On the coupling of models and numerical methods for two-phase flows

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Micro-Macro Modelling and Simulation of Liquid-Vapour Flows
Strasbourg

Context of the work

Collaboration between

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Edwige Godlewski, Frédéric Lagoutière, Pierre-Arnaud Raviart, NS

CEA Saclay (French nuclear agency)
Annalisa Ambroso, Benjamin Boutin, Thomas Galié,
Samuel Kokh, Jacques Segré

Also participating
EDF R&D

Jean-Marc Hérard, Olivier Hurisse

Coupling of numerical codes

Two systems, indexed by (L) and (R), separated by a fixed interface at $\{x = 0\}$:

CODE 1	CODE 2
$\partial_t \mathbf{u}_L + \partial_x \mathbf{f}_L(\mathbf{u}_L) = \mathbf{s}_L(\mathbf{u}_L)$	$\partial_t \mathbf{u}_R + \partial_x \mathbf{f}_R(\mathbf{u}_R) = \mathbf{s}_R(\mathbf{u}_R)$
$x < 0$	$x > 0$

$$x = 0$$

- At $x = 0$: ARTIFICIAL interface coupling.
- Some informations must be transmitted between $\mathbf{u}_L(t, 0_-)$ and $\mathbf{u}_R(t, 0_+)$.
- No modification of the codes: use of boundary conditions.

The models to couple

Models (L) and (R) share

- ▷ the same underlying physics (thermohydraulic flows, multiphase flows...)

BUT come from different modelling

- ▷ different time scales
- ▷ different accuracy of description
- ▷ different space dimension
- ▷ ...

Need to understand the compatibility between the models (L) and (R) in order to couple them

Classical Euler system

Inviscid gas dynamics

$$\begin{cases} \partial_t \rho + \partial_x \rho u = 0 \\ \partial_t \rho u + \partial_x (\rho u^2 + p) = 0 \\ \partial_t \rho E + \partial_x (u(\rho E + p)) = 0 \end{cases}$$

where

$$E = \varepsilon + u^2/2 \quad \text{and} \quad p = p(\rho, \varepsilon)$$

ρ	density
u	velocity
p	pressure
E	total energy
ε	specific energy

Classical Euler system for multiphase mixture

Homogeneous equilibrium model (HEM)

$$\begin{cases} \partial_t \rho + \partial_x \rho u = 0 \\ \partial_t \rho u + \partial_x (\rho u^2 + p) = 0 \\ \partial_t \rho E + \partial_x (u(\rho E + p)) = 0 \end{cases}$$

where

$$E = \varepsilon + u^2/2 \quad \text{and} \quad p = p(\rho, \varepsilon)$$

ρ	density of the multiphase mixture
u	velocity of the multiphase mixture
p	pressure of the multiphase mixture
E	total energy of the multiphase mixture
ε	specific energy of the multiphase mixture

Extended Euler system for multiphase mixture

Homogeneous relaxation model (HRM)

$$\begin{cases} \partial_t \alpha + u \partial_x \alpha = \lambda(\alpha_{eq} - \alpha) \\ \partial_t \rho + \partial_x \rho u = 0 \\ \partial_t \rho u + \partial_x (\rho u^2 + p) = 0 \\ \partial_t \rho E + \partial_x (u(\rho E + p)) = 0 \end{cases}$$

where

$$E = \varepsilon + u^2/2, \quad p = p(\rho, \varepsilon, \alpha) \quad \text{and} \quad \alpha_{eq} = \alpha_{eq}(\rho)$$

ρ	density of the multiphase mixture
u	velocity of the multiphase mixture
p	pressure of the multiphase mixture
E	total energy of the multiphase mixture
ε	specific energy of the multiphase mixture
α	mass fraction of one the phases

Drift-Flux model

Isentropic case

$$\begin{cases} \partial_t \rho Y + \partial_x (\rho u Y + \rho Y(1-Y)\Phi) = 0 \\ \partial_t \rho + \partial_x \rho u = 0 \\ \partial_t \rho u + \partial_x (\rho u^2 + p + \rho Y(1-Y)\Phi^2) = 0 \end{cases}$$

where

$$p = p(\rho) \quad \text{and} \quad \Phi = \Phi(\rho Y, \rho, \rho u)$$

ρ	mean density
u	mean velocity
p	mean pressure
Y	mass fraction of one of the phase
Φ	relative velocity

Bifluid model

Isentropic case, pressure equilibrium

$$\begin{cases} \partial_t \alpha_1 \rho_1 + \partial_x \alpha_1 \rho_1 u_1 = 0 \\ \partial_t \alpha_2 \rho_2 + \partial_x \alpha_2 \rho_2 u_2 = 0 \\ \partial_t \alpha_1 \rho_1 u_1 + \partial_x (\alpha_1 \rho_1 (u_1)^2 + \alpha_1 p) - p_I \partial_x \alpha_1 = \lambda |u_1 - u_2| (u_2 - u_1) \\ \partial_t \alpha_2 \rho_2 u_2 + \partial_x (\alpha_2 \rho_2 (u_2)^2 + \alpha_2 p) - p_I \partial_x \alpha_2 = \lambda |u_1 - u_2| (u_1 - u_2) \end{cases}$$

where

$$p = p_1(\rho_1) = p_2(\rho_2) \quad \text{and} \quad \alpha_1 + \alpha_2 = 1$$

α_k	void fraction of phase k
ρ_k	density of phase k
u_k	velocity of phase k
p	pressure
λ	coefficient of drag force

Bifluid model

Isentropic case, two pressures

$$\begin{cases} \partial_t \alpha_1 + u_2 \partial_x \alpha_1 = \mu(p_1 - p_2) \\ \partial_t \alpha_1 \rho_1 + \partial_x \alpha_1 \rho_1 u_1 = 0 \\ \partial_t \alpha_2 \rho_2 + \partial_x \alpha_2 \rho_2 u_2 = 0 \\ \partial_t \alpha_1 \rho_1 u_1 + \partial_x (\alpha_1 \rho_1 (u_1)^2 + \alpha_1 p_1) - p_1 \partial_x \alpha_1 = \lambda |u_1 - u_2| (u_2 - u_1) \\ \partial_t \alpha_2 \rho_2 u_2 + \partial_x (\alpha_2 \rho_2 (u_2)^2 + \alpha_2 p_2) - p_1 \partial_x \alpha_2 = \lambda |u_1 - u_2| (u_1 - u_2) \end{cases}$$

where

$$p_1(\rho_1), \quad p_2(\rho_2) \quad \text{and} \quad \alpha_1 + \alpha_2 = 1$$

α_k	void fraction of phase k
ρ_k	density of phase k
u_k	velocity of phase k
p_k	pressure of phase k
λ	coefficient of drag force
μ	coefficient for the pressure equilibrium

Compatibility between the different multiphase models

Need to understand the compatibility
between the models (L) and (R)
in order to couple them

- **Understanding of asymptotic behaviours**
  - Relaxation process Chen, Liu, Levermore, Yong...
  - Local expansions Hilbert, Chapman, Enskog...
  - Long-time behaviour $s = \varepsilon t$...
- **Compatibility between the asymptotics for the coupling problem ?**

Compatibility between the different multiphase models

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- **Understanding of asymptotic behaviours**
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- **Compatibility between the asymptotics for the coupling problem ?**

  NO ! The coupling problem is ARTIFICIAL !
- All the models are based on the same **Eulerian structure**
  - How to couple two Euler models coming from different asymptotics ?
  - How to recover the asymptotic compatibility at the interface coupling ?

Coupling of Euler systems with different pressure laws

CODE 1

$$\begin{cases} \partial_t \rho + \partial_x \rho u = 0 \\ \partial_t \rho u + \partial_x (\rho u^2 + p_L) = 0 \\ \partial_t \rho E + \partial_x (u(\rho E + p_L)) = 0 \end{cases}$$

$x < 0$

CODE 2

$$\begin{cases} \partial_t \rho + \partial_x \rho u = 0 \\ \partial_t \rho u + \partial_x (\rho u^2 + p_R) = 0 \\ \partial_t \rho E + \partial_x (u(\rho E + p_R)) = 0 \end{cases}$$

$x > 0$

$x = 0$

The only difference comes from $p_L(.) \neq p_R(.)$

Coupling condition at the interface

$$\partial_t \mathbf{u} + \partial_x \mathbf{f}_L(\mathbf{u}) = 0 \qquad \partial_t \mathbf{u} + \partial_x \mathbf{f}_R(\mathbf{u}) = 0$$

$$x < 0 \qquad\qquad x > 0$$

$$x = 0$$

Different strategies of interface coupling:

Conservative coupling	$\mathbf{f}_L(\mathbf{u}(t,0^-))$ "$=$" $\mathbf{f}_R(\mathbf{u}(t,0^+))$
State coupling	$\mathbf{u}(t,0^-)$ "$=$" $\mathbf{u}(t,0^+)$
Nonconservative state coupling	$\mathbf{v}(t,0^-)$ "$=$" $\mathbf{v}(t,0^+)$

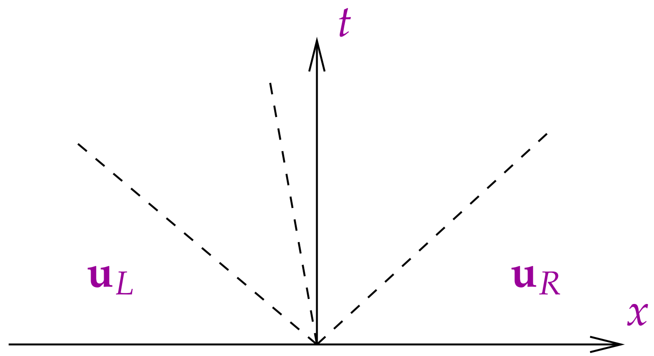
where $\mathbf{v} = \phi_L(\mathbf{u})$ if $x < 0$ and $\mathbf{v} = \phi_R(\mathbf{u})$ if $x > 0$. For instance:

$$(\rho, \rho u, \rho E) \longmapsto (\rho, \rho u, p_\alpha).$$

Riemann problem and boundary conditions

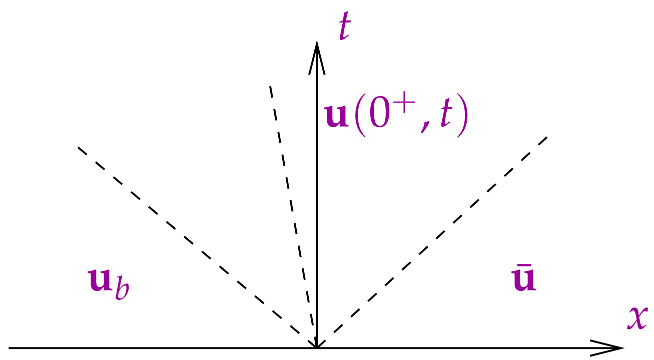
Riemann problem:

$$\begin{cases} \partial_t \mathbf{u} + \partial_x \mathbf{f}(\mathbf{u}) = 0 \\ \mathbf{u}(0,x) = \begin{cases} \mathbf{u}_l & \text{if } x < 0 \\ \mathbf{u}_r & \text{if } x > 0 \end{cases} \end{cases}$$



Weak boundary condition:

$$\begin{cases} \partial_t \mathbf{u} + \partial_x \mathbf{f}(\mathbf{u}) = 0,\ x > 0 \\ \mathbf{u}(0,x) = \mathbf{u}_0(x) \\ \mathbf{u}(t,0)\ \text{``}=\text{''}\ \mathbf{u}_b \end{cases}$$



$$\mathbf{u}(0^+, t) \in \mathcal{O}(\mathbf{u}_b) := \{\mathcal{W}(0^+; \mathbf{u}_b, \bar{\mathbf{u}}), \forall \bar{\mathbf{u}}\}$$

Weak coupling conditions

$$\partial_t \mathbf{u} + \partial_x \mathbf{f}_L(\mathbf{u}) = 0 \qquad\qquad \partial_t \mathbf{u} + \partial_x \mathbf{f}_R(\mathbf{u}) = 0$$

$$x < 0 \qquad\qquad x > 0$$

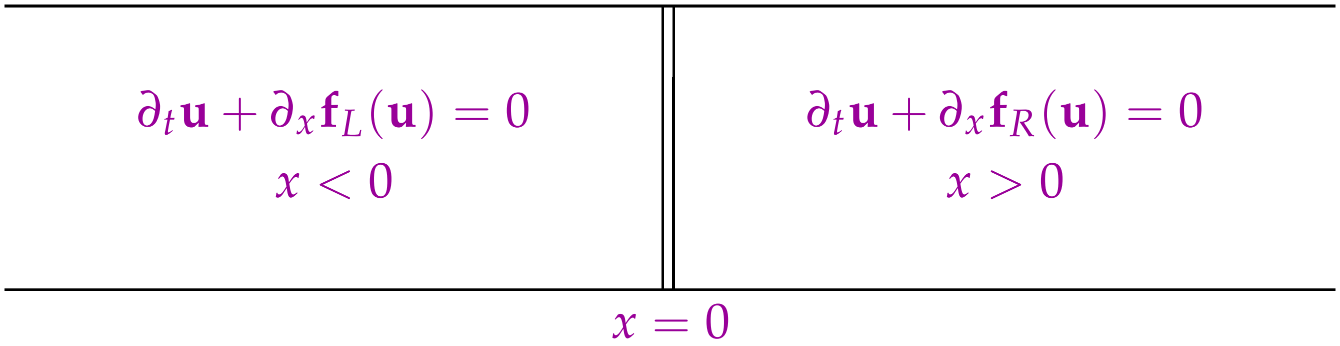
$$x = 0$$

$$\begin{cases} \partial_t \mathbf{u} + \partial_x \mathbf{f}_L(\mathbf{u}) = 0, \\ \mathbf{u}(0, x) = \mathbf{u}_0(x) \\ \mathbf{u}(0^-, t) \in \mathcal{O}_L(\mathbf{u}(0^+, t)) \end{cases} \qquad \begin{cases} \partial_t \mathbf{u} + \partial_x \mathbf{f}_R(\mathbf{u}) = 0, \\ \mathbf{u}(0, x) = \mathbf{u}_0(x) \\ \mathbf{u}(0^+, t) \in \mathcal{O}_R(\mathbf{u}(0^-, t)) \end{cases}$$

Weak coupling conditions for the state coupling

$\mathbf{u}(0^-, t)$ " $=$ " $\mathbf{u}(0^+, t)$ becomes $\begin{cases} \mathbf{u}(0^-, t) \in \mathcal{O}_L(\mathbf{u}(0^+, t)) \\ \mathbf{u}(0^+, t) \in \mathcal{O}_R(\mathbf{u}(0^-, t)) \end{cases}$

Weak coupling conditions



$$\begin{cases} \partial_t \mathbf{u} + \partial_x \mathbf{f}_L(\mathbf{u}) = 0, \\ \mathbf{u}(0,x) = \mathbf{u}_0(x) \\ \mathbf{v}(0^-,t) \in \widetilde{\mathcal{O}}_L(\mathbf{v}(0^+,t)) \end{cases} \qquad \begin{cases} \partial_t \mathbf{u} + \partial_x \mathbf{f}_R(\mathbf{u}) = 0, \\ \mathbf{u}(0,x) = \mathbf{u}_0(x) \\ \mathbf{v}(0^+,t) \in \widetilde{\mathcal{O}}_R(\mathbf{v}(0^-,t)) \end{cases}$$

Weak coupling conditions for the nonconservative state coupling

$\mathbf{v}(0^-,t)$ "$=$" $\mathbf{v}(0^+,t)$ becomes $\begin{cases} \mathbf{v}(0^-,t) \in \widetilde{\mathcal{O}}_L(\mathbf{v}(0^+,t)) \\ \mathbf{v}(0^+,t) \in \widetilde{\mathcal{O}}_R(\mathbf{v}(0^-,t)) \end{cases}$

The Riemann problem for the coupled problem

The Riemann problem for the coupled problem

$$\begin{cases} \mathbf{u} + \partial_x \mathbf{f}_L(\mathbf{u}) = 0, & x < 0,\ t > 0 \\ \mathbf{u}(0,x) = \mathbf{u}_l, & x < 0 \\ \mathbf{u} + \partial_x \mathbf{f}_R(\mathbf{u}) = 0, & x > 0,\ t > 0 \\ \mathbf{u}(0,x) = \mathbf{u}_r, & x > 0 \\ \mathbf{v}(0^-,t) \in \widetilde{\mathcal{O}}_L(\mathbf{v}(0^+,t)), & t > 0 \\ \mathbf{v}(0^+,t) \in \widetilde{\mathcal{O}}_R(\mathbf{v}(0^-,t)), & t > 0 \end{cases}$$

Two kinds of solutions

- Continuous solutions through the coupling interface

  $$\mathbf{v}(0^-,t) = \mathbf{v}(0^+,t)$$

- Discontinuous solutions through the coupling interface

  $$\mathbf{v}(0^-,t) \neq \mathbf{v}(0^+,t)$$

Continuous solutions

Method of resolution

- Compute the set $\mathcal{C}_L(\mathbf{u}_l)$ of states at $x/t = 0^-$ which can be connected to $\mathbf{u}_l$ by waves with negative speeds
- Compute the set $\mathcal{C}_R(\mathbf{u}_r)$ of states at $x/t = 0^-$ which can be connected to $\mathbf{u}_r$ by waves with positive speeds
- Intersect them: $\mathcal{I}(\mathbf{u}_l, \mathbf{u}_r) = \mathcal{C}_L(\mathbf{u}_l) \cap \mathcal{C}_R(\mathbf{u}_r)$

If $\mathcal{I}(\mathbf{u}_l, \mathbf{u}_r)$ is a singleton	Existence and uniqueness
If $\mathcal{I}(\mathbf{u}_l, \mathbf{u}_r)$ is a set	Existence and non uniqueness
If $\mathcal{I}(\mathbf{u}_l, \mathbf{u}_r)$ is empty	Non existence

Theorem

- If $|u| \ll c$ there exists a **unique continuous solution** to the CRP.
- If $|u| \simeq c$ there may exist a **family of continuous solutions** to the CRP.

Discontinuous solutions

Weak coupling conditions

$$\begin{cases} \mathbf{v}(0^-,t) \in \widetilde{\mathcal{O}}_L(\mathbf{v}(0^+,t)) & = \{\mathcal{Z}_L(0^-;\bar{\mathbf{v}},\mathbf{v}(0^+,t)),\ \forall \bar{\mathbf{v}}\} \\ \mathbf{v}(0^+,t) \in \widetilde{\mathcal{O}}_R(\mathbf{v}(0^-,t)) & = \{\mathcal{Z}_R(0^+;\mathbf{v}(0^-,t),\bar{\mathbf{v}}),\ \forall \bar{\mathbf{v}}\} \end{cases}$$

They can be rewritten as

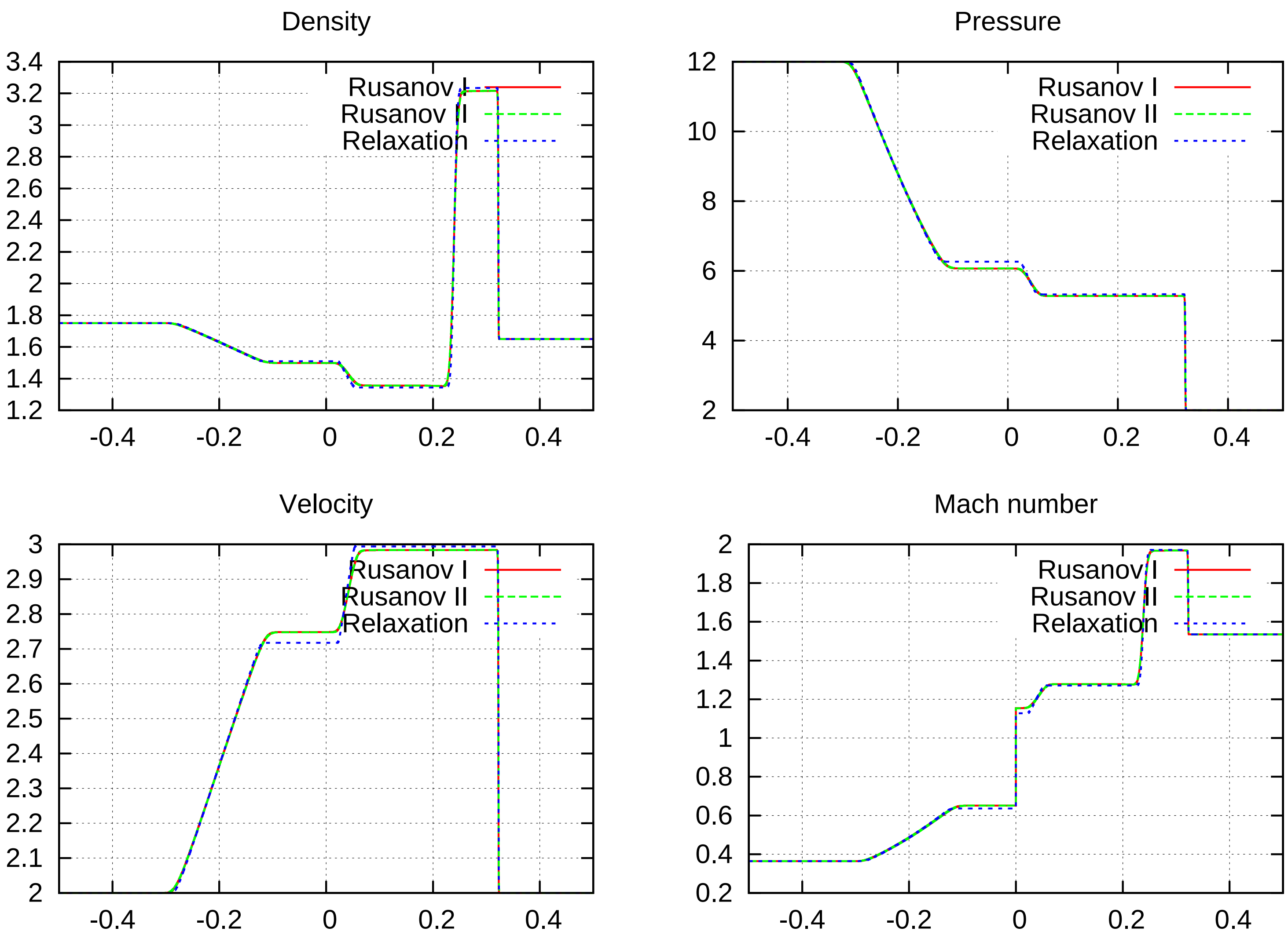
$$\begin{cases} \mathbf{v}(0^-,t) = \mathcal{Z}_L(0^-;\mathbf{v}_l,\mathbf{v}(0^+,t)) \\ \mathbf{v}(0^+,t) = \mathcal{Z}_R(0^+;\mathbf{v}(0^-,t),\mathbf{v}_r) \end{cases}$$

Theorem

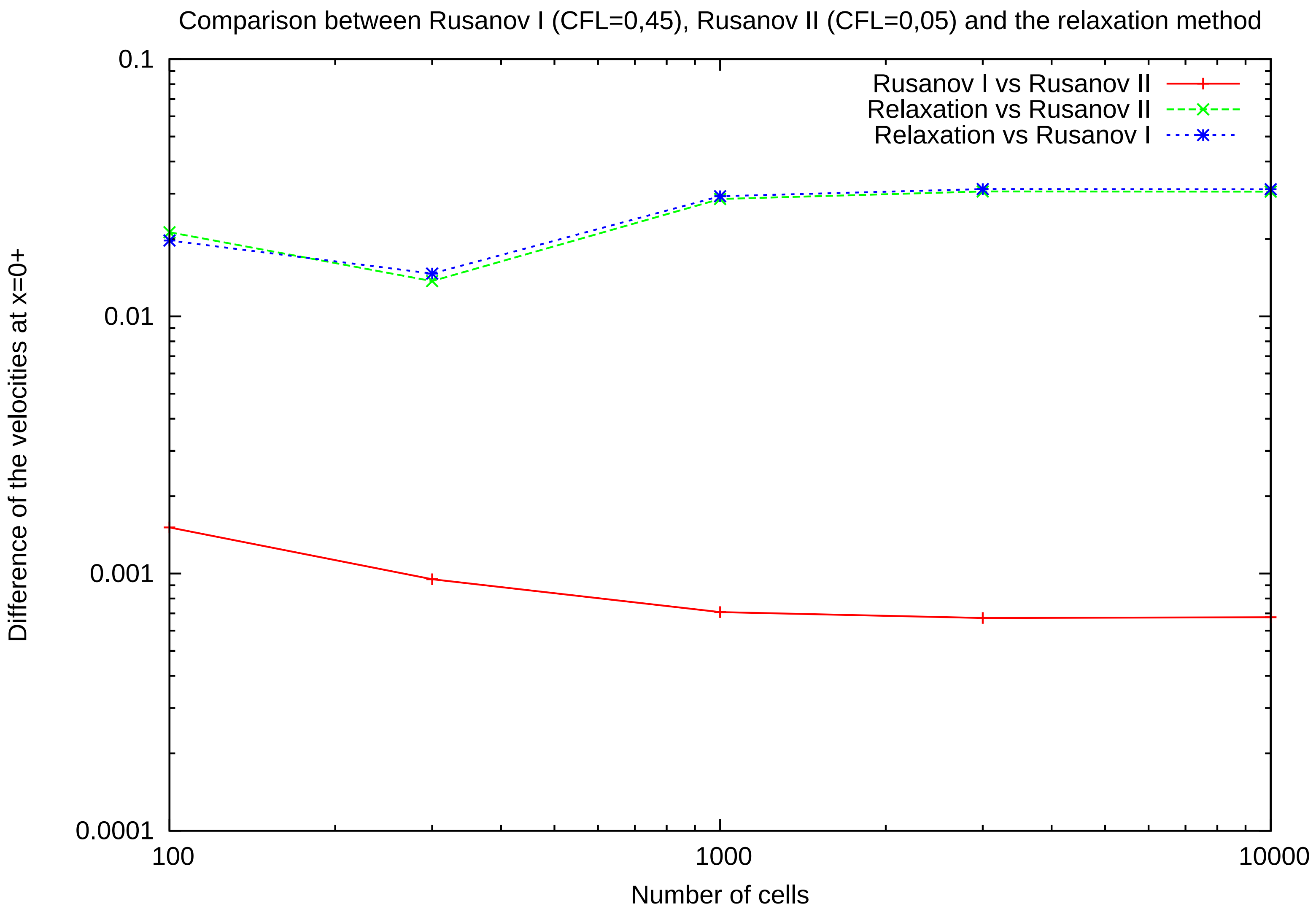
There exists **at most one discontinuous solution** to the CRP.

It appears when $|u| \simeq c$.

Numerical tests with multiple solutions



Numerical tests with multiple solutions



Coupling of compatible models

CODE 1

$$\partial_t \mathbf{u} + \partial_x \mathbf{f}(\mathbf{u}, \mathbf{v}) = 0,$$
$$\partial_t \mathbf{v} + \partial_x \mathbf{g}(\mathbf{u}, \mathbf{v}) = \mathbf{r}(\mathbf{u}, \mathbf{v})/\varepsilon,$$

$x < 0$

CODE 2

$$\partial_t \mathbf{u} + \partial_x \mathbf{f}(\mathbf{u}, \mathbf{v}_{\mathsf{eq}}(\mathbf{u})) = 0,$$

$x > 0$

$x = 0$

where

$$\mathbf{r}(\mathbf{u}, \mathbf{v}) = 0 \Longleftrightarrow \mathbf{v} = \mathbf{v}_{\mathsf{eq}}(\mathbf{u}).$$

Natural coupling conditions:

$$\mathbf{f}(\mathbf{u}, \mathbf{v})(t, 0^-) = \mathbf{f}(\mathbf{u}, \mathbf{v}_{\mathsf{eq}}(\mathbf{u}))(0^+, t),$$
$$\mathbf{v}(t, 0^-) = \mathbf{v}_{\mathsf{eq}}(\mathbf{u})(0^+, t).$$

Coupling of compatible models – HRM and HEM

$$\begin{cases} \partial_t \alpha + u \partial_x \alpha = \lambda(\alpha_{eq}(\rho) - \alpha) \\ \partial_t \rho + \partial_x \rho u = 0 \\ \partial_t \rho u + \partial_x(\rho u^2 + p) = 0 \\ \partial_t \rho E + \partial_x(u(\rho E + p)) = 0 \end{cases}$$

$$p = \mathscr{P}(\rho, \varepsilon, \alpha)$$

$$x < 0$$

$$\begin{cases} \partial_t \rho + \partial_x \rho u = 0 \\ \partial_t \rho u + \partial_x(\rho u^2 + p) = 0 \\ \partial_t \rho E + \partial_x(u(\rho E + p)) = 0 \end{cases}$$

$$p = \mathscr{P}(\rho, \varepsilon, \alpha_{eq}(\rho))$$

$$x > 0$$

$$x = 0$$

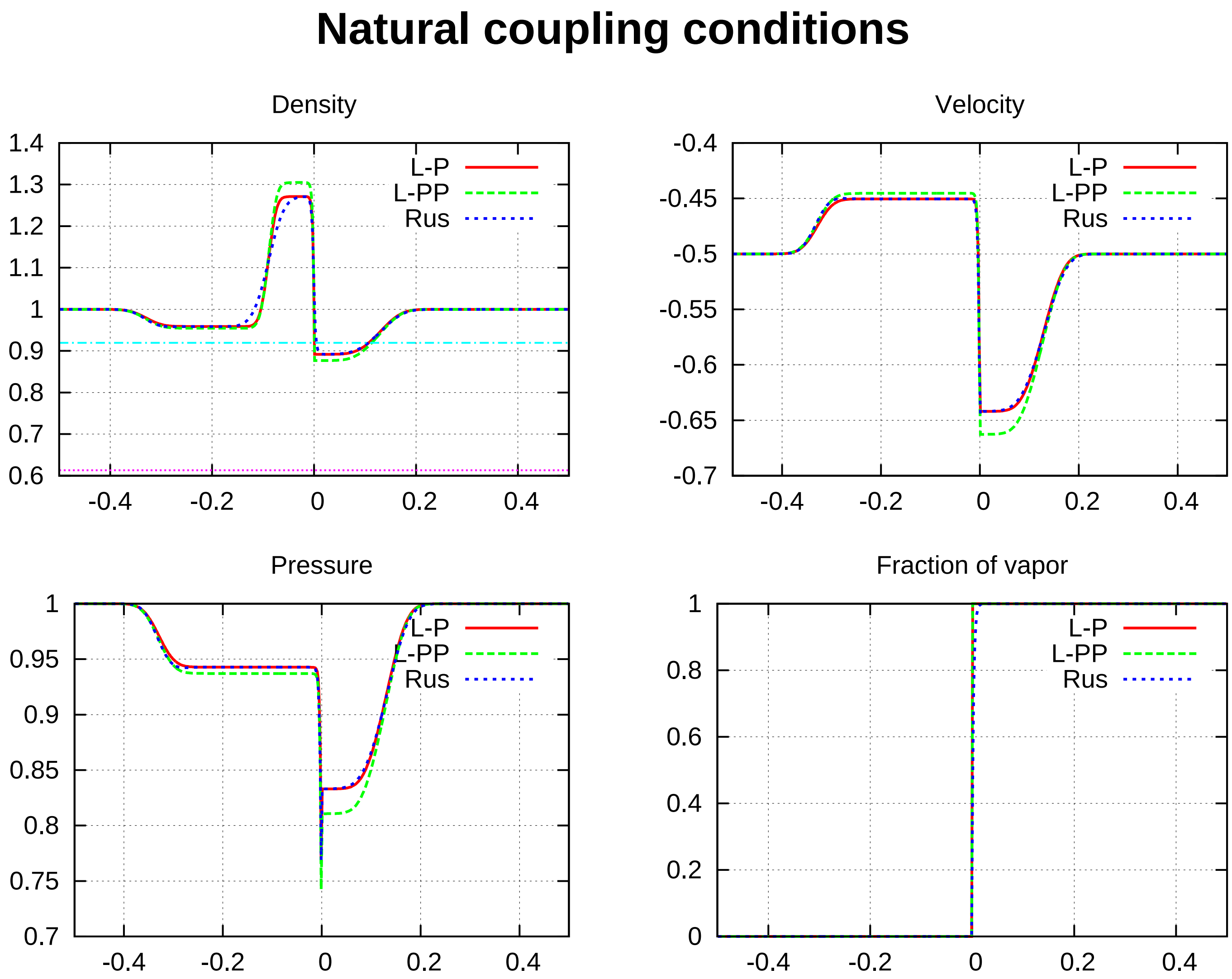
Problem:

The natural coupling does not preserve constant velocity and pressure

New coupling condition:

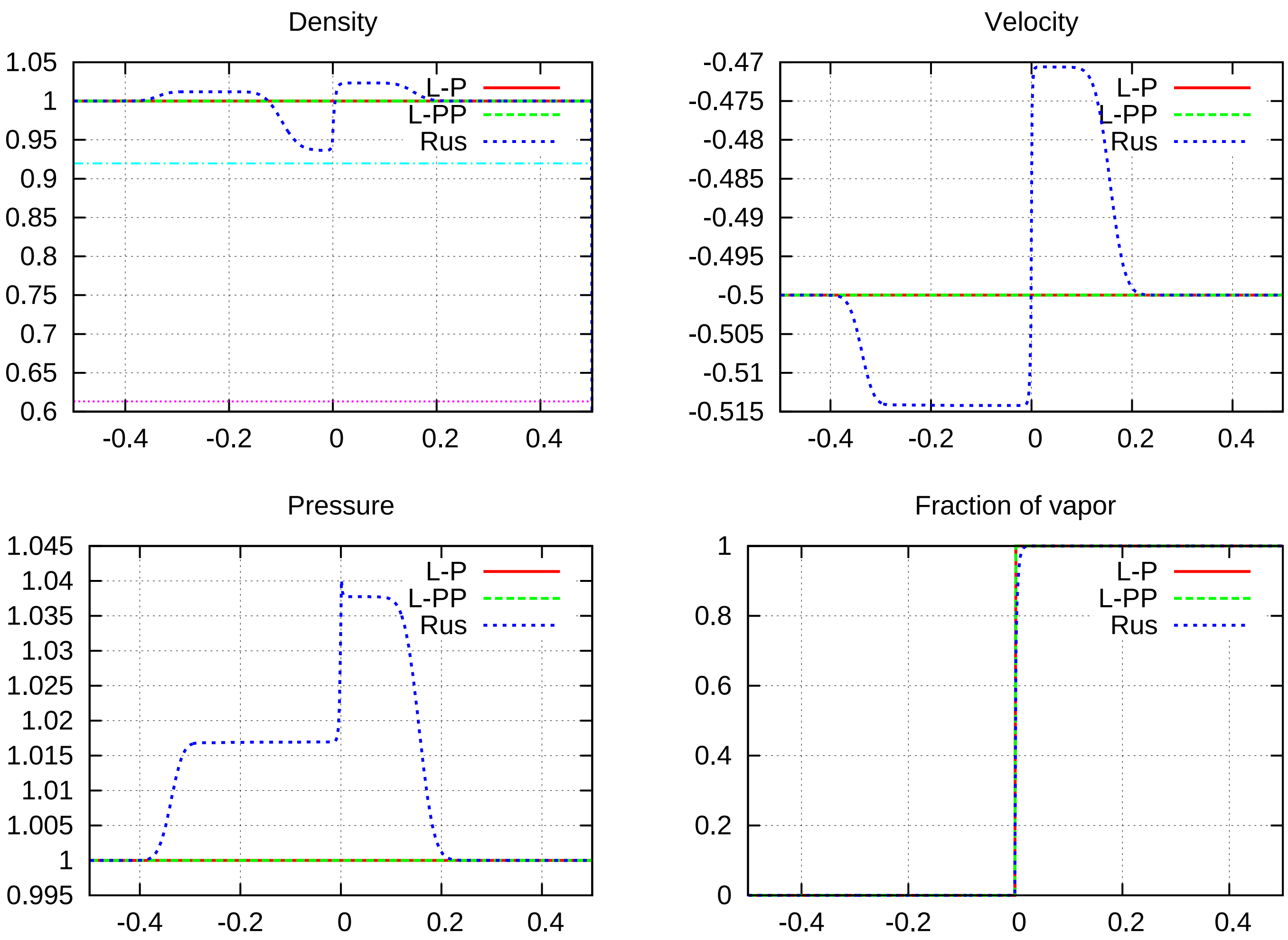
$$(\alpha, \rho, \rho u, p)(t, 0^-) = (\alpha_{eq}(\rho), \rho, \rho u, p)(0^+, t)$$

Coupling of compatible models – HRM and HEM



Coupling of compatible models – HRM and HEM

Modified coupling conditions



Conclusion

Theoritical understanding of the coupling

- Mathematical position of the coupling problem $\rightarrow$ **weak coupling condition**
- Partial resolution of the **Riemann problem** for gas dynamics
- Same behaviour at the numerical point of view
- Study of the **coupling of HEM and HRM**

Some extensions of this work

- Study of the coupling of **more complex models**
- Analysis of numerical methods
- Coupling of different asymptotics:
  - **Scalar case** (Jin-Xin relaxation) by F. Caetano
  - **Two-phase two-pressure model** *vs* **Drift-flux model**
- **Asymptotic preserving schemes** (Euler *vs* Darcy)