Invariant Drinfeld twists on group algebras

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Introduction

- Report on joint work with Pierre Guillot (Strasbourg):
  Cohomology of invariant Drinfeld twists on group algebras
  arXiv:0903.2807

- Our original motivation was to compute the second lazy cohomology group $\mathrm{H}^2_\ell(H)$ of Hopf algebras that are neither cocommutative, nor pointed such as
  the Hopf algebras $\mathcal{O}_k(G)$ of functions on finite non-abelian groups

- We reformulate the problem in terms of invariant Drinfeld twists
  and obtain a method to compute $\mathrm{H}^2_\ell(H)$ when $H = \mathcal{O}_k(G)$

- The answer involves the abelian normal subgroups of central type of G
  as well as the group of class-preserving outer automorphisms of G
  The proof uses tools from quantum group theory, mainly R-matrices

- I shall illustrate all this with several examples

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Drinfeld twists

- Let H be a Hopf algebra over some fixed field k

Definition. *A Drinfeld twist on H is an invertible element F of $H \otimes H$ satisfying the condition*

$$(F \otimes 1)\,(\Delta \otimes \mathrm{id}_H)(F) = (1 \otimes F)\,(\mathrm{id}_H \otimes \Delta)(F) \tag{1}$$

where $\Delta : H \to H \otimes H$ is the coproduct of H

Twists were introduced by Drinfeld in his work on quasi-Hopf algebras, in order to "twist" the coproduct of H without changing its product. They have become an important tool in the classification of finite-dimensional Hopf algebras. There is now an abundant literature on twists

- **"Trivial" solutions of** (1) : $F = (a \otimes a)\,\Delta(a^{-1})$ where a is an invertible element of H

Problem. Find more (all?) solutions of (1)

We shall look for special solutions of (1) as follows. . .

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Invariant twists

- **Definition.** *A twist F is* *invariant*

$$\Delta(a)\,F = F\,\Delta(a) \qquad \text{for all } a \in H$$

In general the product of two twists is not a twist, but...

- **Proposition.** *(a) Invariant twists form a group under multiplication*

(b) The group of invariant twists contains as a central subgroup the group of trivial twists, where a twist F is called trivial if

$$F = (a \otimes a)\,\Delta(a^{-1})$$

for some central invertible element $a \in H$

We can thus consider the quotient group

$$\{\text{invariant twists on } H\}/\{\text{trivial twists}\}\ldots$$

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The group $H^2_\ell(G)$

- . . . which we consider in the special case when $H = k[G]$ is the algebra of a group G

Definition. *Given a group G, set*

$$H^2_\ell(G) = \{\text{invariant twists on } k[G]\}/\{\text{trivial twists}\}$$

Our aim. Determine the group $H^2_\ell(G)$ for any finite group G

Why consider $H^2_\ell(G)$?

- **Because** it is a first step in finding all invariant twists on $k[G]$ and. . .

. . . the group $H^2_\ell(G)$ is isomorphic to the second lazy cohomology group of the Hopf algebra $\mathcal{O}_k(G)$ of k-valued functions on G:

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Lazy cohomology

- In the last decade Schauenburg, Bichon, Carnovale *et al.* defined cohomology groups $\mathrm{H}^1_\ell(H)$ and $\mathrm{H}^2_\ell(H)$ for an arbitrary Hopf algebra H. These groups are called the lazy cohomology groups of H

  • On the class of cocommutative Hopf algebras the lazy cohomology groups coincide with the cohomology groups introduced by Sweedler (1968)

- In particular, if $H = k[G]$ is the algebra of a group G, then lazy cohomology coincides with group cohomology

$$\mathrm{H}^i_\ell(H) \cong H^i(G, k^\times) \qquad (i = 1, 2)$$

  where G acts trivially on $k^\times = k - \{0\}$

- The first lazy cohomology group for a general Hopf algebra is given by

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The second lazy cohomology group

- For the second lazy cohomology group $\mathrm{H}^2_\ell(H)$ only few computations (for non-cocommutative Hopf algebras) had been done

  Our original motivation. Compute $\mathrm{H}^2_\ell(H)$ for more Hopf algebras

  We decided to try to compute $\mathrm{H}^2_\ell(H)$ for the Hopf algebras $\mathcal{O}_k(G)$

- **Proposition.** For any finite group G,

$$\mathrm{H}^2_\ell(\mathcal{O}_k(G)) \cong \mathrm{H}^2_\ell(G)$$

  where $\mathrm{H}^2_\ell(G) = \{\text{invariant twists on } k[G]\}/\{\text{trivial twists}\}$

- **Proof.** The Hopf algebra $\mathcal{O}_k(G)$ is dual of the Hopf algebra $k[G]$
  For a finite-dimensional Hopf algebra H with dual Hopf algebra H^*, we have the identifications

$$\begin{aligned} \text{two-cocycle on } H &\longleftrightarrow \text{twist on } H^* \\ \text{lazy two-cocycle on } H &\longleftrightarrow \text{invariant twist on } H^* \\ \text{cohomologically trivial two-cocycle on } H &\longleftrightarrow \text{trivial twist on } H^* \end{aligned}$$

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Plan

First ingredient: the set $\mathcal{B}(G)$

- For any finite group G
  let $\mathcal{B}(G)$ be the set of pairs (A, b) where

  (i) A is an abelian normal subgroup of G and

  (ii) $b : \widehat{A} \times \widehat{A} \to k^{\times}$ is a $k^{\times}$-valued G-invariant non-degenerate alternating biadditive form on the character group $\widehat{A} = \mathrm{Hom}(A, k^{\times})$

- The set $\mathcal{B}(G)$ is finite and can be "computed in finite time"

  It is non-empty since it always contains the trivial pair $(\{1\}, b \equiv 1)$

- There is a further restriction on the groups A such that $(A, b) \in \mathcal{B}(G)$: Such a group is what is called a group of central type

  In particular, it is a square group: $A \cong A' \times A'$ for some subgroup A'

- **Examples.** $\mathcal{B}(G)$ is trivial if G is simple, or $G = S_n$ (symmetric groups), or $G = GL_n(\mathbb{F}_q)$

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- **Examples.** $\mathcal{B}(G)$ is trivial if G is simple, or $G = S_n$ (symmetric groups), or $G = GL_n(\mathbb{F}_q)$

First ingredient: the set $\mathcal{B}(G)$

- For any finite group G
  let $\mathcal{B}(G)$ be the set of pairs (A, b) where

  (i) A is an abelian normal subgroup of G and

  (ii) $b : \widehat{A} \times \widehat{A} \to k^\times$ is a $k^\times$-valued G-invariant non-degenerate alternating biadditive form on the character group $\widehat{A} = \mathrm{Hom}(A, k^\times)$

- The set $\mathcal{B}(G)$ is finite and can be "computed in finite time"

  It is non-empty since it always contains the trivial pair $(\{1\}, b \equiv 1)$

- There is a further restriction on the groups A such that $(A, b) \in \mathcal{B}(G)$: Such a group is what is called a group of central type

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Second ingredient: class-preserving automorphisms

- Let $\mathrm{Int}_k(G)$ be the group of automorphisms of G induced by the conjugation

$$\mathrm{ad}(a) : g \mapsto aga^{-1} \qquad (g \in G)$$

by some invertible element a of the group algebra $k[G]$:

$$\mathrm{Int}_k(G) = \mathrm{Aut}(G) \bigcap \mathrm{ad}(k[G]^{\times})$$

- **Remarks.** (i) If $\mathrm{ad}(a) \in \mathrm{Int}_k(G)$, then the unit a belongs to the normalizer N of G in $k[G]^{\times}$

(ii) By character theory, if k is of characteristic prime to $|G|$ and is big enough (e.g., algebraically closed), then $\mathrm{Int}_k(G)$ consists of all automorphisms preserving each conjugacy class of G

- The group $\mathrm{Int}_k(G)$ contains the group $\mathrm{Inn}(G)$ of inner automorphisms as a normal subgroup and we may consider the quotient group

$$\mathrm{Int}_k(G)/\,\mathrm{Inn}(G)$$

The main theorem

Assume that the ground field k is algebraically closed of characteristic zero

- **Theorem** (with P. Guillot) *There is a (set-theoretic) map*

$$\Theta : \mathrm{H}^2_\ell(G) \to \mathcal{B}(G)$$

whose fibers are in bijection with $\mathrm{Int}_k(G)/\mathrm{Inn}(G)$ *and*

which is surjective if $|G|$ *is odd*

- **Consequences** (a) The group $\mathrm{H}^2_\ell(G)$ is finite

(b) We have $\mathrm{H}^2_\ell(G) = 1$ if $\mathcal{B}(G)$ is trivial and $\mathrm{Int}_k(G)/\mathrm{Inn}(G) = 1$

There are many groups with $\mathrm{Int}_k(G)/\mathrm{Inn}(G) = 1$ and having no normal abelian square subgroups:

e.g., G simple, $G = S_n$, and $G = SL_n(\mathbb{F}_q)$

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Plan

The abelian case

- Before explaining the construction of the map $\Theta : \mathrm{H}^2_\ell(G) \to \mathcal{B}(G)$ for an arbitrary group, we compute the lazy cohomology of abelian groups

- Let A be an abelian group and $\widehat{A} = \mathrm{Hom}(A, k^\times)$ its group of characters

  The following isomorphisms hold:

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\begin{aligned}
\mathrm{H}^2_\ell(A) &\cong \mathrm{H}^2_\ell(\mathcal{O}_k(A)) && \text{(by definition)}\\
&\cong \mathrm{H}^2_\ell(k[\widehat{A}]) && \text{(discrete Fourier transform)}\\
&\cong H^2(\widehat{A}, k^\times) &&\\
&\cong \mathrm{Hom}(H_2(\widehat{A}, \mathbb{Z}), k^\times) && \text{(universal coefficient theorem)}\\
&\cong \mathrm{Hom}(\Lambda^2_{\mathbb{Z}}(\widehat{A}), k^\times) &&\\
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- Let us give a direct expression for the composite isomorphism

$$\mathrm{H}^2_\ell(A) \xrightarrow{\cong} \{\text{alternating bicharacters on } \widehat{A}\}$$

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Explicit isomorphism

- To any twist $F \in k[A] \otimes k[A]$ representing an element of $\mathrm{H}^2_\ell(A)$ we attach an alternating bicharacter $b_F : \widehat{A} \times \widehat{A} \to k^\times$ as follows:

- Each character $\rho \in \widehat{A}$ defines an idempotent of the algebra $k[A]$ by

$$e_\rho = \frac{1}{|A|} \sum_{g \in A} \rho(g)\, g$$

  The set $\{e_\rho\}_{\rho \in \widehat{A}}$ is a basis of $k[A]$

- Let $F \in k[A] \otimes k[A]$ be a twist representing an element of $\mathrm{H}^2_\ell(A)$

  Consider $R_F = F_{21}\, F^{-1}$ and express it in the basis $\{e_\rho \otimes e_\sigma\}_{\rho,\sigma \in \widehat{A}}$

$$R_F = \sum_{\rho \in \widehat{A}} b_F(\rho, \sigma)\, e_\rho \otimes e_\sigma \in k[A] \otimes k[A]$$

  Then b_F is a bicharacter on $\widehat{A}$; it is alternating because $(R_F)^{-1} = (R_F)_{21}$

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The universal R-matrix attached to a twist

- Let us now explain how to construct the map

$$\Theta : \mathrm{H}^2_\ell(G) \to \mathcal{B}(G)$$

for an arbitrary finite group G

Start with an element of $\mathrm{H}^2_\ell(G)$ and
represent it by an invariant twist $F \in k[G] \otimes k[G]$

- As in the abelian case, consider

$$R_F = F_{21}\, F^{-1} \in k[G] \otimes k[G]$$

It is a universal R-matrix for the Hopf algebra $k[G]$, i.e., an invertible element such that

$$\Delta(a)\, R_F = R_F\, \Delta(a) \qquad \text{for all } a \in k[G]$$

and

$$(\Delta \otimes \mathrm{id}_H)(R_F) = (R_F)_{13}\,(R_F)_{23} \quad \text{and} \quad (\mathrm{id}_H \otimes \Delta)(R_F) = (R_F)_{13}\,(R_F)_{12}$$

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The map Θ

- By work of Radford, there is a minimal Hopf subalgebra $H' \subset k[G]$ such that
$$R_F \in H' \otimes H'$$
Moreover, since $k[G]$ is cocommutative, H' is bicommutative

  • One deduces that $H' = k[A]$ where A is an abelian subgroup of G. Since F is invariant, so is R_F, and A is normal in G

- By duality, the universal R-matrix $R_F \in k[A] \otimes k[A]$ corresponds to a G-invariant alternating bicharacter $b_F : \widehat{A} \times \widehat{A} \to k^\times$

  • The bicharacter b_F is non-degenerate by the minimality of A; one sets
$$\Theta(F) = (A, b_F) \in \mathcal{B}(G)$$

  • It is easy to check that $\Theta(F)$ depends only on the class of F in $H^2_\ell(G)$

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On the proof of the main theorem

- **Fibers of Θ:** To determine them, we use the following fact:

  (Etingof and Gelaki, 2000) If F is a twist such that $R_F = 1 \otimes 1$, equivalently F is symmetric: $F = F_{21}$, then

  $$F = (a \otimes a)\,\Delta(a^{-1})$$

  for some invertible element $a \in k[G]$

  Observation. The element $F = (a \otimes a)\,\Delta(a^{-1})$ is an invariant twist if and only if a belongs to the normalizer N, which is equivalent to $\mathrm{ad}(a)$ belonging to $\mathrm{Int}_k(G)$

- Etingof and Gelaki's result follows from classical Tannakian theory:

  (Deligne and Milne, 1982) Any exact and fully faithful symmetric tensor functor from the category of $k[G]$-modules to the category of k-vector spaces is isomorphic to the forgetful functor

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Plan

Example of groups with non-trivial $\mathrm{H}^2_\ell(G)$

• Let p be an odd prime and let G be the wreath product

$$G = \mathbb{Z}/p \wr \mathbb{Z}/p = A \rtimes \mathbb{Z}/p \qquad \text{with } A = \mathbb{F}_p[\mathbb{Z}/p] \;\; (= (\mathbb{Z}/p)^p)$$

We have $\mathrm{Int}_k(G)/\,\mathrm{Inn}(G) = 1$

• **Proposition.** *We have*

$$\mathrm{H}^2_\ell(G) \;\cong\; H^2(\widehat{A}, k^\times)^G \;\cong\; (\mathbb{Z}/p)^{(p-1)/2}$$

• **Proof.** The second isomorphism follows from a standard cohomological calculation

The first isomorphism is a consequence of the main theorem and the following interpretation of $\mathcal{B}(G)$

The set $\mathcal{B}(G)$ as a colimit

- **Theorem.** *There is a bijection*

$$\mathcal{B}(G) \xrightarrow{\cong} \bigcup_{A} H^2(\widehat{A}, k^\times)^G$$

where the RHS is the colimit in the category whose objects are the abelian normal subgroups A of G and whose arrows are the inclusions

- **Corollary.** *If G is a group of odd order such that $\mathrm{Int}_k(G)/Inn(G) = 1$ and has a unique maximal abelian normal subgroup A, then*

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The alternating group A_4

- We now consider a group of even order, namely the alternating group A_4

  We have $A_4 = V_4 \rtimes \mathbb{Z}/3$ with $V_4 = \mathbb{Z}/2 \times \mathbb{Z}/2$ (Klein's *Vierergruppe*)

  The set $\mathcal{B}(A_4)$ has two elements: $(V_4 = \mathbb{Z}/2 \times \mathbb{Z}/2, \det)$ and the trivial one

  We have $\mathrm{Int}_k(A_4) = \mathrm{Inn}(A_4)$. Thus $\Theta : \mathrm{H}^2_\ell(A_4) \to \mathcal{B}(A_4)$ is injective; since the order of the group is even, we cannot use the main theorem to conclude that Θ is surjective. Nevertheless,...

- **Theorem.** $\mathrm{H}^2_\ell(A_4) \cong \mathbb{Z}/2$

  To prove the surjectivity of Θ, we exhibit an invariant twist F such that $\Theta(F) = (V_4, \det)$, namely

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- In the previous examples the group $\mathrm{Int}_k(G)/\mathrm{Inn}(G)$ was trivial
  What if it is not?

  We have the following addition to the main theorem

  Proposition. *The group* $\mathrm{Int}_k(G)/\mathrm{Inn}(G)$ *is a subgroup of* $\mathrm{H}^2_\ell(G)$

- This result is a consequence of the following facts:

  (a) For any $a \in N$ the element $F = (a \otimes a)\,\Delta(a^{-1})$ in an invariant twist on $k[G]$ and thus defines an element $\delta(a) \in \mathrm{H}^2_\ell(G)$

  (b) The kernel of $\delta : N \to \mathrm{H}^2_\ell(G)$ is the subgroup $G \cdot \mathrm{Center}(k[G]^\times)$ and

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  The proposition has an interesting application (see next slide)

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- *(Burnside, 1912)* There are groups with non-trivial $\mathrm{Int}_k(G)/\,\mathrm{Inn}(G)$ (his smallest example is of order $729 = 3^6$)

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- Since $\mathrm{Int}_k(G)/\,\mathrm{Inn}(G) \subset \mathrm{H}^2_\ell(G)$ we obtain

  Corollary. *There are finite groups G for which $\mathrm{H}^2_\ell(G)$ is non-abelian*

  For such a group G the algebra $\mathcal{O}_k(G)$ is the first known example of a Hopf algebra for which the lazy cohomology is non-abelian

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  Corollary. *There are finite groups G for which $\mathrm{H}^2_\ell(G)$ is non-abelian*

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Wall's group

• G. E. Wall (1947) improved Burnside's result by showing that for

$$G = \mathbb{Z}/8 \rtimes \operatorname{Aut}(\mathbb{Z}/8)$$

(of order 32) the group $\operatorname{Int}_k(G)/\operatorname{Inn}(G)$ is not trivial:

$$\operatorname{Int}_k(G)/\operatorname{Inn}(G) \cong \mathbb{Z}/2$$

• Wall's group has the following presentation:

$$G = \langle s, t, u \mid s^2 = t^2 = u^8 = 1,\ st = ts,\ sus^{-1} = u^3,\ tut^{-1} = u^5 \rangle$$

• The set $\mathcal{B}(G)$ has two elements: $(\langle t, u^4 \rangle = \mathbb{Z}/2 \times \mathbb{Z}/2, \det)$ and the trivial one

• Therefore, $\mathrm{H}^2_\ell(G)$ has order 4 or 2 according as Θ is surjective or not

We were not able to conclude

An invariant twist on Wall's group

- The non-trivial element of $\mathrm{Int}_k(G)/\,\mathrm{Inn}(G)$ may be represented by the non-inner automorphism α given by

$$\alpha(s) = u^4 s\,, \quad \alpha(t) = u^4 t\,, \quad \alpha(u) = u$$

- A computer search gave us the following invertible $a \in k[G]$ such that $\alpha = \mathrm{ad}(a)$:

$$a = \frac{1}{2}\,(1 + u^4) + \frac{\sqrt{2}}{4}\,u\,(1 - u^2 - u^4 + u^6)$$

- Then

$$F = (a \otimes a)\,\Delta(a^{-1})$$

is a symmetric invariant twist representing the non-zero element of $\mathrm{H}^2_\ell(G)$ in the subgroup $\mathrm{Int}_k(G)/\,\mathrm{Inn}(G)$; the twist F is a sum of 52 pure tensors:

$$\begin{aligned} 8F \;&= 2\,(u_{00} + u_{44}) + (u_{11} + u_{33} + u_{55} + u_{77}) \\ &\quad + u_{01} + u_{03} + u_{04} + u_{05} + u_{07} + u_{12} + u_{17} + u_{25} + u_{35} + u_{36} + u_{67} \\ &\quad + u_{10} + u_{30} + u_{40} + u_{50} + u_{70} + u_{21} + u_{71} + u_{52} + u_{53} + u_{63} + u_{76} \\ &\quad - (u_{13} + u_{14} + u_{15} + u_{16} + u_{23} + u_{27} + u_{34} + u_{37} + u_{45} + u_{47} + u_{56} + u_{57}) \\ &\quad - (u_{31} + u_{41} + u_{51} + u_{61} + u_{32} + u_{72} + u_{43} + u_{73} + u_{54} + u_{74} + u_{65} + u_{75}) \end{aligned}$$

where $u_{ij} = u^i \otimes u^j$ $(i, j \in \{0, 1, \ldots, 7\})$

Plan

Rationality

- A word about the case when the ground field k is not algebraically closed (but still of characteristic zero)

  Let $\bar{k}$ be the algebraic closure of k

- **Theorem.** *If all $\bar{k}$-representations of G can be realized over k, then there is an exact sequence*

$$1 \longrightarrow H^1(k, Z(G)) \longrightarrow \mathrm{H}^2_\ell(G/k) \longrightarrow \mathrm{H}^2_\ell(G/\bar{k}) \longrightarrow 1$$

  where $H^1(k, Z(G))$ is the first Galois cohomology group of k with coefficients in the center of G

  In particular, if G is centerless, then $\mathrm{H}^2_\ell(G/k) \cong \mathrm{H}^2_\ell(G/\bar{k})$

- **Ingredients of the proof**

  (a) view the groups defining lazy cohomology as algebraic groups

  (b) compute their tangent Lie algebras

  (c) use Hilbert's Theorem 90

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A final motivation for computing $\mathrm{H}^2_\ell(G)$: torsors

- The group $\mathrm{H}^2_\ell(G)$ classifies analogues of torsors in algebraic geometry

- As in in algebraic geometry such torsors can be used to twist G-algebras (commutative or not)

Unlike what happens in the classical case, we obtain new algebras even over an algebraically closed field

- See forthcoming joint paper with Guillot where we “twist” G-algebras using explicit invariant twists on some group algebras (such as A_4)

Plan

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THANK YOU FOR YOUR ATTENTION