About Ricci curvature in the sub-Riemannian Heisenberg group

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Outline

1. Optimal transport and geodesic interpolation

2. The first Heisenberg group

3. The contraction estimate and two results
Geodesic space

In a metric space \((X,d)\), \(\gamma : [0, 1] \rightarrow X\) is a geodesic if for every \(s, s' \in [0, 1]\)

\[
d(\gamma(s), \gamma(s')) = |s - s'|d(\gamma(0), \gamma(1)).
\]

A metric space \((X, d)\) is geodesic if for all \((p, q)\) there is a geodesic \(\gamma\) from \(p\) to \(q\).

If \(\gamma\) is unique, we define \(M^s(p, q) = \gamma(s)\), the interpolation map.
The $N$-entropy of a probability measure $\mu$ of density $\rho$ is given by

$$\text{Ent}_N(\rho \nu | \nu) = - \int \rho^{1-1/N} d\nu(x).$$

If $\mu$ is singular $\text{Ent}(\mu) = 0$.

Big entropy: $\mu$ concentrated on a small space.
Small entropy: $\mu$ fills a lot of space.
Curvature-dimension $CD(0, N)$

A space $(X, d, \nu)$ satisfies $CD(0, N)$ if

for every absolutely continuous $\mu_0, \mu_1 \in \mathcal{P}_2(X)$, there is a geodesic $(\mu_s)_{s \in [0,1]}$ such that $s \in [0,1] \rightarrow \text{Ent}_N(\mu_s \mid \nu) \in \mathbb{R}$ is convex.

(The exact statement is: there exists a geodesic $(\mu_s)_{s \in [0,1]}$ such that for any $s \in [0,1]$, $\text{Ent}_N(\mu_s \mid \nu) \leq (1-s)\text{Ent}_N(\mu_0 \mid \nu) + s\text{Ent}_N(\mu_1 \mid \nu)$)
Synthetic Ricci curvature

New definitions of positive Ricci curvature for metric measure spaces \((X, d, \nu)\):

- Measure Contraction Property \(MCP(K, N)\) (Sturm; Ohta 2006)
- Curvature-Dimension \(CD(K, N)\) (Lott-Villani; Sturm 2006)

\[ CD(0, N) \Rightarrow \text{Brunn-Minkowski}(0, N) \Rightarrow MCP(0, N). \]

(The second implications is actually only known in the case the number of geodesics between \(p\) and \(q\) of \(X\) is almost surely 1).
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The Heisenberg group $\mathbb{H}$ is $\mathbb{R}^3 = \mathbb{C} \times \mathbb{R}$ with the multiplicative structure:

$$(x, y, t) \cdot (x', y', t') = (z, t) \cdot (z', t') = \left( z + z', t + t' - \frac{\text{Im}(zz')}{2} \right).$$

The Lebesgue measure $\mathcal{L}^3$ is left-invariant.

The left invariant vector fields $X = \frac{\partial}{\partial x} - \frac{y}{2} \frac{\partial}{\partial t}$ and $Y = \frac{\partial}{\partial y} + \frac{x}{2} \frac{\partial}{\partial t}$

and $T = [X, Y] = \frac{\partial}{\partial t}$ span the tangent space in any point.

$$d_c(p, q) = \inf_{\gamma} \int_0^1 \sqrt{a^2(s) + b^2(s)} ds$$

where $\gamma$ is horizontal:

$$\dot{\gamma}(s) = a(s)X(\gamma(s)) + b(s)Y(\gamma(s)).$$
A curve is horizontal if and only if the third coordinate evolves like the algebraic area swept by the complex projection.

The length of the horizontal curves is exactly the length of the projection in $\mathbb{C}$.

The geodesics of $\mathbb{H}_1$ are the horizontal curves whose projection is a circle arc or a line.
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The key estimate

For any $e \in \mathbb{H}$. The contraction map $M^s_e : f \to M^s(e, f)$ is differentiable with

$$\text{Jac}(M^s_e)(f) \geq s^5.$$ 

Equality case: $e$ and $f$ are on a line.

As a consequence $(\mathbb{H}, d_c, L^3)$ satisfies $MCP(0, 5)$:

(Rough) definition of the Measure Contraction Property $MCP(0, N)$ for $(X, d, \nu)$:

for every point $e \in X$, for every $F \subset X$ and for all $s \in [0, 1]$,

$$\nu(M^s(e, F)) \geq s^N \nu(F).$$
First result

**Theorem (Ambrosio, Rigot, 2004)**

Let $\mu_0, \mu_1 \in \mathcal{P}_2(\mathbb{H})$ such that $\mu_0$ is absolutely continuous. Then there is a unique optimal coupling $\pi$. It is $\pi = (\text{Id} \otimes T)\#\mu_0$ for some map $T$. Moreover there is a unique geodesic between $p$ and $T(p)$ ($\mu_0$-almost surely).

Let $T_s(p) = M^s(p, T(p))$. There is a unique geodesic $(\mu_s)_{s \in [0,1]}$ between $\mu_0$ and $\mu_1$.

It is defined by $\mu_s = M^s(\mu_0, \mu_1) = (T_s)\#\mu_0$.

**Open question (Ambrosio, Rigot)**

Let $\mu_0$ be absolutely continuous and $s < 1$. Is $\mu_s$ absolutely continuous as well?

**Theorem (Figalli, J.)**

Let $(\mu_s)_{s \in [0,1]}$ be a geodesic of $\mathcal{P}_2(\mathbb{H})$ and $\mu_0$ absolutely continuous with respect to $L^3$. Then for all $s \in [0,1)$, $\mu_s$ is absolutely continuous too.
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Second result

Theorem (J.)

In \((\mathbb{H}, L^3, d_c)\), the Heisenberg group with the Lebesgue measure and the Carnot-Carathéodory distance

- \(MCP(0, N)\) is true if and only if \(N \geq 5\),
- \(CD(0, N)\) and \(BM(0, N)\) are false for every \(N\).

Theorem (J.)

In \((\mathbb{H}_n, L^{2n+1}, d_c)\), the \(n\)th-Heisenberg group with the Lebesgue measure and the Carnot-Carathéodory distance

- \(MCP(K, N)\) is true if and only if \(N \geq 2n + 3\) and \(K \leq 0\),
- \(CD(K, N)\) and \(BM(K, N)\) are false for every \((K, N)\).
Brunn-Minkowski inequality $BM(0,N)$:

A space $(X,d,\nu)$ satisfies the Brunn-Minkowski inequality $BM(0,N)$ if

For every $E,F \subset X$ and for all $s \in [0,1]$,

$$\nu(M_s^s(E,F))^{1/N} \geq (1-s)\nu(E)^{1/N} + s \nu(F)^{1/N}.$$

In particular if $\nu(E) = \nu(F)$,

$$\nu(M^{1/2}(E,F))^{1/N} \geq \frac{\nu(E)^{1/N} + \nu(F)^{1/N}}{2} = \nu(E)^{1/N} = \nu(F)^{1/N}.$$
Sketch of proof

Let $F$ be a small ball such that $0_H$ and the center of the ball are on a “bad” geodesic.

For $E$ we take the “geodesic inverse” of $F$. 
Let $F$ be a small ball such that $0_\mathbb{H}$ and the center of the ball are on a “bad” geodesic. For $E$ we take the “geodesic inverse” of $F$. It turns out that $L^3(E) = L^3(F)$.

We want to prove $L^3\left(\mathcal{M}^{1/2}(E,F)\right) < L^3(F)$ because it is a contradiction to the $BM(0,N)$. 
For each $e \in E$, the contracted set $\mathcal{M}^{1/2}(e, F)$ is a sort of ellipsoid that contains $0_\mathbb{H}$.

The volume of such an ellipsoid is

$$2^{-5} \mathcal{L}^3(F).$$
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The midstet $\mathcal{M}^{1/2}(E, F)$ is made of the union of these ellipsoids.
All of them contain $0_{\mathbb{H}}$. Then $\mathcal{M}^{1/2}(E, F)$ is an ellipsoid of size 2. Its volume is

$$2^3 \cdot 2^{-5} \mathcal{L}^3(F) = \frac{\mathcal{L}^3(F)}{4}.$$ 

Then $\mathcal{L}^3(\mathcal{M}^{1/2}(E, F)) < \mathcal{L}^3(F) = \mathcal{L}^3(E)$. 