Two-phase flows with granular stress

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January 24, 2008

Context

- flow of weakly compressible grains (powder, sand, *etc.*) inside a compressible gas;
- averaged model (the solid phase is represented by an equivalent continuous media);
- 2 densities, 2 velocities, 2 pressures, 1 volume fraction;
- relaxation approach to return to a 1 pressure model (classical: *cf.* bibliography);
- novelty: granular stress treated in a rigorous way;
- stable approximation;
- application.

Hyperbolicity and stability

Hyperbolicity

Consider $w(x,t) \in R^2$ solution of

$$w_t + Aw_x = 0,$$
$$A = \begin{bmatrix} 0 & \varepsilon \\ 1 & 0 \end{bmatrix}, \quad \varepsilon = \pm 1.$$

Space Fourier transform $\widehat{w}(\xi,t) := \int_{-\infty}^{\infty} e^{-ix\xi} w(x,t)dx$.

$$\widehat{w}(\xi,t) = e^{-i\xi tA}\widehat{w}(\xi,0). \tag{1}$$

- Hyperbolic case ($\varepsilon = 1$): the L^2 norm of $w(.,t)$ is constant
- Elliptic case ($\varepsilon = -1$): the frequency ξ is amplified by a factor $e^{|\xi|t}$ (unstable)

Numerical viscosity

Classical approximations introduce a "numerical viscosity", which can be modelled by

$$w_t + Aw_x - hsw_{xx} = 0.$$

h is the size of the cells, $s = \rho(A)$ the spectral radius of A. The amplification is now

$$\widehat{w}(\xi, t) = e^{-i\xi tA - hs\xi^2 t}\widehat{w}(\xi, 0). \tag{2}$$

But in the elliptic case, the limit system when $h \longrightarrow 0$ is still unstable !
Problem: many models in the two-phase flow community are non-hyperbolic...

A general granular flow model

- flow of compressible grains (powder, sand, etc.) inside a compressible gas;
- averaged model;
- 2 densities, 2 velocities, 2 pressures, 1 volume fraction;
- relaxation approach to return to a 1 pressure model (classical: cf. bibliography);
- novelty: "rigorous" granular stress (tramway);
- stable approximation;
- application.

Two-pressure model

A gaz phase $k = 1$, a solid (powder) phase $k = 2$
7 unknowns: partial densities ρ_k, velocities u_k, internal energies e_k, gas volume fraction α_1.

Pressure law: $p_k = p_k(\rho_k, e_k) = (\gamma_k - 1)\rho_k e_k - \gamma_k \pi_k \quad , \quad \gamma_k > 1$

Other definitions: $m_k = \alpha_k \rho_k \quad \alpha_2 = 1 - \alpha_1 \quad E_k = e_k + \frac{u_k^2}{2}$

The balance of mass, momentum and energy reads

$$
\begin{aligned}
& m_{k,t} + (m_k u_k)_x = 0, \\
& (m_k u_k)_t + (m_k u_k^2 + \alpha_k p_k)_x - p_1 \alpha_{k,x} = 0, \\
& (m_k E_k)_t + ((m_k E_k + \alpha_k p_k) u_k)_x + p_1 \alpha_{k,t} = 0, \\
& \alpha_{k,t} + u_2 \alpha_{k,x} = \pm P,
\end{aligned}
$$

Entropy

The phase entropies satisfy the following PDEs

$$T_1 ds_1 = de_1 - \frac{p_1}{\rho_1^2} d\rho_1$$
$$T_2 ds_2 = de_2 - \frac{p_2}{\rho_2^2} d\rho_2 - \Theta d\alpha_2$$

After some computations, we find the following entropy dissipation equation

$$(\sum m_k s_k)_t + (\sum m_k u_k s_k)_x = \frac{P}{T_2}(p_1 + m_2\Theta - p_2)$$

Natural choice to ensure positive dissipation

$$P = \frac{1}{\varepsilon}(p_1 + m_2\Theta - p_2), \quad \varepsilon \to 0+.$$

$R := m_2\Theta$ is called the *granular stress*.

Hyperbolicity

Let

$$Y = (\alpha_1, \rho_1, u_1, s_1, \rho_2, u_2, s_2)^T.$$

In this set of variables the system becomes

$$Y_t + B(Y)Y_x = S(P),$$

$$B(Y) = \begin{bmatrix} u_2 & & & & & & \\ \frac{\rho_1(u_1-u_2)}{\alpha_1} & u_1 & \rho_1 & & & & \\ & \frac{c_1^2}{\rho_1} & u_1 & \frac{p_{1,s_1}}{\rho_1} & & & \\ & & & u_1 & & & \\ & & & & u_2 & \rho_2 & \\ \frac{p_1-p_2}{m_2} & & & & \frac{c_2^2}{\rho_2} & u_2 & \frac{p_{2,s_2}}{\rho_2} \\ & & & & & & u_2 \end{bmatrix} \qquad c_k = \sqrt{\frac{\gamma_k(p_k+\pi_k)}{\rho_k}}$$

The characteristic polynomial is

$$P(\lambda) = (u_2-\lambda)^2(u_1-\lambda)(u_1-c_1-\lambda)(u_1+c_1-\lambda)(u_2-c_2-\lambda)(u_2+c_2-\lambda)$$

Granular stress

How to choose the granular stress $R = m_2\Theta$?

$$\Theta = \Theta(\alpha_2) \Rightarrow \Theta = 0$$

Thus a more general choice is necessary.
Exemple: for a stiffened gas equation of state

$$p_2 = (\gamma_2 - 1)\rho_2 e_2 - \gamma_2 \pi_2 \quad , \quad \gamma_2 > 1.$$

We suppose $\Theta = \Theta(\rho_2, \alpha_2)$. We find

$$\Theta(\rho_2, \alpha_2) = \rho_2^{\gamma_2 - 1}\theta(\alpha_2)$$

Particular choice

$$\theta(\alpha_2) = \lambda \alpha_2^{\gamma_2 - 1} \quad \Rightarrow \quad R = \lambda m_2^{\gamma_2}.$$

Reduction: one pressure model

When $\varepsilon \longrightarrow 0+$, formally, we end up with a standard one pressure model

$$p_2 = p_1 + m_2\Theta \tag{3}$$

We can remove an equation (for example the volume fraction evolution) and we find a 6 equations system

$$\begin{aligned} &Z = (\rho_1, u_1, s_1, \rho_2, u_2, s_2)^T. \\ &Z_t + C(Z)Z_x = 0. \end{aligned} \tag{4}$$

Let

$$\Delta = \alpha_1\alpha_2 + \delta(\alpha_1\rho_2 a_2^2 + \alpha_2\rho_1 c_1^2), \tag{5}$$

and

$$\delta = \frac{\alpha_2^{1-1/\gamma_2}}{\lambda\gamma_2\rho_2^{\gamma_2}}. \tag{6}$$

Then we find

$$C(Z) = \begin{bmatrix} u_1 + \frac{\rho_1\rho_2 c_1^2 a_2^2 \delta^2 (u_1 - u_2)}{\Delta} & \frac{\alpha_1\rho_1(\alpha_2 + \rho_2 a_2^2 \delta)}{\Delta} & \frac{\rho_1\rho_2 a_2^2 \delta^2 (u_1 - u_2) p_{1,2,}}{\Delta} & \frac{\rho_1 a_2^2 \delta(\alpha_2 + \rho_2 a_2^2 \delta)(u_2 - u_1)}{\Delta} & \frac{\alpha_2\rho_1\rho_2 a_2^2 \delta}{\Delta} & \frac{\rho_1 \delta(\alpha_2 + \rho_2 a_2^2 \delta)(u_2 - u_1) p_{2,2,}}{\Delta} \\ \frac{c_1^2}{\rho_1} & u_1 & \frac{p_{1,2,}}{\rho_1} & 0 & 0 & 0 \\ 0 & 0 & u_1 & 0 & 0 & 0 \\ \frac{\rho_2 c_1^2 \delta(\alpha_1 + \rho_1 c_1^2 \delta)(u_1 - u_2)}{\Delta} & \frac{\alpha_1\rho_1\rho_2 c_1^2 \delta}{\Delta} & \frac{\rho_2 \delta(\alpha_1 + \rho_1 c_1^2 \delta)(u_1 - u_2) p_{1,2,}}{\Delta} & u_2 + \frac{\rho_1\rho_2 c_1^2 a_2^2 \delta^2 (u_2 - u_1)}{\Delta} & \frac{\alpha_2\rho_2(\alpha_1 + \rho_1 c_1^2 \delta)}{\Delta} & \frac{\rho_1\rho_2 c_1^2 \delta^2 (u_2 - u_1) p_{2,2,}}{\Delta} \\ \frac{(p_1 - p_2)\delta c_1^2}{\alpha_2\rho_2} & 0 & \frac{(p_1 - p_2)\delta p_{1,2,}}{\alpha_2\rho_2} & \frac{\delta(p_2 - p_1) + \alpha_2 c_2^2}{\alpha_2\rho_2} & u_2 & \frac{(\alpha_2 + \delta(p_2 - p_1)) p_{2,2,}}{\alpha_2\rho_2} \\ 0 & 0 & 0 & 0 & 0 & u_2 \end{bmatrix}$$

The eigenvalues can be computed only numerically.

We observe that when $\lambda \longrightarrow 0$, the system is generally not hyperbolic.

We observe also that when $\lambda \longrightarrow \infty$, we recover hyperbolicity.

Numerical approximation: splitting method

Approximation of the one-pressure model by the more general two-pressure model.
At the end of each time step, we have to return to the pressure equilibrium
Relaxation approach.

Convection step

Let

$$w = (\alpha_1, m_1, m_1 u_1, m_1 E_1, m_2, m_2 u_2, m_2 E_2)^T$$

The system can be written

$$w_t + f(w)_x + G(w) w_x = \Sigma(P).$$

In the first half step the source term is omitted. We use a standard Rusanov scheme

$$\frac{w_i^* - w_i^n}{\Delta t} + \frac{f_{i+1/2}^n - f_{i-1/2}^n}{\Delta x} + G(w_i^n) \frac{w_{i+1}^n - w_{i-1}^n}{2\Delta x} = 0,$$

$f_{i+1/2}^n = f(w_i^n, w_{i+1}^n)$ numerical conservative flux

$$f(a,b) = \frac{f(a) + f(b)}{2} - \frac{s}{2}(b - a)$$

For s large enough, the scheme is entropy dissipative. Typically, we take

$$s = \max\left(\rho(f'(a)), \rho(f'(b))\right)$$

Relaxation step

In the second half step, we have formally to solve

$$\begin{aligned} &\alpha_{k,t} = \pm P, \\ &m_{k,t} = u_{k,t} = 0, \\ &(m_k e_k)_t + p_1 \alpha_{k,t} = 0. \end{aligned} \tag{7}$$

Because of mass and momentum conservation we have $m_k = m_k^*$ and $u_k = u_k^*$. In each cell we have to compute (α_1, p_1, p_2) from the previous state w^*

$$\begin{aligned} &p_2 = p_1 + \lambda m_2^{\gamma_2}, \\ &m_1 e_1 + m_2 e_2 = m_1^* e_1^* + m_2^* e_2^*, \\ &(m_1 e_1 - m_1^* e_1^*) + p_1(\alpha_1 - \alpha_1^*) = 0. \end{aligned}$$

After some manipulations, we have to solve

$$H(\alpha_2) = (\pi_2 - \pi_1)(\alpha_1 + (\gamma_1 - 1)(\alpha_1 - \alpha_1^*))(\alpha_2 + (\gamma_2 - 1)(\alpha_2 - \alpha_2^*))$$
$$+(\lambda\alpha_2 m_2^{\gamma_2} - A_2)(\alpha_1 + (\gamma_1 - 1)(\alpha_1 - \alpha_1^*))$$
$$+A_1(\alpha_2 + (\gamma_2 - 1)(\alpha_2 - \alpha_2^*)) = 0$$

with

$$\text{with } A_k = \alpha_k^*(p_k^* + \pi_k) > 0.$$

The solution is unique in the interval $[0, 1 - \beta_1]$ with

$$\beta_1 = \frac{\gamma_1 - 1}{\gamma_1}\alpha_1^* \tag{8}$$

Numerical application

We have constructed an entropy dissipative approximation of a non-hyperbolic system !
What happens numerically ?
Consider a simple Riemann problem in the interval $[-1/2, 1/2]$. $\gamma_1 = 1.0924$ and $\gamma_2 = 1.0182$. We compute the solution at time $t = 0.0008$. The CFL number is 0.9.
Data:

$$
\begin{array}{lcc}
 & (L) & (R) \\
\rho_1 & 76.45430093 & 57.34072568 \\
u_1 & 0 & 0 \\
p_1 & 200 \times 10^5 & 150 \times 10^5 \\
\rho_2 & 836.1239718 & 358.8982226 \\
u_2 & 0 & 0 \\
p_2 & 200 \times 10^5 & 150 \times 10^5 \\
\alpha_1 & 0.25 & 0.25
\end{array}
\tag{9}
$$

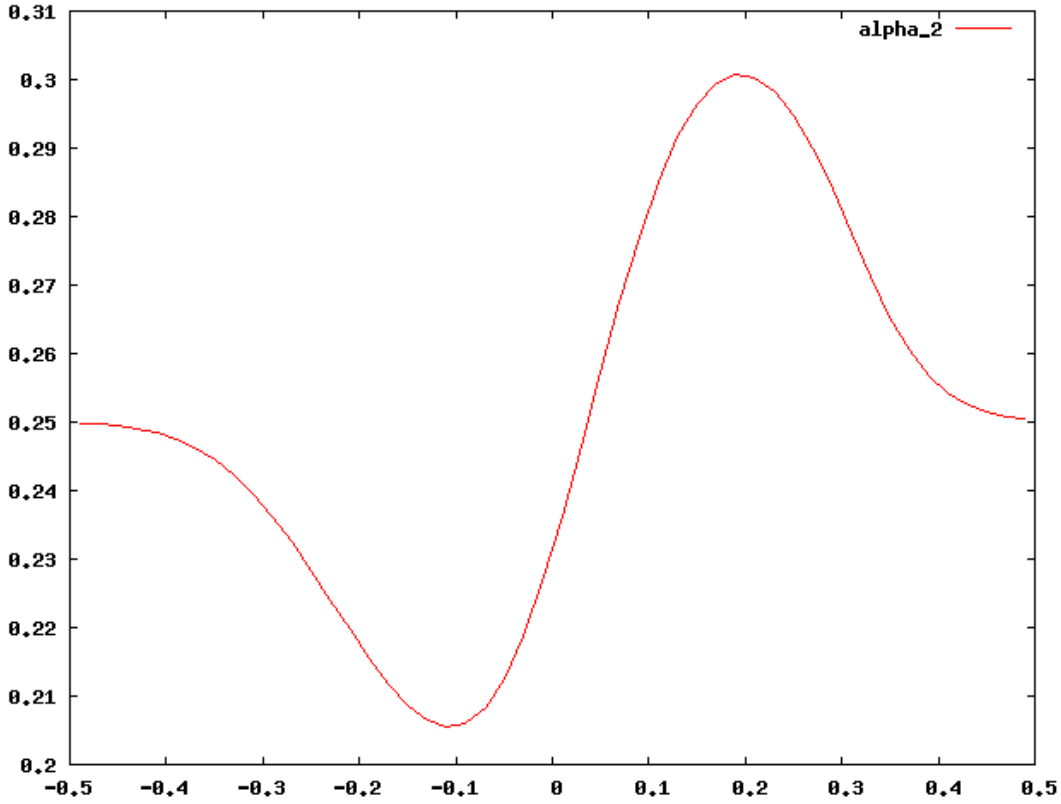

Figure: Void fraction, 50 cells, no granular stress .

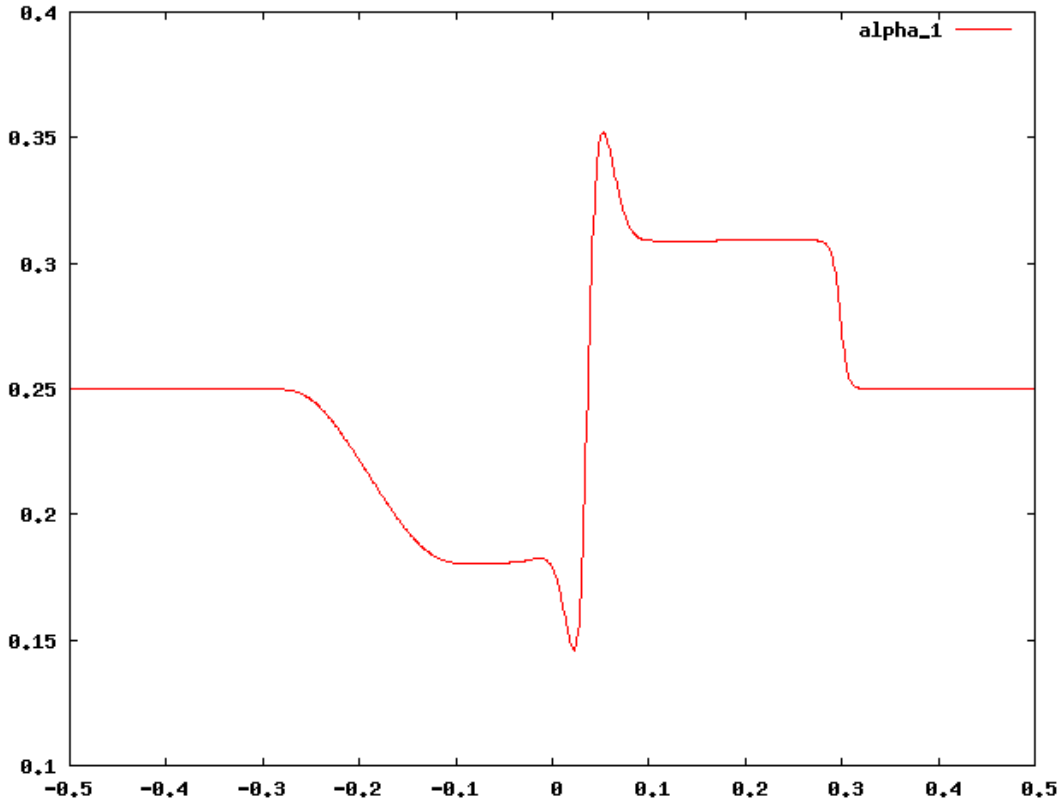

Figure: Void fraction, 1000 cells, no granular stress .

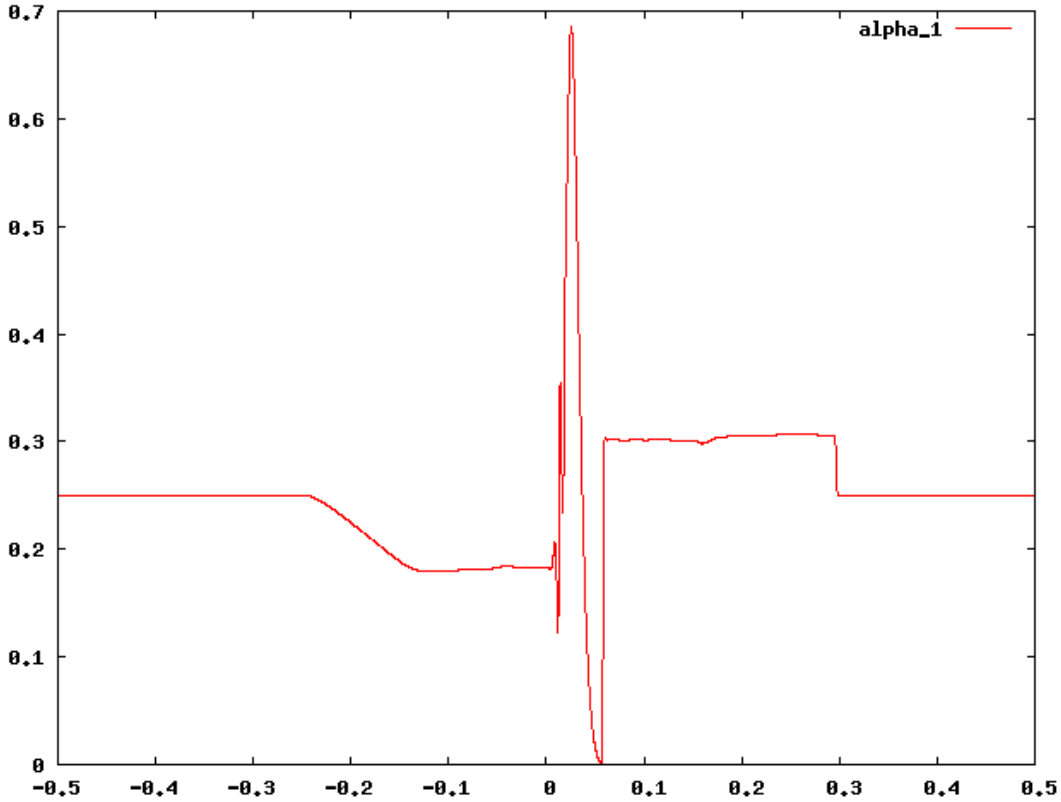

Figure: Void fraction, 10000 cells, no granular stress .

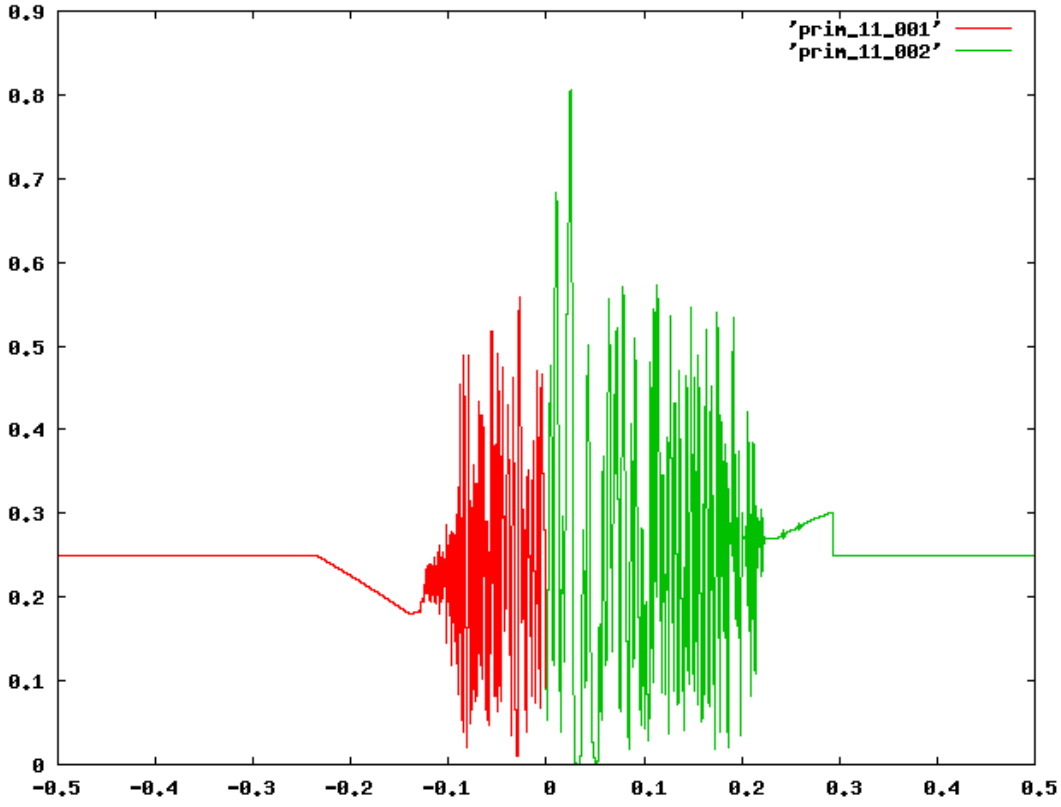

Figure: Void fraction, 100000 cells, no granular stress .

Linearly unstable but non-linearly stable...

Combustion chamber

We consider now a simplified gun. The right boundary of the computational domain is moving. We activate the granular stress and other source terms (chemical reaction and drag), which are all entropy dissipative. The instabilities would occur on much finer grids...

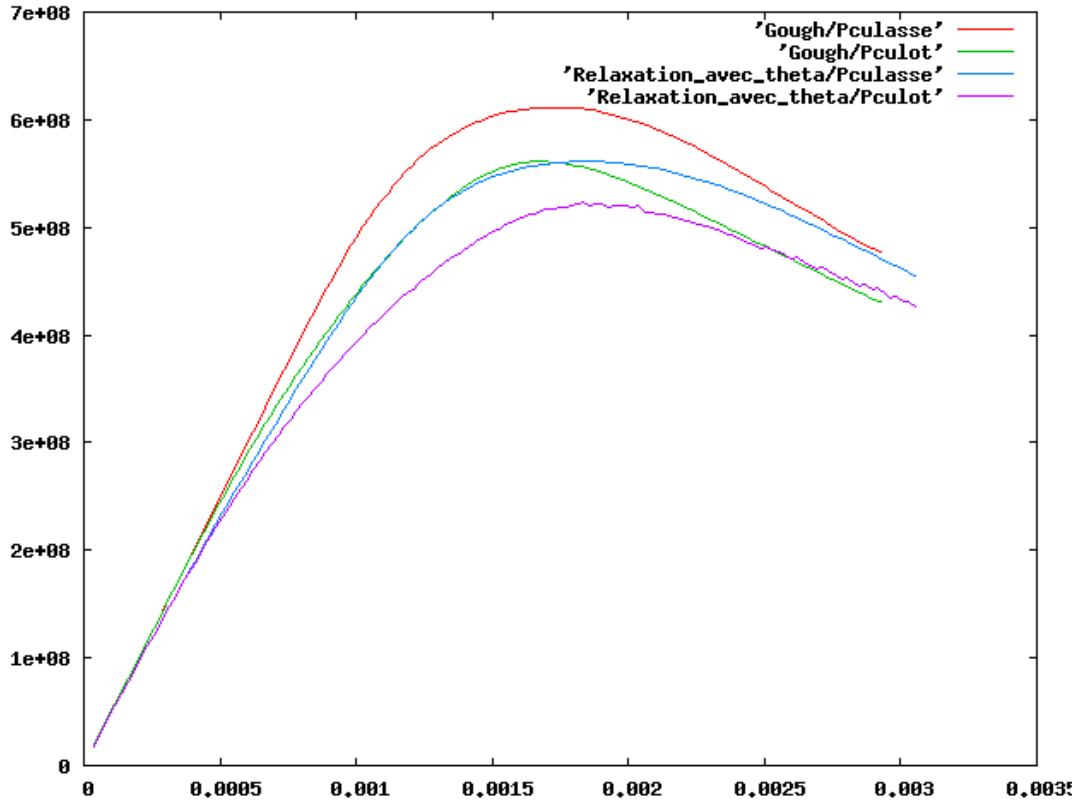

Figure: Pressure evolution at the breech and the shot base during time. Comparison between the Gough and the relaxation model.

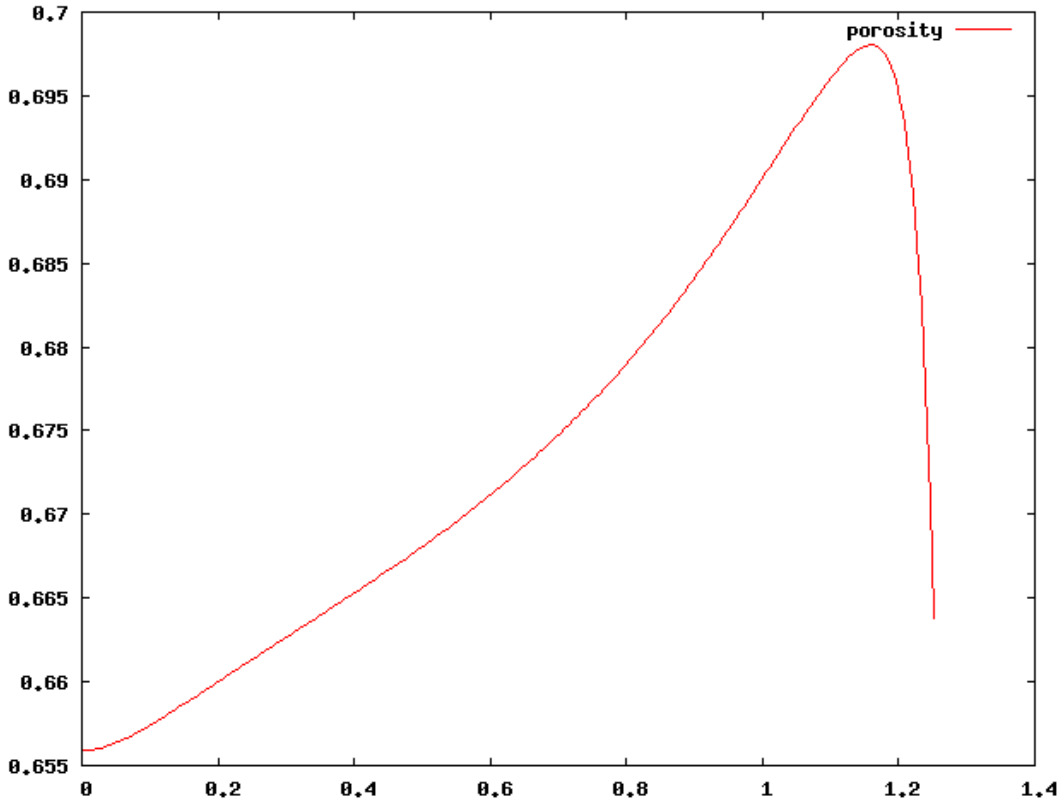

Figure: Porosity at the final time. Relaxation model with granular stress.

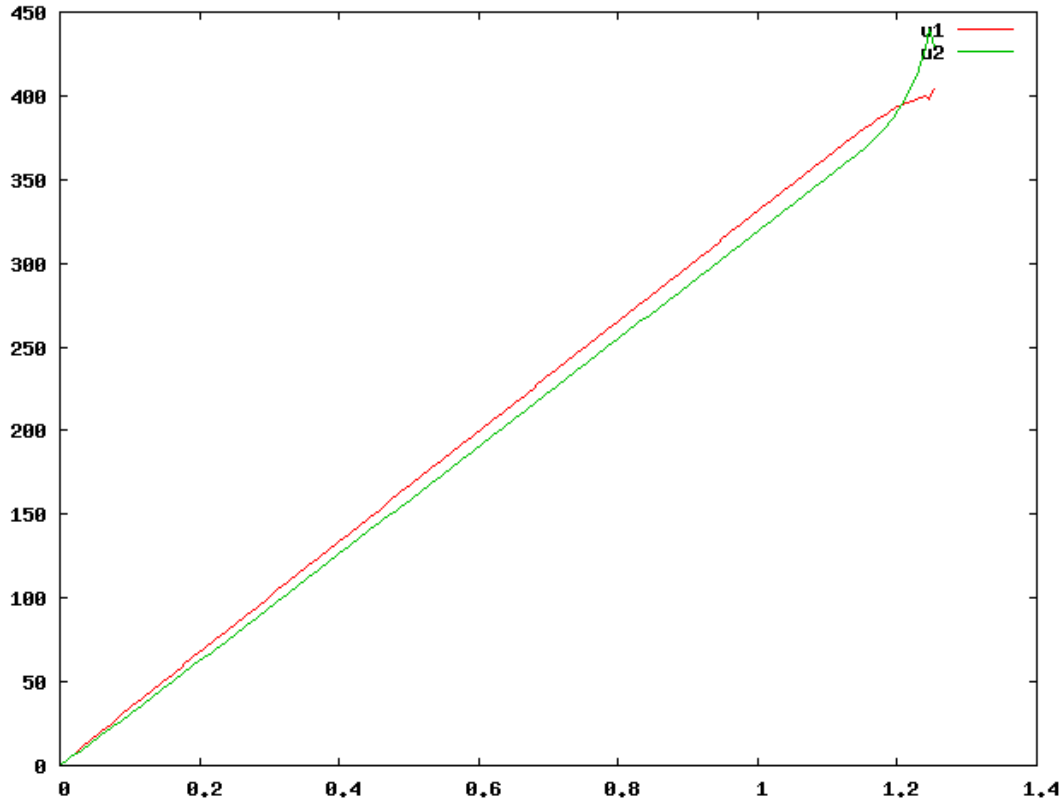

Figure: Velocities at the final time. Relaxation model with granular stress.

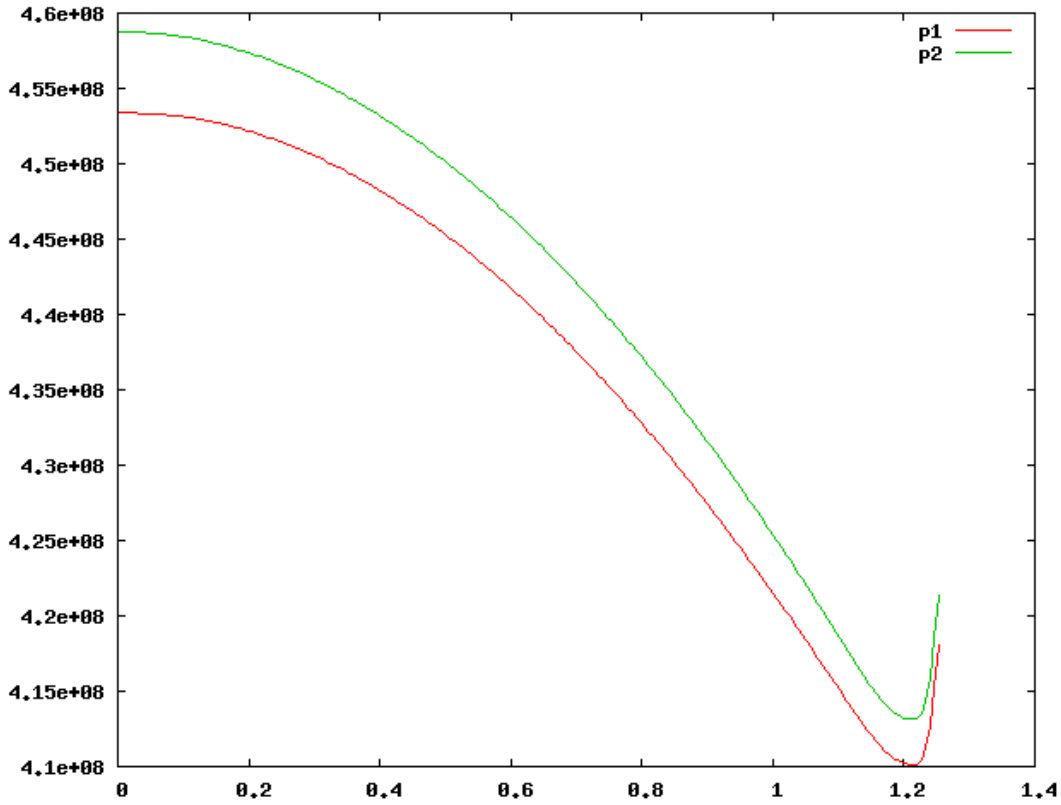

Figure: Pressures at the final time. Relaxation model with granular stress.

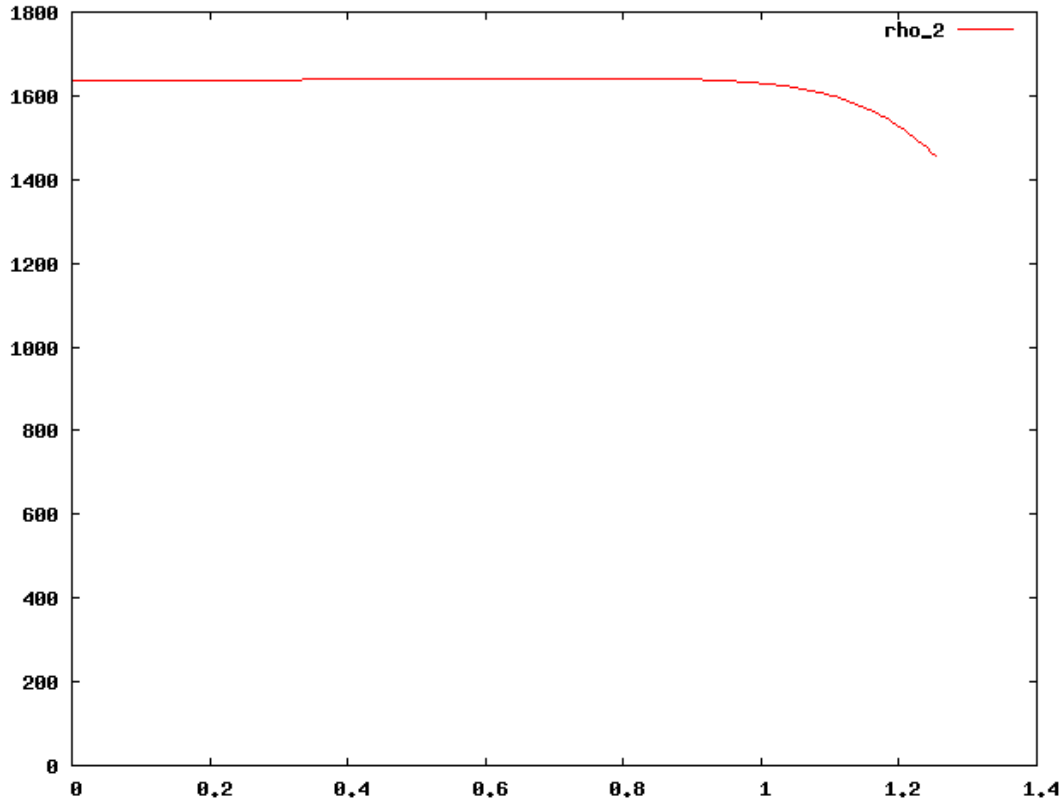

Figure: Density of the solid phase at the final time. Relaxation model with granular stress.

Conclusion

- Good generalization of the one pressure models;
- Rigorous entropy dissipation and maximum principle on the volume fraction;
- Stability for a finite relaxation time;
- The instability is (fortunately) preserved by the scheme for fast pressure equilibrium;
- The model can be used in practical configurations (the solid phase remains almost incompressible).

Biblio

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