A derivation of Abel's name by Postnikov

$$\begin{array}{ccc}
\textbf{Abel} & \longrightarrow & \textbf{Abel} \\
\text{son of Adam and Eva} & & \\
\downarrow & & \\
\textbf{Brother Cain} & \longrightarrow & \textbf{Cainian group}
\end{array}$$

Abelian group: $G = G^{\mathrm{ab}}$; derived subgroup $G' = \{1\}$

Cainian group: $G^{\mathrm{ab}} = \{1\}$; derived subgroup $G' = G$ (= perfect group)

Reference: М. М. Постников, Лекции по геометрии - Группы и алгебры Ли, Изд. Наука, Москва 1982 (French translation: Leçons de géométrie - Groupes et algèbres de Lie, Editions Mir, Moscou 1985).

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Action of the absolute Galois group of the rationals on knot invariants

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C. Kassel, V. Turaev, *Chord diagram invariants of tangles and graphs*, Duke Math. J. 92 (1998), 497–552.

Quantum group $\leftrightarrow$ group in non-commutative geom.
depending on a parameter h,
deformation of alg. of functions
on a group (in usual sense)

Quantum groups have applications in low-dimensional topology: - invariants of knots
- invariants of 3-manifolds
- TQFT's

The concept of a braided category (Joyal-Street) explains this relationship:

(i) Representations of a quantum group form a braided category

(ii) Free braided categories are made of geometric braids (loops in the configuration space of points in the plane)

Braided category (Joyal-Street, 1985): Category $\mathcal{C}$ with associative, commutative, unital functor

$$\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C} \quad \text{(tensor product)}$$

Associative: family of natural isomorphisms

$$a_{U,V,W} : (U \otimes V) \otimes W \xrightarrow{\cong} U \otimes (V \otimes W)$$

Commutative: family of natural isomorphisms

$$c_{U,V} : U \otimes V \xrightarrow{\cong} V \otimes U \quad \text{(braiding)}$$

Unital: $\exists$ an object I with natural isomorphisms

$$U \otimes I \xrightarrow{\cong} U \xrightarrow{\cong} I \otimes U$$

subject to coherence conditions (such as Mac Lane's pentagon).

Braided cat. is *symmetric* if braiding is involutive:

$$c_{V,U} \circ c_{U,V} = \mathrm{id}_{U \otimes V}$$

Examples

(a) Category of vector spaces with usual tensor product and braiding $\sigma_{U,V}$ given by

$$\sigma_{U,V}(u \otimes v) = v \otimes u \quad \text{(flip)}$$

This category is symmetric. So are

- categories of representations of groups
- categories of representations of Lie algebras

(b) Categories of representations of quantum groups are braided with non-involutive braidings such that

$$c_{U,V} = \sigma_{U,V} + O(h)$$

(c) (Joyal-Street) The free braided category $\mathcal{B}$ on one object X has objects $I, X, X^{\otimes 2}, X^{\otimes 3}, \ldots$ and morphisms

$$\mathrm{Hom}_{\mathcal{B}}(X^{\otimes n}, X^{\otimes m}) = \begin{cases} B_n & \text{if } n = m \\ \emptyset & \text{if } n \neq m \end{cases}$$

where B_n is Artin's braid group.

Given a braided category $\mathcal{C}$ and an object V, there is a unique functor $F : \mathcal{B} \to \mathcal{C}$ of braided categories such that $F(X) = V$. It induces braid group actions:

$$F : B_n = \mathrm{Aut}_{\mathcal{B}}(X^{\otimes n}) \to \mathrm{Aut}_{\mathcal{C}}(V^{\otimes n})$$

Braided categories with duality: the corresponding free category $\mathcal{T}$ has more morphisms: not only braids, also knots, links, tangles embedded in $\mathbf{R}^3$.

Completion of a braided category

Let $\mathcal{C}$ be a *braided* category which is linear over a commutative ring R.

Augmentation ideal of $\mathcal{C}$: ideal I of morphisms (w.r.t. $\circ$ and $\otimes$) generated by all morphisms

$$c_{V,U} \circ c_{U,V} - \mathrm{id}_{U \otimes V}$$

Quotient category $\mathcal{C}/I$: same objects as $\mathcal{C}$ and

$$\mathrm{Hom}_{\mathcal{C}/I}(V,W) = \mathrm{Hom}_{\mathcal{C}/I}(V,W)/\big(I \cap \mathrm{Hom}_{\mathcal{C}/I}(V,W)\big).$$

The quotient category $\mathcal{C}/I$ is symmetric.

Inverse system of braided categories:

$$\widehat{\mathcal{C}} = \varprojlim \{\cdots \to \mathcal{C}/I^3 \to \mathcal{C}/I^2 \to \mathcal{C}/I\}$$

Example. Let $R[\mathcal{T}]$ be the linearization of the tangle category: $\mathrm{Ob}(R[\mathcal{T}]) = \mathrm{Ob}(\mathcal{T})$ and

$$\mathrm{Hom}_{R[\mathcal{T}]}(X^{\otimes n}, X^{\otimes m}) = R[\mathrm{Hom}_{\mathcal{T}}(X^{\otimes n}, X^{\otimes m})].$$

Then the completed category $\widehat{R[\mathcal{T}]}$ is universal for the Vassiliev invariants of links in $\mathbf{R}^3$.

Drinfeld's Grothendieck-Teichmüller group

Elements of $GT(R)$ are couples (λ, f), where $\lambda \in R^*$ and $f = \exp F(\log A, \log B) \in R\langle\langle A, B\rangle\rangle$ and F a Lie series. These couples satisfy certain conditions.

Drinfeld (1990): There exists a group morphism

$$\rho : \mathrm{Gal}(\bar{\mathbf{Q}}/\mathbf{Q}) \to GT(\mathbf{Q}_\ell).$$

Definition of ρ*:* $\mathrm{Gal}(\bar{\mathbf{Q}}/\mathbf{Q})$ acts on ℓ-pro-unipotent completion $\widehat{\pi}_1$ of $\pi_1(\mathbf{P}^1(\mathbf{C}) - \{0, 1, \infty\})$.

Action of $\sigma \in \mathrm{Gal}(\bar{\mathbf{Q}}/\mathbf{Q})$ on generators x and y:

$$x \mapsto x^{\chi(\sigma)} \quad \text{and} \quad y \mapsto f_\sigma(x, y)^{-1} y^{\chi(\sigma)} f_\sigma(x, y)$$

$\chi(\sigma)$ is *cyclotomic character:* $\sigma(\zeta) = \zeta^{\chi(\sigma)}$ if $\zeta \in \mu_{\ell^n}$

Drinfeld: $\rho(\sigma) = (\chi(\sigma), f_\sigma(x, y)) \in GT(\mathbf{Q}_\ell)$.

Twisting of a braided category by an element of $GT(R)$

Let $\mathcal{C}$ be a R-linear braided category with associativity isomorphism a and braiding c.

For $g = (\lambda, f) \in GT(R)$, define braided category $\widehat{\mathcal{C}}_g$

- $\widehat{\mathcal{C}}_g = \widehat{\mathcal{C}}$ as a category
- same $\otimes$-product as $\widehat{\mathcal{C}}$
- new associativity isomorphism a'

$$a'_{U,V,W} = a_{U,V,W} \circ f(c_{V,U} c_{U,V} \otimes \mathrm{id}_W, \mathrm{id}_U \otimes c_{W,V} c_{V,W})$$

- new braiding $c' (= c^\lambda)$

$$c'_{U,V} = c_{U,V} \circ \exp\Big(\frac{\lambda - 1}{2} \log(c_{V,U} c_{U,V})\Big).$$

• Formulas make sense in the completion w.r.t. any ideal containing $c_{V,U} c_{U,V} - \mathrm{id}_{U \otimes V}$.

• Coherence conditions satisfied by a' and c' because of defining relations of $GT(R)$.

$GT(R)$ acts on Vassiliev invariants

Let $g \in GT(R)$. Apply twisting to linearization $R[\mathcal{T}]$ of tangle category $\mathcal{T}$.

$$\begin{array}{ccc} \mathcal{T} & \xrightarrow{F_g} & \widehat{R[\mathcal{T}]}_g \\ \big\downarrow & & \big\downarrow \mathrm{id} \\ \widehat{R[\mathcal{T}]} & \xrightarrow{\widehat{F_g}} & \widehat{R[\mathcal{T}]}_g \end{array}$$

$GT(R)$ (and $\mathrm{Gal}(\bar{\mathbf{Q}}/\mathbf{Q})$ if $R = \mathbf{Q}_\ell$) acts on category $\widehat{R[\mathcal{T}]}$.

GT-action is not trivial:

If $g = (-1, f = 1) \in GT(R)$, then $a' = a$, $c' = c^{-1}$, and

$$\widehat{F_g}(K) = \text{mirror image of knot } K.$$

References

- V. G. Drinfeld, *On quasitriangular quasi-Hopf algebras and a group closely connected with* $\mathrm{Gal}(\bar{\mathbf{Q}}/\mathbf{Q})$, Algebra i Analiz 2:4 (1990), 149–181 (= Leningrad Math. J. 2 (1991), 829–860)

- Y. Ihara, *Braids, Galois groups, and some arithmetic functions,* Proc. I.C.M. Kyoto 1990, Math. Soc. of Japan (1991), 99–120.

- A. Joyal, R. Street, *Braided tensor categories*, Adv. Math. 102 (1993), 20–78.

- C. Kassel, *Quantum groups*, Grad. Texts in Math., vol. 155, Springer-Verlag, New York-Heidelberg, 1995.

- C. Kassel, M. Rosso, V. Turaev, *Quantum groups and knot invariants*, Panoramas et Synthèses, vol. 5, Soc. Math. France, Paris, 1997.

- C. Kassel, V. Turaev, *Chord diagram invariants of tangles and graphs*, Duke Math. J. 92 (1998), 497–552.