TWO NEW TRIANGLES OF q-INTEGERS VIA q-EULERIAN POLYNOMIALS OF TYPE A AND B

GUONIU HAN, FRÉDÉRIC JOUHET AND JIANG ZENG

ABSTRACT. The classical Eulerian polynomials can be expanded in the basis $t^{k-1}(1+t)^{n+1-2k}$ ($1 \leq k \leq \lfloor (n+1)/2 \rfloor$) with positive integers coefficients. This formula implies both the symmetry and the unimodality of the Eulerian polynomials. In this paper, we prove a q-analogue of this expansion for Carlitz's q-Eulerian polynomials as well as a similar formula for Chow-Gessel's q-Eulerian polynomials of type B. We shall give some applications of these two formulae, which involve two new sequences of polynomials in q with positive integral coefficients. An open problem is to give a combinatorial interpretation for these polynomials.

1. INTRODUCTION

The *Eulerian polynomials* $A_n(t) := \sum_{k=1}^{n} A_{n,k} t^{k-1}$ (see [FS70, Fo09, St97]) may be defined by

$$\sum_{k\geq 1} k^n t^k = \frac{A_n(t)}{(1-t)^{n+1}} \qquad (n \in \mathbb{N}).$$

It is well known (see [FS70]) that there are nonnegative integers $a_{n,k}$ such that

$$A_n(t) = \sum_{k=1}^{\lfloor (n+1)/2 \rfloor} a_{n,k} t^{k-1}(1+t)^{n+1-2k}. \tag{1.1}$$

For example, for $n = 1, \ldots, 4$, the identity reads

$$A_1(t) = 1, \quad A_2(t) = 1+t, \quad A_3(t) = (1+t)^2 + 2t^2, \quad A_4(t) = (1+t)^3 + 8t(1+t).$$

In particular, this formula implies both the *symmetry* and the *unimodality* of the Eulerian numbers $(A_{n,k})_{1\leq k\leq n}$ for any fixed n. The coefficients $a_{n,k}$ defined by (1.1) satisfy the following recurrence relation:

$$a_{n,k} = k a_{n-1,k} + 2(n+2-2k) a_{n-1,k-1} \tag{1.2}$$

for $n \geq 2$ and $1 \leq k \leq \lfloor (n+1)/2 \rfloor$, with $a_{1,1} = 1$, and $a_{n,k} = 0$ for $k \leq 0$ or $k > \lfloor (n+1)/2 \rfloor$.

Date: March 15, 2012.

$n\backslash k$	1	2	3	4
1	1			
2	1			
3	1	2		
4	1	8		
5	1	22	16	
6	1	52	136	

$n\backslash k$	0	1	2	3
1	1			
2	1	4		
3	1	20		
4	1	72	80	
5	1	232	976	
6	1	716	766	3904

Table 1. The first values of $(a_{n,k})$ and $(b_{n,k})$

The classical Eulerian polynomials are the descent polynomials of symmetric groups or Coxeter group of type A. Analogues of Eulerian polynomials for other Coxeter groups were introduced and studied from combinatorial point of view in the last three decades. Recall that the Eulerian polynomial of type B are defined by

$$\sum_{n\geq 0}(2k+1)^n t^n = \frac{B_n(t)}{(1-t)^{n+1}}. \tag{1.3}$$

The type B version of (1.1) appeared quite recently (see [Pe07, St08, Ch08]) and reads as follows

$$B_n(t) = \sum_{k=0}^{\lfloor \frac{n}{2}\rfloor} b_{n,k} t^k (1+t)^{n-2k}, \tag{1.4}$$

where $b_{n,k}$ are positive integers satisfying the recurrence relation

$$b_{n,k} = (2k+1)b_{n-1,k} + 4(n+1-2k)b_{n-1,k-1}, \tag{1.5}$$

and for $n \geq 2$ and $0 \leq k \leq \lfloor n/2 \rfloor$, with $a_{1,0} = 1$, and $a_{n,k} = 0$ for $k \leq 0$ or $k > \lfloor n/2 \rfloor$.

The numbers $a_{n,k}$ and $4^{-k}b_{n,k}$ appear as $A101280$ and $A008971$ in The *On-Line Encyclopedia of Integer Sequences* : http://oeis.org.

The aim of this paper is to prove a q-analogue of (1.1) with a refinement of the triangle $(a_{n,k})$ for Carlitz's q-Eulerian polynomials [Ca75], and also a q-analogue of (1.4) with a refinement of the triangle $(b_{n,k})$ for Chow-Gessel's q-Eulerian polynomials of type B [CG07]. Note that some other extensions of (1.1) are discussed in [Br08, SW10, SZ10].

The plan of this paper is as follows: we derive in Section 2 a q-analog of (1.1) using Carlitz's q-Eulerian polynomials and derive some results about the q-tangent number $T_{2n+1}(q)$ studied in [FH09], in Section 3, we give a q-analogue of (1.4) using Chow-Gessel's q-Eulerian polynomials of type B, which yields a new q-analogue of secant numbers. In Section 4, we apply the ur constructions to some conjectures on the unimodality from [CG07]. Finally, we will briefly give some concluding remarks in the last section.

2. A q-analogue for type A

The q-binomial coefficient $\begin{bmatrix} n \\ k \end{bmatrix}_q$ is defined by

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{(q;q)_n}{(q;q)_k (q;q)_{n-k}}, \qquad n \geq k \geq 0,$$

where $(x;q)_n = (1-x)(1-xq)\cdots(1-xq^{n-1})$ and $(x;q)_0 = 1$. Recall [Ca54] that Carlitz's q-Eulerian polynomials $A_n(t,q) := \sum_{k=1}^{n-1} A_{n,k}(q)t^k$ can be defined by

$$\sum_{k\geq 0} [k+1]_q^n t^k = \frac{A_n(t,q)}{(t;q)_{n+1}}, \tag{2.1}$$

where $[n]_q = 1 + q + \cdots + q^{n-1}$. It is easy to see that $A_{n,k}(q)$ satisfy the recurrence:

$$A_{n,k}(q) = [k]_q A_{n-1,k}(q) + q^{k-1}[n+1-k]_q A_{n-1,k-1}(q) \qquad (1 \leq k \leq n). \tag{2.2}$$

The following is our q-analog of (1.1).

Theorem 1. *For any positive integer n, there are polynomials $a_{n,k}(q) \in \mathbb{N}[q]$ such that the q-Eulerian polynomials $A_n(t,q)$ can be written as follows:*

$$A_n(t,q) = \sum_{k=1}^{\lfloor (n+1)/2 \rfloor} a_{n,k}(q) t^{k-1} (-tq^k;q)_{n+1-2k}. \tag{2.3}$$

Moreover, the polynomials $a_{n,k}(q)$ satisfy the following recurrence relation

$$a_{n,k}(q) = [k]_q a_{n-1,k}(q) + (1+q^{k-1}) q^{k-1} [n+2-2k]_q a_{n-1,k-1}(q) \tag{2.4}$$

for $n \geq 2$ and $1 \leq k \leq \lfloor (n+1)/2 \rfloor$, with $a_{1,1}(q) = 1$ and $a_{n,k}(q) = 0$ for $k \leq 0$ or $k > \lfloor (n+1)/2 \rfloor$.

Proof. Assume that $a_{n,k}(q)$ are coefficients satisfying (2.4). Then, by the q-binomial formula (cf. [An98, Theorem 3.3]),

$$(z;q)_N = \sum_{j=0}^{N} \begin{bmatrix} N \\ j \end{bmatrix}_q (-z)^j q^{j(j-1)/2}, \tag{2.5}$$

we see that (2.3) is equivalent to:

$$A_{n,k}(q) = \sum_{s\geq 1} \begin{bmatrix} n+1-2s \\ k-s \end{bmatrix}_q q^{(k-s)s + \binom{k-s}{2}} a_{n,s}(q). \tag{2.6}$$

Substituting (2.6) in (2.2), and using (2.4), we derive:

$$\sum_{s\geq 1}\begin{bmatrix} n+1-2s \\ k-s \end{bmatrix}_q q^{(k-s)s+\binom{k-s}{2}}\left([s]_q a_{n-1,s}(q)+(1+q^{s-1})q^{s-1}[n+2-2s]_q a_{n-1,s-1}(q)\right)$$
$$=\sum_{s\geq 1} q^{(k-s)s+\binom{k-s}{2}}\left([k]_q\begin{bmatrix} n-2s \\ k-s \end{bmatrix}_q+[n+1-k]_q\begin{bmatrix} n-2s \\ k-1-s \end{bmatrix}_q\right)a_{n-1,s}(q).$$

Extracting the coefficents of $a_{n-1,s}(q)$ we obtain:

$$\begin{bmatrix} n+1-2s \\ k-s \end{bmatrix}_q [s]_q+\begin{bmatrix} n-1-2s \\ k-s-1 \end{bmatrix}_q (1+q^s)[n-2s]_q$$
$$=[k]_q\begin{bmatrix} n-2s \\ k-s \end{bmatrix}_q+[n+1-k]_q\begin{bmatrix} n-2s \\ k-1-s \end{bmatrix}_q.$$

Canceling the common factors we get:

$$[n+1-2s]_q[s]_q+[n-k-s+1]_q(1+q^s)[k-s]_q=[k]_q[n-k-s+1]_q+[n+1-k]_q[k-s]_q.$$

The last identity is easy to verify, and this shows that (2.3) is satisfied. □

The first values of the coefficients $a_{n,k}(q)$ read as follows:

$n\backslash k$	1	2	3
1	1		
2	1		
3	1	$q+q^2$	
4	1	$2q(1+q)^2$	
5	1	$q(1+q)(3+5q+3q^2)$	$2q^3(1+q)^2(1+q^2)$
6	1	$q(1+q)^2(4+5q+4q^2)$	$q^3(1+q)^2(1+q^2)(5+7q+5q^2)$

In [FH09] Foata and Han defined a new sequence of q-tangent numbers $T_{2n+1}(q)$ by

$$T_{2n+1}(q)=(-1)^n q^{\binom{n}{2}} A_{2n+1}(-q^{-n},q). \tag{2.7}$$

We derive easily the following result from Theorem 1, which is the most difficult part of the main result in [FH09, Theorem 1.1].

Corollary 2. *The q-tangent number $T_{2n+1}(q)$ is a polynomial with positive integral coefficients.*

Proof. Let $a^*_{n,k}(q)=q^{-k(k+1)/2}a_{n,k}(q)$. Then (2.4) becomes

$$a^*_{n,k}(q)=[k]_q a^*_{n-1,k}(q)+(1+q^{k-1})[n+2-2k]_q a^*_{n-1,k-1}(q)$$

with the same initial conditions as $a_{n,k}(q)$. This proves that $a^*_{n,k}(q)$ is a polynomial in q with nonnegative integral coefficients. Now we show that $T_{2n+1}(q)=a^*_{2n+1,n+1}(q)$, which is sufficient to conclude. Let $n := 2n+1, k := n+1, t := -q^{-n}$ in (2.3), we get

$$A_{2n+1}(-q^{-n}, q) = \sum_{k=1}^{n+1} a_{2n+1,k}(q)(-q^{-n})^{k-1}(q^{k-n}; q)_{2n+2-2k} = a_{2n+1,n+1}(q)(-q^{-n})^n,$$

since $(q^{k-n}; q)_{2n+2-2k} = 0$ for $k = 1, 2, \dots, n$. The result follows then from (2.7). □

We can also derive straightforwardly the following result, which was proved in [FH09] using combinatorics of doubloons.

Corollary 3. *The quotient $A_{2n}(t, q)/(1 + tq^n)$ is a polynomial in t and q with positive integral coefficients.*

Proof. Note that

$$A_{2n}(t, q) = \sum_{k=1}^{n} a_{2n,k}(q)t^{k-1}(-tq^k; q)_{2n+1-2k}.$$

The result follows then from the fact that $(-tq^k; q)_{2n+1-2k} = (1 + tq^k) \cdots (1 + tq^{2n-k})$ contains the factor $1 + tq^n$ for $k = 1, \dots, n$. □

For any nonnegative integer n, set

$$f_n(q) := \sum_{k=0}^{2n+1} \binom{2n+1}{k} \frac{(-1)^k}{1 + q^{k-n}}. \tag{2.8}$$

Using doubloon model Foata-Han [FH09] proved that

$$d_n(q) := \frac{T_{2n+1}(q)}{(1+q)(1+q^2)\dots(1+q^n)} = \frac{(-1)^{n+1}(-1; q)_{n+2}}{(1-q)^{2n+1}} f_n(q)$$

is a polynomial in $\mathbb{N}[q]$. Actually we can prove the integrability of $d_n(q)$ without using the combinatorial device.

Proposition 4. *We have $d_n(q) \in \mathbb{Z}[q]$.*

Proof. Let $g_n(q) = (-1)^{n+1}(-1; q)_{n+2}$. Then $f_n(q)g_n(q)$ is clearly a polynomial in $\mathbb{Z}[q]$. We must show that 1 is a zero of order $2n+1$ of the polynomial $f_n(q)g_n(q)$ or

$$d^p(f_n(q)g_n(q))/dq^n|_{q=1} = 0 \quad \text{for} \quad p = 0, \dots, 2n.$$

By the Leibniz rule it suffices to show that $f_n^{(p)}(1) = 0$ for $p = 0, \dots, 2n$.

For any $k \in \mathbb{Z}$ and $m \in \mathbb{N}$, we define the Laurent polynomial $P_{m,k}(x)$ by the relation:

$$h_k^{(m)}(x) = \left(\frac{d}{dx}\right)^m (1 + x^k)^{-1} = \frac{P_{m,k}(x)}{(1 + x^k)^{m+1}}.$$

So $P_{0,k} = 1$, $P_{1,k} = -kx^{k-1}$, and for $m \geq 0$, we have

$$P_{m+1,k}(x) = (1 + x^k)P'_{m,k}(x) - k(m+1)x^{k-1}P_{m,k}(x).$$

Therefore the $P_{m,k}$ can, for $m \geq 1$, be written as follows:

$$P_{m,k}(x) = \sum_{l=1}^{m} \alpha_{l,m} x^{lk-m},$$

where $\alpha_{1,1} = -k$ and for $m \geq 1$, $\alpha_{1,m+1} = (k-m)\alpha_{1,m}$, $\alpha_{m+1,m+1} = (m-k)\alpha_{m,m}$,

$$\alpha_{l,m+1} = (lk-m)\alpha_{l,m} + (lk-mk-2k-m)\alpha_{l-1,m}, \quad 2 \leq l \leq m.$$

This shows that for $m \geq 1$ and $1 \leq l \leq m$, the coefficient $\alpha_{l,m}$ is a polynomial in the variable k, with degree less than or equal to m. We deduce that $P_{m,k}(1) = \sum_{l=1}^{m} \alpha_{l,m}$ is also a polynomial in the variable k, with degree less than or equal to m, therefore we can write for some rational coefficients $a_j(m)$ only depending on m:

$$h_k^{(m)}(1) = \frac{P_{m,k}(1)}{2^{m+1}} = \sum_{j=0}^{m} a_j(m) k^j.$$

Thus, differentiating (2.8) m times ($m \geq 0$) and then setting $q = 1$, we get

$$\begin{aligned} f_n^{(m)}(1) &= \sum_{k=0}^{2n+1} \binom{2n+1}{k} (-1)^k \sum_{j=0}^{m} a_j(m)(k-n)^j \\ &= \sum_{j=0}^{2n} a_j(m) \sum_{k=0}^{2n+1} \binom{2n+1}{k} (-1)^k (k-n)^j. \end{aligned}$$

Now, applying $2n+1$ times the finite difference operator Δ ($\Delta f(x) := f(x+1) - f(x)$) to the polynomial $(n+1-x)^j$ ($0 \leq j \leq 2n$) and setting $x = 0$ we get

$$\Delta^{2n+1}(n+1-x)^j\Big|_{x=0} = \sum_{k=0}^{2n+1} \binom{2n+1}{k} (-1)^k (k-n)^j,$$

which should vanish because $(n+1-x)^j$ is a polynomial in x of degree $j < 2n+1$. □

3. A q-analogue for type B

A B_n-analogue of Carlitz's q-Eulerian polynomials are introduced by Chow and Gessel [CG07]. These polynomials $B(t,q)$ are defined by

$$\sum_{k \geq 0} [2k+1]_q^n t^k = \frac{B_n(t,q)}{(t;q^2)_{n+1}}. \tag{3.1}$$

Let $B(t,q) := \sum_{k=0}^{n} B_{n,k}(q) t^k$. Then, the coefficients $B_{n,k}(q)$ satisfy the recurrence relation [CG07, Prop. 3.2] :

$$B_{n,k}(q) = [2k+1]_q B_{n-1,k}(q) + q^{2k-1}[2n-2k+1]_q B_{n-1,k-1}(q) \qquad 1 \leq k \leq n. \tag{3.2}$$

We have the following B_n-analog of (2.3).

Theorem 5. *For any positive integer n, there are polynomials $b_{n,k}(q) \in \mathbb{N}[q]$ such that the q-Eulerian polynomials of type B can be written as follows:*

$$B_n(t,q) = \sum_{k=0}^{n} B_{n,k}(q)t^k = \sum_{k=0}^{\lfloor n/2 \rfloor} b_{n,k}(q)t^k(-tq^{2k+1};q^2)_{n-2k}. \tag{3.3}$$

Moreover, the coefficients $b_{n,k}(q)$ satisfy the following recurrence relation:

$$b_{n,k}(q) = [2k+1]_q b_{n-1,k}(q) + (1+q)(1+q^{2k-1})q^{2k-1}[n+1-2k]_{q^2} b_{n-1,k-1}(q) \tag{3.4}$$

for $n \geq 2$ and $0 \leq k \leq \lfloor n/2 \rfloor$, with $b_{1,0}(q) = 1$, and $b_{n,k}(q) = 0$ for $k < 0$ or $k > \lfloor n/2 \rfloor$.

Proof. Assume that $b_{n,k}(q)$ satisfy (3.4). Then, by applying (2.5) with substitution $q \leftarrow q^2$, we derive that (3.3) is equivalent to:

$$B_{n,k}(q) = \sum_{s \geq 0} \begin{bmatrix} n-2s \\ k-s \end{bmatrix}_{q^2} q^{k^2-s^2} b_{n,s}(q). \tag{3.5}$$

Substituting (3.5) in (3.2), and using (3.4), we get:

$$\sum_{s\geq 0} \begin{bmatrix} n-2s \\ k-s \end{bmatrix}_{q^2} q^{k^2-s^2} \left([2s+1]_q b_{n-1,s}(q) + (1+q)(1+q^{2s-1})q^{2s-1}[n+1-2s]_{q^2} b_{n-1,s-1}(q)\right)$$

$$= \sum_{s\geq 0} q^{k^2-s^2} \left([2k+1]_q \begin{bmatrix} n-1-2s \\ k-s \end{bmatrix}_{q^2} + [2n+1-2k]_q \begin{bmatrix} n-1-2s \\ k-1-s \end{bmatrix}_{q^2} \right) b_{n-1,s}(q).$$

Extracting the coefficents of $b_{n-1,s}(q)$ we obtain:

$$\begin{bmatrix} n-2s \\ k-s \end{bmatrix}_{q^2} [2s+1]_q + \begin{bmatrix} n-2-2s \\ k-s-1 \end{bmatrix}_{q^2} (1+q)(1+q^{2s+1})[n-1-2s]_{q^2}$$
$$= [2k+1]_q \begin{bmatrix} n-1-2s \\ k-s \end{bmatrix}_{q^2} + [2n+1-2k]_q \begin{bmatrix} n-1-2s \\ k-1-s \end{bmatrix}_{q^2}.$$

Canceling the common factors yields:

$$[n-2s]_{q^2}[2s+1]_q + [n-k-s]_{q^2}(1+q)(1+q^{2s+1})[k-s]_{q^2}$$
$$= [2k+1]_q[n-k-s]_{q^2} + [2n+1-2k]_q[k-s]_{q^2}.$$

The last identity is easy to verify. This proves (3.3). □

For example, for $n = 1, \ldots, 4$, the equation (3.3) reads:

$B_1(t,q) = 1 + qt;$

$B_2(t,q) = (-tq;q^2)_2 + (q + 2q^2 + q^3)t;$

$B_3(t,q) = (-tq;q^2)_3 + (2q + 5q^2 + 6q^3 + 5q^4 + 2q^5)t(1 + tq^3);$

$$B_4(t,q) = (-tq;q^2)_4 + (3q + 9q^2 + 15q^3 + 18q^4 + 15q^5 + 9q^6 + 3q^7)t(-tq^3;q^2)_2$$
$$+ (2q^4 + 7q^5 + 11q^6 + 13q^7 + 14q^8 + 13q^9 + 11q^{10} + 7q^{11} + 2q^{12})t^2.$$

Theorem 5 implies immediately the following result, of which the first was derived in [FH10', Theorem 1.1 (d)] with more work.

Corollary 6. *For $n \geq 0$, we have*

$$B_{2n+1}(-q^{-2n-1}, q) = 0, \tag{3.6}$$

$$B_{2n}(-q^{-2n-1}, q) = (-1)^n q^{-n(2n+1)} b_{2n,n}(q). \tag{3.7}$$

Proof. By (3.3) we get

$$B_{2n+1}(-q^{-2n-1}, q) = \sum_{k=0}^{n} b_{2n+1,k}(q)(-q^{-2n-1})^k (q^{-2n+2k}; q^2)_{2n+1-2k} = 0.$$

Substituting n by $2n$ and t by $-q^{-2n-1}$ in (3.3) yields

$$\begin{aligned} B_{2n}(-q^{-2n-1}, q) &= \sum_{k=0}^{n} b_{2n,k}(q)(-q^{-2n-1})^k (q^{-2n+2k}; q^2)_{2n-2k} \\ &= (-1)^n q^{-n(2n+1)} b_{2n,n}(q). \end{aligned}$$

□

The above result leads to define a q-analogue of $B_n(1) = 4^n E_{2n}$ by

$$E^*_{2n}(q) := (-1)^n q^{n(n+1)} B_{2n}(-q^{-2n-1}, q). \tag{3.8}$$

Theorem 7. *There is a polynomial $G^*_{2n}(q) \in \mathbb{Z}[q]$ such that $G^*_{2n}(1) = E_{2n}$ and*

$$E^*_{2n}(q) = (1+q)(1+q^3)(1+q^5)\cdots(1+q^{2n-1}) \cdot (1+q)^n \cdot G^*_{2n}(q).$$

Proof. Recall that $E^*_{2n}(q) = (-1)^n q^{n(n+1)} B_{2n}(-q^{-2n-1}, q)$. From (3.1) we derive

$$\begin{aligned} \frac{B_n(t,q)}{(t;q^2)_{n+1}} &= (1-q)^{-n} \sum_{j\geq 0} (1-q^{2j+1})^n t^j \\ &= (1-q)^{-n} \sum_{j\geq 0} t^j \sum_{k=0}^{n} \binom{n}{k} (-q^{2j+1})^k \\ &= (1-q)^{-n} \sum_{k=0}^{n} \binom{n}{k} \frac{(-q)^k}{1-tq^{2k}}. \end{aligned}$$

Substituting n by $2n$ and $t = -q^{-2n-1}$ we obtain

$$E^*_{2n}(q) = (-1)^n q^{n(n+1)} \frac{(-q^{-2n-1}; q^2)_{2n+1}}{(1-q)^{2n}} \sum_{k=0}^{2n} \binom{2n}{k} \frac{(-q)^k}{1+q^{2k-2n-1}}.$$

Let

$$G^*_{2n}(q) := \frac{E^*_{2n}(q)}{(1+q)(1+q^3)\dots(1+q^{2n-1})(1+q)^n}$$
$$= (-1)^n q^{-n-1}\frac{(-q;q^2)_{n+1}}{(1+q)^n(1-q)^{2n}}\sum_{k=0}^{2n}\binom{2n}{k}\frac{(-q)^k}{1+q^{2k-2n-1}}.$$

For any nonnegative integer n, set

$$f^*_n(q) := \sum_{k=0}^{2n}\binom{2n}{k}\frac{(-q)^k}{1+q^{2k-2n-1}}. \tag{3.9}$$

Let $g^*_n(q) = (-1)^n q^{-n-1}(-q;q^2)_{n+1}/(1+q)^n$. Then $f^*_n(q)g^*_n(q)$ is clearly a polynomial in $\mathbb{Z}[q]$. We must show that 1 is a zero of order $2n$ of the polynomial $f^*_n(q)g^*_n(q)$ or

$$d^p(f^*_n(q)g^*_n(q))/dq^n|_{q=1} = 0 \quad \text{for} \quad p = 0,\dots,2n-1.$$

By the Leibniz rule it suffices to show that $d^p(f^*_n(q))/dq^p|_{q=1} = 0$ for $p = 0,\dots,2n-1$. The rest of the proof is almost the same as that of Proposition 4. □

Conjecture 8. *All the coefficients of the polynomials $G^*_{2n}(q)$ are positive.*

Since $G^*_{2n}(1) = E_{2n}$, the above conjecture would yield a new refinement of the secant number.

4. Application to unimodal problems

A sequence $\{\alpha_0,\dots,\alpha_d\}$ is *unimodal* if there exists an index $0 \le j \le d$ such that $\alpha_i \le \alpha_{i+1}$ for $i = 1,\dots,j-1$ and $\alpha_i \ge \alpha_{i+1}$ for $i = j,\dots,d$. Chow and Gessel [CG07] studied a kind of unimodality property of the q-Eulerian numbers assuming that q is a real number. In this section, we derive some unimodal properties of the sequence $(A_{n,k}(q))_{1\le k\le n}$ and $(B_{n,k}(q))_{1\le k\le n}$ from our previous results. From Theorem 1, we are able to deduce the following corollary, which provides a further support to the Conjecture 4.8 in [CG07].

Proposition 9. *Let $n \ge 2$ be an integer and $j = \lfloor (n+1)/2 \rfloor$. Then for $k = 1,\dots,j-1$, we have $A_{n,k+1}(q) > A_{n,k}(q)$ if $q > 1$ and $A_{n,n-k+1}(q) < A_{n,n-k}(q)$ if $q < 1$.*

Proof. We start from (2.6), which can be rewritten

$$A_{n,k}(q) = \sum_{s=1}^{k}\begin{bmatrix} n+1-2s \\ k-s \end{bmatrix}_q q^{(k-s)(k+s-1)/2}a_{n,s}(q),$$

for $k = 1, \ldots, n$, where we assume $a_{n,s}(q) = 0$ for $s > j$. Thus we can write for $k = 1, \ldots, j-1$:

$$A_{n,k+1}(q) - A_{n,k}(q) = a_{n,k+1}(q) + \sum_{s=1}^{k} \begin{bmatrix} n+1-2s \\ k+1-s \end{bmatrix}_q q^{(k+1-s)(k+s)/2} a_{n,s}(q) \left(1 - q^{-k} \frac{1-q^{k+1-s}}{1-q^{n+1-k-s}}\right).$$

We know that the q-binomial coefficient is a polynomial in q with nonnegative integer coefficients, and from Theorem 1 that this is also true for $a_{n,s}(q)$, $s = 1, \ldots, k+1$. Therefore it is enough to show that the coefficient between brackets is nonegative for $1 \le s \le k \le j-1$. This coefficient can be rewritten as:

$$\frac{q^{n+1} - q^{k+s} + q^s - q^{k+1}}{q^{n+1} - q^{k+s}}.$$

Assume first that $q > 1$. As $k+s \le 2j-2 \le n-1 < n+1$, the denominator of this fraction is positive. Moreover, it is not difficult to see that under the conditions $1 \le s \le k \le j-1$, and by using $(n-1)/2 \le j \le (n+1)/2$, we have the following inequalities:

$$\begin{aligned} q^{n+1} - q^{k+s} + q^s - q^{k+1} &\ge q^{n+1} - q^{2k} + q^k - q^{k+1} \\ &\ge q^{n+1} - q^{2j-2} + q^{j-1} - q^j \\ &\ge q^{n+1} - q^{n-1} + q^{(n-3)/2} - q^{(n+1)/2}. \end{aligned}$$

This last expression can be rewritten $(q^{(n+1)/2} - 1)(q^{(n+1)/2} - q^{(n-3)/2})$ and is nonnegative, which shows that $A_{n,k+1}(q) \ge A_{n,k}(q)$ for $k = 1, \ldots, j-1$.

In the case $0 < q < 1$, we only need to use the well-known relation $A_{n,n-k+1}(q) = q^{n(n-1)/2} A_{n,k}(1/q)$ for any $k = 1, \ldots, n$, and the result is obvious from the case $q > 1$. □

In the type B case, it is conjectured in [CG07, Conjecture 4.6] that the sequence $(B_{n,k}(q))_{0 \le k \le n}$ is unimodal. By Theorem 5, we are able to confirm partially this conjecture.

Proposition 10. *Let $n \ge 2$ be an integer and $j = \lfloor n/2 \rfloor$. Then for $k = 1, \ldots, j-1$, we have $B_{n,k+1}(q) > B_{n,k}(q)$ if $q > 1$ and $B_{n,n-k}(q) < B_{n,n-k-1}(q)$ if $q < 1$.*

Proof. We start from (3.5), which can be rewritten

$$B_{n,k}(q) = \sum_{s=0}^{k} \begin{bmatrix} n-2s \\ k-s \end{bmatrix}_{q^2} q^{k^2 - s^2} b_{n,s}(q),$$

for $k = 0, \ldots, n$, where we assume $b_{n,s}(q) = 0$ for $s > j$. Thus we can write for $k = 0, \ldots, j-1$:

$$B_{n,k+1}(q) - B_{n,k}(q) = b_{n,k+1}(q) + \sum_{s=0}^{k} \begin{bmatrix} n-2s \\ k+1-s \end{bmatrix}_{q^2} q^{(k+1)^2 - s^2} b_{n,s}(q) \left(1 - q^{-2k-1} \frac{1-q^{2(k+1-s)}}{1-q^{2(n-k-s)}}\right).$$

We know that the q-binomial coefficient is a polynomial in q with nonnegative integer coefficients, and from Theorem 5 that this is also true for $b_{n,s}(q)$, $s = 0, \ldots, k+1$. Therefore it is enough to show that the coefficient between brackets is nonegative for $0 \le s \le k \le j-1$. This coefficient can be rewritten as:

$$\frac{q^{2n} - q^{2s+2k} + q^{2s-1} - q^{2k+1}}{q^{2n} - q^{2k+2s}}.$$

Assume first that $q > 1$. As $k + s \le 2j - 2 \le n - 2 < n$, the denominator of this fraction is positive. Moreover, it is not difficult to see that under the conditions $0 \le s \le k \le j-1$, and by using $n/2 - 1 \le j \le n/2$, we have the following inequalities:

$$\begin{aligned} q^{2n} - q^{2s+2k} + q^{2s-1} - q^{2k+1} &\ge q^{2n} - q^{4k} + q^{2k-1} - q^{2k+1} \\ &\ge q^{2n} - q^{4j-4} + q^{2j-3} - q^{2j-1} \\ &\ge q^{2n} - q^{2n-4} + q^{n-5} - q^{n-1}. \end{aligned}$$

This last expression can be rewritten $(q^{2n} - q^{n-1})(1 - q^{-4})$ and is nonnegative, which shows that $B_{n,k+1}(q) \ge B_{n,k}(q)$ for $k = 0, \ldots, j-1$.

In the case $0 < q < 1$, we only need to use the well-known relation $B_{n,n-k}(q) = q^{n^2} B_{n,k}(1/q)$ for any $k = 0, \ldots, n$, and the result is obvious from the case $q > 1$. □

5. An open problem on the combinatorial interpretations

By Theorems 1 and 5, the polynomials $a_{n,k}(q)$ and $b_{n,k}(q)$ have positive integral coefficients. It is then natural to ask the following question.

Problem 11. *What are the combinatorial interpretations for $a_{n,k}(q)$ and $b_{n,k}(q)$?*

We can give a combinatorial interpretation for the *odd central terms* $a_{2n+1,n+1}(q)$ by using the *doubloon* model. Recall [FH09] that a *doubloon* of order $(2n+1)$ is defined to be a permutation of the word $012\cdots(2n+1)$, represented as a $2\times(n+1)$-matrix $\delta = \binom{a_0 \cdots a_n}{b_0 \cdots b_n}$. Define

$$\operatorname{cmaj}' \delta := \operatorname{maj}(a_0 \cdots a_n b_n \cdots b_0) - (n+1)\operatorname{des}(a_0 \cdots a_n b_n \cdots b_0) + n^2,$$

where "des" and "maj" are the usual *number of descents* and *major index* defined for words. A doubloon $\delta = \binom{a_0 \cdots a_n}{b_0 \cdots b_n}$ is said to be *interlaced*, if for every $k = 1, 2, \ldots, n$ the sequence $(a_{k-1}, a_k, b_{k-1}, b_k)$ or one of its three *cyclic rearrangements* is monotonic increasing or decreasing. By Theorem 1.5 in [FH09] we have the following result.

Proposition 12. *The polynomial* $a_{2n+1,n+1}(q)$ *is the generating function for the set of interlaced doubloons of order* $2n+1$ *by the statistic* cmaj'.

Another sequence of q-secant numbers is introduced in [FH10'] by

$$E_{2n}(q) = (-1)^n q^{n^2} B_{2n}(-q^{-2n}, q).$$

Unfortunately, it seems not easy to relate our coefficients $b_{n,k}(q)$ from Section 3 to the doubloons of type B, even for the central cases.

Acknowledgement

This work was partially supported by the grant ANR-08-BLAN-0243-03. The second author is grateful to Frédéric Chapoton for several discussions at the initial stage of this work.

References

[An98] G. E. Andrews, *The Theory of Partitions*, Cambridge Mathematical Library, Cambridge University Press, Cambridge, U.K., 1998.

[Br08] P. Brändén, *Actions on permutations and unimodality of descent polynomials*, European J. Combin. **29** (2008), no. 2, 514–531.

[Ca54] L. Carlitz, *q-Bernoulli and Eulerian numbers*, Trans. Amer. Math. Soc., 76 (1954), 332–350. Adv. in Appl. Math. **41** (2008), no. 2, 133–157.

[Ca75] L. Carlitz, *A combinatorial property of q-Eulerian number*, Amer. Math. Monthly, 82(1975), 51–54.

[Ch08] C.-O. Chow, *On certain combinatorial expansions of the Eulerian polynomials*, Adv. in Appl. Math. **41** (2008), no. 2, 133–157.

[CG07] C.-O. Chow and I.M. Gessel, *On the descent numbers and major indices for the hyperoctahedral group*, Adv. in Appl. Math. **38** (2007), 275–301.

[Fo09] D. Foata, *Eulerian polynomials: from Euler's time to the present*, preprint (2009), 20 pages.

[FH09] D. Foata and G.-N. Han, *Doubloons and new q-tangent numbers*, Quart. J. Math. 00 (2009), 1–16; doi:10.1093/qmath/hap043.

[FH10] D. Foata and G.-N. Han, *The doubloon polynomial triangle*, Raman. J. **23** (2010), 107–126.

[FH10'] D. Foata and G.-N. Han, *Doubloons and q-secant numbers*, Munster J. Math. **3** (2010), 89–110.

[FS70] D. Foata and M.-P. Schützenberger, *Théorie géométrique des polynômes eulériens*, Lecture Notes in Mathematics, Vol. 138, Springer-Verlag, Berlin, 1970.

[Pe07] T. K. Petersen, *Enriched P-partitions and peak algebras*, Adv. Math. **209** (2007), 561–610.

[SGW83] L.W. Shapiro, S. Getu and W.-J. Woan , *Runs, slides and moments*, SIAM J. Algebr. Discrete Methods **4** (1983), 459–466.

[SW10] J. Shareshian and M. L. Wachs, *Eulerian quasisymmetric functions*, Adv. in Math., **225** (2010), 2921–2966.

[SZ10] H. Shin and J. Zeng, *The symmetric and unimodal expansion of Eulerian polynomials via continued fractions*, preprint (2010).

[St97] R.P. Stanley, Enumerative Combinatorics, Vol. I, Cambridge University Press, Cambridge, 1997.

[St08] J.R. Stembridge, *Coxeter cones and their h-vectors*, Adv. Math. **217** (2008), no. 5, 1935–1961.

Guoniu Han,
Institut de Recherche Mathématique Avancée,
Université de Strasbourg et CNRS,
7, rue René-Descartes, F-67084 Strasbourg, France.
http://www-irma.u-strasbg.fr/~guoniu/

Frédéric Jouhet,
Université de Lyon, Université Lyon I,
CNRS, UMR 5208 Institut Camille Jordan,
43, bd du 11 Novembre 1918, 69622 Villeurbanne Cedex, France.
http://math.univ-lyon1.fr/~jouhet/

Jiang Zeng,
Université de Lyon, Université Lyon I,
CNRS, UMR 5208 Institut Camille Jordan,
43, bd du 11 Novembre 1918, 69622 Villeurbanne Cedex, France.
http://math.univ-lyon1.fr/~zeng/