 21, 24, 26, 28, 35, 36, \ldots$$

by Klein and Fricke for $s = 8$, Atkin for $s = 14, 26$, Winquist for $s = 10$, and Dyson for $s = 24, \ldots$ [Wi69; Dy72]. The paper entitled "Affine root systems and Dedekind's η-function", written by Macdonald in 1972, is a milestone in the study of powers of Euler Product [Ma72]. The review of this paper for MathSciNet, written by Verma [Ve], contains seven pages! It has also inspired several followers, see [Ka74; Mo75; Ko76; Ko04; Mi85; AF02; CFP05; RS06]. The main achievement of Macdonald was to unify all the well-known formulas for the integers s listed in (3.4), except for $s = 1$ and $s = 26$. He obtained an expansion formula of

$$(3.5) \qquad \prod_{m \geq 0} (1 - x^m)^{\dim \mathfrak{g}}$$

for every semi-simple Lie algebra $\mathfrak{g}$.

A variation of the Euler Product, called the *Dedekind η-function*, is defined by

$$(3.6) \qquad \eta(x) = x^{1/24} \prod_{m \geq 1} (1 - x^m).$$

We are ready to state the Macdonald identities for $A_\ell^{(a)}$ [Ma72], which play a fundamental role in the following proof of Thorem 1.2.

Theorem 3.1 [Macdonald]. *Let $t = 2t' + 1$ be an odd integer. We have*

$$(3.7) \qquad \eta(x)^{t^2-1} = c_0 \sum_{(v_0, \ldots, v_{t-1})} \prod_{i<j} (v_i - v_j) x^{(v_0^2 + v_1^2 + \cdots + v_{t-1}^2)/(2t)},$$

where the sum ranges over all V-codings $(v_0, v_1, \ldots, v_{t-1})$ (see Definition 1.1) and c_0 is a numerical constant.

Consider the term of lowest degree in the above power series. We immediately get

$$(3.8) \qquad c_0 = \frac{(-1)^{t'}}{1! \cdot 2! \cdot 3! \cdots (t-1)!}.$$

Proof of Theorem 1.2. Using the following identity

$$(3.9) \qquad \prod_{m \geq 1} \frac{1}{1 - x^m} = \exp \Big(\sum_{k \geq 1} \frac{x^k}{k(1 - x^k)} \Big),$$

the right-hand side of equation (1.3) can be written:

$$(3.10) \qquad \prod_{m \geq 1} \frac{1}{1 - x^m} \times \exp\left(-z \sum_{k \geq 1} \frac{x^k}{k(1 - x^k)}\right).$$

Let $n \geq 0$ be a positive integer. The coefficient $C_n(z)$ of x^n on the left-hand side of (1.3) is a polynomial in z of degree n. The coefficient $D_n(z)$ of x^n on the right-hand side of (1.3) is also a polynomial in z of degree n thanks to (3.10). For proving $C_n(z) = D_n(z)$, it suffices to find $n + 1$ explicit numerical values $z_0, z_1, \ldots, z_n$ such that $C_n(z_i) = D_n(z_i)$ for $0 \leq i \leq n$ by using the Lagrange interpolation formula. The basic fact is that

$$\prod_{v \in \lambda} \left(1 - \frac{t^2}{h_v^2}\right) = 0$$

for every partition λ which is not a t-core. By comparing Theorems 1.1 and 3.1 we see that equation (1.3) is true when $z = t^2$ for every odd integer t, i.e.,

$$\sum_{\lambda \in \mathcal{P}} x^{|\lambda|} \prod_{v \in \lambda} \left(1 - \frac{t^2}{h_v^2}\right) = \prod_{m \geq 1} (1 - x^m)^{t^2 - 1},$$

so that $C_n(z) = D_n(z)$ for every complex number z. $\square$

Note that Kostant already observed that $D_n(z)$ is a polynomial in z, but did not mention any explicit expression [Ko04].

4. A unified hook formula via t-cores

In this section we prove Theorems 1.3 and 1.4 by using the properties of a classical bijection which maps each partition to its t-core and t-quotient [Ma95, p.12; St99, p.468; JK81, p.75; GSK90]. Let $\mathcal{W}$ be the set of bi-infinite binary sequences beginning with infinitely many 0's and ending with infinitely many 1's. Each element w of $\mathcal{W}$ can be represented by $(b_i)_i = \cdots b_{-3}b_{-2}b_{-1}b_0b_1b_2b_3 \cdots$, but the representation is not unique. Actually, for any fixed integer k the sequence $(b_{i+k})_i$ also represents w. The *canonical representation* of w is the unique sequence $(c_i)_i = \cdots c_{-3}c_{-2}c_{-1}c_0c_1c_2c_3 \cdots$ such that

$$\#\{i \leq -1, c_i = 1\} = \#\{i \geq 0, c_i = 0\}.$$

We put a dot symbol "." between the letters c_{-1} and c_0 in the bi-infinite sequence $(c_i)_i$ when it is the canonical representation.

There is a natural one-to-one correspondence between $\mathcal{P}$ and $\mathcal{W}$ (see, e.g. [St99, p.468; AF02] for more detail). Let λ be a partition. We encode each horizontal edge of λ by 1 and each vertical edge by 0. Reading these (0,1)-encodings from top to bottom and from left to right yields a binary word u. By adding infinitely many 0's to the left and infinitely many 1's to the right of u we get an element $w = \cdots 000u111\cdots \in \mathcal{W}$. Clearly the map $\psi : \lambda \mapsto w$ is a one-to-one correspondence between $\mathcal{P}$ and $\mathcal{W}$. The canonical representation of $\psi(\lambda)$ will be denoted by C_λ. For example, take $\lambda = (6, 5, 3, 3)$; we have $u = 1110011010$, so that $w = \cdots 0001110011010111\cdots$ and $C_\lambda = (c_i)_i = \cdots 0001110.011010111\cdots$

Fig. 4.1. Partition and (0,1)-sequence

Let t be a positive integer. It is known [Ma95, p.12; St99, p.468; JK81, p.75; GSK90] that there is a bijection Ω which maps a partition λ to $(\mu; \lambda^0, \lambda^1, \ldots, \lambda^{t-1})$ such that

(P1) μ is a t-core and $\lambda^0, \lambda^1, \ldots, \lambda^{t-1}$ are partitions;

(P2) $|\lambda| = |\mu| + t(|\lambda^0| + |\lambda^1| + \cdots + |\lambda^{t-1}|)$;

(P3) $\{h/t \mid h \in \mathcal{H}_t(\lambda)\} = \mathcal{H}(\lambda^0) \cup \mathcal{H}(\lambda^1) \cup \cdots \cup \mathcal{H}(\lambda^{t-1})$.

The vector $(\lambda^0, \lambda^1, \ldots, \lambda^{t-1})$ is usually called the *t-quotient* of the partition λ. Let us briefly describe the bijection Ω (see, e.g., [AF02; St99, p.468]). We split the canonical representation $C_\lambda = (c_i)_i$ of the partition λ into t sections. This means that we form the subsequence $w^k = (c_{it+k})_i$ for each $k = 0, 1, \ldots, t - 1$. The k-th entry λ^k of the t-quotient of λ is defined to be the inverse image $\psi^{-1}(w^k)$ of the subsequence w^k. With the above example and $t = 2$ we have $w^0 = \cdots 00110111\cdots$ and $w^1 = 00010100111\cdots$, so that $\lambda^0 = (2)$ and $\lambda^1 = (2, 2, 1)$. Property (P3) holds since $\mathcal{H}(\lambda^0) = \{2, 1\}$, $\mathcal{H}(\lambda^1) = \{1, 3, 1, 4, 2\}$ and $\mathcal{H}_2(\lambda) = \{2, 4, 2, 6, 2, 8, 4\}$ (See Fig. 4.1-4). Notice that the subsequence w^k defined by $w^k = (c_{it+k})_i$ is not necessarily the canonical representation. For that reason we do not reproduce the dot symbol "." in the corresponding rows in the following tableau.

C_λ	$\cdots$	0	0	0	0	0	1	1	1	0	.	0	1	1	0	1	0	1	1	1	$\cdots$
w^0	$\cdots$		0		0		1		1			0		1		1		1		1	$\cdots$
v^0	$\cdots$		0		0		0		1			1		1		1		1		1	$\cdots$
w^1	$\cdots$	0		0		0		1		0		1		0		0		1			$\cdots$
v^1	$\cdots$	0		0		0		0		0		0		1		1		1			$\cdots$
C_μ	$\cdots$	0	0	0	0	0	0	0	1	0	.	1	0	1	1	1	1	1	1	1	$\cdots$

Fig. 4.2. Partition λ^0 Fig. 4.3. Partition λ^1 Fig. 4.4. The 2-core μ

For each subsequence w^k we continually replace the subword 10 by 01. The final resulting sequence is of the form $\cdots 000111 \cdots$ and is denoted by v^k. The t-core of the partition λ is defined to be the partition μ such that the t sections of the canonical representation C_μ are exactly $v^0, v^1, \ldots, v^{t-1}$. For the above example we have $\mu = (2,1)$. Properties (P2) and (P3) can be derived from the following basic fact: each box of λ is in one-to-one correspondence with the ordered pair of integers (i,j) such that $i < j$ and $c_i = 1, c_j = 0$. Moreover the hook length of that box is equal to $j - i$.

Proof of Theorem 1.3. By the properties of the bijection Ω we get

$$\sum_{\lambda \in \mathcal{P}} x^{|\lambda|} \prod_{h \in \mathcal{H}_t(\lambda)} \left(y - \frac{tyz}{h^2}\right) = \prod_{k \geq 1} \frac{(1 - x^{tk})^t}{1 - x^k} \left(\sum_{\lambda \in \mathcal{P}} x^{t|\lambda|} \prod_{h \in \mathcal{H}(\lambda)} \left(y - \frac{tyz}{(th)^2}\right)\right)^t$$

$$(4.1) \qquad = \prod_{k \geq 1} \frac{(1 - x^{tk})^t}{1 - x^k} \left(\sum_{\lambda \in \mathcal{P}} (yx^t)^{|\lambda|} \prod_{h \in \mathcal{H}(\lambda)} \left(1 - \frac{z/t}{h^2}\right)\right)^t.$$

By Theorem 1.2

$$(4.2) \qquad \sum_{\lambda \in \mathcal{P}} (yx^t)^{|\lambda|} \prod_{h \in \mathcal{H}(\lambda)} \left(1 - \frac{z/t}{h^2}\right) = \prod_{m \geq 1} (1 - (yx^t)^m)^{z/t - 1}.$$

We obtain (1.4) when reporting (4.2) into (4.1). $\quad \square$

Proof of Theorem 1.4. It is easy to see that

$$\sum_{\lambda \in \mathcal{P}} x^{|\lambda|} y^{\#\{h \in \mathcal{H}(\lambda), h = 1\}} = \prod_{m \geq 1} \frac{1 + (y-1)x^m}{1 - x^m}.$$

By the properties of the bijection Ω we get

$$\sum_{\lambda \in \mathcal{P}} x^{|\lambda|} y^{\#\{h \in \mathcal{H}(\lambda), h=t\}} = \prod_{k \geq 1} \frac{(1-x^{tk})^t}{1-x^k} \left(\sum_{\lambda \in \mathcal{P}} x^{t|\lambda|} y^{\#\{h \in \mathcal{H}(\lambda), h=1\}} \right)^t$$

$$= \prod_{k \geq 1} \frac{(1-x^{tk})^t}{1-x^k} \left(\prod_{m \geq 1} \frac{1+(y-1)x^{tm}}{1-x^{tm}} \right)^t$$

$$= \prod_{k \geq 1} \frac{(1+(y-1)x^{tk})^t}{1-x^k}. \quad \square$$

5. Other Specializations

Some specializations are given in the introduction and in [Ha08a]. In this section we collect other specializations of Theorem 1.3. When the specialization is easy to derive, a simple comment is written between brackets.

Corollary 5.1 $[z = 0]$. We have

$$(5.1) \qquad \sum_{\lambda \in \mathcal{P}} x^{|\lambda|} y^{\# \mathcal{H}_t(\lambda)} = \prod_{k \geq 1} \frac{(1-x^{tk})^t}{(1-(yx^t)^k)^t(1-x^k)}.$$

Corollary 5.2 $[z = 0, y = -1]$. We have

$$(5.2) \qquad \sum_{\lambda \in \mathcal{P}} x^{|\lambda|} (-1)^{\# \mathcal{H}_t(\lambda)} = \prod_{k \geq 1} \frac{(1-x^{4tk})^t(1-x^{tk})^{2t}}{(1-x^{2tk})^{3t}(1-x^k)}.$$

Proof. First, we have

$$\prod_k \frac{1}{1-(-x)^k} = \prod_k \frac{1}{1-x^{2k}} \prod_{k \text{ odd}} \frac{1-x^k}{1-x^{2k}}$$

$$= \prod_k \frac{1-x^{4k}}{(1-x^{2k})(1-x^{4k})} \prod_{k \text{ odd}} \frac{1-x^k}{1-x^{2k}}$$

$$= \prod_k \frac{1-x^{4k}}{(1-x^{2k})^2} \prod_{k \text{ odd}} (1-x^k)$$

$$= \prod_k \frac{(1-x^{4k})(1-x^k)}{(1-x^{2k})^3}.$$

By Corollary 5.1

$$\sum_{\lambda \in \mathcal{P}} x^{|\lambda|} (-1)^{\# \mathcal{H}_t(\lambda)} = \left(\prod_{k \geq 1} \frac{1}{1-(-x^t)^k} \right)^t \times \prod_{k \geq 1} \frac{(1-x^{tk})^t}{1-x^k}$$

$$= \Big(\prod_{k\geq 1} \frac{(1-x^{4tk})(1-x^{tk})}{(1-x^{2tk})^3}\Big)^t \times \prod_{k\geq 1} \frac{(1-x^{tk})^t}{1-x^k}$$

$$= \prod_{k\geq 1} \frac{(1-x^{4tk})^t(1-x^{tk})^{2t}}{(1-x^{2tk})^{3t}(1-x^k)}. \quad \square$$

Corollary 5.3 $[y=1]$**.** *We have*

$$(5.3) \qquad \sum_{\lambda\in\mathcal{P}} x^{|\lambda|} \prod_{h\in\mathcal{H}_t(\lambda)} \Big(1-\frac{tz}{h^2}\Big) = \prod_{k\geq 1} \frac{(1-x^{tk})^z}{1-x^k}.$$

Corollary 5.3 can be seen as a discrete interpolation between formulas (1.10) and (1.11). For example, we have

$$\sum_{\lambda} x^{|\lambda|} \prod_{h\in\mathcal{H}_1(\lambda)} \Big(1-\frac{36}{h^2}\Big) = \prod_{k\geq 1} \frac{(1-x^k)^{36}}{1-x^k};$$

$$\sum_{\lambda} x^{|\lambda|} \prod_{h\in\mathcal{H}_2(\lambda)} \Big(1-\frac{36}{h^2}\Big) = \prod_{k\geq 1} \frac{(1-x^{2k})^{18}}{1-x^k};$$

$$\sum_{\lambda} x^{|\lambda|} \prod_{h\in\mathcal{H}_3(\lambda)} \Big(1-\frac{36}{h^2}\Big) = \prod_{k\geq 1} \frac{(1-x^{3k})^{12}}{1-x^k};$$

$$\sum_{\lambda} x^{|\lambda|} \prod_{h\in\mathcal{H}_6(\lambda)} \Big(1-\frac{36}{h^2}\Big) = \prod_{k\geq 1} \frac{(1-x^{6k})^{6}}{1-x^k},$$

where each sum is over all 6-cores λ.

Corollary 5.4 $[z=-b/y, y\to 0]$**.** *We have*

$$(5.4) \qquad \sum_{\lambda\in\mathcal{P}} x^{|\lambda|} \prod_{h\in\mathcal{H}_t(\lambda)} \frac{tb}{h^2} = e^{bx^t} \prod_{k\geq 1} \frac{(1-x^{tk})^t}{1-x^k}.$$

Proof. Using identity (3.9) the right-hand side of (1.4) can be written:

$$(5.5) \qquad \prod_{k\geq 1} \frac{(1-x^{tk})^t}{(1-(yx^t)^k)^t(1-x^k)} \exp\Big(-z\sum_{m\geq 1}\frac{(yx^t)^m}{m(1-(yx^t)^m)}\Big).$$

Since

$$\exp\Big(\frac{b}{y}\sum_{m\geq 1}\frac{(yx^t)^m}{m(1-(yx^t)^m)}\Big) = \exp\Big(\frac{b}{y}\big(\frac{yx^t}{1-yx^t}+O(y^2)\big)\Big)$$

$$= e^{bx^t} + O(y),$$

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we obtain (5.4) when $z = -b/y$ and $y \to 0$ in Theorem 1.3 under the form (5.5). $\square$

Corollary 5.5 [Compare the coefficients of $b^n x^{tn}$ in (5.4)]. *We have*

$$(5.6) \qquad \sum_{\lambda \vdash tn, \# \, \mathcal{H}_t(\lambda)=n} \quad \prod_{h \in \mathcal{H}_t(\lambda)} \frac{1}{h^2} = \frac{1}{t^n n!}.$$

Formula (5.6) is a classical result (see, e.g., [St99, p.469]).

Corollary 5.6 [Compare the coefficients of $b^n x^{tn+m}$ in (5.4)]. *We have*

$$(5.7) \qquad \sum_{\lambda \vdash tn+m, \# \, \mathcal{H}_t(\lambda)=n} \quad \prod_{h \in \mathcal{H}_t(\lambda)} \frac{1}{h^2} = \frac{c_t(m)}{t^n n!},$$

where $c_t(m)$ is the number of t-cores of size m.

Corollary 5.7 [Compare the coefficients of $(-z)^{n-1} x^{nt} y^n$]. *We have*

$$(5.8) \qquad \sum_{\lambda \vdash nt, \# \, \mathcal{H}_t(\lambda)=n} \quad \prod_{h \in \mathcal{H}_t(\lambda)} \frac{1}{h^2} \sum_{h \in \mathcal{H}_t(\lambda)} h^2 = \frac{3n-3+2t}{2(n-1)! \, t^{n-1}}.$$

Proof. Let R be the right-hand side of (1.4). As R is equal to (5.5), we have

$$
\begin{aligned}
&[(-z)^{n-1} x^{nt} y^n] R \\
={}&[x^{nt} y^n] \frac{1}{(n-1)!} \prod_{k \geq 1} \frac{(1-x^{tk})^t}{(1-(yx^t)^k)^t (1-x^k)} \Big(\sum_{m \geq 1} \frac{(yx^t)^m}{m(1-(yx^t)^m)} \Big)^{n-1} \\
={}&[x^t y] \frac{1}{(n-1)!} \frac{1}{(1-yx^t)^t} \Big(\frac{1}{(1-(yx^t))} + \frac{yx^t}{2(1-(yx^t)^2)} \Big)^{n-1} \\
={}&[x^t y] \frac{1}{(n-1)!} (1+tyx^t) \Big((1+yx^t) + \frac{yx^t}{2} \Big)^{n-1} \\
={}&\frac{1}{(n-1)!} \Big((n-1)\frac{3}{2} + t \Big). \quad \square
\end{aligned}
$$

The above corollary is the t-core analogue of the marked hook formula [Ha08a]. When $t = 1$ Formula (5.8) reduces to

$$\sum_{\lambda \vdash n} \quad \prod_{h \in \mathcal{H}(\lambda)} \frac{1}{h^2} \sum_{h \in \mathcal{H}(\lambda)} h^2 = \frac{3n-1}{2(n-1)!}.$$

We recover the marked hook formula (Theorem 1.5) thanks to the famous hook formula due to Frame, Robinson and Thrall [FRT54]

$$f_\lambda = \frac{n!}{\prod_{v \in \lambda} h_v(\lambda)},$$

where f_λ is the number of standard Young tableaux of shape λ (see [St99, p.376; Kn98, p.59; Kr99; Ze84; GNW79; GV85; NPS97; RW83]).

Corollary 5.8 [$y = 1$; compare the coefficients of z]. *We have*

$$(5.9) \qquad \sum_{\lambda \in \mathcal{P}} x^{|\lambda|} \sum_{h \in \mathcal{H}_t(\lambda)} \frac{1}{h^2} = \frac{1}{t} \prod_{m \geq 1} \frac{1}{1 - x^m} \sum_{k \geq 1} \frac{x^{tk}}{k(1 - x^{tk})}.$$

Proof. Let $y = 1$. Using (3.9) we have

$$\sum_{\lambda \in \mathcal{P}} x^{|\lambda|} \prod_{h \in \mathcal{H}_t(\lambda)} \left(1 - \frac{tz}{h^2}\right) = \prod_{k \geq 1} \frac{1}{(1 - x^k)} \exp \left(-z \sum_{m \geq 1} \frac{x^{tm}}{m(1 - x^{tm})}\right).$$

Comparing the coefficients of z in the above identity yields (5.9). □

Corollary 5.9. *We have*

$$(5.10) \qquad \sum_{\lambda \in \mathcal{P}} x^{|\lambda|} \sum_{h \in \mathcal{H}(\lambda), h \text{ odd}} \frac{1}{h^2} = \prod_{m \geq 1} \frac{1}{1 - x^m} \sum_{k \geq 1} \frac{x^{2k} + 2x^k}{2k(1 - x^{2k})}.$$

Proof. Let $t = 1$ and $t = 2$ in Corollary 5.8. We obtain respectively

$$(5.11) \qquad \sum_{\lambda \in \mathcal{P}} x^{|\lambda|} \sum_{h \in \mathcal{H}(\lambda)} \frac{1}{h^2} = \prod_{m \geq 1} \frac{1}{1 - x^m} \sum_{k \geq 1} \frac{x^k}{k(1 - x^k)}.$$

$$(5.12) \qquad \sum_{\lambda \in \mathcal{P}} x^{|\lambda|} \sum_{h \in \mathcal{H}_2(\lambda)} \frac{1}{h^2} = \frac{1}{2} \prod_{m \geq 1} \frac{1}{1 - x^m} \sum_{k \geq 1} \frac{x^{2k}}{k(1 - x^{2k})}.$$

Taking the difference between identities (5.11) and (5.12) yields (5.10). □

Remark. Identity (5.11) has a direct proof, which makes use of an elegant result on multi-sets of hook lengths and multi-sets of partition parts obtained by Stanley, Elder, Bessenrodt, Bacher and Manivel et al. [Be98, BM02, Ho86, St08, KS82, We1, We2]. See [Ha08a] for more details and applications.

6. Improvement of a result due to Kostant

Let

$$(6.1) \qquad \prod_{n \geq 1} (1 - x^n)^s = \sum_{k \geq 0} f_k(s) x^k.$$

Kostant proved the following result [Ko04, Th. 4.28].

Theorem 6.1 [Kostant]. *Let k and m be two positive integers such that $m \geq \max(k, 4)$. Then $f_k(m^2 - 1) \neq 0$.*

The condition $m > 1$ in the original statement of Kostant's Theorem should be replaced by $m \geq 4$, as, for example, $f_3(8) = 0$ (see Theorem 6.2). Our Theorem 1.6 extends Kostant's result in two directions: first, we claim that $(-1)^k f_k(s) > 0$ instead of $f_k(s) \neq 0$; second, s is any real number instead of an integer of the form $m^2 - 1$.

Proof of Theorem 1.6. By identity (1.3) we may write

$$(6.2) \qquad (-1)^k f_k(s) = \sum_{\lambda \vdash k} W(\lambda),$$

where

$$(6.3) \qquad W(\lambda) = \prod_{v \in \lambda} \left(\frac{s+1}{h_v^2} - 1 \right) = \prod_{v \in \lambda} \left(\frac{s+1-h_v^2}{h_v^2} \right).$$

For each $\lambda \vdash k$ and $v \in \lambda$ we have $h_v(\lambda) \leq k$, so that $W(\lambda) \geq 0$. This means that there is *no cancellation* in the sum (6.2). If $s > k^2 - 1$, then $W(\lambda) > 0$. If $s = k^2 - 1 \geq 15$, we have $k \geq 4$. In that case there is at least one partition λ, whose hook lengths are strictly less than k. Hence $W(\lambda) > 0$. ☐

Here is another result of Kostant [Ko04, Th.4.27] on which we will make some comments.

Theorem 6.2 [Kostant]. *We have*

$$f_4(s) = 1/4! \ s(s-1)(s-3)(s-14);$$
$$-f_3(s) = 1/3! \ s(s-1)(s-8);$$
$$f_2(s) = 1/2! \ s(s-3).$$

Even though we do not see how to factorize each $f_k(s)$, the occurrences of some factors in the above formulas have some relevance in terms of hook lengths. Every partition contains one hook length $h_v = 1$, so that $f_k(s)$ (for $k \geq 1$) has the factor $s + 1 - h_v^2 = s$ (see (6.3)). Every partition of 3 contains a hook length $h_v = 3$, so that $f_3(s)$ has the factor $s - 8$. Every partition of 2 or 4 has a hook length $h_v = 2$, so that $s - 3$ is a factor of $f_2(s)$ and $f_4(s)$. Note that Lehmer's conjecture claims that 24 is never a root of $f_k(s)$ for any positive integer k (see [Se70]).

7. Reversion of the Euler Product

Let $y(x)$ be a formal power series satisfying the following relation

(7.1)
$$x = y(1-y)(1-y^2)(1-y^3)\cdots$$
$$= y - y^2 - y^3 + y^6 + y^8 - y^{13} - y^{16} + \cdots$$

The first coeficients of the reversion series in (7.1) are the following

(7.2) $\quad y(x) = x + x^2 + 3\,x^3 + 10\,x^4 + 38\,x^5 + 153\,x^6 + 646\,x^7 + \cdots$

They are referred to as the first values of the sequence A109085 in The On-Line Encyclopedia of Integer Sequences [Slo].

Theorem 7.1. *We have the following explicit formula for the reversion of (7.1) in terms of hook lengths:*

(7.3)
$$y(x) = \sum_{n\geq 1} \frac{x^n}{n} \sum_{\lambda\vdash n-1} \prod_{v\in\lambda}\left(1 + \frac{n-1}{h_v^2}\right).$$

Proof. Rewrite (7.1) as $y = x\phi(y)$ where $\phi(y) = \prod_{m\geq 1}(1-y^m)^{-1}$. By the Lagrange inversion formula and identity (1.3) we have

$$[x^n]\,y = \frac{1}{n}[x^{n-1}]\,\phi(x)^n$$
$$= \frac{1}{n}[x^{n-1}]\prod_{m\geq 1}(1-y^m)^{-n}$$
$$= \frac{1}{n}[x^{n-1}]\sum_{\lambda\in\mathcal{P}}\prod_{v\in\lambda}\left(1 + \frac{n-1}{h_v^2}\right)x$$
$$= \frac{1}{n}\sum_{\lambda\vdash n-1}\prod_{v\in\lambda}\left(1 + \frac{n-1}{h_v^2}\right). \quad \Box$$

Proof of Theorem 1.7. The first part of Theorem 1.7 is easy to verify by using identity (1.3). As the coefficients of $y(x)$ defined by (7.1) are all positive integers, the above theorem implies that the expression

$$\frac{1}{n+1}\sum_{\lambda\vdash n}\prod_{v\in\lambda}\left(1 + \frac{n}{h_v^2}\right)$$

is a positive integer. $\quad \Box$

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