From Christoffel words to braids

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What is it about?

- Construct words from simple planar geometry
- Words on two letters
  - Christoffel words: finite words
  - Sturmian sequences: infinite words
- In relation to
  - bases of the free group F_2 on two generators
  - the braid group B_4 on four strands

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I. HISTORICAL BACKGROUND

Elwin Bruno Christoffel (1829–1900)

- Worked on conformal mappings, geometry and tensor analysis (Christoffel symbols), theory of invariants, orthogonal polynomials, continued fractions, and applications to the theory of shock waves, to the dispersion of light.
- Held positions at Polytechnicum Zurich, TU Berlin, and…
- in Strasbourg
  After French-Prussian War in 1870, France lost Alsace-Lorraine to the German Empire
  The Prussians created a new university in Strasbourg
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- *Designantibus a, b numeros positivos integros et primos inter se, sint $r_1, r_2, r_3, \ldots$ residua minima non negativa numerorum a, $2a$, $3a, \ldots$ secundum modulum b,...*
- Denoting coprime positive integers a, b, let $r_1, r_2, r_3, \ldots$ be the minimal non-negative remainders of the numbers a, $2a$, $3a, \ldots$ modulo b,...
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Observatio arithmetica 2

- *. . . notatur littera c vel d, prout r_m crescit vel decrescit. Hoc modo nova series nascitur, e duabus tantum literis c, d, sed certo quodam ordine composita. . .*

- . . . write down the letter c or d according as r_m increases or decreases. In this way a new sequence arises, made of the two letters c, d arranged in a certain precise order. . .

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Observatio arithmetica 3

- *Exemplum I. Sit $a = 4$, $b = 11$, erit series $(r_.)$ notis c, d ornata*

$r_. =$	4	8	1	5	9	2	6	10	3	7	0
$g_. =$	c	d	c	c	d	c	c	d	c	d	c

- The integers a and b can be recovered from the sequence $(g_.)$:
  - a is the number of d's in $(g_.)$
  - b is the length of $(g_.)$
- Main result of the paper tells how to recover the continued fraction expansion of a/b from $(g_.)$

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Observatio arithmetica 4

Sequences of letters c et d thus obtained, e.g.,

$$cdccdccdcdc\,,$$

are called Christoffel words

Christoffel words (and infinite variants) come up in

- mathematics
  - ⋆ symbolic dynamics (Morse)
  - ⋆ continued fractions
- computer science
  - ⋆ formal language theory
  - ⋆ algorithms on words
  - ⋆ pattern recognition
- physics
  - ⋆ crystallography
- biology

II. A GEOMETRICAL CONSTRUCTION OF CHRISTOFFEL WORDS

Primitive vectors of $\mathbb{Z}^2$ and Christoffel words

- $\binom{p}{q} \in \mathbb{Z}^2$ is primitive if p, q are coprime
- To $\binom{p}{q}$ we shall attach a Christoffel word

$$w = w\begin{pmatrix} p \\ q \end{pmatrix}$$

  in the letters a, a^{-1}, b, b^{-1}

  - w represents an element of free group $F_2 = F(a, b)$
  - the image of w in $\mathbb{Z}^2$ is $\binom{p}{q}$

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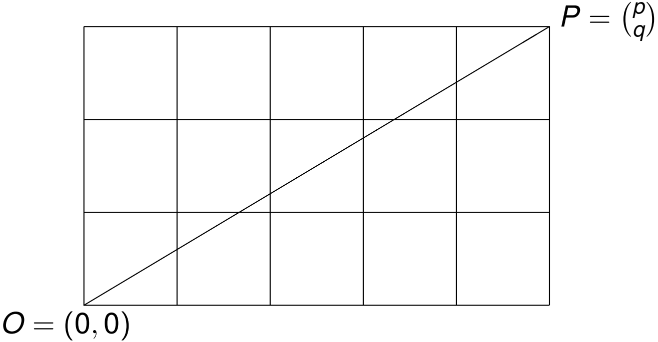
Planar representation of a primitive vector

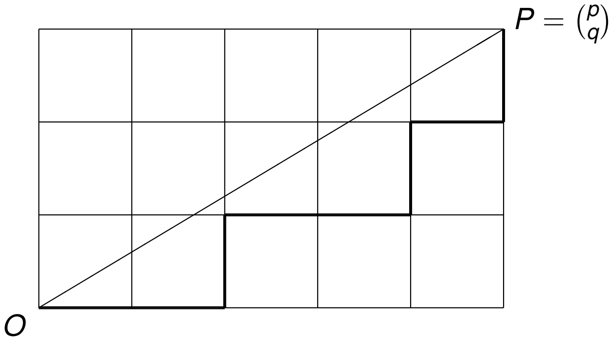
If $P = \binom{p}{q}$ is primitive, then $[OP] \bigcap \mathbb{Z}^2 = \{O, P\}$



Stair-case approximation

If p, $q \geq 0$, then approximate segment OP

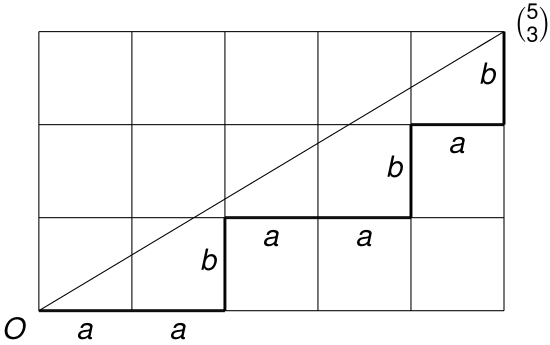
by closest stair-case path from beneath



Christoffel word of a positive primitive vector

To the vector $\binom{5}{3} \in \mathbb{N}^2$ we attach the word

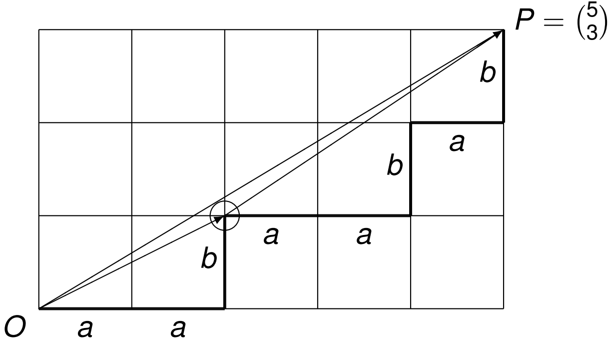
$$w\binom{5}{3} = aabaabab = a^2ba^2bab \in F_2$$



Christoffel words: factorization property 1

◯: closest point of $\mathbb{Z}^2$ to segment OP

$$w\binom{5}{3} = (a^2b)(a^2bab) = w\binom{2}{1}\, w\binom{3}{2}$$



Christoffel words: factorization property 2

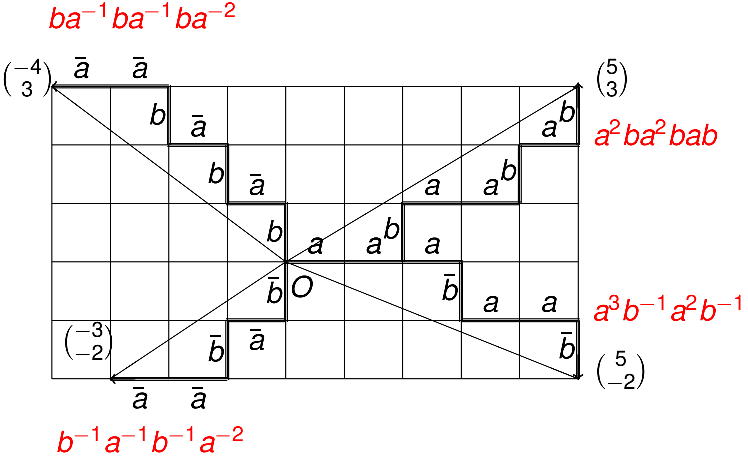
Theorem

If $\binom{p}{q}, \binom{r}{s} \in \mathbb{N}^2$ *satisfy* $\begin{vmatrix} p & r \\ q & s \end{vmatrix} = 1$, *then*

$$w\begin{pmatrix} p+r \\ q+s \end{pmatrix} = w\begin{pmatrix} p \\ q \end{pmatrix} w\begin{pmatrix} r \\ s \end{pmatrix}$$

General Christoffel words

$$\bar{a} = a^{-1}$$
$$\bar{b} = b^{-1}$$



Bases of the free group F_2

Let $F_2 = F(a, b)$ be the free group on two generators a, b

- $\{u, v\} \subset F_2$ is a *basis* if $\{u, v\}$ generates F_2

  Equivalently, $\{u, v\}$ is a basis if there is $\varphi \in \mathrm{Aut}(F_2)$ with

$$\varphi(a) = u \text{ and } \varphi(b) = v$$

- Two bases $\{u, v\}$ and $\{u', v'\}$ of F_2 are *conjugate* if there is $w \in F_2$ such that $u' = wuw^{-1}$ and $v' = wvw^{-1}$

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Lifting bases of $\mathbb{Z}^2$ to F_2

- *Recall:* $\{\binom{p}{q}, \binom{r}{s}\}$ is a basis of $\mathbb{Z}^2 \iff ps - qr = \pm 1$
- *Question:*

  Can we lift $\{\binom{p}{q}, \binom{r}{s}\}$ to a basis of F_2?
- *Example:*
  - $a^2b \in F_2$ is a lift of $\binom{2}{1}$
  - a^3b^2 is a lift of $\binom{3}{2}$
  - $\{a^2b, a^3b^2\}$ is not a basis of F_2,
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Christoffel bases of F_2

Theorem

- *If* $\left\{ \binom{p}{q}, \binom{r}{s} \right\}$ *is a basis of* $\mathbb{Z}^2$*, then*

$$\left\{ w\begin{pmatrix} p \\ q \end{pmatrix}, w\begin{pmatrix} r \\ s \end{pmatrix} \right\}$$

  is a basis of F_2 *(called a Christoffel basis). It lifts the basis* $\left\{ \binom{p}{q}, \binom{r}{s} \right\}$ *to* F_2

- *Any basis of* F_2 *is conjugate of a (unique) Christoffel basis*

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- *Any basis of* F_2 *is conjugate of a (unique) Christoffel basis*

Palindromes

- The *reverse* of a word w is the word $\widetilde{w}$ obtained by reading w from right to left
- A *palindrome* is a word w such that $\widetilde{w} = w$
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Existence of palindromic bases

Let $|w|$ be the length of a word w with respect to the alphabet $\{a, b, a^{-1}, b^{-1}\}$

Theorem

Any basis $\{u, v\}$ of F_2 with $|u|$, $|v|$ odd is the conjugate of a unique (cyclically reduced) palindromic basis

Example:

- $\{aba^2b, a^2b\}$ is a non-palindromic basis of F_2,
- but is conjugated to the palindromic basis $\{ababa, aba\}$

III. STURMIAN SEQUENCES AND MORPHISMS

Infinite version of Christoffel words

- Let $L \subset \mathbb{R}^2_+$ be a half-line originating from O and satisfying $L \bigcap \mathbb{Z}^2 = \{O\}$

  The slope of L is irrational

- The infinite sequence in a, b encoding the stair-case approximation of L is a Sturmian sequence

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Sturmian sequences: formal definition

- Infinite word in a, b: map $\{0, 1, 2, \ldots\} \to \{a, b\}$
  Example: $abaababaabaab \cdots$

- Sturmian sequence: infinite word w in a, b such that the number of distinct factors of w of length n is $n + 1$ for each $n \geq 1$ (w_1 is a factor of $w = w_0 w_1 w_2$)
  - $n = 1$: Number of distinct letters is 2
  - $n = 2$: Three possible length-two factors out of four:

$$aa, \quad ab, \quad ba, \quad bb$$

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The Fibonacci sequence

- Obtain Sturmian sequences dynamically by iterating an appropriate substitution $\varphi : \{a, b\}^* \to \{a, b\}^*$
- *Example*: Let $\varphi(a) = ab$ and $\varphi(b) = a$. Then $w_n = \varphi^n(a)$ converges to a Sturmian sequence w
  - $w_0 = a$
  - $w_1 = ab$
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- A substitution replaces a et b by (positive) words in a and b (i.e., it is an endomorphism of free monoid $\{a, b\}^*$)

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Examples of Sturmian morphisms

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  We have $E^2 = \mathrm{id}$

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The Sturmian monoid as a submonoid of Aut(F_2)

- Mignosi & Séébold (1993): The monoid St is generated by $\{E, L_a, R_a\}$ or by $\{E, L_b, R_b\}$
- Considered as endomorphisms of the free group F_2, the generators E, L_a, R_a of St are invertible in End(F_2)
- Therefore, St is a submonoid of the group Aut(F_2)

$$\mathrm{St} \subset \mathrm{Aut}(F_2)$$

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Projecting onto $GL_2(\mathbb{Z})$

Linearization map: $\pi : \mathrm{Aut}(F_2) \to \mathrm{Aut}(\mathbb{Z}^2) = GL_2(\mathbb{Z})$

- $$\pi(E) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$
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Presentation of $SL_2(\mathbb{Z})$

- Generators: $A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$
- Defining relations:
  - Braid relation: $AB^{-1}A = B^{-1}AB^{-1}$
  - Torsion relation: $(AB^{-1}A)^4 = 1$
- Question: *Can we lift these relations to* $Aut(F_2)$ *using the lifts* L_a, R_a *of* A *and the lifts* L_b, R_b *of* B*?*
- Answer: *Yes,* see next slide

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IV. BRAIDS

The braid group B_n on n strands

Presentation of B_n:

Generators:

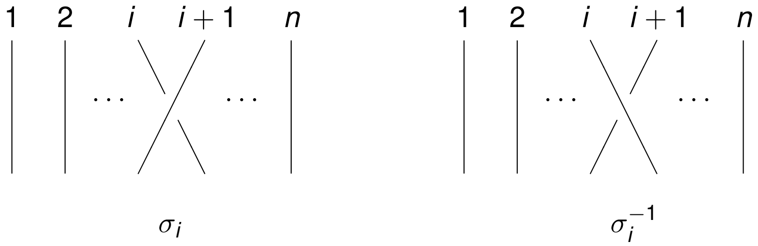
$$\sigma_1, \ldots, \sigma_{n-1}$$

Defining relations:

$$\sigma_i \sigma_j = \sigma_j \sigma_i \qquad \text{if } |i-j| \geq 2$$

$$\sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1}$$

Generators of the braid group on n strands



Relation between B_4 and $\mathrm{Aut}(F_2)$

- Group homomorphism $f : B_4 \to \mathrm{Aut}(F_2)$ defined by

$$f(\sigma_1) = L_a\,, \quad f(\sigma_2) = L_b^{-1}\,, \quad f(\sigma_3) = R_a\,.$$

- **Theorem.** *The following sequence is exact:*

$$1 \longrightarrow Z_4 \longrightarrow B_4 \xrightarrow{f} \mathrm{Aut}(F_2) \longrightarrow \mathbb{Z}/2 \longrightarrow 0$$

• Z_4 = center of B_4, infinite cyclic generated by $(\sigma_1\sigma_2\sigma_3)^4$

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The complete picture

Theorem. *There is a map of exact sequences*

$$\begin{array}{ccccccccccc} 1 & \longrightarrow & Z_4 & \longrightarrow & B_4 & \xrightarrow{f} & \mathrm{Aut}(F_2) & \xrightarrow{\det} & \mathbb{Z}/2 & \longrightarrow & 1 \\ & & \cong\downarrow & & \pi'\downarrow & & \pi\downarrow & & =\downarrow & & \\ 1 & \longrightarrow & 2Z_3 & \longrightarrow & B_3 & \longrightarrow & GL_2(\mathbb{Z}) & \xrightarrow{\det} & \mathbb{Z}/2 & \longrightarrow & 1 \end{array}$$

- $B_3 = \langle A, B \mid AB^{-1}A = B^{-1}AB^{-1} \rangle$
- $2Z_3$ subgroup of center of B_3, generated by $(AB^{-1}A)^4$
- $\pi' : B_4 \to B_3$ defined by $\pi'(\sigma_1) = \pi'(\sigma_3) = A$ and $\pi'(\sigma_2) = B^{-1}$

- Z_4 = center of B_4, generated by $(\sigma_1\sigma_2\sigma_3)^4$

The special Sturmian monoid St_0

- Recall: Monoid St generated by E, L_a, R_a or by E, L_b, R_b
- **Definition**

$$\mathrm{St}_0 = \{\varphi \in \mathrm{St} \mid \det(\varphi) = 1\} \subset \mathrm{Aut}(F_2)$$

- The substitutions L_a, R_a, L_b, R_b belong to St_0
- The monoid St_0 is generated by L_a, R_a, L_b, R_b

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Presentation of the monoid St_0

Theorem. *The monoid St_0 has the following presentation:*

Generators: L_a, R_a, L_b, R_b

Defining relations: $L_a\,R_a = R_a\,L_a\,, \quad L_b\,R_b = R_b\,L_b\,,$
and

$$L_a\,L_b^k\,R_a = R_a\,R_b^k\,L_a\,, \quad L_b\,L_a^k\,R_b = R_b\,R_a^k\,L_b$$

for all $k \geq 1$.

(This presentation is infinite)

St_0 is a submonoid of B_4

The monoid $\mathrm{St}_0 \subset \mathrm{Aut}(F_2)$ can be lifted to a monoid in B_4

- **Theorem.** *There is a monoid morphism $i : St_0 \to B_4$ such that*

  $$\mathrm{St}_0 \xrightarrow{i} B_4 \xrightarrow{f} \mathrm{Aut}(F_2)$$

  is the inclusion.
- The monoid embedding $i : \mathrm{St}_0 \to B_4$ is defined by

  $$i(L_a) = \sigma_1\,, \quad i(L_b) = \sigma_2^{-1}\,, \quad i(R_a) = \sigma_3\,, \quad i(R_b) = \sigma_4^{-1}\,.$$
- What is the braid σ_4^{-1}?

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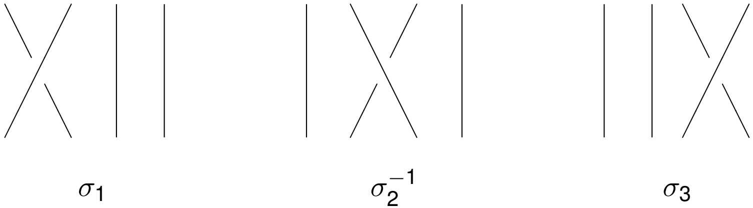
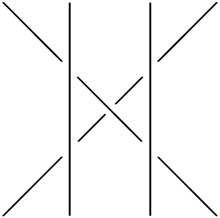
- What is the braid σ_4^{-1}?

The braids σ_1, σ_2^{-1}, σ_3



The braid σ_4^{-1}

It braids the 1st and the 4th strands with a negative crossing behind the 2nd and 3rd strands



$$\sigma_4^{-1} = (\sigma_1 \sigma_3^{-1})\, \sigma_2\, (\sigma_1 \sigma_3^{-1})^{-1}$$

Conclusion

- The monoid generated by σ_1, σ_2^{-1}, σ_3, σ_4^{-1} in B_4 is isomorphic to the special Sturmian monoid St_0

$$\langle \sigma_1 , \sigma_2^{-1} , \sigma_3 , \sigma_4^{-1} \rangle^+ \cong \mathrm{St}_0$$

- Note: The subgroup generated by σ_1, σ_2^{-1}, σ_3, σ_4^{-1} is B_4

Conclusion

- The monoid generated by $\sigma_1, \sigma_2^{-1}, \sigma_3, \sigma_4^{-1}$ in B_4 is isomorphic to the special Sturmian monoid St_0

$$\langle \sigma_1, \sigma_2^{-1}, \sigma_3, \sigma_4^{-1} \rangle^+ \cong \mathrm{St}_0$$

- Note: The subgroup generated by $\sigma_1, \sigma_2^{-1}, \sigma_3, \sigma_4^{-1}$ is B_4

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THANK YOU FOR YOUR ATTENTION