Homology (of) Hopf algebras

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Quantum Group Workshop
Université Paul Sabatier
Toulouse
16 May 2009

Introduction

- Report on joint work with Julien Bichon (Clermont-Ferrand): *The lazy homology of a Hopf algebra*, arXiv:0807.1651

- **Original motivation:** *The classification of Hopf Galois extensions*, which are noncommutative analogues of principal fiber bundles

  The rôle of the structural group for these noncommutative bundles is played by a Hopf algebra

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Previous work 1

- In joint work with Eli Aljadeff (*Polynomial identities and noncommutative versal torsors*, Adv. Math. 218 (2008), 1453–1495), we concentrated on a class of Hopf Galois extensions obtained from a given Hopf algebra by twisting its product using a two-cocycle, and we constructed "universal spaces" using polynomial identities

- The special case where the Hopf algebra is a group algebra had been worked out by E. Aljadeff, D. Haile, M. Natapov (*Graded identities of matrix algebras and the universal graded algebra*, to appear in Trans. Amer. Math. Soc., 2008)

  They make use of the second cohomology group $H^2(G, k^\times)$ of a group G and the universal coefficient theorem relating the cohomology of G to the integral homology of G via an exact sequence of the form

$$1 \to \mathrm{Ext}^1(H_1(G), k^\times) \to H^2(G, k^\times) \to \mathrm{Hom}(H_2(G), k^\times) \to 1$$

  In particular, if k is algebraically closed, then

$$H^2(G, k^\times) \cong \mathrm{Hom}(H_2(G), k^\times)$$

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Previous work 2

- **Our aim:** Construct homology and cohomology groups for general Hopf algebras and a similar universal coefficient theorem connecting them

- Now Sweedler (1968) constructed a cohomology theory for cocommutative Hopf algebras extending group cohomology

- For general Hopf algebras, Schauenburg, Chen, Bichon, Carnovale *et al.* recently constructed two cohomology groups

$$H^1_\ell(H) \quad \text{and} \quad H^2_\ell(H)$$

  called "lazy cohomology" groups

- Lazy cohomology coincides with Sweedler cohomology when H is cocommutative.

  In particular, it extends group cohomology: for any group algebra $H = k[G]$

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Summary of joint work with Julien Bichon

- To any Hopf algebra H we associate their "lazy homology"

$$H_1^\ell(H) \quad \text{and} \quad H_2^\ell(H)$$

which are commutative Hopf algebras (not groups!)

- (i) together with a group isomorphism

$$H^1_\ell(H) \xrightarrow{\cong} \mathrm{Alg}(H_1^\ell(H), k)$$

(ii) and an exact sequence of groups (Universal coefficient theorem)

$$1 \longrightarrow \mathrm{Ext}^1(H,k) \longrightarrow H^2_\ell(H) \xrightarrow{\kappa} \mathrm{Alg}(H_2^\ell(H), k)$$

When the ground field k is algebraically closed, κ is an isomorphism

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Part One

- **Part One:** Sweedler and lazy cohomology
- Part Two: A homological version of Sweedler cohomology
- Part Three: Lazy homology
- References

Convolution groups

- Let H be a coalgebra over some fixed field k with coproduct $\Delta : H \to H \otimes H$ and counit $\varepsilon : H \to k$

  The dual vector space $\mathrm{Hom}(H, k)$ is an associative unital algebra whose product is the convolution product given for $f, g \in \mathrm{Hom}(H, k)$ and $x \in H$ by
  $$(f * g)(x) = \sum_{(x)} f(x')\, g(x'')$$
  where $\Delta(x) = \sum_{(x)} x' \otimes x''$ (Sweedler's sigma notation)

  The counit $\varepsilon : H \to k$ is the unit for the convolution product

- Let $\mathrm{Reg}(H)$ be the group of invertible elements of $\mathrm{Hom}(H, k)$

  The group $\mathrm{Reg}(H)$ is abelian if H is cocommutative, i.e., if
  $$\sum_{(x)} x' \otimes x'' = \sum_{(x)} x'' \otimes x'$$

  Since $H^{\otimes n} = H \otimes \cdots \otimes H$ (n copies of H) is a coalgebra for any $n \geq 2$, we may also consider $\mathrm{Reg}(H^{\otimes n})$

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Sweedler's cosimplicial group

Let H be a Hopf algebra

- For any $n \geq 1$ define coface maps

$$\delta_0, \delta_1, \dots, \delta_{n+1} : \mathrm{Reg}(H^{\otimes n}) \to \mathrm{Reg}(H^{\otimes(n+1)})$$

for $f \in \mathrm{Reg}(H^{\otimes n})$ and $x_0, x_1, \dots, x_n \in H$ by

$$\delta_i(f)(x_0 \otimes x_1 \otimes \cdots \otimes x_n) = \begin{cases} \varepsilon(x_0)\, f(x_1 \otimes \cdots \otimes x_n) & \text{for } i = 0 \\ f(x_0 \otimes \cdots \otimes x_{i-1}x_i \otimes \cdots \otimes x_n) & \text{for } i = 1, \dots, n \\ f(x_0 \otimes \cdots \otimes x_{n-1})\, \varepsilon(x_n) & \text{for } i = n+1 \end{cases}$$

- The maps $\delta_0, \delta_1, \dots, \delta_{n+1}$ are group homomorphisms satisfying the standard simplicial relations

- Consider the alternating convolution product

$$\delta^n = \delta_0 * \delta_1^{-1} * \delta_2 * \cdots * \delta_{n+1}^{(-1)^{n+1}} : \mathrm{Reg}(H^{\otimes n}) \to \mathrm{Reg}(H^{\otimes(n+1)})$$

We have $\delta^{n+1} \circ \delta^n = \varepsilon$. The maps δ^n are group homomorphisms if H is cocommutative

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Sweedler cohomology

- Let H be a cocommutative Hopf algebra.
  The Sweedler cohomology $H^*_{\mathrm{Sw}}(H)$ of H is the cohomology of the cochain complex of abelian groups

$$\mathrm{Reg}(k) = k^\times \to \mathrm{Reg}(H) \xrightarrow{\delta^1} \mathrm{Reg}(H \otimes H) \xrightarrow{\delta^2} \mathrm{Reg}(H \otimes H \otimes H) \xrightarrow{\delta^3} \cdots$$

- Let G be a group and $H = k[G]$ the group algebra. Equipped with the coproduct

$$\Delta(g) = g \otimes g \qquad (g \in G)$$

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  The cochain complex $(\mathrm{Reg}(H^{\otimes *}, \delta^*)$ coincides with the standard complex computing the cohomology of the group G acting trivially on $k^\times = k - \{0\}$

  Hence,

$$H^n_{\mathrm{Sw}}(k[G]) \cong H^n(G, k^\times) \qquad (n \geq 1)$$

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Sweedler cohomology in low degree

- The Sweedler cohomology groups $H^1_{\mathrm{Sw}}(H)$ and $H^2_{\mathrm{Sw}}(H)$ fit into the exact sequence of abelian groups

$$0 \to H^1_{\mathrm{Sw}}(H) \longrightarrow \mathrm{Reg}(H) \xrightarrow{\delta^1} Z^2(H) \longrightarrow H^2_{\mathrm{Sw}}(H) \to 0$$

where

$$Z^2(H) = \mathrm{Ker}(\delta^2)$$

is the group of two-cocycles, i.e., of elements $\alpha \in \mathrm{Reg}(H \otimes H)$ satisfying

$$\sum_{(x),(y)} \alpha(x' \otimes y')\, \alpha(x''y'' \otimes z) = \sum_{(y),(z)} \alpha(y' \otimes z')\, \alpha(x \otimes y''z'')$$

for all $x, y, z \in H$

- The differential $\delta^1 : \mathrm{Reg}(H) \to Z^2(H)$ is given by

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Towards cohomology for general Hopf algebras

- When the Hopf algebra H is not cocommutative, the sequence

$$0 \to H^1_{\mathrm{Sw}}(H) \longrightarrow \mathrm{Reg}(H) \xrightarrow{\delta^1} Z^2(H) \longrightarrow H^2_{\mathrm{Sw}}(H) \to 0 \qquad (1)$$

may no longer be a sequence of groups

- **Remedy:** Replace (1) by a new exact sequence of groups (not abelian!)

$$1 \to H^1_\ell(H) \longrightarrow \mathrm{Reg}_\ell(H) \xrightarrow{\delta^1} Z^2_\ell(H) \longrightarrow H^2_\ell(H) \to 1$$

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$$\mathrm{Reg}_\ell(H) \subset \mathrm{Reg}(H) \qquad \text{and} \qquad Z^2_\ell(H) \subset Z^2(H)$$

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Lazy cocycles

- An element $\mu \in \mathrm{Reg}(H)$ is called lazy if for all $x \in H$,

$$\sum_{(x)} \mu(x')\, x'' = \sum_{(x)} \mu(x'')\, x' \in H$$

The set of lazy elements of $\mathrm{Reg}(H)$ is an abelian subgroup $\mathrm{Reg}_\ell(H)$

- An element $\alpha \in \mathrm{Reg}(H \otimes H)$ is called lazy if for all $x, y \in H$,

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- A lazy two-cocycle is an element of

$$Z_\ell^2(H) = Z^2(H) \cap \mathrm{Reg}_\ell^{(2)}(H)$$

(Chen) The set $Z_\ell^2(H)$ is a group

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Lazy cohomology groups

- The differential $\delta^1 : \mathrm{Reg}(H) \to Z^2(H)$ restricts to a homomorphism of groups
$$\delta^1 : \mathrm{Reg}_\ell(H) \to Z^2_\ell(H)$$
whose image is a central subgroup of $Z^2_\ell(H)$

- **Definition.** *The lazy cohomology groups $H^1_\ell(H)$ and $H^2_\ell(H)$ are defined by the exact sequence of groups*
$$1 \longrightarrow H^1_\ell(H) \longrightarrow \mathrm{Reg}_\ell(H) \xrightarrow{\delta^1} Z^2_\ell(H) \longrightarrow H^2_\ell(H) \longrightarrow 1$$

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The lazy cohomology of group algebras

- If H is cocommutative, then laziness is no restriction:

$$\mathrm{Reg}_\ell(H) = \mathrm{Reg}(H) \qquad \text{and} \qquad Z^2_\ell(H) = Z^2(H)$$

Hence, lazy cohomology coincides with the Sweedler cohomology:

$$H^i_\ell(H) = H^i_{\mathrm{Sw}}(H) \qquad (i = 1, 2)$$

- In particular, if $H = k[G]$ is a group algebra, then

$$H^i_\ell(H) \cong H^i(G, k^\times) \qquad (i = 1, 2)$$

Lazy cohomology groups may be non-abelian

- The group $H^1_\ell(H)$ is abelian for all Hopf algebras, but $H^2_\ell(H)$ is not always abelian

- **Example.** Let G be a finite group and $H = \mathcal{O}(G)$ the Hopf algebra of k-valued functions on G (the Hopf algebra $\mathcal{O}(G)$ is dual to the Hopf algebra $k[G]$)

  In the case when $H = \mathcal{O}(G)$

  (a) $H^1_\ell(H)$ is isomorphic to the center of G

  (b) a method to compute $H^2_\ell(H)$ was obtained jointly with Pierre Guillot

  (c) there are finite groups G (in particular one of order 2^{15}) for which the group $H^2_\ell(H)$ is not abelian

  See P. Guillot, C. Kassel, *Cohomology of invariant Drinfeld twists on group algebras* (arXiv:0903.2807)

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The lazy cohomology of the Sweedler algebra

Assume that k has characteristic $\neq 2$

- The Sweedler algebra is the four-dimensional algebra

$$H_4 = k \langle g, x \mid g^2 = 1 \,, \;\; gx + xg = 0 \,, \;\; x^2 = 0 \rangle$$

It is the smallest noncommutative noncocommutative Hopf algebra with

Coproduct: $\Delta(g) = g \otimes g, \quad \Delta(x) = 1 \otimes x + x \otimes g$

Coünit: $\varepsilon(g) = 1, \quad \varepsilon(x) = 0$

Antipode: $S(g) = g, \quad S(x) = gx$

- **Lazy cohomology:**

$$H^1_\ell(H_4) \cong \{1\} \qquad \text{and} \qquad H^2_\ell(H_4) \cong (k, +)$$

Part Two

- Part One: Sweedler and lazy cohomology
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- Part Three: Lazy homology
- References

Predualizing $\mathrm{Reg}(H)$

• Before constructing lazy homology, we present a homological version of Sweedler cohomology for cocommutative Hopf algebras

• We need to associate to any coalgebra H a commutative Hopf algebra $F(H)$ together with a functorial group isomorphism

$$\mathrm{Alg}(F(H), k) \cong \mathrm{Reg}(H)$$

where $\mathrm{Alg}(F(H), k)$ is the group of characters of $F(H)$, i.e., algebra morphisms $F(H) \rightarrow k$

Takeuchi's free commutative Hopf algebra

• Takeuchi (1971) constructed such a Hopf algebra: as an algebra, it is generated by symbols t_x and t_x^{-1} ($x \in H$) and the relations such that

(a) the maps $x \mapsto t_x$ and $x \mapsto t_x^{-1} : H \to F(H)$ are linear

(b) for all $x \in H$,

$$\sum_{(x)} t_{x'}\, t_{x''}^{-1} = \varepsilon(x)\, 1 = \sum_{(x)} t_{x'}^{-1}\, t_{x''}$$

The coproduct Δ, counit ε, and (involutive) antipode S are given by

$$\Delta(t_x) = \sum_{(x)} t_{x'} \otimes t_{x''}\,, \qquad \varepsilon(t_x) = \varepsilon(x)\,, \qquad S(t_x) = t_x^{-1}$$

The algebra $F(H)$ is a commutative Hopf algebra

• If H is cocommutative, then $F(H)$ is bicommutative (i.e., commutative and cocommutative)

Predualizing Sweedler's cochain complex

• Let H be a cocommutative Hopf algebra
There is a chain complex of bicommutative Hopf algebras

$$\cdots \xrightarrow{\partial_4} F(H \otimes H \otimes H) \xrightarrow{\partial_3} F(H \otimes H) \xrightarrow{\partial_2} F(H) \xrightarrow{\partial_1} F(k)$$

such that applying the functor $\mathrm{Alg}(-, k)$ we obtain Sweedler's cochain complex:

$$\mathrm{Alg}(F(H^{\otimes *}), k) \cong (\mathrm{Reg}(H^{\otimes *}), \delta^*)$$

• We have

$$\partial_1(t_x) = \varepsilon(x)\, t_1$$

$$\partial_2(t_{x \otimes y}) = \sum_{(x)(y)} t_{x'}\, t_{y'}\, t^{-1}_{x'' y''}$$

$$\partial_3(t_{x \otimes y \otimes z}) = \sum_{(x)(y)(z)} t_{y' \otimes z'}\, t_{x' \otimes y'' z''}\, t^{-1}_{x'' y''' \otimes z'''}\, t^{-1}_{x''' \otimes y''''}$$

• By Takeuchi the category of bicommutative Hopf algebras is an abelian category, so that we can take the homology of the above chain complex

Hopf kernels

- A Hopf algebra morphism $\pi : H \to H'$ is normal if

$$\Big\{x \in H \,\Big|\, \sum_{(x)} \pi(x') \otimes x'' = 1 \otimes x\Big\} = \Big\{x \in H \,\Big|\, \sum_{(x)} \pi(x'') \otimes x' = 1 \otimes x\Big\} \tag{2}$$

  Condition (2) is always satisfied if H is cocommutative

  When π is normal, then we denote both sides of (2) by $\mathrm{HKer}(\pi)$: it is the Hopf kernel of π.

- **Properties.** (a) The Hopf kernel $\mathrm{HKer}(\pi)$ is a Hopf subalgebra of H

  (b) If $u : G \to G'$ is a group homomorphism, then the induced Hopf algebra morphism $k[u] : k[G] \to k[G']$ is normal and

$$\mathrm{HKer}(k[u]) = k[\mathrm{Ker}(u)]$$

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Hopf quotients

- Let $H_0 \subset H$ be a Hopf subalgebra and let $H_0^+ = \mathrm{Ker}(\varepsilon : H_0 \to k)$ be the augmentation ideal of H_0

  If $H_0^+ H = H H_0^+$ (always satisfied if H is commutative), then we define the Hopf quotient to be

$$H /\!/ H_0 = H / H_0^+ H$$

- **Properties.** (a) The quotient $H /\!/ H_0$ is a Hopf algebra

  (b) If G_0 is a normal subgroup of G, then $k[G_0]$ is a Hopf subalgebra of $k[G]$ such that $k[G_0]^+ k[G] = k[G]\, k[G_0]^+$, and

$$k[G] /\!/ k[G_0] = k[G/G_0]$$

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A homology theory for cocommutative Hopf algebras

Let H be a cocommutative Hopf algebra

- Recall the chain complex of bicommutative Hopf algebras

$$\cdots \xrightarrow{\partial_4} F(H \otimes H \otimes H) \xrightarrow{\partial_3} F(H \otimes H) \xrightarrow{\partial_2} F(H) \xrightarrow{\partial_1} F(k)$$

In the abelian category of bicommutative Hopf algebras we can take kernels and quotients as above

- **Definition**. *The Sweedler-type homology of H is given by*

$$H_n^{\mathrm{Sw}}(H) = \mathrm{HKer}\big(\partial_n : F(H^{\otimes n}) \to F(H^{\otimes (n-1)})\big) /\!\!/ \mathrm{Im}(\partial_{n+1})$$

These “homology groups” are bicommutative Hopf algebras

Properties

• The homology Hopf algebras $H_n^{\mathrm{Sw}}(H)$ are related to Sweedler cohomology by natural homomorphisms of abelian groups

$$H^n_{\mathrm{Sw}}(H) \to \mathrm{Alg}(H_n^{\mathrm{Sw}}(H), k) \qquad (n \geq 1)$$

These maps are isomorphisms if k is algebraically closed

• They extend group homology: for any group G,

$$H_*^{\mathrm{Sw}}(k[G]) \cong k[H_*(G, \mathbb{Z})] \, ,$$

where $H_*(G, \mathbb{Z})$ is the homology of G with integral coefficients

Part Three

- Part One: Sweedler and lazy cohomology
- Part Two: A homological version of Sweedler cohomology
- **Part Three:** Lazy homology
- References

How to construct lazy homology

- We derived the homology of a cocommutative Hopf algebra H from the chain complex of bicommutative Hopf algebras

$$\cdots \xrightarrow{\partial_4} F(H \otimes H \otimes H) \xrightarrow{\partial_3} F(H \otimes H) \xrightarrow{\partial_2} F(H) \xrightarrow{\partial_1} F(k) \qquad (3)$$

- We replace (3) by a short sequence of commutative Hopf algebras

$$H \otimes H \otimes H \xrightarrow{\partial_3} F(H^{[2]}) \xrightarrow{\partial_2} F(H^{[1]}) \xrightarrow{\partial_1} F(k)$$

where $H^{[1]}$ and $H^{[2]}$ are coalgebras such that

$$\mathrm{Reg}_\ell(H) \cong \mathrm{Reg}(H^{[1]}) \quad \text{and} \quad \mathrm{Reg}_\ell^{(2)}(H) \cong \mathrm{Reg}(H^{[2]})$$

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The first lazy quotient

- Given a coalgebra H, let $H^{[1]}$ be the quotient of H by the subspace spanned by the elements

$$\sum_{(x)} \varphi(x')\, x'' - \sum_{(x)} \varphi(x'')\, x' \qquad (x \in H,\ \varphi \in \mathrm{Hom}(H,k))$$

  The projection $H \to H^{[1]}$ turns $H^{[1]}$ into a cocommutative coalgebra

- **Proposition.** *The projection $H \to H^{[1]}$ induces a group isomorphism*

$$\mathrm{Reg}(H^{[1]}) \cong \mathrm{Reg}_\ell(H)$$

  Corollary. *There is a group isomorphism*

$$\mathrm{Reg}_\ell(H) \cong \mathrm{Alg}(F(H^{[1]}), k)$$

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The second lazy quotient

- Given a Hopf algebra H, let $H^{[2]}$ be the quotient of $H \otimes H$ by the subspace spanned by the elements

$$\sum_{(x)(y)} \varphi(x'y')\, x'' \otimes y'' - \sum_{(x)(y)} \varphi(x''y'')\, x' \otimes y' \qquad (x, y \in H,\ \varphi \in \mathrm{Hom}(H, k))$$

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- **Proposition.** *The projection $H \to H^{[2]}$ induces a group isomorphism*

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The first lazy homology Hopf algebra

Let H be a Hopf algebra

- **Proposition.** *There are morphisms of commutative Hopf algebras*

$$F(H^{[2]}) \xrightarrow{\partial_2} F(H^{[1]}) \xrightarrow{\partial_1} F(k)$$

induced by

$$\partial_2 : t_{x\otimes y} \mapsto \sum_{(x)(y)} t_{x'}\, t_{y'}\, t^{-1}_{x''y''} \qquad \text{and} \qquad \partial_1 : t_x \mapsto \varepsilon(x)\, t_1$$

- **Definition.** *The first lazy homology Hopf algebra of H is given by*

$$H_1^{\ell}(H) = \mathrm{HKer}(\partial_1) /\!\!/ \mathrm{Im}(\partial_2)$$

- **Theorem.** *The Hopf algebra $H_1^{\ell}(H)$ is bicommutative and there is a natural isomorphism of abelian groups*

$$H^1_{\ell}(H) \xrightarrow{\cong} \mathrm{Alg}(H_1^{\ell}(H), k)$$

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where $\partial_3(x \otimes y \otimes z)$ is the image in $F(H^{[2]})$ of

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- **Lemma** *(a) The Hopf algebra morphism ∂_2 is normal*

(b) For all $x, y, z \in H$, $\partial_3(x \otimes y \otimes z)$ belongs to the Hopf kernel $\mathrm{HKer}(d_2)$

(c) The ideal $B_2^\ell(H)$ of $\mathrm{HKer}(\partial_2)$ *generated by the elements*

$$\partial_3(x \otimes y \otimes z) - \varepsilon(xyz)\,1 \qquad \text{and} \qquad S(\partial_3(x \otimes y \otimes z)) - \varepsilon(xyz)$$

is a Hopf ideal of the Hopf algebra $\mathrm{HKer}(\partial_2)$

- **Definition.** *The second lazy homology Hopf algebra of a Hopf algebra H is given by*

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Some computations of lazy homology

- **Proposition.** *If $H = k[G]$ is the Hopf algebra of a group G, then*

$$H_i^\ell(k[G]) \cong k[H_i(G, \mathbb{Z})] \qquad (i = 1, 2)$$

Remark. $H_i^\ell(k[G])$ is a group algebra from which we can recover the homology group $H_i(G, \mathbb{Z})$ by considering the group-like elements

- **Proposition.** *If $H = \mathcal{O}(G)$ is the Hopf algebra of k-valued functions on a finite group G, then*

$$H_1^\ell(\mathcal{O}(G)) \cong \mathcal{O}(Z(G))$$

(the Hopf algebra of functions on the center of G)

Remark. When k is algebraically closed of characteristic zero, then $\mathcal{O}(G)$ is a cosemisimple Hopf algebra

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The cosemisimple case

Assume that the ground field k is algebraically closed of characteristic zero

- Let H be a cosemisimple Hopf algebra: the category of H-comodules is semisimple

  Definition (Baumgärtel & Lledo, Müger, Gelaki & Nikshych, Petit). *The universal abelian grading group Γ_H is the abelian group generated by the isomorphism classes of finite-dimensional simple H-comodules and the relations (U, V, W simple H-comodules)*

  $$U + V = W \qquad \text{whenever} \quad W \subset U \otimes V$$

- **Theorem.** *If H is a cosemisimple Hopf algebra, then*

  $$H_1^\ell(H) \cong k[\Gamma_H]$$

  Remark. If G is a finite group, then $H = \mathcal{O}(G)$ is cosemisimple and

  $$\Gamma_H \cong \widehat{Z(G)} \overset{\text{def}}{=} \mathrm{Hom}(Z(G), k^\times) \qquad \text{(Pontryagin dual of the center)}$$

  We recover $H_1^\ell(H) \cong k[\Gamma_H] \cong k[\widehat{Z(G)}] \cong \mathcal{O}(Z(G))$

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The lazy homology of the Sweedler algebra 1

- **Theorem.** *For the Sweedler algebra H_4 we have*

$$H_1^{\ell}(H_4) \cong k \qquad \text{and} \qquad H_2^{\ell}(H_4) \cong k[X]$$

where X is a primitive element, i.e., $\Delta(X) = 1 \otimes X + X \otimes 1$

Remark. In this case $H_2^{\ell}(H)$ is not a group algebra and the group of group-like elements is trivial

- To prove the theorem, we first determine the lazy quotients $H_4^{[1]}$ and $H_4^{[1]}$

We have $H_4^{[1]} = k$ and $H_4^{[2]}$ is a five-dimensional coalgebra:

$$H_4^{[2]} = ky_0 \oplus ky_1 \oplus ky_2 \oplus ky_3 \oplus ky_4$$

with $\quad \Delta(y_0) = y_0 \otimes y_0 \quad$ and $\quad \Delta(y_i) = y_0 \otimes y_i + y_i \otimes y_0 \quad (i = 1, 2, 3, 4)$

Therefore, $F(H_4^{[1]}) = k[T, T^{-1}]$ and $F(H_4^{[2]}) = k[Y_0, Y_0^{-1}, Y_1, Y_2, Y_3, Y_4]$

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The lazy homology of the Sweedler algebra 2

- The differential ∂_2 is the Hopf algebra map

$$\partial_2 : F(H_4^{[2]}) = k[Y_0, Y_0^{-1}, Y_1, Y_2, Y_3, Y_4] \longrightarrow F(H_4^{[1]}) = k[T, T^{-1}]$$

given by $\partial_2(Y_0) = T$ and $\partial_2(Y_i) = 0$ if $i = 1, 2, 3, 4$

One deduces the first lazy cohomology group and the computation of $\mathrm{HKer}(\partial_2)$, namely

$$H_1^{\ell}(H_4) = k \quad \text{and} \quad \mathrm{HKer}(\partial_2) = k[X_1, X_2, X_3, X_4]$$

where X_1, X_2, X_3, X_4 are the primitive elements $X_i = Y_i/Y_0$ $(i = 1, 2, 3, 4)$

- Computing the values of $\partial_3 : H_4^{\otimes 3} \to \mathrm{HKer}(\partial_2)$ yields the second lazy cohomology:

$$H_2^{\ell}(H_4) = k[X_1, X_2, X_3, X_4]/(X_1 = X_2 = -X_3 = -X_4) \cong k[X]$$

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Short exact sequences of Hopf algebras

To obtain a universal coefficient theorem, we need to define what short exact sequence of Hopf algebras are and how they behave under the contravariant functor $\mathrm{Alg}(-, k)$

- Let $H_0 \subset H$ be a Hopf subalgebra such that $H_0^+ H = H H_0^+$ so that the Hopf quotient $H /\!\!/ H_0$ makes sense

  An exact sequence of Hopf algebras is a sequence of the form

$$k \longrightarrow H_0 \longrightarrow H \longrightarrow H /\!\!/ H_0 \longrightarrow k \tag{4}$$

- **Proposition.** *(a) The sequence* (4) *induces an exact sequence of groups*

$$1 \longrightarrow \mathrm{Alg}(H /\!\!/ H_0, k) \longrightarrow \mathrm{Alg}(H, k) \longrightarrow \mathrm{Alg}(H_0, k)$$

  (b) If furthermore H is commutative and k is algebraically closed, then

$$\mathrm{Alg}(H, k) \longrightarrow \mathrm{Alg}(H_0, k)$$

  is surjective

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$$k \longrightarrow H_0 \longrightarrow H \longrightarrow H /\!/ H_0 \longrightarrow k \tag{4}$$

- **Proposition.** *(a) The sequence* (4) *induces an exact sequence of groups*

$$1 \longrightarrow \mathrm{Alg}(H /\!/ H_0, k) \longrightarrow \mathrm{Alg}(H, k) \longrightarrow \mathrm{Alg}(H_0, k)$$

  (b) If furthermore H is commutative and k is algebraically closed, then

$$\mathrm{Alg}(H, k) \longrightarrow \mathrm{Alg}(H_0, k)$$

  is surjective

A universal coefficient theorem

- **Theorem.** *For any Hopf algebra H there is an exact sequence of groups*

$$1 \longrightarrow \mathrm{Ext}^1(H,k) \longrightarrow H^2_\ell(H) \xrightarrow{\kappa} \mathrm{Alg}(H^\ell_2(H),k)$$

If in addition k is algebraically closed, then κ is an isomorphism:

$$\kappa : H^2_\ell(H) \xrightarrow{\cong} \mathrm{Alg}(H^\ell_2(H),k)$$

- The exact sequence of commutative Hopf algebras

$$k \longrightarrow \mathrm{Im}(\partial_2) \xrightarrow{\iota} \mathrm{HKer}(\partial_1) \longrightarrow H^\ell_1(H) \longrightarrow k$$

defining the first lazy homology group $H^\ell_1(H)$ induces the exact sequence of groups

$$1 \longrightarrow \mathrm{Alg}(H^\ell_1(H),k) \longrightarrow \mathrm{Alg}(\mathrm{HKer}(\partial_1),k) \xrightarrow{\iota^*} \mathrm{Alg}(\mathrm{Im}(\partial_2),k)$$

Definition. $\mathrm{Ext}^1(H,k)$ *is defined by*

$$\mathrm{Ext}^1(H,k) = \mathrm{Coker}\big(\mathrm{Alg}(\mathrm{HKer}(\partial_1),k) \xrightarrow{\iota^*} \mathrm{Alg}(\mathrm{Im}(\partial_2),k)\big)$$

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