Semaine speciale/Summer school

Stable homotopy theory:
classical calculations and modern structures

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Organizers:

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`http://www-irma.u-strasbg.fr/~henn/semaine_speciale/summer_school.html`

Student talks:

Besides three lecture series, the summer school will also have a series of lectures given by student participants. The following is a list of possible talks, together with references and some explanation of what should be covered.

(1) The Landweber exact functor theorem

Many important generalized homology theories can be constructed from complex bordism theory MU_* in a way which at first looks quite surprising. Namely, given an MU_*-module R (in pratice this tends to be even an MU_*-algebra), one can consider the functor $X \mapsto R \otimes_{MU_*} MU_*(X)$. This is clearly a homology theory if this functor is exact which is trivially the case if R is a free or more generally a flat MU_*-module. The Landweber exact funcor theorem (LEFT) gives a workable criterion which guarantees exactness under conditions on the MU_*-module structure of R which are much weaker than flatness. Consequences of LEFT are that classical homology theories such as complex K-theory and rational homology can be obtained by such a procedure. Furthermore LEFT is of central importance in chromatic stable homotopy theory on which there will be more in one of the three lecture series. It leads to the construction of the so-called Johnson-Wilson-theories $E(n)_*$ and the Lubin-Tate theories $(E_n)_*$ which will be discussed in a separate talk.

The classical approach [10] relies on the Landweber filtration theorem ([11], [12]) which is a generalization of the classical filtration of finitely generated modules over a noetherian ring in the context of comodules over the Hopf algebroid (MU_*, MU_*MU). An interesting recent alternative proof is presented in [14] and the speaker is encouraged to follow this approach.

(2) Lubin-Tate deformation theory and Morava E-theories

Lubin-Tate deformation theory is part of the theory of formal group laws. Formal group laws enter algebraic topology via complex oriented cohomology theories and the cohomology of infinite dimensional complex projective spaces. Given a formal group law Γ over a finite field of characteristic $p > 0$, one can ask how Γ can be 'deformed' (lifted) to local algebras B with residue field k. (Suitable equivalence classes of) deformations to B turn out to be in bijection with homomorphisms of local algebras $LT(\Gamma, k) \to B$ where $LT(\Gamma, k)$ is the Lubin-Tate ring associated to (Γ, k) which is equipped with a universal deformation. This purely algebraic result has important consequences in stable homotopy theory via the Landweber exact functor theorem which guarantees the existence of a homology theory whose coefficients are a graded version of $LT(\Gamma, k)$.

This talk should outline the algebraic deformation theory and explain why Landweber's criterion holds in this case. The classical reference is [13]. A more recent reference which explains the relations of this work to stable homotopy theory are sections 3-6 of [21].

(3) Derived categories in algebra

One of the most important advances in stable homotopy theory in the last few decades is the proof of Ravenel's nilpotence conjecture [18] and related conjectures like the classification of thick subcategories of the category of p-local finite spectra by Devinatz, Hopkins and Smith [3] and Hopkins-Smith [7].

To fully appreciate the philosophical relevance of these results it is very instructive to consider their analogues in the case of the homotopy category of chain complexes over a commutative ring. The analogy suggests that the nilpotence theorem determines the set of all 'prime ideals' in stable homotopy theory.

The proof of Ravenel's conjectures in the topological context is quite difficult and very deep. The algebraic analogues are still deep but much more accessible.

The talk should state and explain the classification theorem for thick subcategories of the category of p-local finite spectra (Theorem 7 in [6]) and then it should proceed to outline the proof in the algebraic case which is discussed in section 4 of [6]. A more thorough discussion of the algebraic case and its correct hypothesis is given in [17].

(4) Equivalence of $\mathcal{D}(R)$ and Ho(HR-mod)

A ring R gives rise to triangulated categories in two very different ways. On the one hand side one can consider chain complexes of R-modules and construct the derived category $\mathcal{D}(R)$ as a localization (e.g. by formally inverting the class of quasi-isomorphisms). On the other hand, R gives rise to an *Eilenberg-Mac Lane ring spectrum* HR, a structured ring spectrum whose homotopy groups are concentrated in dimension 0, where we have a ring isomorphism $\pi_0 HR \cong R$. Like any structured ring spectrum, HR has a stable model category of module spectra, whose homotopy category Ho(HR-mod) is then another triangulated category. The aim of this talk is to introduce $\mathcal{D}(R)$ and Ho(HR-mod) and prove that they are equivalent as triangulated categories.

This theorem can approached in different ways, and the speaker has to make certain expository decisions. The first result of this kind in the literature is Robinson's paper [22]. Another proof is given in [5, IV Thm. 2.4] in the context of 'S-algebras' and in [25, App. B] in the context of symmetric spectra. The latter result is somewhat stronger because the equivalence of triangulated homotopy categories arises from a zigzag of Quillen equivalences of model categories.

(5) Monoidal uniqueness of stable homotopy theory

One of the lecture series in this summer school explains the rigidity theorem [24] which rules out exotic models for the stable homotopy category. More precisely, any model for the stable homotopy category (i.e., stable model category in the sense of Quillen) is necessarily 'standard' (i.e., Quillen equivalent to the category of sequential spectra with the Bousfield-Friedlander stable model structure [2, §2]). This rigidity result is about the triangulated stable homotopy category but does not take the smash product into account. An interesting variation of the theme is thus: given a stable *symmetric monoidal* model category $(\mathcal{C}, \otimes, I)$ such that the homotopy category Ho($\mathcal{C}$) is equivalent to the stable homotopy category *as a tensor triangulated category* (i.e., the equivalence preserves the monoidal structures), does it follow that $\mathcal{C}$ and the category of symmetric spectra [8] are monoidally equivalent? In the paper [26] Shipley provides a positive answer under some mild technical hypothesis. This talk should explain the main results of Shipley's paper.

(6) Smith-Toda complexes

The so called Smith-Toda complexes $V(n)$ are certain finite p-local spectra (for p a prime number) which are characterized by their mod-p cohomology. For given p and n, there need not exist a Smith-Toda complex $V(n)$, and in fact it is an open question whether any $V(n)$ exists for $n \geq 4$. The Smith-Toda complexes which do exist are of interest because they admit periodic selfmaps which give rise to infinite families of non-trivial elements in the homotopy groups of spheres, the so called *Greek letter elements*. For example, $V(0)$ is a mod-p Moore spectrum, and for odd primes the iterates of the Adams selfmap $v_1 : \Sigma^{2p-2}V(0) \to V(0)$ gives rise to non-trivial p-primary elements $\alpha_n \in \pi^s_{n(2p-2)-1}S^0$ (and others) which are in the image of the J-homomorphism. A Smith-Toda complex $V(1)$ exists for odd primes and it admits a periodic 'v_2-selfmap' for $p \geq 5$ which produces the β-family. Finally, $V(2)$ exists for $p \geq 5$ and leads to a γ-family for primes at least 7.

The talk should define Smith-Toda complexes and explain the equivalence of the two definitions using mod-p cohomology or BP-homology. The constructions of the known Smith-Toda complexes and their periodic selfmaps along with the procedure to construct infinite families of stable homotopy classes should be given. Explain why $V(1)$ does not exist for $p = 2$ and $V(2)$ does not exists for $p = 3$, following Toda. Time permitting, the talk could end with some words about Nave's general non-existence theorem [16] and what makes the existence question for $V(4)$ different from the previous cases [19, Ch. 5, Sec. 6]. Section 2.4 of Ravenel's book [20] can serve as a starting point, and further references are [19, Ch. 1, Sec. 3], [28] and [29].

(7) Morita theory for stable model categories

Morita theory started out with the paper [15] by Kiiti Morita who initiated a systematic study of equivalences between module categories over rings. This theory has later been extended by Rickard, Keller and others to equivalences between derived categories of rings and is an important tool in representation theory. More recently, various authors have adapted the ideas of Morita theory to stable homotopy theory, where the question is about Quillen equivalences of stable model categories or module categories over ring spectra.

This talk should give a summary of 'classical' and 'derived' Morita theory and then explain how the main ideas can be adapted to stable model categories. Places to start reading could be the survey articles [23] and [27] which also contain many more references. Some of the original sources are [1], [4] and [25], and Keller's article [9] is worth looking at for a more algebraic perspective.

References

[1] A. Baker, A. Lazarev, *Topological Hochschild cohomology and generalized Morita equivalence*. Algebr. Geom. Topol. **4** (2004), 623–645

[2] A. K. Bousfield and E. M. Friedlander, *Homotopy theory of* Γ*-spaces, spectra, and bisimplicial sets*. Geometric applications of homotopy theory (Proc. Conf., Evanston, Ill., 1977), II Lecture Notes in Math., vol. 658, Springer, Berlin, 1978, pp. 80–130

[3] E. Devinatz, M. J. Hopkins, J. H. Smith, *Nilpotence and Stable Homotopy Theory*. Annals of Math. **128** (1988), 207–242

[4] W. G. Dwyer, J. P. C. Greenlees, S. B. Iyengar, *Duality in algebra and topology*. Adv. Math. **200** (2006), 357-402

[5] A. D. Elmendorf, I. Kriz, M. A. Mandell, J. P. May, *Rings, modules, and algebras in stable homotopy theory. With an appendix by M. Cole*, Mathematical Surveys and Monographs, **47**, American Mathematical Society, Providence, RI, 1997, xii+249 pp.

[6] M. J. Hopkins, *Global methods in Homotopy Theory*. In Homotopy Theory (Durham 1985) 73–96, London Math. Soc. Lecture Notes Series **117**, Cambridge University Press, Cambridge 1987

[7] M. J. Hopkins, J. H. Smith, *Nilpotence and Stable Homotopy Theory II*. Annals of Math. **148** (1998), 1–49

[8] M. Hovey, B. Shipley, J. Smith, *Symmetric spectra*. J. Amer. Math. Soc. **13** (2000), 149–208

[9] B. Keller, *On differential graded categories*. `math.KT/0601185`

[10] P. Landweber, *Homological properites of comodules over* MU_*MU *and* BP_*BP. American J. Math. **98** (1976), 591–610

[11] P. Landweber, *Annihilator ideals and primitive elements in complex cobordism*. Illinois J. Math. **17** (1973), 273–284

[12] P. Landweber, *Associated prime ideals and Hopf algebras*. J. Pure and Applied Algebra **3** (1973), 43–58

[13] J. Lubin and J. Tate, *Formal moduli for one parameter formal Lie groups*. Bull. Soc. Math. France **(94)** (1966), 49-60

[14] H. Miller, *Sheaves, gradings and the exact functor theorem.* Preprint `www-math.mit.edu/~hrm/papers/papers.html`

[15] K. Morita, *Duality for modules and its applications to the theory of rings with minimum condition.* Sci. Rep. Tokyo Kyoiku Daigaku Sect. A 6 (1958), 83–142

[16] L. S. Nave, *On the nonexistence of Smith-Toda complexes.* `math.AT/9810178`

[17] A. Neeman, *The chromatic tower for $D(R)$.* Topology **31** (1992), 519–532

[18] D. C. Ravenel, *Localization with respect to certain periodi homology theories.* Amer. J. Math **106** (1984), 351–414

[19] D. C. Ravenel, *Complex cobordism and stable homotopy groups of spheres.* Pure and Applied Mathematics, 121. Academic Press, Inc., Orlando, FL, 1986. xx+413 pp.

[20] D. C. Ravenel, *Nilpotence and periodicity in stable homotopy theory.* Appendix C by Jeff Smith. Annals of Mathematics Studies, 128. Princeton University Press, Princeton, NJ, 1992. xiv+209 pp.

[21] C. Rezk, *Notes on the Hopkins-Miller Theorem.* In *Homotopy Theory via Algebraic Geometry and Group Representations* (Evanston, IL 1997), 313–366, Amer. Math. Soc., Providence RI 1998

[22] A. Robinson, *The extraordinary derived category.* Math. Z. **196** (1987), no.2, 231–238.

[23] S. Schwede, *Morita theory in abelian, derived and stable model categories.* Structured ring spectra, 33–86, London Math. Soc. Lecture Note Ser., 315, Cambridge Univ. Press, Cambridge, 2004.

[24] S. Schwede, *The stable homotopy category is rigid.* To appear in *Annals of Math.* `http://www.math.uni-bonn.de/people/schwede`

[25] S. Schwede, B. Shipley, *Stable model categories are categories of modules.* Topology **42** (2003), no. 1, 103–153.

[26] B. Shipley, *Monoidal uniqueness of stable homotopy theory.* Adv. Math. **160** (2001), 217-240.

[27] B. Shipley, *Morita theory in stable homotopy theory.* Handbook of Tilting Theory (eds. Angeleri Huegel, Happel, Krause) London Math. Soc. Lecture Notes Series. **332** (2007) 393-409.

[28] L. Smith, *On realizing complex bordism modules. Applications to stable homotopy of spheres*. Amer. J. Math. **92** (1970), 793-856.

[29] H. Toda, *On spectra realizing exterior parts of the Steenrod algebra*. Topology **10** (1971), 53-65.