Several numerical models for the simulation of bubble oscillations

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25 janvier 2008

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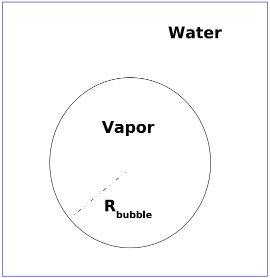
Introduction



- $p_{bubble} \ll p_{liquid}$
  $\Rightarrow$ Oscillations and/or collapse

- Several models (energy model / isothermal model, bifluid /monofluid...)
- Different numerical approximations

Plan

Plan

1 Studied models
- Euler System
- Isothermal Euler equations
- Boundary conditions and initial data

2 Numerical scheme

3 Numerical results

Euler System

Unknows:

- Density: $\rho(r, t)$
- Radial velocity: $v(r, t)$
- Internal energy: $e(r, t)$
- Fraction of gas: $\varphi(r, t)$

Pressure of the mixture: $p = p(\rho, e, \varphi)$

$$p(\rho, e, \varphi) = \begin{cases} (\gamma_1 - 1)\rho e & \text{if} \quad \varphi = 1, \\ (\gamma_2 - 1)\rho e - \gamma_2 \pi_2 & \text{if} \quad \varphi = 0. \end{cases} \tag{1}$$

with $\gamma_1 = 1.4$, $\gamma_2 = 1.1$ and $\pi_2 = \rho_0 c_0^2 / \gamma_2$.

Euler System

Mass conservation of gas:

$$(r^{d-1}\rho\varphi)_t + (r^{d-1}\rho\varphi u)_r = 0,$$

Mass conservation of liquid:

$$(r^{d-1}\rho(1-\varphi))_t + (r^{d-1}\rho(1-\varphi)u)_r = 0,$$

Momentuum conservation: (2)

$$(r^{d-1}\rho u)_t + (r^{d-1}(\rho u^2 + p))_x = (d-1)pr^{(d-2)},$$

Conservation of the total energy:

$$(r^{d-1}\rho E)_t + (r^{d-1}(\rho E + p)u)_x = 0,$$

Total energy: $E = e + u^2/2.$

with $d = 1, 2$ or 3

Isothermal Euler equations

φ is constant on the whole domain

Unknows:

- Density: $\rho(r,t)$
- Radial velocity: $v(r,t)$

$$
\begin{aligned}
&(r^{d-1}\rho)_t + (r^{d-1}\rho v)_r = 0, \\
&(r^{d-1}\rho v)_t + (r^{d-1}(\rho v^2 + p))_x = (d-1)pr^{(d-2)}
\end{aligned}
\qquad (3)
$$

with $d = 1, 2$ or 3

Isothermal Euler equations + monofluid model

Isobaric pressure law:

$$\begin{aligned} p &= p_{vapor} = p_{water} \\ &= p_0 + c^2(\rho - (\alpha(\rho,\varphi)\rho_A + (1-\alpha(\rho,\varphi))\rho_W)) \end{aligned} \tag{4}$$

where

$$\alpha(\rho,\varphi) = \frac{\theta - 1 + \sqrt{(\theta-1)^2 + 4\theta\varphi}}{2\theta}$$

and

$$\theta = \frac{\rho_W - \rho_A}{\rho}$$

Volume fraction of gas: $\alpha(\rho,\varphi)$
Reference pressure: $p_0 = 10^5 Pa$
Reference density for water: $\rho_W = 1000 kg/m^3$
Reference density for air: $\rho_A = 1 kg/m^3$
Sound of speed: $c = 1500 m/s$

Boundary conditions

- $r \in [0, AR_b]$, R_b =initial bubble radius
- $$\begin{aligned} Y &= (\rho, u, p, \varphi), \\ Y(r,0) &= \begin{cases} Y_L & \text{if} \quad r < R_b, \\ Y_R & \text{if} \quad r > R_b. \end{cases} \end{aligned} \tag{5}$$
- $Y(0,t) = Y_L$ and $Y(AR_b, t) = Y_R$
- $d > 1 \Rightarrow$ left boundary useless (because the first edge surface is 0)

Initial data

Reference parameters:

- Sound speed: $c_0 = 1500 m/s$
- Pressure: $p_0 = 10^5 Pa$
- Density: $\rho_0 = 1000 kg/m^3$

Vapor: $\gamma_1 = 1.4$ $(\pi_1 = 0)$

Water: $\gamma_2 = 1.1$, $\pi_2 = \dfrac{\rho_0 c_0^2}{\gamma_2}$

Left Side	Right Side
$\varphi_L = 1$	$\varphi_R = 0$
$u_L = 0$	$u_R = 0$
$p_L = p_0 \left(\frac{R_{eq}}{R_b}\right)^{3\gamma_1} = 75208.8 Pa$	$p_R = p_0$
$\rho_L = \rho_{eq} \left(\frac{p_L}{p_0}\right)^{1/\gamma_1} = 0.948 kg/m^3$	$\rho_R = \rho_0$

Plan

Numerical scheme

Eulerian approach: poor precision, smears the interface

Lagrangian approach: more precise at the interface,
constant states (u, p) when d=1,

if $R_b \longrightarrow 0 \Rightarrow \Delta t \longrightarrow 0$
(because of the CFL condition)

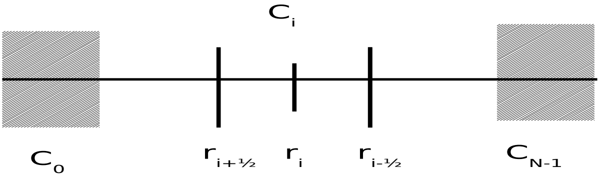
$\Rightarrow$ **Lagrangian approach only at the interface**

Numerical scheme

Conservative variables:

$$W = \begin{pmatrix} \rho\varphi \\ \rho(1-\varphi) \\ \rho u \\ \rho E \end{pmatrix} \qquad \text{and} \qquad W_i^n = W(r_i^n, t^n)$$

Left boundary motionless
Time step: $\tau^n = t^{n+1} - t^n$



Numerical scheme

$$r_{i+1/2}^{n+1} = r_{i+1/2}^{n} + \tau^n u_{i+1/2}^{n}. \tag{6}$$

$u_{i+1/2}^{n} = 0 \quad \longrightarrow \quad$ Eulerian scheme

$u_{i+1/2}^{n} = v_{i+1/2}^{n} \quad \longrightarrow \quad$ Lagrangian scheme

Numerical scheme

Integrating the balance laws on the space-time trapezoid

$$\{(r^n_{i-1/2}, t^n), (r^n_{i+1/2}, t^n), (r^{n+1}_{i+1/2}, t^n), (r^{n+1}_{i-1/2}, t^{n+1})\},$$

$$V_i^{n+1} W_i^{n+1} - V_i^n W_i^n + \tau^n \left(S^n_{i+1/2} F^n_{i+1/2} - S^n_{i-1/2} F^n_{i-1/2} \right) = \tau^n G_i^n. \tag{7}$$

Volume of ce cell i : $V_i^n = \int_{r^n_{i-1/2}}^{r^n_{i+1/2}} r^{d-1} dr$

Surface of edge : $i+1/2$: $S^n_{i+1/2} = \left(r^n_{i+1/2} \right)^{d-1}$

Lagrangian flux : $F^n_{i+1/2}$

Dimensional source term : G_i^n

Numerical scheme

1 Solve de Riemann problem $R(W_L, W_R, x/t) = W(x,t)$

$$\begin{aligned} & W_t + f(W)_x = 0, \\ & W(0,t) = \begin{cases} W_L & \text{if} \quad x < 0, \\ W_R & \text{if} \quad x > 0. \end{cases} \end{aligned} \tag{8}$$

To take into account the edges velocity, we compute:

$$W^n_{i+1/2} = R(W^n_i, W^n_{i+1}, u^n_{i+1/2})$$

2 Compute the Lagrangian flux $F^n_{i+1/2} = f(W^n_{i+1/2}) - v^n_{i+1/2} W^n_{i+1/2}$

3 Compute the dimensional source term :

$$G^n_i \simeq \int_{r^n_{i-1/2}}^{r^n_{i+1/2}} (d-1)p(r)r^{d-2}dr$$

given by $G^n_i = p^n_i \left(S^n_{i+1/2} - S^n_{i-1/2} \right)$

Stability condition

$$h_i^n = \frac{V_i^n}{S_{i-1/2}^n + S_{i+1/2}^n}$$

The time step has to respect two conditions :

Non vanishing condition:

$$\boxed{\tau^n \leq \max_i \left(\frac{h_i^n}{\left| v_{i\pm1/2}^n \right|} \right)} \tag{9}$$

Stability condition :

$$\boxed{\tau^n \leq \max_i \left(\frac{h_i^n}{\left| u_{i\pm1/2}^n - v_{i\pm1/2}^n \right| + c_{i\pm1/2}^n} \right)} \tag{10}$$

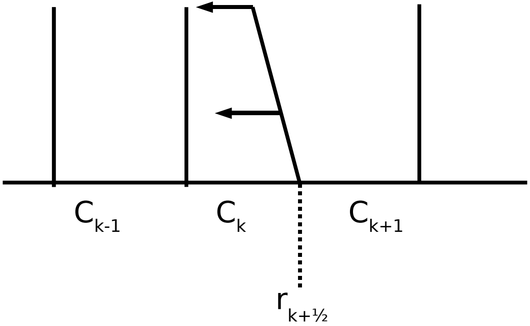
Cell averaging

Only the bubble interface move

$$\begin{aligned} &\varphi_k^n \neq \varphi_{k+1}^n \Rightarrow v_{k+1/2}^n = u_{k+1/2}^n, \\ &v_{i+1/2}^n = 0 \quad \text{if} \quad i \neq k. \end{aligned} \tag{11}$$

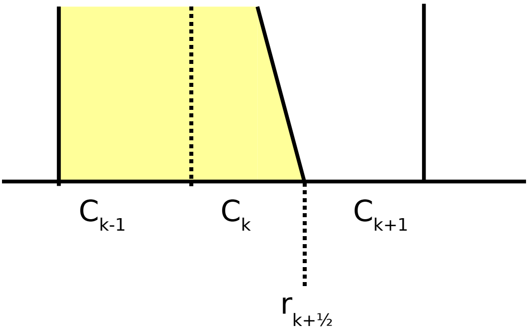
Cell averaging

The interface moves from right to left



Cell averaging

The interface moves from right to left



Cell averaging

1. Gather the two left cells
2. $\tau^n = \min(\tau^n, \Delta t')$ where

$$\Delta t' = \frac{r^n_{k+1/2} - r^n_{k-1/2}}{\left| v^n_{k+1/2} \right|}. \tag{12}$$

In the isothermal case

1. φ constant $\Rightarrow$ Eulerian scheme
2. Godunov scheme is not employed (because the exact Riemann solver is too much CPU consuming) $\Rightarrow$ Rusanov scheme

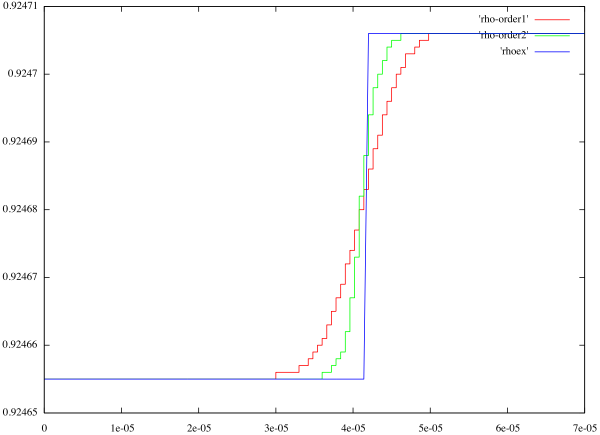
$$f^n_{i+\frac{1}{2}} = \frac{f^n_i + f^n_{i+1}}{2} - \frac{\sigma^n_{i+\frac{1}{2}}}{2}(w^n_{i+1} - w^n_i)$$

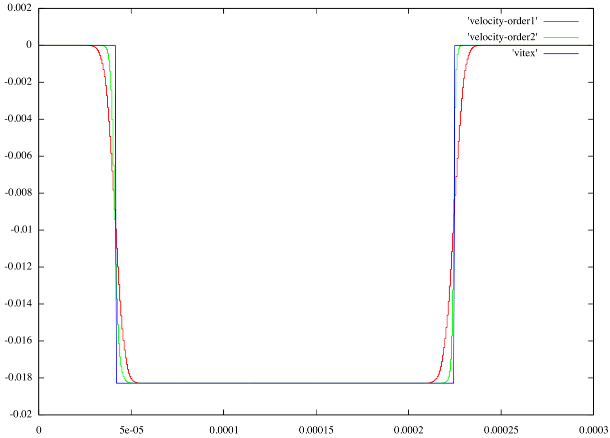
Plan

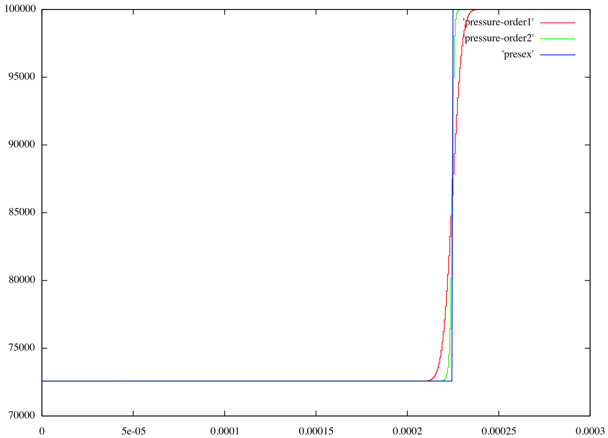
Validate ALE + averaging approach in 1D

500 cells, CFL=0.8

Initial bubble radius$R_b = 0.7469, \times 10^{-4}, t_{final} = 1 \times 10^{-7} s,$
$r \in [0, 4 * R_b]$
Comparison between: Exact Riemann Solver and Godunov with ALE and stiffened gas law


0.92471
0.9247
0.92469
0.92468
0.92467
0.92466
0.92465
0
1e-05
2e-05
3e-05
4e-05
5e-05
6e-05
7e-05
'rho-order1'
'rho-order2'
'rhoex'


0.002
0
-0.002
-0.004
-0.006
-0.008
-0.01
-0.012
-0.014
-0.016
-0.018
-0.02
0
5e-05
0.0001
0.00015
0.0002
0.00025
0.0003
'velocity-order1'
'velocity-order2'
'vitex'


'pressure-order1'
'pressure-order2'
'presex'
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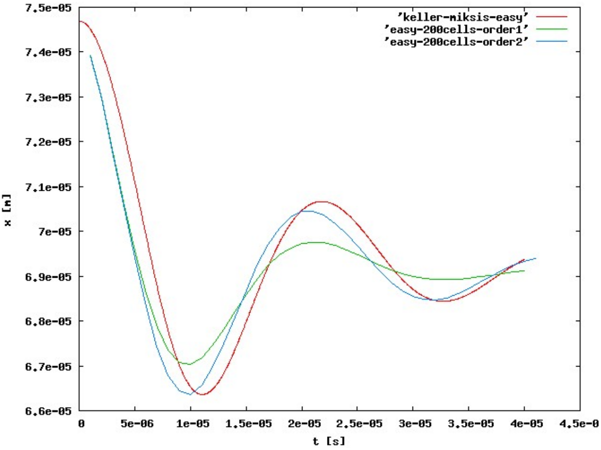
95000
90000
85000
80000
75000
70000
0
5e-05
0.0001
0.00015
0.0002
0.00025
0.0003

Easy bubble test with d=3

$c_0 = 50m/s$: decrease the numerical viscosity of the scheme
$R_b = 0.7469 \times 10^{-4}$: oscillations of smaller amplitude

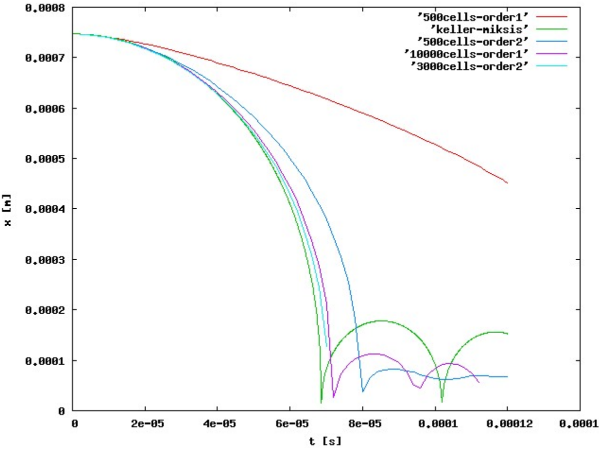
Left Side	Right Side
$\varphi_L = 1$	$\varphi_R = 0$
$u_L = 0$	$u_R = 0$
$p_L = p_0 \left(\frac{R_{eq}}{R_b}\right)^{3\gamma_1}$	$p_R = P_0$
$\rho_L = \rho_{eq} \left(\frac{p_L}{p_0}\right)^{1/\gamma_1}$	$\rho_R = \rho_0$

N=100 cells, CFL=0.8, $t_{final} = 1 \times 10^{-7} s$.
Comparison between: Keller-Miksis model and Godunov with ALE and stiffened gas law


7.5e-05
7.4e-05
7.3e-05
7.2e-05
7.1e-05
7e-05
6.9e-05
6.8e-05
6.7e-05
6.6e-05
x [m]
0
5e-06
1e-05
1.5e-05
2e-05
2.5e-05
3e-05
3.5e-05
4e-05
4.5e-0
t [s]
'keller-miksis-easy'
'easy-200cells-order1'
'easy-200cells-order2'

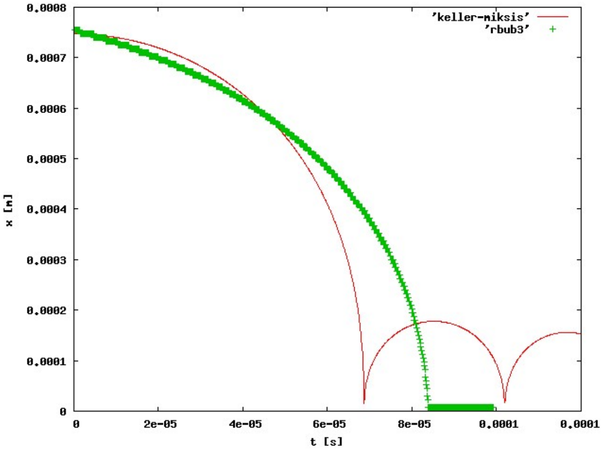
Hard bubble test with d=3

$c_0 = 1500 m/s$: decrease the numerical viscosity of the scheme
$R_b = 0.7469 \times 10^{-3}$: oscillations of smaller amplitude
3000 and 10000 cells
Comparison between: Keller-Miksis model and Godunov with ALE and stiffened gas law


'500cells-order1'
'keller-miksis'
'500cells-order2'
'10000cells-order1'
'3000cells-order2'
0.0008
0.0007
0.0006
0.0005
0.0004
0.0003
0.0002
0.0001
0
x [m]
0
2e-05
4e-05
6e-05
8e-05
0.0001
0.00012
0.0001
t [s]

Isothermal case with d=3

2000 cells
$\varphi = 10^{-3}$
Comparison between: Keller-Miksis model and Rusanov with isobaric pressure law


'keller-miksis'
'rbub3'
x [m]
t [s]
0.0008
0.0007
0.0006
0.0005
0.0004
0.0003
0.0002
0.0001
0
0
2e-05
4e-05
6e-05
8e-05
0.0001
0.0001

Conclusion

Future works

1. Second order in time
2. Employ the VFRoe scheme
3. Isothermal two-fluid model with “linear” pressure law

References

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