Any Hopf algebra fibers over an affine variety of the same dimension

Christian Kassel

Institut de Recherche Mathématique Avancée
CNRS - Université de Strasbourg
Strasbourg, France

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Introduction 1

This lecture is based on joint work with Eli Aljadeff (Technion, Haifa) and with Akira Masuoka (University of Tsukuba)

• E. ALJADEFF, C. KASSEL, *Polynomial identities and noncommutative versal torsors*, Adv. Math. 218 (2008), 1453–1495

• C. KASSEL, A. MASUOKA, *Flatness and freeness properties of the generic Hopf Galois extensions*, Rev. Un. Mat. Argentina 51:1 (2010), 79–94
(see also arXiv:0911.3719)

Introduction 2

Fix a field k and let H be a Hopf algebra over k

• Aljadeff and I constructed an H-Galois extension $\mathcal{B}_H \subset \mathcal{A}_H$ whose algebra of coinvariants $\mathcal{B}_H$ is central in $\mathcal{A}_H$

• This Galois extension possesses the following properties:

- The commutative algebra $\mathcal{B}_H$ is a domain of Krull dimension $\leq \dim_k H$
- There is a maximal ideal $\mathfrak{m}_0$ of $\mathcal{B}_H$ such that

$$\mathcal{A}_H/\mathfrak{m}_0\mathcal{A}_H \cong H$$

  as H-comodule algebras
- Under some Condition (FP) [see below], for any maximal ideal $\mathfrak{m}$ of $\mathcal{B}_H$, the H-comodule algebra $\mathcal{A}_H/\mathfrak{m}\mathcal{A}_H$ is a form of H, meaning that $\mathcal{A}_H/\mathfrak{m}\mathcal{A}_H$ and H become isomorphic after extension of scalars to the algebraic closure of the field $\mathcal{B}_H/\mathfrak{m}$
- Conversely, any form of the H-comodule algebra H can be obtained in this way

Thus, $\mathcal{B}_H \subset \mathcal{A}_H$ is a (weak) moduli space for forms of H

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Summary of Part I

• In geometric terms, $\mathcal{B}_H \subset \mathcal{A}_H$ is a quantum principal fiber bundle with "structural group" H over the domain $\mathcal{B}_H$

Under Condition (FP) the fibers of this bundle are forms of H, and all forms of H are isomorphic to such fibers

• In Part I

- ∗ we recall how to construct the domain $\mathcal{B}_H$
- ∗ we state Condition (FP)
- ∗ we give examples for which the algebra $\mathcal{B}_H$ has been determined
- ∗ we state a simple open problem

Summary of Part II

• Masuoka and I determined classes of Hopf algebras for which Condition (FP) is verified

• In Part II we show that Condition (FP) is verified for all

- ∗ finite-dimensional Hopf algebras
- ∗ commutative Hopf algebras
- ∗ cocommutative Hopf algebras
- ∗ pointed Hopf algebras whose group-like elements are of finite order

Summary of Part III

• There is a version of the previous constructions starting from a Hopf algebra H together with a two-cocycle

$$\alpha : H \times H \to k$$

In an analogous way we obtain a (weak) moduli space for the forms of any given cleft Galois object of H

• In Part III we show how to reduce this more general case to the one considered in Parts I & II

Plan

I. The algebra $\mathcal{B}_H$ and Condition (FP)

II. Joint results with Masuoka

III. Twisting with a cocycle

References

Plan

A construction by Takeuchi (1971)

• Let C be a coalgebra with coproduct $\Delta : C \to C \otimes C$ and counit $\varepsilon : C \to k$

Let t_C be a copy of the underlying vector space of C and let $x \mapsto t_x$ denote the identity map from C to t_C

• Consider the symmetric algebra $\mathrm{Sym}(t_C)$ over the vector space t_C: if $\{x_i\}_{i \in I}$ is a linear basis of C, then $\mathrm{Sym}(t_C)$ is a polynomial algebra over the symbols $\{t_{x_i}\}_{i \in I}$

• There is a unique linear map $x \mapsto t_x^{-1}$ from C to the fraction field $\mathrm{Frac}\,\mathrm{Sym}(t_C)$ of $\mathrm{Sym}(t_C)$ such that for all $x \in C$,

$$\sum_{(x)} t_{x_1}\, t_{x_2}^{-1} = \varepsilon(x)\, 1 = \sum_{(x)} t_{x_1}^{-1}\, t_{x_2}$$

where $\Delta(x) = \sum_{(x)} x_1 \otimes x_2$ (Sweedler's notation)

• **Definition.** Let $\mathcal{S}_C$ be the subalgebra of $\mathrm{Frac}\,\mathrm{Sym}(t_C)$ generated by t_x and t_x^{-1}, $(x \in C)$:

$$\mathcal{S}_C = \mathrm{Sym}(t_C)[t_x^{-1} \,|\, x \in C]$$

The algebra $\mathcal{B}_H$ and Condition (FP)

Now let $C = H$ be a Hopf algebra

• For $x, y \in H$ consider the following elements of $\mathcal{S}_H$:

$$\sigma(x,y) = \sum_{(x)(y)} t_{x_1}\, t_{y_1}\, t^{-1}_{x_2 y_2} \quad \text{and} \quad \sigma^{-1}(x,y) = \sum_{(x)(y)} t_{x_1 y_1}\, t^{-1}_{x_2}\, t^{-1}_{y_2}$$

• **Definition (AK, 2008).** Let $\mathcal{B}_H$ be the subalgebra of $\mathcal{S}_H$ generated by all elements $\sigma(x,y)$ and $\sigma^{-1}(x,y)$

• Since $\mathcal{B}_H \subset \mathcal{S}_C \subset \operatorname{Frac}\operatorname{Sym}(t_H)$, the commutative algebra $\mathcal{B}_H$ is a domain and its Krull dimension does not exceed $\dim_k H$

• We now state the above-mentioned Condition (FP):

Condition (FP): As a $\mathcal{B}_H$-module, $\mathcal{S}_H$ is faithfully flat

This means that a sequence $0 \to M' \to M \to M'' \to 0$ of $\mathcal{B}_H$-modules is exact if and only if the induced sequence

$$0 \to \mathcal{S}_H \otimes_{\mathcal{B}_H} M' \to \mathcal{S}_H \otimes_{\mathcal{B}_H} M \to \mathcal{S}_H \otimes_{\mathcal{B}_H} M'' \to 0$$

of $\mathcal{S}_H$-modules is exact

First example: group algebras

• The algebra $H = kG$ of a group G is a Hopf algebra with

$$\Delta(g) = g \otimes g \quad \text{and} \quad \varepsilon(g) = 1 \qquad (g \in G)$$

• In this case, $t_g^{-1} = 1/t_g$ for all $g \in G$ and

$$\mathcal{S}_H = k[t_g, t_g^{-1} \,|\, g \in G]$$

is an algebra of Laurent polynomials

• The algebra $\mathcal{B}_H$ is the subalgebra generated by the elements

$$\sigma(g,h) = \frac{t_g t_h}{t_{gh}} \quad \text{and} \quad \sigma^{-1}(g,h) = \frac{t_{gh}}{t_g t_h} \qquad (g, h \in G)$$

• From this description of $\mathcal{S}_H$ and $\mathcal{B}_H$, it is easy to check that when G is finite, then the algebra $\mathcal{S}_H$ is integral over $\mathcal{B}_H$, and Condition (FP) is satisfied

[We shall see later that Condition (FP) is satisfied for any group algebra and more generally for any cocommutative Hopf algebra]

Special cases

- Let $G = \mathbb{Z}$ be the group of integers. If $y_m = t_m/t_1^m$ for each $m \in \mathbb{Z}$, then

$$\mathcal{B}_H = k[y_m, y_m^{-1} \,|\, m \in \mathbb{Z} - \{1\}]$$

is an algebra of Laurent polynomials and

$$\mathcal{S}_H = \mathcal{B}_H[t_1, t^{-1}]$$

As a $\mathcal{B}_H$-module, $\mathcal{S}_H$ is free and Condition (FP) is satisfied

- If $G = \mathbb{Z}/N$ is a cyclic group of order $N \geq 2$, then

$$\mathcal{B}_H = k[y_0^{\pm 1}, y_2^{\pm 1}, \ldots, y_{N-1}^{\pm 1}, y_N^{\pm 1}]$$

where $y_0, , y_2, \ldots, y_{N-1}$ are as above and $y_N = t_0/t_1^N$, and

$$\mathcal{S}_H = \mathcal{B}_H[t_1]/(t_1^N - y_0/y_N)$$

is an integral extension of $\mathcal{B}_H$. Condition (FP) is satisfied

Second example: the Sweedler algebra

- The Sweedler algebra is the Hopf algebra

$$H = k\langle x, y \mid x^2 = 1,\ y^2 = 0,\ yx = -xy\rangle$$

with coproduct

$$\Delta(x) = x \otimes x \quad \text{and} \quad \Delta(y) = 1 \otimes y + y \otimes x$$

It is four-dimensional with basis $\{1, x, y, z\}$, where $z = xy$

- In this case,

$$\mathcal{S}_H = k[t_1, t_1^{-1}, t_x, t_x^{-1}, t_y, t_z]$$

and $\mathcal{B}_H$ is generated by the elements $e, e^{-1}, a, a^{-1}, b, c, d$ where

$$e = t_1, \quad a = t_x^2, \quad b = 2t_x t_y, \quad c = t_y^2, \quad d = t_z$$

- The algebra $\mathcal{B}_H$ has the following presentation (AK, 2008):

$$\mathcal{B}_H \cong k[e^{\pm 1}, a^{\pm 1}, b, c, d]/(b^2 - 4ac)$$

The algebra $\mathcal{S}_H$ is integral over $\mathcal{B}_H$ and Condition (FP) is satisfied

An open question

The reader is encouraged to solve the open problem below

• Let n be an integer ≥ 2. The Taft algebra H_{n^2} is the algebra

$$H_{n^2} = k\langle x, y \mid x^n = 1,\ y^n = 0,\ yx = qxy\rangle$$

where q is a root of unity of order n. The Taft algebra H_4 is Sweedler's algebra

It has a Hopf algebra structure with coproduct

$$\Delta(x) = x \otimes x \quad \text{and} \quad \Delta(y) = 1 \otimes y + y \otimes x$$

As a vector space, it is of dimension n^2

• **Problem:** Give a presentation by generators and relations of the algebra $\mathcal{B}_H$ when $H = H_{n^2}$

Plan

Generators of the $\mathcal{B}_H$-module $\mathcal{S}_H$

We start with an observation

Lemma 1. *As a $\mathcal{B}_H$-module, $\mathcal{S}_H$ is generated by the elements t_x ($x \in H$)*

Proof.

• Let M be the $\mathcal{B}_H$-submodule generated by the elements t_x ($x \in H$). The relation $\sigma(x, y) = \sum_{(x)(y)} t_{x_1}\, t_{y_1}\, t^{-1}_{x_2 y_2}$ implies

$$t_x t_y = \sum_{(x)(y)} \sigma(x_1, y_1)\, t_{x_2 y_2}$$

Hence, $t_x t_y \in M$ for all $x, y \in H$

By an easy induction, any finite product of elements t_x belong to M

• To prove that each t_x^{-1} belongs to M, we use the antipode S of H and the identities

$$t_1^{-1} = \sigma^{-1}(1, 1) \in \mathcal{B}_H$$

and

$$t_x^{-1} = \sum_{(x)} t_{S(x_1)}\, \sigma^{-1}(S(x_2), x_3)\, t_1^{-1}$$

A Hopf algebra structure on $\mathcal{S}_H$

Takeuchi (1971): The algebra $\mathcal{S}_H$ is a commutative Hopf algebra with coproduct Δ, counit ε, and (involutive) antipode S given by

$$\Delta(t_x) = \sum_{(x)} t_{x_1} \otimes t_{x_2} \qquad \text{and} \qquad \Delta(t_x^{-1}) = \sum_{(x)} t_{x_2}^{-1} \otimes t_{x_1}^{-1}$$

$$\varepsilon(t_x) = \varepsilon(t_x^{-1}) = \varepsilon(x)$$

$$S(t_x) = t_x^{-1} \quad \text{and} \quad S(t_x^{-1}) = t_x$$

A quotient of $\mathcal{S}_H$

• Inside $\mathcal{S}_H$ consider the ideal $(\mathcal{B}_H^+)$ generated by $\mathcal{B}_H \cap \ker(\varepsilon)$. It is easy to prove the following

Proposition 1. *There is an isomorphism of commutative Hopf algebras*

$$\mathcal{S}_H/(\mathcal{B}_H^+) \cong H_{\text{ab}}$$

where H_{ab} is the largest commutative quotient of H

As a consequence, $\mathcal{S}_H$ is an H_{ab}-comodule algebra

• Using a result by Takeuchi (1979), we obtain the following

Proposition 2. *If Condition (FP) is satisfied, then $\mathcal{B}_H$ is the subalgebra of H_{ab}-coinvariants of $\mathcal{S}_H$:*

$$\mathcal{B}_H = (\mathcal{S}_H)^{\text{co}-H_{\text{ab}}}$$

This result allows to identity $\mathcal{B}_H$ inside $\mathcal{S}_H$ (see examples below)

Flatness

Now comes a first step towards Condition (FP)

• **Theorem 1.** *For any Hopf algebra H, the $\mathcal{B}_H$-module $\mathcal{S}_H$ is flat*

By Masuoka and Wigner (1994), any commutative Hopf algebra is flat over any left coideal subalgebra. We apply their result to the following proposition.

• **Proposition 3.** *For any Hopf algebra H, the algebra $\mathcal{B}_H$ is a left coideal subalgebra of the commutative Hopf algebra $\mathcal{S}_H$*

Proof. This is a consequence of the following identities:

$$\Delta(\sigma(x,y)) = t_{x_1}\, t_{y_1}\, t^{-1}_{x_3 y_3} \otimes \sigma(x_2, y_2)$$

and

$$\Delta(\sigma^{-1}(x,y)) = t_{x_1 y_1}\, t^{-1}_{x_3}\, t^{-1}_{y_3} \otimes \sigma^{-1}(x_2, y_2)$$

Faithful flatness

We next list examples of Hopf algebras satisfying Condition (FP)

• **Theorem 2.** *For any cocommutative Hopf algebra H, the $\mathcal{B}_H$-module $\mathcal{S}_H$ is faithfully flat*

Proof. If H is cocommutative, then so is $\mathcal{S}_H$. In this case the identities in the proof of Proposition 3 can be rewritten as

$$\Delta(\sigma(x,y)) = \sigma(x_1,y_1) \otimes \sigma(x_2,y_2)$$

and

$$\Delta(\sigma^{-1}(x,y)) = \sigma^{-1}(x_1,y_1) \otimes \sigma^{-1}(x_2,y_2)$$

Consequently, $\mathcal{B}_H$ is a Hopf subalgebra of $\mathcal{S}_H$. We conclude by following Takeuchi who observed that any cocommutative Hopf algebra is faithfully flat over any Hopf subalgebra.

Finite-dimensional Hopf algebras

Condition (FP) is satisfied for all finite-dimensional Hopf algebras as a consequence of the following

• **Theorem 3.** *If H is a finite-dimensional Hopf algebra, then the $\mathcal{B}_H$-module $\mathcal{S}_H$ is finitely generated projective*

• **Corollary 1.** *If H is a finite-dimensional Hopf algebra, then the Krull dimension of $\mathcal{B}_H$ is equal to* $\dim_K H$

In other words, $\mathcal{B}_H \subset \mathcal{A}_H$ represents a deformation of H as an H-comodule algebra over an affine algebraic variety that has the same dimension as H

• **Recall from above:** If H is Sweedler's four-dimensional algebra, then the spectrum of $\mathcal{B}_H$ is a quadric of dimension 4

Freeness

The following implies that Condition (FP) is satisfied for the Hopf algebras listed below (no condition is assumed on the dimension)

- **Theorem 4.** *The $\mathcal{B}_H$-module $\mathcal{S}_H$ is free in each of the following cases:*

  (i) *H is commutative*

  (ii) *H is pointed and cocommutative*

  (iii) *H is pointed and the natural embedding $kG(H) \to H$ splits as an algebra map*

  (iv) *H is pointed and each element of the kernel of the homomorphism*

$$G(H)_{\mathrm{ab}} \to G(H_{\mathrm{ab}})$$

  is of finite order

Here $G(H)$ is the group of group-like elements of H:

$$G(H) = \{g \in H \,|\, \Delta(g) = g \otimes g \ \text{ and } \ \varepsilon(g) = 1\}$$

and G_{ab} is the largest abelian quotient of G

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Group algebras again

• Consider the Hopf algebra $H = kG$, where G is a group. We have

$$H_{ab} = kG_{ab}$$

• By Proposition 1, the Laurent polynomial algebra $\mathcal{S}_H$ is a kG_{ab}-comodule algebra. This means that the algebra $\mathcal{S}_H$ is graded by the abelian group G_{ab}

Explicitly, if $\bar{g}$ is the image of $g \in G$ in G_{ab}, then the generator $t_g^{\pm 1}$ of

$$\mathcal{S}_H = k[t_g, t_g^{-1} \,|\, g \in G]$$

is of degree $\pm\bar{g}$ (we write the composition law of G_{ab} additively)

• Since the Hopf algebra $H = kG$ is cocommutative, Condition (FP) is satisfied by Theorem 2, and by Proposition 2 the subalgebra $\mathcal{B}_H$ is the degree 0 summand of $\mathcal{S}_H$

In other words, $\mathcal{B}_H$ is spanned by the monomials $t_{g_1}^{\varepsilon_1} \cdots t_{g_k}^{\varepsilon_k}$ such that

$$\varepsilon_1 \bar{t}_{g_1} + \cdots + \varepsilon_k \bar{t}_{g_k} = 0 \ \text{ in } G_{ab}$$

Function algebras

• Let G be a finite group and $H = (kG)^*$ be the dual of the group algebra. It is a commutative Hopf algebra whose elements can be identified with scalar-valued functions on G. In this case,

$$\mathcal{S}_H = k[t_g \,|\, g \in G] \left[\frac{1}{\Theta_G} \right]$$

where Θ_G is the Dedekind determinant of G:

$$\Theta_G = \det \left(t_{gh^{-1}} \right)_{g,h \in G}$$

• By Proposition 1, the algebra $\mathcal{S}_H$ is a H-comodule algebra, which means that it is an algebra with a G-action. Explicitly, $g \in G$ acts on t_h by

$$g \cdot t_h = t_{gh}$$

• Condition (FP) is satisfied by Theorem 3, and by Proposition 2 the subalgebra $\mathcal{B}_H$ is the subalgebra of G-invariant elements of $\mathcal{S}_H$:

$$\mathcal{B}_H = (\mathcal{S}_H)^G = k[t_g \,|\, g \in G]^G \left[\frac{1}{\Theta_G^2} \right]$$

Plan

Twisted comodule algebras

Let H be a Hopf algebra

- A two-cocycle α on H is a bilinear form $\alpha : H \times H \to k$ such that

$$\sum_{(x)(y)} \alpha(x_1, y_1)\, \alpha(x_2 y_2, z) = \sum_{(y)(z)} \alpha(y_1, z_1)\, \alpha(x, y_2 z_2) \qquad (x, y, z \in H)$$

We assume that α is convolution-invertible and denote its inverse by α^{-1}

- Consider the "twisted" H-comodule algebra ${}^{\alpha}H$ defined as follows:
  - (a) as a right H-comodule, ${}^{\alpha}H = H$
  - (b) as an algebra it is equipped with the associative product

$$x \cdot_{\alpha} y = \sum_{(x)(y)} \alpha(x_1, y_1)\, x_2 y_2$$

All cleft Galois objects of H are of the form ${}^{\alpha}H$

- If α is the trivial cocycle $\alpha(x, y) = \varepsilon(x)\varepsilon(y)$, then ${}^{\alpha}H = H$

The algebra $\mathcal{B}_H^\alpha$

Let H be a Hopf algebra and α a two-cocycle on H

- For $x, y \in H$ consider the following elements of $\mathcal{S}_H$:

$$\sigma_\alpha(x, y) = \sum_{(x)(y)} t_{x_1}\, t_{y_1}\, \alpha(x_2, y_2)\, t^{-1}_{x_3 y_3}$$

and

$$\sigma_\alpha^{-1}(x, y) = \sum_{(x)(y)} t_{x_1 y_1}\, \alpha^{-1}(x_2, y_2)\, t^{-1}_{x_3}\, t^{-1}_{y_3}$$

- **Definition.** Let $\mathcal{B}_H^\alpha$ be the subalgebra of $\mathcal{S}_H$ generated by all elements $\sigma_\alpha(x, y)$ and $\sigma_\alpha^{-1}(x, y)$

- **Remark.** If H is cocommutative or more generally if α is lazy, i.e.

$$\sum_{(x)(y)} \alpha(x_1, y_1)\, x_2 y_2 = \sum_{(x)(y)} \alpha(x_2, y_2)\, x_1 y_1$$

for all $x, y \in H$, then $\mathcal{B}_H^\alpha = \mathcal{B}_H$

Reduction to the trivial cocycle

Let H be a Hopf algebra and α a two-cocycle on H

- Consider the Hopf algebra $L = {}^{\alpha}H^{\alpha^{-1}}$ defined as follows:
  (a) as a coalgebra, $L = H$
  (b) as an algebra, it is equipped with the associative product

$$x * y = \sum_{(x)(y)} \alpha(x_1, y_1)\, x_2 y_2\, \alpha^{-1}(x_3, y_3)\,.$$

- **Proposition.** *We have $\mathcal{B}_H^{\alpha} = \mathcal{B}_L$ inside $S(t_H)_{\ominus}$*

- **Remark.** If α and β are cohomologous two-cocycles, then

$$\mathcal{B}_H^{\alpha} \cong \mathcal{B}_H^{\beta}$$

Hence by the remark on the previous slide, if a two-cocycle α is cohomologous to a lazy two-cocycle, then

$$\mathcal{B}_H^{\alpha} \cong \mathcal{B}_H$$

Note that all two-cocycles on a Taft algebra (on the Sweedler algebra) are cohomologous to lazy two-cocycles

Plan

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THANK YOU FOR YOUR ATTENTION