TD1- M2 Strasbourg 2017-2018
Exercices tirés du livre de Atiyah-Macdonald, Introduction to commutative algebra.

1. Let x be a nilpotent element of a ring A. Show that $1 + x$ is a unit of A. Deduce that the sum of a nilpotent element and a unit is a unit.

2. Let A be a ring and let $A[x]$ be the ring of polynomials in an indeterminate x, with coefficients in A. Let $f = a_0 + a_1x + \cdots + a_nx^n \in A[x]$. Prove that

i) f is a unit in $A[x] \Leftrightarrow a_0$ is a unit in A and $a_1, \ldots, a_n$ are nilpotent. [If $b_0 + b_1x + \cdots + b_mx^m$ is the inverse of f, prove by induction on r that $a_n^{r+1}b_{m-r} = 0$. Hence show that a_n is nilpotent, and then use Ex. 1.]

ii) f is nilpotent $\Leftrightarrow a_0, a_1, \ldots, a_n$ are nilpotent.

iii) f is a zero-divisor $\Leftrightarrow$ there exists $a \neq 0$ in A such that $af = 0$. [Choose a polynomial $g = b_0 + b_1x + \cdots + b_mx^m$ of least degree m such that $fg = 0$. Then $a_nb_m = 0$, hence $a_ng = 0$ (because a_ng annihilates f and has degree $< m$). Now show by induction that $a_{n-r}g = 0$ $(0 \leqslant r \leqslant n)$.]

iv) f is said to be *primitive* if $(a_0, a_1, \ldots, a_n) = (1)$. Prove that if $f, g \in A[x]$, then fg is primitive $\Leftrightarrow f$ and g are primitive.

4. In the ring $A[x]$, the Jacobson radical is equal to the nilradical.

5. Let A be a ring and let $A[[x]]$ be the ring of formal power series $f = \sum_{n=0}^{\infty} a_nx^n$ with coefficients in A. Show that

i) f is a unit in $A[[x]] \Leftrightarrow a_0$ is a unit in A.

ii) If f is nilpotent, then a_n is nilpotent for all $n \geqslant 0$. Is the converse true? (See Chapter 7, Exercise 2.)

iii) f belongs to the Jacobson radical of $A[[x]] \Leftrightarrow a_0$ belongs to the Jacobson radical of A.

6. A ring A is such that every ideal not contained in the nilradical contains a non-zero idempotent (that is, an element e such that $e^2 = e \neq 0$). Prove that the nilradical and Jacobson radical of A are equal.

7. Let A be a ring in which every element x satisfies $x^n = x$ for some $n > 1$ (depending on x). Show that every prime ideal in A is maximal.

8. Let A be a ring $\neq 0$. Show that the set of prime ideals of A has minimal elements with respect to inclusion.

10. Let A be a ring, $\mathfrak{R}$ its nilradical. Show that the following are equivalent:
i) A has exactly one prime ideal;
ii) every element of A is either a unit or nilpotent;
iii) $A/\mathfrak{R}$ is a field.

11. A ring A is *Boolean* if $x^2 = x$ for all $x \in A$. In a Boolean ring A, show that
i) $2x = 0$ for all $x \in A$;
ii) every prime ideal $\mathfrak{p}$ is maximal, and $A/\mathfrak{p}$ is a field with two elements;
iii) every finitely generated ideal in A is principal.

16. Draw pictures of Spec ($\mathbf{Z}$), Spec ($\mathbf{R}$), Spec ($\mathbf{C}[x]$), Spec ($\mathbf{R}[x]$).

17. For each $f \in A$, let $D(f)$ denote the complement of $V(f)$ in $X = \text{Spec}(A)$. The sets $D(f)$ are open. Show that they form a basis of open sets for the Zariski topology, and that
i) $D(f) \cap D(g) = D(fg)$;
ii) $D(f) = \varnothing \Leftrightarrow f$ is nilpotent;
iii) $D(f) = X \Leftrightarrow f$ is a unit;

v) X is quasi-compact (that is, every open covering of X has a finite sub-covering).

[To prove (v), remark that it is enough to consider a covering of X by basic open sets $D(f_i)$ $(i \in I)$. Show that the f_i generate the unit ideal and hence that there is an equation of the form

$$1 = \sum_{i \in J} g_i f_i \qquad (g_i \in A)$$

where J is some *finite* subset of I. Then the $D(f_i)$ $(i \in J)$ cover X.]