A cohomological invariant for groups coming from quantum group theory

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Introduction

• Report on joint work with Pierre Guillot (Strasbourg):
Cohomology of invariant Drinfeld twists on group algebras,
Internat. Math. Res. Notices 2010, 1894–1939; arXiv:0903.2807

• Given a finite group G and a field k, we define a 'cohomology group'

$$\mathcal{H}^2(G/k)$$

with the following properties:

- If G is abelian, then

$$\mathcal{H}^2(G/k) = H^2(\widehat{G}, k^\times) \qquad \text{(group cohomology)}$$

  where $k^\times = k - \{0\}$ and $\widehat{G} = \mathrm{Hom}(G, k^\times)$

- There are groups G such that $\mathcal{H}^2(G/k)$ is not abelian

  Remark. $\mathcal{H}^2(G/k) \not\cong H^2(G, k^\times)$, the latter being abelian

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I. Origin: Where $\mathcal{H}^2(G/k)$ comes from

II. Definition of $\mathcal{H}^2(G/k)$

III. Main results

IV. A proof using quantum group theory

V. Examples of computation

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Extending group cohomology to Hopf algebras

Start from the cohomology groups $H^i(G, k^\times)$ of G acting trivially on $k^\times$

- **Sweedler (1968)** associated to any cocommutative Hopf algebra H cohomology groups
$$H^i_{\mathrm{Sw}}(H) \qquad (i \geq 1)$$
such that
$$H^i_{\mathrm{Sw}}(kG) = H^i(G, k^\times)$$
for any group G

- **Schauenburg (2002)** extended this to arbitrary Hopf algebras; he defined groups
$$H^i_{\mathrm{Sch}}(H) \qquad (i = 1, 2)$$
such that
$$H^i_{\mathrm{Sch}}(H) = H^i_{\mathrm{Sw}}(H)$$
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Price to pay: these groups are defined only for $i = 1$ and $i = 2$

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The group $\mathcal{H}^2(G/k)$

• **First Schauenburg cohomology:** $H^1_{\mathrm{Sch}}(H)$ is easy to compute: it is the abelian group

$$H^1_{\mathrm{Sch}}(H) = \{\chi \in \mathrm{Alg}(H,k) \mid \chi\alpha = \alpha\chi \text{ for all } \alpha \in \mathrm{Hom}(H,k)\}$$

So we are left with $H^2_{\mathrm{Sch}}(H)$

• **Second Schauenburg cohomology:** $H^2_{\mathrm{Sch}}(H)$ has been computed for very few non-cocommutative Hopf algebras

• **Our initial aim:**

Compute $H^2_{\mathrm{Sch}}(H)$ for a large class of non-cocommutative Hopf algebras, namely for the dual Hopf algebras $H = (kG)^*$, where G is a finite group

Observation: The Hopf algebra $(kG)^*$ is not cocommutative if and only if G is not abelian

• For simplicity, we set

$$H^2_{\mathrm{Sch}}((kG)^*) = \mathcal{H}^2(G/k)$$

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The abelian case

If A is a finite abelian group and k is algebraically closed, then the discrete Fourier transform induces a Hopf algebra isomorphism

$$(kA)^* \cong k\widehat{A}$$

Thus,

$$\begin{aligned}
\mathcal{H}^2(A/k) &= H^2_{\mathrm{Sch}}((kA)^*) && \text{(by definition)} \\
&\cong \mathrm{H}^2_{\mathrm{Sch}}(k\widehat{A}) && \text{(discrete Fourier transform)} \\
&\cong H^2(\widehat{A}, k^\times) && \\
&\cong \mathrm{Hom}(H_2(\widehat{A}, \mathbb{Z}), k^\times) && \text{(universal coefficient theorem)} \\
&\cong \mathrm{Hom}(\Lambda^2_{\mathbb{Z}}(\widehat{A}), k^\times) && \\
&= \{\text{alternating bicharacters on } \widehat{A}\} &&
\end{aligned}$$

A bicharacter is a bimultiplicative map $b : \widehat{A} \times \widehat{A} \to k^\times$, i.e.,

$$b(u_1 u_2, v) = b(u_1, v)\, b(u_2, v) \quad \text{and} \quad b(u, v_1 v_2) = b(u, v_1)\, b(u, v_2)$$

Alternating means that $b(u, u) = 1$ for all $u \in \widehat{A}$

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Towards a geometric interpretation of $\mathcal{H}^2(G/k)$

We had another motivation to try to understand $\mathcal{H}^2(G/k)$

• **Recall:** A G-torsor is a right G-variety T such that the map

$$\begin{array}{ccc} T \times G & \longrightarrow & T \times T \\ (t, g) & \longmapsto & (t, tg) \end{array}$$

is an isomorphism

• If $k = \bar{k}$ is algebraically closed, then any torsor is isomorphic to $T = G$ with G acting by right translations

• Over a general ground field k:

torsors are classified by Serre's non-abelian Galois cohomology set

$$H^1(\mathrm{Gal}(\bar{k}/k), G)$$

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Bitorsors

• To obtain a group structure, we consider bitorsors

• A bitorsor T is both a left and right G-torsor such the left and right actions commute

• The product of two bitorsors T, T' is the bitorsor

$$T * T' = T \times_G T' = (T \times T')/\langle (tg, t') = (t, gt') \rangle$$

• In this way, the set of isomorphism classes of bitorsors becomes a group, which is isomorphic to the Galois cohomology group

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Noncommutative bitorsors

- In the spirit of noncommutative geometry, replace

varieties	$\longleftrightarrow$	algebras
groups	$\longleftrightarrow$	Hopf algebras
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Reminder on group cohomology

• **Define** $Z^2(G, k^\times)$ as the set of all maps $f : G \times G \to k^\times$ satisfying the cocycle condition

$$f(x,y)\, f(xy,z) = f(y,z)\, f(x,yz) \qquad (x,y,z \in G)$$

Cocycles form a group under pointwise multiplication

• Two cocycles are cohomologous—we write $f \sim f'$—if there is a map $g : G \to k^\times$ such that

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Let us extend this to cocommutative Hopf algebras

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Sweedler cohomology 1

Let H be a Hopf algebra with coproduct $\Delta : H \to H \otimes H$

• Define the convolution product of two bilinear maps $f, f' : H \times H \to k$ by

$$(f * f')(x, y) = \sum_{(x)(y)} f(x_1, y_1)\, f'(x_2, y_2)$$

where $\Delta(x) = \sum_{(x)} x_1 \otimes x_2$ and $\Delta(y) = \sum_{(y)} y_1 \otimes y_2$

• Define $Z^2(H)$ as the set of all convolution-invertible bilinear maps $f : H \times H \to k$ satisfying the cocycle condition

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Sweedler cohomology 2

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where g^{-1} is the convolution-inverse of g

Convention: $(\Delta \otimes \mathrm{id})\Delta(x) = \sum_{(x)} x_1 \otimes x_2 \otimes x_3$

• By definition, the Sweedler cohomology group of a cocommutative Hopf algebra is

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Schauenburg cohomology

Let H be an arbitrary Hopf algebra

• We restrict to special cocycles, namely cocycles satisfying the additional condition

$$\sum_{(x)(y)} f(x_1, y_1)\, x_2 y_2 = \sum_{(x)(y)} f(x_2, y_2)\, x_1 y_1 \qquad (x, y \in H)$$

If f, f' are special cocycles, then so is $f * f'$

and thus the set $Z^2_s(H)$ of special cocycles becomes a group

Remark. If H is cocommutative, then any cocycle is special

• Two special cocycles are cohomologous if there is a $g : H \to k$ as above, but also satisfying

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Rationality

Assume that k is of characteristic zero with algebraic closure $\bar{k}$

• **Theorem 1.** *If all irreducible $\bar{k}$-representations of G can be realized over k, then there is an exact sequence of groups*

$$1 \longrightarrow H^1(\mathrm{Gal}(\bar{k}/k), Z(G)) \longrightarrow \mathcal{H}^2(G/k) \longrightarrow \mathcal{H}^2(G/\bar{k}) \longrightarrow 1$$

(bitorsors) *(noncomm. bitorsors)*

(ARITHMETIC) (GEOMETRIC)

In particular, if G is centerless, then $\mathcal{H}^2(G/k) \cong \mathcal{H}^2(G/\bar{k})$

• We are now reduced to computing $\mathcal{H}^2(G/\bar{k})$

• From now on, assume k is algebraically closed: $\bar{k} = k$

• To compute $\mathcal{H}^2(G/\bar{k})$, we need the following two ingredients. . .

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First ingredient: class-preserving automorphisms

- Let $\mathrm{Aut}_c(G)$ be the group of automorphisms φ of G such that for all $g \in G$

$$\varphi(g) = hgh^{-1} \qquad \text{for some } h \in G$$

The group $\mathrm{Inn}(G)$ of inner automorphism is normal in $\mathrm{Aut}_c(G)$

- **Definition.**

$$\mathrm{Out}_c(G) = \mathrm{Aut}_c(G)/\,\mathrm{Inn}(G)$$

- We have $\mathrm{Out}_c(G) = 1$ if

  (a) $G = S_n$ (symmetric group)

  (b) $G = SL_n(\mathbb{F}_q)$

  (c) G is simple (Feit-Seitz, 1989)

- Finding finite groups with $\mathrm{Out}_c(G) \neq 1$ is not straightforward

Nevertheless, there are groups with non-trivial $\mathrm{Out}_c(G)$,
even with non-abelian $\mathrm{Out}_c(G)$

First ingredient: class-preserving automorphisms

- Let $\mathrm{Aut}_c(G)$ be the group of automorphisms φ of G such that for all $g \in G$

$$\varphi(g) = hgh^{-1} \qquad \text{for some } h \in G$$

The group $\mathrm{Inn}(G)$ of inner automorphism is normal in $\mathrm{Aut}_c(G)$

- **Definition.**

$$\mathrm{Out}_c(G) = \mathrm{Aut}_c(G)/\,\mathrm{Inn}(G)$$

- We have $\mathrm{Out}_c(G) = 1$ if

  (a) $G = S_n$ (symmetric group)

  (b) $G = SL_n(\mathbb{F}_q)$

  (c) G is simple (Feit-Seitz, 1989)

- Finding finite groups with $\mathrm{Out}_c(G) \neq 1$ is not straightforward

Nevertheless, there are groups with non-trivial $\mathrm{Out}_c(G)$,
even with non-abelian $\mathrm{Out}_c(G)$

Second ingredient: the pointed set $\mathcal{B}(G)$

- For any finite group G
  let $\mathcal{B}(G)$ be the set of pairs (A, b) where

  (i) A is an abelian normal subgroup of G and

  (ii) $b : \widehat{A} \times \widehat{A} \to k^\times$ is a G-invariant non-degenerate alternating bicharacter

- The set $\mathcal{B}(G)$ is finite and can be "computed in finite time"

  It is non-empty since it always contains the trivial pair $(\{1\}, b_0 \equiv 1)$

- If $(A, b) \in \mathcal{B}(G)$, then necessarily

$$A \cong A_0 \times \widehat{A_0}$$

  for some subgroup A_0. Hence, $|A| = |A_0|^2$ is a square

- **Examples.** $\mathcal{B}(G)$ is trivial if $G = S_n$, $G = SL_n(\mathbb{F}_q)$, or G is simple

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Determining $\mathcal{H}^2(G/\bar{k})$

Assume that the ground field $k = \bar{k}$ is of characteristic prime to $|G|$

We set $\mathcal{H}^2(G) = \mathcal{H}^2(G/\bar{k})$

- **Theorem 2** *There is a set-theoretic map $\Theta : \mathcal{H}^2(G) \to \mathcal{B}(G)$ such that*

  (a) $\mathcal{H}_0 = \Theta^{-1}((\{1\}, b_0))$ is a subgroup of $\mathcal{H}^2(G)$ such that

$$\mathcal{H}_0 \cong \mathrm{Out}_c(G)$$

  (b) $\Theta(\alpha) = \Theta(\beta)$ if and only if $\beta \in \alpha\mathcal{H}_0$

  (c) If $|G|$ is odd, then Θ is surjective

- **Remark.** If $|G|$ is even, then Θ may or may not be surjective

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Consequences of Theorem 2

- The group $\mathcal{H}^2(G)$ is finite: it sits between $\mathrm{Out}_c(G)$ and $\mathcal{B}(G)$
- The group $\mathcal{H}^2(G)$ can be non-abelian since $\mathcal{H}^2(G) \supset \mathcal{H}_0 \cong \mathrm{Out}_c(G)$
- If $\mathrm{Out}_c(G) = 1$ and $\mathcal{B}(G)$ is trivial, then $\mathcal{H}^2(G) = 1$
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Representing elements of $\mathcal{H}^2(G)$ by Drinfeld twists

- Represent an element of $\mathcal{H}^2(G) = H^2_{\mathrm{Sch}}((kG)^*)$ by a special cocycle

$$f : (kG)^* \times (kG)^* \to k$$

- Dualizing, we obtain an element $F \in kG \otimes kG$ such that

  (a) F is invertible in the algebra $kG \otimes kG$ (invertibility of f)

  (b) F is a Drinfeld twist, i.e.,

$$(F \otimes 1)\,(\mathrm{id} \otimes \Delta)(F) = (1 \otimes F)\,(\Delta \otimes \mathrm{id})(F) \in kG \otimes kG \otimes kG$$

(translation of the cocycle condition)

  (c) F is G-invariant, i.e.,

$$(g \otimes g)\,F = F\,(g \otimes g) \qquad (g \in G)$$

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The universal R-matrix attached to a Drinfeld twist

- Given an invertible invariant Drinfeld twist F, consider

$$R_F = F_{21}\, F^{-1} \in kG \otimes kG$$

This is a universal R-matrix for kG, i.e., an invertible element satisfying

$$\Delta(a)\, R_F = R_F\, \Delta(a) \qquad \text{for all } a \in kG$$

and

$$(\Delta \otimes \mathrm{id})(R_F) = (R_F)_{13}\, (R_F)_{23} \quad \text{and} \quad (\mathrm{id} \otimes \Delta)(R_F) = (R_F)_{13}\, (R_F)_{12}$$

- The universal R-matrix induces braidings on the tensor category of G-modules

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Constructing the map Θ: Part 1

• By work of Radford, there is a minimal Hopf subalgebra $H \subset kG$ such that

$$R_F \in H \otimes H$$

This Hopf algebra is self-dual: $H^* \cong H$

• Since kG is cocommutative, so is H. From self-duality, H is bicommutative

One deduces that $H = kA$ for some abelian subgroup A of G

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Restricting to $\widehat{A} \times \widehat{A}$, we obtain a G-invariant bicharacter $b_F : \widehat{A} \times \widehat{A} \to k^\times$

One shows that b_F is non-degenerate and alternating

- We define

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On the proof of Theorem 2

- To determine $\mathcal{H}_0 = \Theta^{-1}((\{1\}, b_0))$, we use the following result:

Etingof and Gelaki (2000): *If F is a Drinfeld twist such that $R_F = 1 \otimes 1$, or equivalently F is symmetric: $F = F_{21}$, then*

$$F = (a \otimes a)\,\Delta(a^{-1})$$

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Observation. The symmetric twist $F = (a \otimes a)\,\Delta(a^{-1})$ is invariant if and only if the automorphism $x \mapsto axa^{-1}$ preserves G and is an element of $\mathrm{Aut}_c(G)$

From this it is easy to deduce that $\mathcal{H}_0 \cong \mathrm{Out}_c(G)$

- Etingof and Gelaki's result follows from classical Tannakian theory:

Deligne and Milne (1982): *Any exact and fully faithful symmetric tensor functor from the category of kG-modules to the category of k-vector spaces is isomorphic to the forgetful functor*

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Example of groups with non-trivial $\mathcal{H}^2(G)$

• Let p be an odd prime and let G be the wreath product

$$G = \mathbb{Z}/p \wr \mathbb{Z}/p = A \rtimes \mathbb{Z}/p$$

with $\mathbb{Z}/p$ acting cyclically on $A = (\mathbb{Z}/p)^p$

• We have $\mathrm{Out}_c(G) = 1$ and $|G|$ is odd; we deduce from Theorem 2 that the map $\Theta : \mathcal{H}^2(G) \to \mathcal{B}(G)$ is bijective

• **Proposition.** *We have*

$$\mathcal{H}^2(G) \;\cong\; H^2(\widehat{A}, k^\times)^G \;\cong\; (\mathbb{Z}/p)^{(p-1)/2}$$

• **Proof.** The second isomorphism follows from a standard cohomological calculation

The first isomorphism is a consequence of the following interpretation of $\mathcal{B}(G)$

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The set $\mathcal{B}(G)$ as a colimit

- **Theorem.** *There is a bijection*

$$\mathcal{B}(G) \xrightarrow{\cong} \bigcup_A H^2(\widehat{A}, k^\times)^G$$

where the RHS is the colimit in the category whose objects are the abelian normal subgroups A of G and whose arrows are the inclusions

- **Corollary.** *If G has a unique maximal abelian normal subgroup A, then*

$$\mathcal{B}(G) \cong H^2(\widehat{A}, k^\times)^G$$

- **An example where Θ is not surjective:** Take $G = \mathbb{Z}/2 \wr \mathbb{Z}/2$, which is the dihedral group of order 8

  The map Θ is injective since $\mathrm{Out}_c(G) = 1$

  G having two maximal abelian normal subgroups, we obtain $|\mathcal{B}(G)| = 3$

  Proposition. $\mathcal{H}^2(G) = 1$

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The order of A_4 being even, we cannot conclude that Θ is surjective.

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- The group $G = \mathbb{Z}/8 \rtimes \mathrm{Aut}(\mathbb{Z}/8)$ of order 32 has the presentation

$$G = \langle s, t, u \mid s^2 = t^2 = u^8 = 1,\ st = ts,\ sus^{-1} = u^3,\ tut^{-1} = u^5 \rangle$$

- G. E. Wall (1947) proved that $\mathrm{Out}_c(G) = \mathbb{Z}/2$, generated by the automorphism α defined by

$$\alpha(s) = u^4 s = u^2 s u^{-2}, \quad \alpha(t) = u^4 t = utu^{-1}, \quad \alpha(u) = u$$

- The set $\mathcal{B}(G)$ has two elements:
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Danke für Ihre Aufmerksamkeit!

Appendix 1. Wall's group (continued)

- The automorphism α is of the form

$$\alpha(g) = aga^{-1}$$

for some invertible $a \in kG$. A computer search gave us

$$a = \frac{1}{2}(1 + u^4) + \frac{\sqrt{2}}{4}\, u\,(1 - u^2 - u^4 + u^6)$$

- Then

$$F = (a \otimes a)\,\Delta(a^{-1})$$

is a symmetric invariant twist representing the non-zero element of $\mathcal{H}_0 \subset \mathcal{H}^2(G)$

The Drinfeld twist F is a sum of 52 pure tensors:

$$\begin{aligned}
8F \;&= 2\,(u_{00} + u_{44}) + (u_{11} + u_{33} + u_{55} + u_{77})\\
&+ u_{01} + u_{03} + u_{04} + u_{05} + u_{07} + u_{12} + u_{17} + u_{25} + u_{35} + u_{36} + u_{67}\\
&+ u_{10} + u_{30} + u_{40} + u_{50} + u_{70} + u_{21} + u_{71} + u_{52} + u_{53} + u_{63} + u_{76}\\
&- (u_{13} + u_{14} + u_{15} + u_{16} + u_{23} + u_{27} + u_{34} + u_{37} + u_{45} + u_{47} + u_{56} + u_{57})\\
&- (u_{31} + u_{41} + u_{51} + u_{61} + u_{32} + u_{72} + u_{43} + u_{73} + u_{54} + u_{74} + u_{65} + u_{75})
\end{aligned}$$

where $u_{ij} = u^i \otimes u^j$ $(i, j \in \{0, 1, \ldots, 7\})$

Appendix 2. On the proof of Theorem 1

• **Theorem 1.** *If all irreducible $\bar{k}$-representations of G can be realized over k, then there is an exact sequence of groups*

$$1 \longrightarrow H^1(\mathrm{Gal}(\bar{k}/k), Z(G)) \longrightarrow \mathcal{H}^2(G/k) \longrightarrow \mathcal{H}^2(G/\bar{k}) \longrightarrow 1$$

• **Ingredients of the proof**

(a) Observe that Schauenburg cohomology is defined as a complex of algebraic groups

(b) Compute their tangent Lie algebras and show that the corresponding complex is acyclic

(c) Use Hilbert's Theorem 90 twice