Sharp-Interface Asymptotics for Navier-Stokes-Korteweg Equations

Jan Giesselmann

IANS, Universität Stuttgart

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Outline

1 Motivation
2 Sharp interface limits
3 The Navier-Stokes-Korteweg model
4 Asymptotic analysis and the main result
5 Prospects
Bubbles

single substance liquid vapour flow resolving single bubbles/droplets

sharp interface models  diffuse interface models
## Sharp Interface vs Diffuse Interface models

<table>
<thead>
<tr>
<th>Sharp Interface</th>
<th>Diffuse Interface</th>
</tr>
</thead>
<tbody>
<tr>
<td>interface is modeled as a surface</td>
<td>interface has a nonzero thickness $\varepsilon$</td>
</tr>
<tr>
<td>different PDEs / parameters in both bulk domains</td>
<td>one set of PDEs for the whole domain</td>
</tr>
<tr>
<td>jump conditions at the interface</td>
<td>smooth transition of the variables over the interfacial zone</td>
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<tr>
<td>(i.e. (dis)continuities are enforced)</td>
<td></td>
</tr>
<tr>
<td>physically well founded</td>
<td>physical derivation is not clear</td>
</tr>
<tr>
<td>cannot cope with topology</td>
<td>have no problem with topology changes</td>
</tr>
<tr>
<td>changes of the interface</td>
<td></td>
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**Aim:** Justify a diffuse interface model by showing that its solutions converge to solutions of a sharp interface model for $\varepsilon \to 0$. 
Sharp interface models

A set of PDEs in each bulk domain, e.g.

Euler equations

\[
\begin{align*}
\rho_t + \text{div}(\rho \mathbf{u}) &= 0, \\
(\rho \mathbf{u})_t + \text{div}(\rho \mathbf{u} \otimes \mathbf{u}) + \nabla p(\rho) &= 0,
\end{align*}
\]

where \( \rho \) is the density, \( \mathbf{u} \) the velocity and \( p(\rho) \) is the pressure given by a constitutive relation.

Rankine-Hugoniot conditions at the interface

\[
\begin{align*}
\rho^+(\mathbf{u}^+ \cdot \nu - w_\nu) &= \rho^-(\mathbf{u}^- \cdot \nu - w_\nu), \\
\rho^+ \mathbf{u}^+(\mathbf{u}^+ \cdot \nu - w_\nu) + \nu p(\rho^+) &= \rho^- \mathbf{u}^-(\mathbf{u}^- \cdot \nu - w_\nu) + \nu p(\rho^-) - \nu \sigma \kappa,
\end{align*}
\]

- \( \nu \) unit normal vector to the interface,
- \( w_\nu \) normal velocity of the interface,
- \( \kappa \) is the sum of the principal curvatures.
Van der Waals energy and pressure

For phase transitions we need a non-monotone pressure function. Common choice:

<table>
<thead>
<tr>
<th>$\rho \in (0, \alpha_1]$</th>
<th>vapour</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\rho \in (\alpha_1, \alpha_2)$</td>
<td>elliptic region</td>
</tr>
<tr>
<td>$\rho \in [\alpha_2, b)$</td>
<td>liquid</td>
</tr>
</tbody>
</table>

$$p(\rho) := \rho W'(\rho) - W(\rho).$$

The Euler equations are hyperbolic provided $p'(\rho) > 0.$
Motivation
Sharp Interface
NSK model
Asymptotics
Prospects

Uniqueness of solutions

Overview on well-posedness in 1D, see LeFloch, Hyperbolic systems of conservation laws.

Rankine Hugoniot conditions + entropy inequality $\iff$ uniqueness

Numerical simulations by C. Märlke
Uniqueness of solutions

We need an additional condition called **kinetic relation**, which prescribes the state on the right of the phase boundary given the state on the left of the phase boundary.

**Theorem (Benzoni-Gavage, Freistühler ’04):**
The free boundary value problem for the Euler equations with a van-der-Waals pressure function is locally well-posed, when one imposes the Rankine Hugoniot conditions and zero entropy dissipation at the interface, i.e.

\[
W'(\rho^+) + \frac{1}{2}|u^+ - w|^2 = W'(\rho^-) + \frac{1}{2}|u^- - w|^2
\]

Are there other possible kinetic relations?
General jump conditions

In general the jump conditions have to take the form

\[
\left[ \rho (u_\nu - w_\nu) \right] = 0
\]

\[
=: j
\]

\[
\left[ \rho (u_\nu - w_\nu)(u - w) + \nu p(\rho) \right] = 0
\]

and satisfy

\[
+\{j\} \left[ W'(\rho) + \frac{|u - w|^2}{2} \right] = 0,
\]

where

\[
[a] := a^+ - a^- \quad \{a\} := \frac{a^+ + a^-}{2},
\]
General jump conditions

In general the jump conditions have to take the form

\[
\left[ \rho (u_\nu - w_\nu) \right] = -\frac{\partial \rho_\Gamma}{\partial t} - \rho_\Gamma \left( \text{div}_\Gamma (w_\theta) - \kappa w_\nu \right),
\]

\[
\left[ \rho (u_\nu - w_\nu) (u - w) + \nu p(\rho) - j \right] = 0,
\]

and satisfy

\[
+ \{j\} \left[ W'(\rho) + \frac{|u - w|^2}{2} \right] = 0,
\]

where

\[
\left[ a \right] := a^+ - a^- \quad \{ a \} := \frac{a^+ + a^-}{2},
\]

- $\rho_\Gamma$ surface mass density,
- $\sigma_\Gamma$ the surface tension vector,
- $\sigma$ trace free part of the stress tensor.
General jump conditions

In general the jump conditions have to take the form

\[
\begin{align*}
\mathcal{J} = & \frac{\partial \rho}{\partial t} - \rho \left( \text{div}(w_\theta) - \kappa w_\nu \right), \\
\mathcal{J} = & \frac{\partial \rho}{\partial t} \rho \Gamma + \text{div}(\sigma \Gamma), \\
\{ \rho(u - w) \} + \{ \nu p(\rho) - \sigma \nu \} = & -\frac{\partial w}{\partial t} \rho \Gamma + \text{div}(\sigma \Gamma),
\end{align*}
\]

and satisfy

\[
\begin{align*}
+\{j\} \left[ W'(\rho) + \frac{|u - w|^2}{2} \right] = 0,
\end{align*}
\]

where

\[
\begin{align*}
\mathcal{J} := & \rho \Gamma - a^- \\
\{ a \} := & \frac{a^+ + a^-}{2},
\end{align*}
\]

- $\rho\Gamma$ surface mass density,
- $\sigma\Gamma$ the surface tension vector,
- $\sigma$ trace free part of the stress tensor.
General jump conditions

In general the jump conditions have to take the form

\[
\begin{bmatrix} \rho(u - w) \end{bmatrix} = - \frac{\partial \rho}{\partial t} - \rho \left( \text{div}_\Gamma (w_\theta - \kappa w) \right),
\]

\[
\begin{bmatrix} \rho(u - w)(u - w) + \nu p - \sigma \nu \end{bmatrix} = - \frac{\partial w}{\partial t} \rho + \text{div}_\Gamma (\sigma),
\]

and satisfy

\[
\begin{bmatrix} j \end{bmatrix} \left\{ W'(\rho) - \frac{\sigma^{ij} \nu^i \nu^j}{\rho} + \frac{|u - w|^2}{2} \right\} + \{ j \} \left\{ W'(\rho) - \frac{\sigma^{ij} \nu^i \nu^j}{\rho} + \frac{|u - w|^2}{2} \right\} \leq 0,
\]

where

\[
\begin{bmatrix} a \end{bmatrix} := a^+ - a^- \quad \{ a \} := \frac{a^+ + a^-}{2},
\]

- \( \rho \) \( \Gamma \) surface mass density,
- \( \sigma \) \( \Gamma \) the surface tension vector,
- \( \sigma \) trace free part of the stress tensor.
Stationary states

To find kinetic relations we consider regularisations of the Euler equations: Stationary solutions can be characterised as functions $\rho \in L^1(\Omega, (0, b))$ minimising

$$\int_{\Omega} W(\rho(x)) \, dx \to \min \text{ with constraint } \int_{\Omega} \rho(x) \, dx = m \quad (MIN)$$

Due to the total mass constraint we can change $W$ such that we obtain:

$\rho$ is a minimiser iff

- $\rho(x) \in \{\beta_1, \beta_2\}$ a.e.
- it satisfies the mass constraint
Minimisers

Theorem (Modica ’87):
Let $|\Omega| = 1$, $\varepsilon_n \to 0$ and let $\rho_{\varepsilon_n}$ be a sequence of minimisers of

$$F_{\varepsilon_n}[\rho] := \int_{\Omega} \tilde{W}(\rho(x)) + \gamma \varepsilon_n^2 |\nabla \rho|^2 \, dx \to \min, \quad \int_{\Omega} \rho(x) \, dx = m \quad (\text{MIN- } \varepsilon_n)$$

such that $\rho_{\varepsilon_n} \to \rho_0$ in $L^1(\Omega)$ then

- $\rho_0$ is a minimiser of (MIN),
- $\rho_0 \in BV(\Omega)$ and the length of the boundary of $S_{\text{min}} := \{x \in \Omega : \rho_0(x) = \beta_1\}$ can be defined by the perimeter functional

$$P_{\Omega}[S_{\text{min}}] := \int_{\Omega} |D \chi_{S_{\text{min}}}| \, dx.$$
Minimisers

Theorem (Modica ’87) continued:

- the set $S_{\text{min}} := \{ x \in \Omega : \rho_0(x) = \beta_1 \}$ is a minimiser of the functional $P_\Omega = P_\Omega[S]$ among all measurable sets $S$ satisfying $|S| = \frac{\beta_2 - m}{\beta_2 - \beta_1}$,

- for $\sigma := \int_{\beta_1}^{\beta_2} \sqrt{2\gamma(W(s) - \ell(s))} \, ds$ the relation

$$\lim_{n \to \infty} \frac{1}{\varepsilon_n} F_{\varepsilon_n}[\rho_{\varepsilon_n}] \to \sigma|\partial S_{\text{min}}|$$

holds.

Theorem (Luckhaus, Modica ’89, Dreyer, Kraus ’07):

The minimiser $\rho_0$ satisfies $\| \rho(\rho_0) \| = \varepsilon \kappa \sigma + o(\varepsilon)$. 
The Navier-Stokes-Korteweg model

The Euler-Lagrange equations of the regularised potential motivate the Navier-Stokes-Korteweg equations

\[
\begin{align*}
\rho_t + \text{div}(\rho u) &= 0 \\
(\rho u)_t + \text{div}(\rho u \otimes u) + \nabla p(\rho) &= \text{div}(\sigma_{NS}) + \gamma \rho \nabla \Delta \rho,
\end{align*}
\]

where

\[
\sigma_{NS}^{ij} = \lambda (\text{div } u) \delta^{ij} + \mu \left( \frac{\partial u^i}{\partial x^j} + \frac{\partial u^j}{\partial x^i} \right).
\]

- Hattori, Li '94: Local existence and uniqueness of classical solutions for the initial value problem,
- Kotschote '06: Local existence and uniqueness of classical solutions for the initial boundary value problem
- Bresch, Desjardins, Lin '03; Feireisl '04: Global existence of weak solutions
Numerical experiments

Numerical simulation by J. Haink
Travelling wave solutions

Travelling wave solutions for the non-viscous problem

\[
\rho(x, t) = R(x \cdot \nu - wt)
\]
\[
u(x, t) = U(x \cdot \nu - wt),
\]

where \( w \) is the wave velocity and \( \nu \) the direction of wave propagation, have to satisfy

\[
-wR_z + (R\nu \cdot U)_z = 0,
\]
\[
((\nu \cdot U - w)\nu \cdot U)_z + W'(R)_z = \gamma R_{zzz}.
\]

The first equation implies that the mass flux

\[
j := R(\nu \cdot U - w)
\]

is constant along the wave.
Travelling wave solutions

Theorem (Benzoni-Gavage, Danchin, Descombes, Jamet, ’07):
For $|j| << 1$ there exist $\rho_0^\pm(j) > 0$ such that

$$
\begin{align*}
\mathcal{W}'(\rho_0) + \frac{1}{2} \frac{j^2}{(\rho_0)^2} &= 0, \\
p(\rho_0) + \frac{j^2}{\rho_0} &= 0.
\end{align*}
$$

Furthermore there exists a solution $R(j)$ of (TW) satisfying

$$
R(j) \overset{z \to \pm \infty}{\to} \rho_0^\pm(j).
$$

The speed $w$ can be computed from mass flux and density.
Scaled NSK system

Non-dimensionalising the equations and choosing

\[ M := u_r \sqrt{\frac{\rho_r}{\rho}} = O(1), \quad \frac{1}{\text{Re}} := \frac{\mu_r}{\rho_r u_x} = O(\varepsilon^2), \quad \frac{t_r^2 \rho_r}{x^4} \gamma_r = O(\varepsilon^2), \]

we get

Scaled version of the NSK system

\[ \rho_t + \nabla \cdot (\rho u) = 0 \]
\[ (\rho u)_t + \text{div}(\rho u \otimes u) + \nabla p(\rho) = \varepsilon^2 \frac{1}{\mu} \text{div}(\sigma_{NS}) + \gamma \varepsilon^2 \rho \nabla \Delta \rho, \]

which means that the magnitudes of viscosity and capillarity are of the same (small) order.
For a different scaling, see Hermsdörfer, Kraus, Kröner ’09
Asymptotic Analysis

We assume: (backed by numerical experiments)

- Solutions of the Navier-Stokes-Korteweg equations which nearly are in phase equilibrium have an interfacial layer structure with width $\mathcal{O}(\varepsilon)$.
- These boundary layers approach sharp interfaces for $0 < \varepsilon \rightarrow 0$.

Hence we decompose the problem into

- "outer problems" away from the interfaces,
- "inner problem" inside the interfacial layer.

Matching conditions link both problems.
Coordinate change in the interfacial zone

New coordinates \((z, s, \tau)\) in the interfacial layer

\[
(x, t) = (r(s, \tau) + \varepsilon z\nu(s, \tau), \tau),
\]

where \(r\) denotes the position of the interface and \(s\) is some parametrisation of the interface.
Expansions

Quantities in inner coordinates will be denoted by capital letters and we assume there exist expansions in $\varepsilon$

$$U(\tau, s, z; \varepsilon) = \sum_{i=0}^{\infty} \varepsilon^i U_i(\tau, s, z) \quad \text{and} \quad R(\tau, s, z; \varepsilon) = \sum_{i=0}^{\infty} \varepsilon^i R_i(\tau, s, z).$$

Similarly we assume the existence of expansions for the quantities in the outer coordinates

$$\rho^{\pm}(x, t; \varepsilon) = \sum_{i=0}^{\infty} \varepsilon^i \rho_i^{\pm}(x, t) \quad \text{and} \quad u^{\pm}(x, t; \varepsilon) = \sum_{i=0}^{\infty} \varepsilon^i u_i^{\pm}(x, t)$$

and matching conditions linking boundary values of inner quantities to traces of outer quantities

$$R_0(z, s, \tau) \xrightarrow{z \to \pm \infty} \rho_0^{\pm}(r(s, \tau), \tau) \quad (\text{MATCH}).$$
Inner equations: leading order

We insert the inner expansions into (NSK) and change the coordinates. Collecting the terms of order $\varepsilon^{-1}$ yields

$$-w_{\nu}R_{0,z} + (R_0\nu \cdot U_0)_z = 0,$$

$$((\nu \cdot U_0 - w_{\nu})\nu \cdot U_0)_z + W'(R_0)_z = \gamma R_{0,zzz},$$

where $w_{\nu}$ is the velocity of the phase boundary. These equations are the same as for travelling wave solutions of the non-viscous (NSK).

$$\left[\rho_0(u_0 \cdot \nu - w_{\nu})\right]_{j_0} = 0,$$

$$\left[W'(\rho_0) + \frac{1}{2} \frac{j_0^2}{(\rho_0)^2}\right] = 0,$$

$$\left[p(\rho_0) + \frac{j_0^2}{\rho_0}\right] = 0,$$

$$\left[\theta \cdot u_0\right] = 0.$$
Jump Conditions up to $O(\varepsilon)$ I

Theorem (Dreyer, Giesselmann, Kraus, Rohde, '09):

Provided the inner and outer quantities have expansions as above, solve (NSK) for every $\varepsilon > 0$ and their boundary values are linked by (MATCH) then the outer quantities satisfy the following jump conditions at the interface:

\[
\left[ \rho (\mathbf{u} \cdot \mathbf{v} - w_\nu) \right] = -\frac{\partial \rho_\Gamma}{\partial \tau} - \rho_\Gamma (\text{div}_\Gamma (w_\theta) - \kappa w_\nu) + O(\varepsilon^2),
\]

where $\rho_\Gamma$ is the mass attributed to the interface and $\text{div}_\Gamma$ the surface divergence:

\[
\rho_\Gamma := \varepsilon \int_0^\infty R_0 - \rho_0^+ \, dz + \varepsilon \int_{-\infty}^0 R_0 - \rho_0^- \, dz,
\]

\[
\text{div}_\Gamma (w_\theta) := \frac{1}{\| \theta \|} \left( \| \theta \| w_\theta \right)_s.
\]
Jump Conditions up to $O(\varepsilon)$ II

Theorem: continued

$$\left[ \rho (u \cdot \nu - w \nu)(u - w) + \nu p(\rho) \right] = - \frac{\partial \mathbf{w}}{\partial \tau} \rho \Gamma + \text{div}_{\Gamma} (\sigma_{\Gamma}) + O(\varepsilon^2),$$

where $\sigma_{\Gamma}$ is the surface stress $(2 \times 1)$-vector given by

$$\sigma_{j\alpha}^{\Gamma} = \varepsilon \int_{-\infty}^{0} \left( p(\rho_0^+) - p(R_0) - \frac{\gamma}{2} (R_0, z)^2 \right) \frac{\theta_j}{\|\theta\|^2} \, dz$$

$$+ \varepsilon \int_{0}^{\infty} \left( p(\rho_0^-) - p(R_0) - \frac{\gamma}{2} (R_0, z)^2 \right) \frac{\theta_j}{\|\theta\|^2} \, dz.$$
Jump Conditions up to $O(\varepsilon)$ III

Theorem: continued

$$\left[ \frac{1}{2} (\boldsymbol{\nu} \cdot \mathbf{u} - w_\nu)^2 + W'(\rho) \right] = -\varepsilon \int_0^\infty \boldsymbol{\nu} \cdot (\mathbf{U}_0 - \mathbf{u}_0^+) \tau \, dz$$

$$-\varepsilon \int_{-\infty}^0 \boldsymbol{\nu} \cdot (\mathbf{U}_0 - \mathbf{u}_0^-) \tau \, dz$$

$$-\varepsilon \left( \frac{\lambda}{\mu} + 2 \right) j_0 \int_{-\infty}^\infty \left( \left( \frac{1}{R_0} \right)_z \right)^2 \, dz$$

$$+ O(\varepsilon^2).$$
Prospects

- Consider equations with stronger viscous effect, i.e. $\frac{1}{\text{Re}} = O(\epsilon)$.
- To obtain surface tension of order $O(1)$ we probably have to introduce a new phase field variable.
- Rigorous convergence proofs using $\Gamma$-convergence.
- Examine the local well-posedness with this kinetic relation.