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Homology and cohomology of associative algebras

- A concise introduction to cyclic homology -

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Noncommutative geometry can be considered as the study of (not necessarily commutative) associative algebras with ideas coming from differential geometry.

The aim of this course is to define certain (co)homological invariants for associative algebras such as Hochschild (co)homology, cyclic (co)homology, algebraic K-theory, and to exhibit the connections between these invariants. In the dictionary between manifolds (or varieties) and associative algebras, which is at the heart of Alain Connes's noncommutative geometry, differential forms correspond to elements of the Hochschild homology groups, de Rham cohomology to cyclic homology, vector bundles to elements of the algebraic K_0-group, and the Chern character of a vector bundle to the algebraic Chern character of Section 8.

An interesting feature of these invariants is that they are related to the trace map. It is therefore not surprising that we shall deal with functionals $\tau(a_0, a_1, \ldots, a_n)$ invariant (up to sign) under *cyclic* permutations of the variables.

In the world of noncommutative algebras, there is a concept that does not appear for commutative algebras, namely the concept of Morita equivalence. For instance, every algebra is Morita equivalent to the algebra $M_n(A)$ of $n \times n$-matrices with entries in A. The (co)homological invariants we shall define are invariant under Morita equivalence.

The ideas underlying noncommutative geometry as presented here are essentially due to Alain Connes. The basic references for this course are Connes [1985], [1994], Feigin & Tsygan [1983], [1985], [1987], Karoubi [1987], Loday & Quillen [1984], Tsygan [1983].

1. Associative algebras

We start by defining precisely the main objects of interest in noncommutative geometry, namely the associative algebras.

We shall work over a fixed field k (although it would often be possible to assume that k is a commutative ring). The symbol $\otimes$ will denote the tensor product of vector spaces over k.

1.1. Definition.— *(a) An associative k-algebra is a triple (A, μ, η), where A is a vector space over the field k, $\mu : A \otimes A \to A$ and $\eta : k \to A$ are linear maps satisfying the conditions*

- *(Associativity)* $\mu(\mu \otimes \mathrm{id}_A) = \mu(\mathrm{id}_A \otimes \mu)$,
- *(Unitality)* $\mu(\eta \otimes \mathrm{id}_A) = \mathrm{id}_A = \mu(\mathrm{id}_A \otimes \eta)$.

(b) A morphism of algebras $f : (A, \mu, \eta) \to (A', \mu', \eta')$ is a linear map $f : A \to A'$ such that $f\mu = \mu'(f \otimes f)$ and $f\eta = \eta'$.

For simplicity we shall simply say *algebra* for an associative k-algebra. Using the notation $ab = \mu(a \otimes b)$ for a, $b \in A$, and $1 = \mu(1)$, we see that the associativity condition is equivalent to

$$(ab)c = a(bc) \tag{1.1}$$

for all a, b, $c \in A$. The unitality condition is equivalent to

$$1a = a = a1 \tag{1.2}$$

for all $a \in A$. The linear map μ is called the *product* of the algebra whereas the element $1 = \eta(1)$ is called its *unit*.

With this notation we see that $f : A \to A'$ is a morphism of algebras if and only if $f(1) = 1$ and

$$f(ab) = f(a)f(b) \tag{1.3}$$

for all a, $b \in A$.

An algebra $A = (A, \mu, \eta)$ is said to be *commutative* if

$$ab = ba \tag{1.4}$$

for all a, $b \in A$. The simplest examples of commutative algebras are the field k itself and the *polynomial algebras* $k[x_1, \ldots, x_N]$ where N is a positive integer. As a vector space, $k[x_1, \ldots, x_N]$ has a basis formed by the monomials $x_1^{a_1} \cdots x_N^{a_N}$, where $a_1, \ldots, a_N$ run over all nonnegative integers. The product is given on this basis by the formula

$$(x_1^{a_1} \cdots x_N^{a_N})(x_1^{b_1} \cdots x_N^{b_N}) = x_1^{a_1+b_1} \cdots x_N^{a_N+b_N}. \tag{1.5}$$

We now give examples of *noncommutative algebras*, that is of algebras that are not commutative.

1.2. Examples. (a) *(Opposite algebra)* If $A = (A, \mu, \eta)$ is an algebra, let μ^{op} be the composite map $A \otimes A \xrightarrow{\sigma} A \otimes A \xrightarrow{\mu} A$, where $\sigma : A \otimes A \to A \otimes A$ is the flip defined by $\sigma(a_0 \otimes a_1) = a_1 \otimes a_0$ for all a_0, $a_1 \in A$. The triple $A^{\mathrm{op}} = (A, \mu^{\mathrm{op}}, \eta)$ is an algebra, called the opposite algebra of A. Observe that $\mu^{\mathrm{op}} = \mu$ if and only if A is commutative.

(b) *(Tensor product of algebras)* If A and B are algebra, then the tensor product $A \otimes B$ becomes an algebra with product

$$(a_0 \otimes b_0)(a_1 \otimes b_1) = a_0 a_1 \otimes b_0 b_1 \tag{1.6}$$

(a_0, $a_1 \in A$ and b_0, $b_1 \in B$) and unit $1 \otimes 1$.

(c) *(Endomorphism algebra and matrix algebra)* If M is a vector space over k, then the vector space $\mathrm{End}_k(M)$ of k-linear endomorphisms of M is an algebra whose product is given by the composition of endomorphisms and the unit is the identity map id_M. If M is finite-dimensional of dimension n over k, then the algebra $\mathrm{End}_k(M)$ is isomorphic to the matrix algebra $M_n(A)$ of $n \times n$-matrices with entries in A.

(d) *(Tensor algebras)* For any vector space V over k and any integer $n \geq 1$ set $T^n(V) = V^{\otimes n}$, where the latter means the tensor product of n copies of V. We also set $T^0(V) = k$. Observe that there are natural isomorphisms

$$T^p(V) \otimes T^q(V) \cong T^{p+q}(V)$$

for all p, $q \geq 1$, and

$$T^0(V) \otimes T^n(V) \cong T^n(V) \cong T^n(V) \otimes T^0(V)$$

for all $n \geq 0$. There is a unique algebra structure on the direct sum

$$T(V) = \bigoplus_{n \geq 0} T^n(V)$$

whose product $\mu : T(V) \otimes T(V) \to T(V)$ is induced by the natural isomorphisms $T^p(V) \otimes T^q(V) \cong T^{p+q}(V)$ (p, $q \geq 0$) and the unit is the natural inclusion $k = T^0(V) \to T(V)$.

Now suppose that the vector space V is finite-dimensional of dimension N and let $\{v_1, \ldots, v_N\}$ be a basis of V. For any $n \geq 0$ we denote W_n the set of all words $v_{i_1} \cdots v_{i_n}$ of length n in the alphabet $\{v_1, \ldots, v_N\}$. By convention there is only one word of length 0, namely the empty word. It is easy to check that W_n is a basis of $T^n(V)$. Therefore $T(V)$ has a basis consisting of the union of all sets W_n ($n \geq 0$). Under the product structure of $T(V)$ the product of two words w and w' is their *concatenation*, namely the word ww' obtained by putting w' after w and gluing them together. The empty word is the unit of $T(V)$.

(e) *(Quantum plane)* Let q be an element of k. On the polynomial algebra $k[x, y]$ we consider the following product

$$x^a y^b *_q x^c y^d = q^{bc} x^{a+c} y^{b+d} \tag{1.7}$$

for all a, b, c, $d \geq 0$. This product is associative and its unit is $1 = x^0y^0$. We denote $k_q[x, y]$ the algebra equipped with this product and call it the "quantum plane". By (1.5) we have $y *_q x = q\, x *_q y$, which shows that the quantum plane is not commutative unless $q = 1$. For more details on the quantum plane, see KASSEL [1994], Chapter IV.

(f) *(Weyl algebra)* It is the algebra $A_1(k)$ generated by two noncommuting variables p and q subject to the relation $pq - qp = 1$. The algebra $A_1(k)$ acts on the polynomial algebra $k[X]$ as follows: p acts by derivation and q by multiplication by X:

$$p(f) = \frac{df}{dx} \quad \text{and} \quad q(f) = Xf$$

for all $f \in k[X]$. Through this action, $A_1(k)$ coincides with the algebra of polynomial differential operators of $k[X]$. It can be checked that $(p^m q^n)_{m,\, n \in \mathbf{N}}$ is a basis of $A_1(k)$ as a vector space.

1.3. EXERCISES. (a) Given $N \geq 1$ show that for any *commutative* algebra A and any N-uple $(a_1, \ldots, a_n) \in A^N$ there is a unique morphism of algebras $f : k[x_1, \ldots, x_N] \to A$ such that $f(x_k) = a_k$ for all $k = 1, \ldots, N$.

(b) Let $u : V \to A$ be a linear map from a vector space V to an algebra A. Prove the existence of a unique morphism of algebras $f : T(V) \to A$ whose restriction to $T^1(V) = V$ is u.

(c) Fix $q \in k$. Show that for any algebra A and any couple $(a, b) \in A^2$ satisfying $ba = qab$ there is a unique morphism of algebras $f : k_q[x, y] \to A$ such that $f(x) = a$ and $f(y) = b$.

2. Modules and bimodules

The natural objects linking an algebra to another algebra are bimodules. We start with the definition of left and right modules.

2.1. DEFINITION.— *(a) Let $A = (A, \mu, \eta)$ be an algebra. A left A-module is a couple $M = (M, \mu_M)$, where M is a vector space over the field k and $\mu_M : A \otimes M \to M$ is a linear map satisfying the conditions*

- *(Associativity)* $\mu_M(\mu \otimes \mathrm{id}_M) = \mu_M(\mathrm{id}_A \otimes \mu_M)$,
- *(Unitarity)* $\mu_M(\eta \otimes \mathrm{id}_M) = \mathrm{id}_M$.

(b) A morphism of left A-modules $f : (M, \mu_M) \to (M', \mu_{M'})$ is a linear map $f : M \to M'$ such that $f\mu_M = \mu_{M'}(\mathrm{id}_A \otimes f)$.

Using the notation $am = \mu_M(a \otimes m)$ for $a \in A$ and $m \in M$, we see that the associativity condition is equivalent to

$$(ab)m = a(bm) \tag{2.1}$$

for all a, $b \in A$, $m \in M$. The unitarity condition is equivalent to

$$1m = m \tag{2.2}$$

for all $m \in M$. The map μ_M is called the *action* of the algebra on the module.

With this notation we see that $f : M \to M'$ is a morphism of left A-modules if and only if

$$f(am) = af(m) \tag{2.3}$$

for all $a \in A$ and $m \in M$.

2.2. Definition.— *(a) A right A-module is a couple $M = (M, \mu'_M)$ where M is a vector space over the field k and $\mu'_M : M \otimes A \to M$ is a linear map satisfying the conditions*

- *(Associativity)* $\mu'_M(\mathrm{id}_M \otimes \mu) = \mu'_M(\mu'_M \otimes \mathrm{id}_A)$,
- *(Unitarity)* $\mu'_M(\mathrm{id}_M \otimes \eta) = \mathrm{id}_M$.

(b) A morphism of right A-modules $f : (M, \mu'_M) \to (M', \mu'_{M'})$ is a linear map $f : M \to M'$ such that $f\mu'_M = \mu'_{M'}(f \otimes \mathrm{id}_A)$.

Recall the opposite algebra A^{op}. Any right A-module (M, μ'_M) is a left A^{op}-module (M, μ_M), where μ_M is the composite map $A \otimes M \xrightarrow{\sigma} M \otimes A \xrightarrow{\mu'_M} M$. We can therefore pass easily from right module structures to left module structures and *vice versa*.

2.3. Definition.— *(a) Let A and B be algebras. An A-B-bimodule is a vector space M equipped with a left A-module structure and a right B-module structure such that*

$$a(mb) = (am)b$$

for all $a \in A$, $m \in M$, $b \in B$.

(b) A linear map $f : M \to M'$ between A-B-bimodules is a morphism of A-B-bimodules if it is a morphism of left A-modules and a morphism of right B-modules.

An A-B-bimodule structure is equivalent to a left $A \otimes B^{\mathrm{op}}$-module structure and to a right $A^{\mathrm{op}} \otimes B$-module structure. One passes from one structure to the others by the formulas

$$amb = (a \otimes b)m = m(a \otimes b) \tag{2.4}$$

$(a \in A, m \in M, b \in B)$.

Note that an A-k-bimodule is the same as a left A-module. A k-B-bimodule is the same as a right B-module. Therefore we could have defined the concept of bimodule before defining the concept of a module.

2.4. Examples. (a) *(Direct sum of modules)* If M and N are left A-modules, then so is the direct sum $M \oplus N$, the action of A on $M \oplus N$ being given by

$$a(m, n) = (am, an)$$

for all $a \in A$, $m \in M$, $n \in N$.

(b) Let $f : A \to B$ be a morphism of algebras. Then B becomes an A-B-bimodule by the formula $abb' = f(a)bb'$ for all $a \in A$, $b, b' \in B$. We shall denote this bimodule by ${}_fB$.

Similarly, B becomes a B-A-bimodule by $b'ba = b'bf(a)$ for $a \in A$, $b, b' \in B$. We denote this bimodule by B_f.

In particular, if $B = A$, we have the natural A-A-bimodule ${}_{\mathrm{id}_A}A = A_{\mathrm{id}_A}$.

2.5. Exercises. (a) Let M be a left A-module, where A is an algebra. Define a linear map $\rho : A \to \mathrm{End}_k(M)$ by $\rho(a)(m) = am$ for $a \in A$ and $m \in M$. Show that ρ is a morphism of algebras. Conversely, show that, if there is a morphism of algebras $\rho : A \to \mathrm{End}_k(M)$, where M is a vector space, then there is a left A-module structure on M.

(b) Let M be a right B-module and let $\mathrm{End}_B(M)$ be the vector space of all endomorphisms of B-modules of M, i.e., of all k-linear endomorphisms of M satisfying $f(mb) = f(m)b$ for all $m \in M$ and $b \in B$. Show that $\mathrm{End}_B(M)$ is an algebra for the composition of endomorphisms.

Suppose there is a morphism of algebra $\rho : A \to \mathrm{End}_B(M)$. Show that the formula $amb = \rho(a)(mb)$ ($a \in A$, $m \in M$, $b \in B$) defines a A-B-bimodule structure on M, and *vice versa*.

2.6. Tensor Product of Bimodules. Let A, B, C be algebras, M be an A-B-bimodule, N be a B-C-bimodule. Consider the tensor product $M \otimes_B N$, that is the quotient of the vector space $M \otimes N$ by the subspace spanned by all tensors of the form

$$mb \otimes n - m \otimes bn,$$

where $m \in M$, $b \in B$, $n \in N$. We shall keep the notation $m \otimes n$ for the image of $m \otimes n$ in the quotient space $M \otimes_B N$. The vector space $M \otimes_B N$ is an A-C-bimodule by

$$a(m \otimes n)c = (am) \otimes (nc)$$

for all $a \in A$, $m \in M$, $n \in N$, $c \in C$.

2.7. Two Categories of Algebras. It is natural to consider the category $\mathcal{A}$ whose objects are associative k-algebras and whose arrows are the morphisms of algebras. But there is a more interesting category, which we denote $\widetilde{\mathcal{A}}$; it has the same objects as $\mathcal{A}$, namely the associative k-algebras, but different objects.

Given algebras A and B, we define the class $\widetilde{\mathcal{A}}(A, B)$ of morphisms from A to B in $\widetilde{\mathcal{A}}$ to be the set of isomorphism classes of all A-B-bimodules. In order to make $\widetilde{\mathcal{A}}$ into a category, we have to define a composition map

$$\widetilde{\mathcal{A}}(A, B) \times \widetilde{\mathcal{A}}(B, C) \to \widetilde{\mathcal{A}}(A, C).$$

This map is given by the tensor product of bimodules defined above. The isomorphism class of the natural A-A-bimodule ${}_{\mathrm{id}_A}A = A_{\mathrm{id}_A}$ is the identity of the object A in this category. There is a natural functor $\mathcal{A} \to \widetilde{\mathcal{A}}$, which is the identity on objects and sends a morphism of algebras $f : A \to B$ to the class of the A-B-bimodule ${}_fB \in \widetilde{\mathcal{A}}(A, B)$. We leave as an exercise to check that, if $f : A \to B$ and $g : B \to C$ are morphisms of algebras, then

$$({}_fB) \otimes_B ({}_gC) = {}_{(g \circ f)}C.$$

The categories $\mathcal{A}$ and $\widetilde{\mathcal{A}}$ have very different features. For instance, whereas $\mathcal{A}(k, B)$ consists of the unique morphism of algebras $\eta : k \to B$, the class $\widetilde{\mathcal{A}}(k, B)$ consists of all isomorphism classes of right B-modules.

The set $\mathcal{A}(A, k)$ consists of all morphisms of algebras $A \to k$; it may be empty (which happens for simple algebras). The class $\widetilde{\mathcal{A}}(A, k)$ is never empty: it consists of all isomorphism classes of left A-modules.

2.8. MORITA EQUIVALENCE. *What is an isomorphism in the category* $\mathcal{A}$? It is a morphism of algebras $f : A \to B$ such that there exists a morphism of algebras $g : B \to A$ satisfying $g \circ f = \mathrm{id}_A$ and $f \circ g = \mathrm{id}_B$. Clearly, a morphism of algebras f is an isomorphism if and only if it is bijective. If $\mathcal{A}(A, B)$ contains an isomorphism, we say that A and B are *isomorphic algebras* and we write $A \cong B$.

The situation is very different with $\widetilde{\mathcal{A}}$. *An isomorphism in the category* $\widetilde{\mathcal{A}}$ is the isomorphism class of an A-B-bimodule P such there is a B-A-bimodule Q together with bimodule isomorphisms $\alpha : P \otimes_B Q \to A$ and $\beta : Q \otimes_A P \to B$.

The sextuple $(A, B, P, Q, \alpha, \beta)$ is called a *Morita context* and A and B are said to be *Morita equivalent* if there is a Morita context $(A, B, P, Q, \alpha, \beta)$ or, equivalently, if there is an isomorphism $A \to B$ in the category $\widetilde{\mathcal{A}}$. The bimodules P and Q have an important additional property: they are projective over A and over B (for a definition of a projective module, see Section 4.2).

We also point out that Morita equivalent algebras have isomorphic centres. Therefore, Morita equivalence reduces to isomorphism in the case of commutative algebras. For a complete treatment of Morita equivalence, see BASS [1968], JACOBSON [1989].

Let us present the basic example of a Morita context. Let A be an algebra and for any integers $p, q \geq 1$ let $M_{p,q}(A)$ be the vector space of $p \times q$-matrices with entries in A. We write $M_p(A)$ for $M_{p,p}(A)$. Consider the matrix multiplication

$$\mu_{p,q,r} : M_{p,q}(A) \otimes M_{q,r}(A) \to M_{p,r}(A).$$

It allows to equip $M_{p,q}(A)$ with a $M_p(A)$-$M_q(A)$-bimodule structure.

2.9. PROPOSITION.— *The induced map*

$$\mu_{p,q,r} : M_{p,q}(A) \otimes_{M_q(A)} M_{q,r}(A) \to M_{p,r}(A)$$

is an isomorphism of $M_p(A)$-$M_r(A)$-*bimodules.*

PROOF.— Clearly, $\mu_{p,q,r}$ is the direct sum of pr copies of the map

$$\mu_{1,q,1} : M_{1,q}(A) \otimes_{M_q(A)} M_{q,1}(A) \to M_1(A) = A.$$

Now in the vector space on the left-hand side we have for $a_1, \ldots, a_q, b_1, \ldots, b_q \in A$,

$$\begin{aligned}
(\, a_1, \ldots, a_q \,) \otimes \begin{pmatrix} b_1 \\ \vdots \\ b_q \end{pmatrix} &= (\, a_1, \ldots, a_q \,) \otimes \begin{pmatrix} b_1 & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ b_q & 0 & \cdots & 0 \end{pmatrix} \begin{pmatrix} 1 \\ \vdots \\ 0 \end{pmatrix} \\
&= (\, a_1, \ldots, a_q \,) \begin{pmatrix} b_1 & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ b_q & 0 & \cdots & 0 \end{pmatrix} \otimes \begin{pmatrix} 1 \\ \vdots \\ 0 \end{pmatrix} \\
&= (\, c, \ldots, 0 \,) \otimes \begin{pmatrix} 1 \\ \vdots \\ 0 \end{pmatrix},
\end{aligned}$$

where $c = a_1 b_1 + \cdots a_q b_q$. From this one concludes that $\mu_{1,q,1}$ is bijective. $\square$

2.10. COROLLARY.— *For any algebra A and any integer $q \geq 2$ the sextuple*

$$(A, M_q(A), M_{1,q}(A), M_{q,1}(A), \mu_{1,q,1}, \mu_{q,1,q})$$

is a Morita context. Hence $M_q(A)$ and A are Morita equivalent.

2.11. EXERCISE. Let A be an algebra and $e \in A$ an *idempotent*, i.e., an element such that $e^2 = e$. Show that the set eAe of all elements of the form eae ($a \in A$) is an algebra with unit e. Assuming that for any element $c \in A$ there are elements $a_1, \ldots, a_q$, $b_1, \ldots, b_q \in A$ such that $c = a_1 e b_1 + \cdots a_q e b_q$ (which can be shortened to $A = AeA$), prove that the algebras A and eAe are Morita equivalent.

3. Trace maps

Trace maps are important in noncommutative geometry. The cyclic cohomology groups introduced in Section 6 can be considered as higher analogues of the group of trace maps.

3.1. THE GROUP $H_0(R)$. Let R be a ring and C an abelian group. We say that a homomorphism of groups $\tau : R \to C$ is a *trace* if

$$\tau(rs) = \tau(sr)$$

for all r, $s \in R$. In other words, τ vanishes on the subgroup $[R, R]$ of R spanned by all *commutators* $[r, s] = rs - sr$ ($r, s \in R$). Hence, a trace factors through the quotient group $R/[R, R]$. Conversely, any homomorphism of groups $\tau : R \to C$ factoring through $[R, R]$ is a trace. It follows from these considerations that the group of traces $\tau : R \to C$ is in bijection with the group $\mathrm{Hom}(R/[R, R], C)$ of all homomorphisms of groups from $R/[R, R]$ to C, and the canonical surjection $R \to R/[R, R]$ is the universal trace on R.

For reasons that will become clear later, we denote $R/[R, R]$ by $H_0(R)$.

3.2. EXAMPLES. (a) *(Commutative ring)* Observe that a ring R is commutative if and only if $[R, R] = 0$. Hence, $H_0(R) = R$ when R is a commutative ring.

(b) *(Matrix algebra)* For a matrix $M \in M_p(R)$ denote its trace, i.e., the sum of its diagonal entries by $\mathrm{tr}(M)$. Since the ring R is not necessarily commutative, we do not have $\mathrm{tr}(MN) = \mathrm{tr}(NM)$ in general, but we always have

$$\mathrm{tr}(MN) \equiv \mathrm{tr}(NM) \quad \text{modulo } [R, R]. \tag{3.1}$$

Therefore, the map $\mathrm{tr} : M_p(R) \to R$ sends $[M_p(R), M_p(R)]$ into $[R, R]$, and thus induces an homomorphism $H_0(M_p(R)) \to H_0(R)$. It will be shown in Exercise 3.3 below that this homomorphism is an isomorphism

$$H_0(M_p(R)) \cong H_0(R). \tag{3.2}$$

(c) *(Tensor algebra)* Let V be a vector space and $n \in \mathbf{N}$. Consider the cyclic operator τ_n acting on the space $T^n(V) = V^{\otimes n}$ by

$$\tau_n(v_1 \otimes v_2 \otimes \cdots \otimes v_{n-1} \otimes v_n) = v_n \otimes v_1 \otimes v_2 \otimes \cdots \otimes v_{n-1} \tag{3.3}$$

for all $v_1, v_2, \ldots, v_{n-1}, v_n \in V$. Then

$$H_0(T(V)) = k \oplus V \oplus \bigoplus_{n \geq 2} V^{\otimes n}/(\mathrm{id} - \tau_n)V^{\otimes n}. \tag{3.4}$$

(d) *(Weyl algebra)* Let $A_1(k)$ be the algebra introduced in Example 1.2 (f). The defining relation $[p, q] = 1$ induces commutation relations such as

$$[p, p^m q^n] = np^m q^{n-1} \quad \text{and} \quad [p^m q^n, q] = mp^{m-1} q^n$$

for all m, n. From these relations it follows that $[A_1(k), A_1(k)] = A_1(k)$, hence $H_0(A_1(k)) = 0$, if the ground field k is of characteristic zero.

3.3. Exercise. Fix a ring R and an integer $p \geq 2$. For any $r \in R$ and integers i, j satisfying $1 \leq i \leq p$ and $1 \leq j \leq p$, define $E_{ij}(r)$ as the $p \times p$-matrix whose entries are all zero, except the (i, j)-entry, which is r.

(a) Compute the commutator $[E_{ij}(r), E_{k\ell}(s)]$.

(b) For $2 \leq i \leq p$ and $r \in R$, set $F_i(r) = E_{ii}(r) - E_{11}(r)$. Show that $F_i(r)$ can be written as a commutator in $M_p(R)$.

(c) Show that any matrix $M = (r_{ij})_{1 \leq i \leq p,\, 1 \leq j \leq p}$ can be written uniquely under the form

$$M = \sum_{1 \leq i \neq j \leq p} E_{ij}(r_{ij}) + \sum_{2 \leq i \leq p} F_i(r_{ii}) + E_{11}(\mathrm{tr}(M)).$$

(d) Deduce that the trace induces an isomorphism $H_0(M_p(R)) \cong H_0(R)$.

4. The group K_0 of algebraic K-theory

We give a concise introduction to the algebraic K-theory group K_0 and relate it by the Hattori-Stallings trace to the group $H_0(R)$ introduced in the previous section.

4.1. Free Modules. Let R be a ring. We say that a left R-module M is *free* if it the direct sum of copies of the left R-module R. In other words, a free R-module M has a basis, i.e., a family $(m_i)_{i \in I}$ of elements of M such that any element $m \in M$ can be written in a unique way

$$m = \sum_{i \in I} r_i m_i,$$

where $(r_i)_{i \in I}$ is a family of elements of the ring R satisfying the condition that all of them are zero, except a finite number of them.

A free R-module is said to be *finitely generated* if it is the direct sum of a finite number of copies of R. A finitely generated free module has a finite basis.

4.2. Projective Modules. We say that a left R-module P is *projective* it is a direct summand of a free module. In other words, P is projective if there is a R-module P' such that $P \oplus P'$ is a free module. A projective R-module is said to be *finitely generated* if it is a direct summand of a finitely generated free module.

Suppose that $e \in M_n(R)$ $(n \geq 1)$ is an idempotent matrix $(e^2 = e)$. Then $P = R^n e$ is a left R-module. We claim it is finitely generated projective. Indeed, consider the left R-module $P' = R^n(1-e)$ and the linear map $i : P \oplus P' \to R^n$ defined by $i(p, p') = p + p'$, where $p \in P$ and $p' \in P'$. It suffices to check that i is an isomorphism. Firstly, i is surjective: any $m \in R^n$ can be written

$$m = me + m(1-e),$$

and we have $me \in P$ and $m(1-e) \in P'$. Secondly, if m_0, $m_1 \in R^n$ satisfy $m_0 e + m_1(1-e) = 0$, then multiplying the latter on the right by e, we have $0 = m_0 e^2 + m_1(e - e^2) = m_0 e$. Hence, $m_1(1-e) = 0$. This proves the injectivity of i.

Conversely, let us show that for any finitely generated projective left R-module P there is a natural integer n and an idempotent $e \in M_n(R)$ such that $P \cong R^n e$. Indeed, since P is a direct summand of a free module R^n for some n, there are R-linear maps $u : P \to R^n$ and $v : R^n \to P$ such that $vu = \mathrm{id}_P$. Consider the linear endomorphism uv of R^n. It corresponds to a matrix $e \in M_n(R)$ with respect to the canonical basis of R^n. This matrix is idempotent since

$$(uv)(uv) = u(vu)v = u\mathrm{id}_P v = uv.$$

It is easy to check that v induces an isomorphism from $R^n e = \mathrm{Im}(uv)$ to P.

If R is a field, then any R-module is a vector space. It is well known that all vector spaces have bases. Therefore, in this case any R-module is free, and such a module is finitely generated if and only if it is finite-dimensional.

4.3. The Group $K_0(R)$. For a ring R we define $K_0(R)$ to be the abelian group generated by symbols of the form $[P]$, one for each finitely generated projective left R-module P, subject to the relations $[P] = [Q]$ whenever P and Q are isomorphic left modules and

$$[P \oplus Q] = [P] + [Q]$$

for all finitely generated projective left R-modules P, Q.

It follows from the definition that for any integer $n \geq 1$ we have

$$[R^n] = \underbrace{[R] + \cdots + [R]}_{n} = n[R] \tag{4.1}$$

in $K_0(R)$. For more details on $K_0(R)$, see Bass [1968] and Weibel [2004].

Quillen [1973] defined higher analogues $K_i(R)$ of the group $K_0(R)$ for all $i > 0$. These groups are called the *algebraic K-groups* of the ring R. They are closely related to the homology of the general linear groups $GL_n(R)$ and are very difficult to compute.

Let us give the computation of $K_0(R)$ in the easiest possible case.

4.4. Proposition.— *If R is a field, or the ring $\mathbf{Z}$ of integers, or the ring $K[X]$ of polynomials in one variable over a field K, then*

$$K_0(R) \cong \mathbf{Z}.$$

Proof.— Consider the map $i : \mathbf{Z} \to K_0(R)$ given by $i(n) = n[R]$ ($n \in \mathbf{Z}$). For the rings under consideration any submodule of a free module is free. Therefore, any finitely generated projective left R-module P is isomorphic to R^n for some $n \in \mathbf{N}$. It follows from (4.1) that $[P] = n[R] = i(n)$. This shows that i is surjective.

Any finitely generated free left R-module P has a well-defined rank $\mathrm{rk}(P) \in \mathbf{N}$: it is the unique natural integer n such that $P \cong R^n$. Moreover,

$$\mathrm{rk}(P \oplus Q) = \mathrm{rk}(P) + \mathrm{rk}(Q)$$

for all finitely generated free left R-modules P, Q. Therefore there exists a linear map $j : K_0(R) \to \mathbf{Z}$ such that $j([P]) = \mathrm{rk}(P)$ for every finitely generated free left R-module P. We have

$$(ji)(n) = j(n[R]) = n\,\mathrm{rk}[R] = n$$

for all $n \in \mathbf{N}$. This proves the injectivity of i. □

For general rings the group $K_0(R)$ is different from $\mathbf{Z}$. Combining Proposition 4.4 and Exercise 4.5, we can find examples of rings R with $K_0(R) \cong \mathbf{Z}^2$.

4.5. Exercise. For two rings R_0 and R_1 define the product ring $R = R_0 \times R_1$ whose product is given by

$$(r_0, r_1)(r_0', r_1') = (r_0 r_0', r_1 r_1') \tag{4.2}$$

($r_0, r_0' \in R_0$ and $r_1, r_1' \in R_1$) and with unit $1 = (1,1) \in R$. Observe that $e_0 = (1,0)$ and $e_1 = (0,1)$ are idempotents of R such that $e_0 + e_1 = 1$.

(a) Prove that for any R-module M we have $M \cong Me_0 \oplus Me_1$.

(b) Deduce the product formula

$$K_0(R) \cong K_0(R_0) \oplus K_0(R_1). \tag{4.3}$$

4.6. The Hattori-Stallings Trace. We now connect Sections 3 and 4 by constructing a group homomorphism

$$\tau : K_0(R) \to H_0(R)$$

for any ring R. This map was first defined by Hattori [1965] and Stallings [1965].

Let P be a finitely generated projective left R-module. We know that there exist an integer $n \geq 1$, an idempotent matrix $e \in M_n(R)$ and an isomorphism $\alpha : P \to R^n e$ of R-modules. Define $T(P)$ as the image in $H_0(R) = R/[R,R]$ of the trace $\mathrm{tr}(e)$ of the matrix e.

4.7. THEOREM.— *For any finitely generated projective left R-module P the element $T(P) \in H_0(R)$ depends only on the isomorphism class of P. If Q is another finitely generated projective left R-module, then*

$$T(P \oplus Q) = T(P) + T(Q).$$

It follows from this theorem that there is a unique homomorphism of groups $\tau : K_0(R) \to H_0(R)$ such that $\tau([P]) = T(P)$ for any finitely generated projective left R-module P. This homomorphism is called the *Hattori-Stallings trace*.

PROOF.— Suppose that $P \cong R^n e$ and $P \cong R^m f$ for some idempotent matrices $e \in M_n(R)$ and $f \in M_m(R)$. We have to prove that $\mathrm{tr}(e) \equiv \mathrm{tr}(f)$ modulo $[R, R]$. A moment's thought shows that there is an invertible matrix $u \in M_{n+m}(R)$ such that

$$\begin{pmatrix} e & 0 \\ 0 & 0 \end{pmatrix} = u \begin{pmatrix} f & 0 \\ 0 & 0 \end{pmatrix} u^{-1}.$$

Then modulo $[R, R]$ we have

$$\begin{aligned} \mathrm{tr}(e) = \mathrm{tr} \begin{pmatrix} e & 0 \\ 0 & 0 \end{pmatrix} &= \mathrm{tr} \left(u \begin{pmatrix} f & 0 \\ 0 & 0 \end{pmatrix} u^{-1} \right) \\ &\equiv \mathrm{tr} \left(\begin{pmatrix} f & 0 \\ 0 & 0 \end{pmatrix} u^{-1} u \right) = \mathrm{tr} \begin{pmatrix} f & 0 \\ 0 & 0 \end{pmatrix} = \mathrm{tr}(f). \end{aligned}$$

Finally, suppose that $P \cong R^n e$ and $Q \cong R^m f$ for some idempotent matrices $e \in M_n(R)$ and $f \in M_m(R)$. Then clearly, $P \oplus Q \cong R^{n+m} g$, where $g \in M_{n+m}(R)$ is the idempotent matrix

$$g = \begin{pmatrix} e & 0 \\ 0 & f \end{pmatrix}.$$

Since $\mathrm{tr}(g) = \mathrm{tr}(e) + \mathrm{tr}(f)$, we obtain $T(P \oplus Q) = T(P) + T(Q)$. □

4.8. EXAMPLE. We now give an example which shows the usefulness of the Hattori-Stallings trace. Let $\lambda \in k$ be a nonzero scalar. Consider the following 2×2-matrix with coefficients in the Weyl algebra $A_1(k)$ (see Example 1.2 (f)):

$$e_\lambda = \frac{1}{\lambda} \begin{pmatrix} pq & -p \\ (qp - \lambda)q & -(qp - \lambda) \end{pmatrix}.$$

Check that e_λ is a idempotent matrix. We have

$$\mathrm{tr}(e_\lambda) = \frac{(pq - qp) + \lambda}{\lambda} = 1 + \frac{1}{\lambda}. \tag{4.4}$$

Unfortunately, by Example 3.2 (d), if the characteristic of k is zero, $H_0(A_1(k)) = 0$, so that the Hattori-Stallings trace will not enable us to distinguish the projective modules obtained as the images of the various idempotents e_λ.

Now, let A_λ be the algebra generated by E, F, H subject to the relations

$$HE - EH = 2E, \quad HF - FH = -2F, \quad EF - FE = H,$$

$$2EF + 2FE + H^2 = \lambda^2 - 1.$$

Dixmier showed that $A_\lambda = k1 \oplus [A_\lambda, A_\lambda]$. It follows that $H_0(A_\lambda) = k$. There is an injective morphism of algebras $\varphi_\lambda : A_\lambda \to A_1(k)$ defined by

$$\varphi_\lambda(E) = -p, \quad \varphi_\lambda(F) = q(qp - \lambda + 1), \quad \varphi_\lambda(H) = -2qp + \lambda - 1.$$

The idempotent $e_\lambda \in M_2(A_1(k))$ is the image under φ_λ of the idempotent matrix

$$f_\lambda = \frac{1}{2\lambda}\begin{pmatrix} \lambda + 1 - H & 2E \\ 2F & \lambda + 1 + H \end{pmatrix} \in M_2(A_\lambda).$$

It follows that that the Hattori-Stallings trace of the projective A_λ-module $A_\lambda^2 f_\lambda$ is given by

$$\tau[A_\lambda^2 f_\lambda] = \mathrm{tr}(f_\lambda) = 1 + \frac{1}{\lambda} \in k = H_0(A_\lambda). \tag{4.5}$$

This shows that the modules $A_\lambda^2 f_\lambda$ are pairwise nonisomorphic when λ varies over $k - \{0\}$. For details on $K_0(A_\lambda)$, see Kassel & Vigué-Poirrier [1992].

5. Hochschild (co)homology

In this section we define homology groups $H_*(A, M)$ and cohomology groups $H^*(A, M)$ attached to pairs (A, M) where A is an algebra and M is an A-A-bimodule (for simplicity we shall say that M is an A-bimodule). Recall from (2.4) that a A-bimodule is the same as a left $A \otimes A^{\mathrm{op}}$-module or a right $A^{\mathrm{op}} \otimes A$-module. The definition of the *Hochschild (co)homology groups* is the following:

$$H_*(A, M) = \mathrm{Tor}_*^{A \otimes A^{\mathrm{op}}}(M, A) \quad \text{and} \quad H^*(A, M) = \mathrm{Ext}^*_{A \otimes A^{\mathrm{op}}}(A, M).$$

See, for instance, Jacobson [1989], Weibel [1994] for the definition of Tor- and Ext-groups.

Let us compute the groups $H_0(A, M)$ and $H^0(A, M)$. By definition of Tor_0 and Ext^0 we have

$$H_0(A, M) = M \otimes_{A \otimes A^{\mathrm{op}}} A$$

and

$$H^0(A, M) = \mathrm{Hom}_{A \otimes A^{\mathrm{op}}}(A, M).$$

For any A-bimodule M, define $[A, M]$ as the subspace of M generated by the elements $am - ma$, where $a \in A$ and $m \in M$. Set also

$$M^A = \{m \in M \mid am = ma \text{ for all } a \in A\}.$$

5.1. Lemma.— *We have*

$$H_0(A, M) \cong M/[A, M] \quad \textit{and} \quad H^0(A, M) \cong M^A.$$

Proof.— On one hand, $M \otimes_{A \otimes A^{\mathrm{op}}} A$ is the quotient of $M \otimes A$ by the subspace spanned by elements of the form $am \otimes b - m \otimes ba$ and $ma \otimes b - m \otimes ab$ for all $a, b \in A$ and $m \in M$. The linear map $M \otimes A \to M/[A, M]$ sending $m \otimes a$ to ma factors through $M \otimes_{A \otimes A^{\mathrm{op}}} A$. Conversely, the map $m \mapsto m \otimes 1$ from M to $M \otimes_{A \otimes A^{\mathrm{op}}} A$ factors through $M/[A, M]$. The induced maps between $M \otimes_{A \otimes A^{\mathrm{op}}} A$ and $M/[A, M]$ are inverse of each other.

To any element $m \in M^A$ associate the map $f_m : A \to M$ defined by $f_m(a) = am = ma$. This map is a morphism of A-bimodules. It thus defines a map $M^A \to \mathrm{Hom}_{A \otimes A^{\mathrm{op}}}(A, M)$. Conversely, to any morphism of A-bimodules $f : A \to M$ associate the element $m = f(1) \in M$. We have

$$am = af(1) = f(a1) = f(1a) = f(1)a = ma,$$

which shows that $m \in M^A$. The conclusion follows immediately. □

Recall that, in order to compute Tor- and Ext-groups, it suffices to exhibit free resolutions of the algebra A in the category of A-bimodules. We first construct a free resolution working for any algebra; this is the so-called Hochschild standard resolution. We shall also exhibit special resolutions for special algebras, which will allow us to compute the Hochschild groups for these algebras.

5.2. The Hochschild Standard Resolution. For an algebra A consider the positively graded A-bimodule $C'_*(A)$ defined for $q \in \mathbf{N}$ by

$$C'_q(A) = A \otimes A^{\otimes q} \otimes A.$$

For $q = 0$ set $C'_0(A) = A \otimes A$. The vector space $C'_q(A)$ is an A-bimodule, where A acts by left (resp. right) multiplication on the leftmost (resp. rightmost) tensorand A; it is clearly a free A-bimodule.

For $0 \leq i \leq q$ define $d_i : C'_q(A) \to C'_{q-1}(A)$ by

$$d_i(a_0 \otimes a_1 \otimes \cdots \otimes a_q \otimes a_{q+1}) = a_0 \otimes \cdots \otimes a_i a_{i+1} \otimes \cdots \otimes a_{q+1}. \tag{5.1}$$

The maps $d_i : C'_q(A) \to C'_{q-1}(A)$ are morphisms of bimodules. Define also $s : C'_q(A) \to C'_{q+1}(A)$ by

$$s(a_0 \otimes a_1 \otimes \cdots \otimes a_q \otimes a_{q+1}) = 1 \otimes a_0 \otimes a_1 \otimes \cdots \otimes a_q \otimes a_{q+1}. \tag{5.2}$$

Observe that s is not a morphism of A-bimodules (but it is a morphism of right A-modules). The following lemma follows from straightforward computations.

5.3. LEMMA. *(a) We have* $d_i d_j = d_{j-1} d_i$ *for* $i < j$, *and* $d_0 s = \mathrm{id}$ *and* $d_i s = s d_{i-1}$ *for* $i > 0$.

(b) Set $b' = \sum_{i=0}^{q} (-1)^i d_i : C'_q(A) \to C'_{q-1}(A)$. *Then*

$$b'^2 = 0 \quad \textit{and} \quad b's + sb' = \mathrm{id}.$$

5.4. COROLLARY. *The complex* $(C'_*(A), b')$ *is a resolution of* A *by free* A*-bimodules.*

PROOF.— By Lemma 5.3 the complex $(C'_*(A), b')$ is acyclic. We have already observed that it consists of free bimodules and morphisms of bimodules. It suffices now to compute the cokernel of $b' : C'_1(A) = A \otimes A \otimes A \to C'_0(A) = A \otimes A$. Now

$$b'(a_0 \otimes a_1 \otimes a_2) = a_0 a_1 \otimes a_2 - a_0 \otimes a_1 a_2.$$

There is a map from the cokernel of b' to A, sending $a_0 \otimes a_1$ to the product $a_0 a_1$. Conversely, there is a map from A to the cokernel of b' that sends a to $1 \otimes a$. They are clearly inverse of each other. □

5.5. THE STANDARD COMPLEXES. By definition of the Tor- and Ext-groups, $H_*(A, M)$ are the homology groups of the chain complex

$$(M \otimes_{A \otimes A^{\mathrm{op}}} C'_*(A), \mathrm{id} \otimes b'),$$

and $H^*(A, M)$ are the cohomology groups of the cochain complex

$$(\mathrm{Hom}_{A \otimes A^{\mathrm{op}}}(C'_*(A), M), \mathrm{Hom}(b', \mathrm{id})).$$

Let us simplify these complexes. There is a linear isomorphism

$$\varphi : M \otimes_{A \otimes A^{\mathrm{op}}} C'_q(A) \to C_q(A, M) = M \otimes A^{\otimes q}$$

defined by $\varphi(m \otimes a_0 \otimes a_1 \otimes \cdots \otimes a_q \otimes a_{q+1}) = a_{q+1} m a_0 \otimes a_1 \otimes \cdots \otimes a_q$. If we push the differential $\mathrm{id} \otimes b'$ to $C_*(A, M)$, we obtain a differential $b : C_q(A, M) \to C_{q-1}(A, M)$ given by

$$\begin{aligned} b(m \otimes a_1 \otimes \cdots \otimes a_q) = {} & m a_1 \otimes \cdots \otimes a_q \\ & + \sum_{i=1}^{q-1} (-1)^i m \otimes a_1 \otimes \cdots \otimes a_i a_{i+1} \otimes \cdots \otimes a_q \\ & + (-1)^q a_q m \otimes a_1 \otimes \cdots a_{q-1}. \end{aligned} \tag{5.3}$$

The complex $(C_*(A, M), b)$ is called the *Hochschild standard chain complex.*

Similarly, there is a linear isomorphism

$$\psi : C^q(A, M) = \mathrm{Hom}_k(A^{\otimes q}, M) \to \mathrm{Hom}_{A \otimes A^{\mathrm{op}}}(C'_q(A), M)$$

given by $\psi(f)(a_0 \otimes a_1 \otimes \cdots \otimes a_q \otimes a_{q+1}) = a_0 f(a_1 \otimes \cdots \otimes a_q)a_{q+1}$. If we pull the differential $\mathrm{Hom}_k(b', \mathrm{id})$ back to $C^q(A, M)$, we obtain a degree $+1$ differential $\delta : C^q(A, M) \to C^{q+1}(A, M)$ given by

$$\begin{aligned}\delta(f)(a_0 \otimes a_1 \otimes \cdots \otimes a_q)1259 = {} & a_0 f(a_1 \otimes \cdots \otimes a_q) \\ & + \sum_{i=1}^{q} (-1)^i f(a_0 \otimes a_1 \otimes \cdots \otimes a_{i-1}a_i \otimes \cdots \otimes a_q) \\ & + (-1)^{q+1} f(a_0 \otimes a_1 \otimes \cdots \otimes a_{q-1})a_q.\end{aligned} \tag{5.4}$$

The complex $(C^*(A, M), \delta)$ is called the *Hochschild standard cochain complex*.

5.6. Low-Dimensional Computations. By (5.3) the map

$$b : C_1(A, M) = M \otimes A \to C_0(A, M) = M$$

is given by

$$b(m \otimes a) = ma - am. \tag{5.5}$$

Its cokernel is clearly $H_0(A, M) = M/[A, M]$.

By (5.4) the map $\delta : C^0(A, M) = M \to C^1(A, M) = \mathrm{Hom}_k(A, M)$ is given by

$$\delta(m)(a) = am - ma. \tag{5.6}$$

Its kernel is $H^0(A, M) = M^A$.

The map $\delta : C^1(A, M) = \mathrm{Hom}_k(A, M) \to C^2(A, M) = \mathrm{Hom}_k(A \otimes A, M)$ is given by

$$\delta(f)(a_0 \otimes a_1) = a_0 f(a_1) - f(a_0 a_1) + f(a_0)a_1. \tag{5.7}$$

Therefore, the 1-cocycles for the cochain complex $(C^*(A, M), \delta)$ are the linear maps $f : A \to M$ satisfying the condition

$$f(a_0 a_1) = a_0 f(a_1) + f(a_0)a_1 \tag{5.8}$$

for all a_0, $a_1 \in A$. A map f satisfying (5.8) is called a *bimodule derivation*. The 1-coboundaries are the bimodule derivations of the form $a \mapsto am - ma$; they are called *inner derivations*. Consequently, $H^1(A, M)$ is the space of all bimodule derivations $f : A \to M$ modulo inner derivations.

5.7. Idempotents and Homology of the Ground Field. Let $e \in A$ be an idempotent in an algebra: $e^2 = e$. For $n \geq 0$ set $e_n = \underbrace{e \otimes \cdots \otimes e}_{n+1} \in C_n(A, A)$. An easy computation shows that

$$b(e_n) = \begin{cases} e_{n-1} & \text{if } n \text{ is even} > 0, \\ 0 & \text{otherwise.} \end{cases} \tag{5.9}$$

Let us apply this to the case when $A = k$ is the ground field. Then $C_n(A)$ is the one-dimensional vector space spanned by $1_n = \underbrace{1 \otimes \cdots \otimes 1}_{n+1}$. By (5.9),

$$H_n(k,k) = \begin{cases} k & \text{if } n = 0, \\ 0 & \text{otherwise.} \end{cases} \tag{5.10}$$

The case $A = k$ is essentially the only one when we can use the standard complex to compute Hochschild homology. In general, when we want to compute the Hochschild homology of a specific algebra A, we have to find a specific resolution of A by free A-bimodules. Let us illustrate this method on three examples.

5.8. The Tensor Algebra. Let $A = T(V)$ be the tensor algebra on a vector space V. We claim that the complex

$$0 \to A \otimes V \otimes A \xrightarrow{\partial} A \otimes A \to 0$$

with $\partial(a \otimes v \otimes a') = av \otimes a' - a \otimes va'$ $(a, a' \in A,\ v \in V)$ is a resolution of A by free A-bimodules. Indeed, it suffices to check that the complex

$$0 \to A \otimes V \otimes A \xrightarrow{\partial} A \otimes A \xrightarrow{\mu} A \to 0 \tag{5.11}$$

is acyclic. Define $s : A \to A \otimes A$ by $s(a) = a \otimes 1$ and $s : A \otimes A \to A \otimes V \otimes A$ by $s(a \otimes 1) = 0$ and for $a \in A$, $v_1, \ldots, v_n \in V$

$$\begin{aligned} s(a \otimes v_1 \cdots v_n) = & - a \otimes v_1 \otimes v_2 \cdots v_n \\ & - \sum_{i=2}^{n-1} av_1 \cdots v_{i-1} \otimes v_i \otimes v_{i+1} \cdots v_n \\ & - av_1 \cdots v_{n-1} \otimes v_n \otimes 1. \end{aligned}$$

Check that $\mu s = \mathrm{id}_A$, $\partial s + s\mu = \mathrm{id}_{A \otimes A}$, and $s\partial = \mathrm{id}_{A \otimes V \otimes A}$, which proves that the complex (5.11) is acyclic. Tensoring by $A \otimes_{A \otimes A^{\mathrm{op}}} -$, we obtain the complex

$$0 \to A \otimes V \xrightarrow{b} A \to 0 \tag{5.12}$$

whose homology is the Hochschild homology of the tensor algebra. Here $b(a \otimes v) = av - va$ $(a \in A,\ v \in V)$. As a consequence, we obtain

$$H_n(T(V), T(V)) = \begin{cases} k \oplus V \oplus \bigoplus_{n \geq 2} V^{\otimes n}/(\mathrm{id} - \tau_n)V^{\otimes n} & \text{if } n = 0, \\ V \oplus \bigoplus_{n \geq 2} (V^{\otimes n})^{\tau_n} & \text{if } n = 1, \\ 0 & \text{otherwise,} \end{cases} \tag{5.13}$$

where $\tau_n : V^{\otimes n} \to V^{\otimes n}$ is the cyclic operator (3.3) of Example 3.2 (c).

Consider now the special case when $\dim V = 1$. Then $T(V)$ is isomorphic to the *polynomial algebra* $k[X]$ in one variable X. It follows from (5.13) that

$$H_n(k[X], k[X]) = \begin{cases} k[X] & \text{if } n = 0, \\ Xk[X] & \text{if } n = 1, \\ 0 & \text{otherwise.} \end{cases} \tag{5.14}$$

5.9. THE TRUNCATED POLYNOMIAL ALGEBRA. Let P be a polynomial of degree d in one variable X with coefficients in k. We wish to compute the Hochschild homology of the *truncated polynomial algebra* $A = k[X]/(P)$. We leave the following as an exercise. The complex

$$\cdots \xrightarrow{Q} A \otimes A \xrightarrow{X\otimes 1 - 1\otimes X} A \otimes A \xrightarrow{Q} A \otimes A \xrightarrow{X\otimes 1 - 1\otimes X} A \otimes A \to 0 \tag{5.15}$$

with $Q = (P \otimes 1 - 1 \otimes P)/(X \otimes 1 - 1 \otimes X)$ is a resolution of A by free rank one A-bimodules. It follows that $H_*(A, A)$ is the homology of the complex

$$\cdots \xrightarrow{P'} A \xrightarrow{0} A \xrightarrow{P'} A \xrightarrow{0} A \to 0, \tag{5.16}$$

where P' is the derivative of the polynomial P.

5.10. THE WEYL ALGEBRA. Recall the algebra $A_1(k)$ of Example 1.2 (f). Let V be a two-dimensional vector space with basis $\{u, v\}$. There is a resolution of $A = A_1(k)$ by free A-bimodules given by

$$0 \to A \otimes A^{\mathrm{op}} \otimes \Lambda^2 V \xrightarrow{\beta'} A \otimes A^{\mathrm{op}} \otimes V \xrightarrow{\beta'} A \otimes A^{\mathrm{op}} \to 0, \tag{5.17}$$

where β' is the degree -1 morphism of $A \otimes A^{\mathrm{op}}$-modules defined by

$$\beta'(1 \otimes 1 \otimes u \wedge v) = (1 \otimes q - q \otimes 1) \otimes v - (1 \otimes p - p \otimes 1) \otimes u,$$

$$\beta'(1 \otimes 1 \otimes u) = 1 \otimes q - q \otimes 1,$$
$$\beta'(1 \otimes 1 \otimes v) = 1 \otimes p - p \otimes 1.$$

Tensoring with $A \otimes_{A\otimes A^{\mathrm{op}}} -$, we obtain the following complex, whose homology are the Hochschild groups $H_*(A, A)$:

$$0 \to A \otimes \Lambda^2 V \xrightarrow{\beta} A \otimes V \xrightarrow{\beta} A \to 0. \tag{5.18}$$

Here β is given for all $a \in A$ by

$$\beta(a \otimes u \wedge v) = (aq - qa) \otimes v - (ap - pa) \otimes u,$$

$$\beta(a \otimes u) = aq - qa, \quad \beta(a \otimes v) = ap - pa.$$

When the ground field k is of characteristic zero, we obtain

$$H_n(A_1(k), A_1(k)) = \begin{cases} k & \text{if } n = 2, \\ 0 & \text{otherwise.} \end{cases} \tag{5.19}$$

This computation is consistent with the computation of $H_0(A_1(k))$ in Example 3.2 (d), but what is more striking is that all Hochschild homology groups of $A_1(k)$ are zero, except the one in degree 2. This is related to the existence of a symplectic structure on the plane. For more on this subject, see WODZICKI [1987], KASSEL [1988a], [1992], WAMBST [1996].

5.11. Morita Invariance. Let $(A, B, P, Q, \alpha, \beta)$ be a Morita context in the sense of Section 2.8. Let Φ be the functor from the category of A-bimodules to the category of B-bimodules defined by

$$\Phi(M) = Q \otimes_A M \otimes_A P.$$

Similarly, let Ψ be the functor from the category of B-bimodules to the category of A-bimodules defined by

$$\Psi(N) = P \otimes_B N \otimes_B Q.$$

Since P and Q are projective both as A-modules and as B-modules, these functors are exact and send any projective resolution of A-bimodules to a projective resolution of B-bimodules, and *vice versa*. It then follows from basic arguments in homological algebra that there are isomorphisms between the Hochschild homology groups

$$H_*(A, M) \cong H_*(B, \Phi(M)) \quad \text{and} \quad H_*(B, N) \cong H_*(A, \Psi(N)) \tag{5.20}$$

for all A-bimodules M and all B-bimodules N. In particular, since

$$\Phi(A) = Q \otimes_A A \otimes_A P \cong Q \otimes_A P \cong B,$$

we obtain

$$H_*(A, A) \cong H_*(B, B) \tag{5.21}$$

for all pairs (A, B) of Morita equivalent algebras. We then say that Hochschild homology is *Morita-invariant*. Hochschild cohomology groups are Morita-invariant in a similar way.

5.12. Exercise. Let A, B be algebras, M an A-bimodule, N a B-bimodule. If P_* is a projective resolution of M by A-bimodules and Q_* is a projective resolution of N by B-bimodules, show that $P_* \times Q_*$ is a projective resolution of $M \times N$ by $A \times B$-bimodules. Deduce that

$$H_*(A \times B, M \times N) \cong H_*(A, M) \times H_*(B, N).$$

6. Cyclic (co)homology

In this section we define cyclic (co)homology groups using various (co)chain complexes.

6.1. Hochschild Groups Revisited. Any algebra A has two natural A-bimodules, namely A itself, where the left and right A-modules structures are given by multiplication, and the dual vector space $A^* = \mathrm{Hom}_k(A, k)$ of linear forms on A.

The latter has the following A-bimodule structure: if a_0, $a_1 \in A$ and $f \in A^*$, then $a_0 f a_1$ is the linear form defined by

$$(a_0 f a_1)(a) = f(a_1 a a_0) \tag{6.1}$$

for all $a \in A$. To simply notation, we shall henceforth write $HH_*(A)$ for the Hochschild homology groups $H_*(A, A)$, and $HH^*(A)$ for the Hochschild cohomology groups $H^*(A, A^*)$.

From Section 5.5 and from (5.3) it follows that $HH_*(A)$ are the homology groups of the chain complex $(C_*(A), b)$ defined by $C_q(A) = A^{\otimes(q+1)}$ for all $q \geq 0$, with differential

$$\begin{aligned} b(a_0 \otimes a_1 \otimes \cdots \otimes a_q) = & \sum_{i=0}^{q-1} (-1)^i a_0 \otimes a_1 \otimes \cdots \otimes a_i a_{i+1} \otimes \cdots \otimes a_q \\ & + (-1)^q a_q a_0 \otimes a_1 \otimes \cdots a_{q-1} \end{aligned} \tag{6.2}$$

for all $a_0, a_1, \ldots, a_q \in A$.

We leave as an exercise for the reader to check from Section 5.5 and (5.4) that $HH^*(A)$ are the cohomology groups of the cochain complex $(C^*(A), \delta)$ dual to the chain complex $(C_*(A), b)$. In particular, $C^q(A) = \mathrm{Hom}(A^{\otimes(q+1)}, k)$ consists of all q-multilinear forms on A, and the differential of $f \in C^{q-1}(A)$ is given by

$$\begin{aligned} \delta(f)(a_0 \otimes a_1 \otimes \cdots \otimes a_q) = & \sum_{i=0}^{q-1} (-1)^i f(a_0 \otimes a_1 \otimes \cdots \otimes a_i a_{i+1} \otimes \cdots \otimes a_q) \\ & + (-1)^q f(a_q a_0 \otimes a_1 \otimes \cdots \otimes a_{q-1}). \end{aligned} \tag{6.3}$$

Observe that the differential $\delta : C^0(A) = A^* \to C^1(A)$ is given for $f \in A^*$ by

$$\delta(f)(a_0 \otimes a_1) = f(a_0 a_1 - a_1 a_0) \qquad (6.4).$$

Hence $HH^0(A)$ consists of all trace maps from A to k in the sense of Section 3.1.

Since the cochain complex $(C^*(A), \delta)$ is dual to the chain complex $(C_*(A), b)$, we have the following duality between $HH_*(A)$ and $HH^*(A)$:

$$HH^*(A) \cong \mathrm{Hom}_k(HH_*(A), k). \tag{6.5}$$

It also follows from the above that $HH_*(A)$ and $HH^*(A)$ are functorial in A, meaning that for any morphism of algebra $f : A \to B$ we have graded linear maps $f_* : HH_*(A) \to HH_*(B)$ and $f^* : HH^*(B) \to HH^*(A)$ compatible with composition.

6.2. The Cyclic Operator. For any $q \geq 0$ consider the endomorphism t of $C_q(A) = A^{\otimes(q+1)}$ defined for all $a_0, \ldots, a_{q-1}, a_q \in A$ by

$$t(a_0 \otimes \cdots \otimes a_{q-1} \otimes a_q) = (-1)^q \, a_q \otimes a_0 \otimes \cdots \otimes a_{q-1} \tag{6.6}$$

The operator t is of order $q+1$ on $C_q(A)$ and therefore defines an action of the cyclic group of order $q+1$ on $C_q(A)$ and on the dual space $C^q(A)$. We denote by N the associated norm map

$$N = \mathrm{id} + t + t^2 + \cdots + t^q : C_q(A) \to C_q(A). \tag{6.7}$$

We have

$$(\mathrm{id} - t)N = N(\mathrm{id} - t) = \mathrm{id} - t^{q+1} = 0. \tag{6.8}$$

Now recall the degree -1 differential

$$b' : C'_q(A) = C_{q+1}(A) \to C'_{q-1}(A) = C_q(A)$$

of Lemma 5.3. The following lemma follows from a straightforward computation.

6.3. Lemma.— *We have*

$$b(\mathrm{id} - t) = (\mathrm{id} - t)b' \quad \textit{and} \quad NB = b'N.$$

The first relation shows that the differential b sends the image of $\mathrm{id} - t$ in the vector space $C_q(A)$ to the image of $\mathrm{id} - t$ in $C_{q-1}(A)$. Therefore, if we define

$$C_q^{\mathrm{cyc}}(A) = C_q(A)/\mathrm{Im}(\mathrm{id} - t), \tag{6.9}$$

we obtain a chain complex $(C_*^{\mathrm{cyc}}(A), b)$, which is a quotient of the Hochschild chain complex $(C_*(A), b)$.

In the dual setting, the elements f of $C^q(A) = \mathrm{Hom}_k(A^{\otimes(q+1)}, k)$ satisfying the condition

$$f(a_q \otimes a_0 \otimes \cdots \otimes a_{q-1}) = (-1)^q \, f(a_0 \otimes \cdots \otimes a_{q-1} \otimes a_q) \tag{6.10}$$

for all $a_0, \ldots, a_{q-1}, a_q \in A$ form a cochain subcomplex $(C^*_{\mathrm{cyc}}(A), \delta)$ of the Hochschild cochain complex $(C^*(A), \delta)$. The observation that the Hochschild cocycles satisfying (6.10) are preserved by the differential δ was first made by Connes [1985], who used this to give the first historical definition of cyclic cohomology.

6.4. The Cyclic Bicomplex. Consider the following diagram, where for simplicity we set C_q for $C_q(A)$:

$$\begin{array}{ccccccc}
\vdots & & \vdots & & \vdots & & \\
b\downarrow & & -b'\downarrow & & b\downarrow & & \\
C_3 & \xleftarrow{\mathrm{id}-t} & C_3 & \xleftarrow{N} & C_3 & \xleftarrow{\mathrm{id}-t} & \cdots \\
b\downarrow & & -b'\downarrow & & b\downarrow & & \\
C_2 & \xleftarrow{\mathrm{id}-t} & C_2 & \xleftarrow{N} & C_2 & \xleftarrow{\mathrm{id}-t} & \cdots \\
b\downarrow & & -b'\downarrow & & b\downarrow & & \\
C_1 & \xleftarrow{\mathrm{id}-t} & C_1 & \xleftarrow{N} & C_1 & \xleftarrow{\mathrm{id}-t} & \cdots \\
b\downarrow & & -b'\downarrow & & b\downarrow & & \\
C_0 & \xleftarrow{\mathrm{id}-t} & C_0 & \xleftarrow{N} & C_0 & \xleftarrow{\mathrm{id}-t} & \cdots
\end{array} \tag{6.11}$$

By (6.8) and Lemma 6.3 this forms a bicomplex, which we denote $CC_{**}(A)$. Let $CC_*(A)$ be the associated chain complex; it is defined by

$$CC_q(A) = C_q(A) \oplus C_{q-1}(A) \oplus C_{q-2}(A) \oplus C_{q-3}(A) \oplus \cdots \oplus C_0(A) \tag{6.12}$$

with differential ∂ mapping $(a_q, a_{q-1}, a_{q-2}, a_{q-3}, \ldots)$ to

$$\big(b(a_q) + (\mathrm{id} - t)(a_{q-1}), -b'(a_{q-1}) + N(a_{q-2}), b(a_{q-2}) + (\mathrm{id} - t)(a_{q-3}), \ldots\big). \tag{6.13}$$

We define the *cyclic homology groups* $HC_*(A)$ of the algebra A as the homology groups of the chain complex $(CC_*(A), \partial)$. Taking the dual complex, we obtain the *cyclic cohomology groups* $HC^*(A)$.

6.5. The Characteristic Zero Case. Let V be a vector space on which an operator t of order $q+1$ acts. Define the operator $N : V \to V$ by (6.7) as above. Assume that the ground field k is of characteristic zero. The identity

$$\mathrm{id} = \frac{\mathrm{id}}{q+1} N - (\mathrm{id} - t) \frac{t + 2t^2 + \cdots + qt^q}{q+1} \tag{6.14}$$

implies $\mathrm{Ker}(\mathrm{id} - t) = \mathrm{Im}(N)$ and $\mathrm{Ker}(N) = \mathrm{Im}(\mathrm{id} - t)$. Therefore all rows in the bicomplex $CC_{**}(A)$ are exact, except for the group in the leftmost column; hence, the homology of a row is equal to $C_*(A)/\mathrm{Im}(\mathrm{id} - t)$, which we denoted above by $C_*^{\mathrm{cyc}}(A)$. This observation implies the following result.

6.6. Proposition.— *When the algebra A contains the field $\mathbf{Q}$ of rational numbers, then*

$$H_*(C_*^{\mathrm{cyc}}(A), b) \cong HC_*(A) \quad \textit{and} \quad H^*(C^*_{\mathrm{cyc}}(A), \delta) \cong HC^*(A).$$

The use of the adjective "cyclic" comes from this interpretation of cyclic (co)homology.

6.7. Connes's Operator B. Recall the map

$$s : C'_{q-1}(A) = C_q(A) \to C'_q(A) = C_{q+1}(A)$$

from Section 5.2. By Lemma 5.3 we have

$$b's + sb' = \mathrm{id}. \tag{6.15}$$

This implies that all odd-numbered columns in the bicomplex $CC_{**}(A)$ are acyclic. In a sense they are useless, and we want to get rid of them. Before we do this, let us introduce the degree $+1$ operator

$$B = (\mathrm{id} - t)sN : C_q(A) \to C_{q+1}(A), \tag{6.16}$$

called *Connes's operator* B. The following is a consequence of (6.8) and (6.15).

6.8. Lemma.— *We have* $B^2 = Bb + bB = 0$.

6.9. The $b-B$-Bicomplex. In view of the previous lemma we may consider the following bicomplex, which we denote $\mathcal{B}_{**}(A)$:

$$\begin{array}{ccccccccc}
\vdots & & \vdots & & \vdots & & \vdots & & \\
{\scriptstyle b}\downarrow & & {\scriptstyle b}\downarrow & & {\scriptstyle b}\downarrow & & {\scriptstyle b}\downarrow & & \\
C_3 & \xleftarrow{B} & C_2 & \xleftarrow{B} & C_1 & \xleftarrow{B} & C_0 & \longleftarrow & \cdots \\
{\scriptstyle b}\downarrow & & {\scriptstyle b}\downarrow & & {\scriptstyle b}\downarrow & & \downarrow & & \\
C_2 & \xleftarrow{B} & C_1 & \xleftarrow{B} & C_0 & \longleftarrow & 0 & \longleftarrow & \cdots \\
{\scriptstyle b}\downarrow & & {\scriptstyle b}\downarrow & & \downarrow & & \downarrow & & \\
C_1 & \xleftarrow{B} & C_0 & \longleftarrow & 0 & \longleftarrow & 0 & \longleftarrow & \cdots \\
{\scriptstyle b}\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
C_0 & \longleftarrow & 0 & \longleftarrow & 0 & \longleftarrow & 0 & \longleftarrow & \cdots
\end{array} \tag{6.17}$$

where for simplicity we have written C_q instead of $C_q(A)$. Let $\mathcal{B}_*(A)$ be the associated chain complex; it is defined by

$$\mathcal{B}_q(A) = C_q(A) \oplus C_{q-2}(A) \oplus C_{q-4}(A) \oplus \cdots \tag{6.18}$$

with differential ∇ mapping $(a_q, a_{q-2}, a_{q-4}, a_{q-6}, \ldots)$ to

$$\big(b(a_q) + B(a_{q-2}), b(a_{q-2}) + B(a_{q-4}), b(a_{q-4}) + B(a_{q-6}), \ldots\big). \tag{6.19}$$

6.10. Proposition.— *There is a morphism of complexes* $i : \mathcal{B}_*(A), \nabla \to CC_*(A), \partial$ *inducing an isomorphism in homology*

$$H_*(\mathcal{B}_*(A), \nabla) \cong HC_*(A) = H_*(CC_*(A), \partial).$$

The formulas for the morphism of complexes $i : \mathcal{B}_*(A) \to CC_*(A)$ and the proof of the proposition can be found in Loday-Quillen [1984] and in Loday [1992a]. A formula for a morphism of complexes $j : CC_*(A), \partial \to \mathcal{B}_*(A), \nabla$ and for homotopies between the composition of i and j and the identity can be found in Kassel [1990] (the latter paper is among other devoted to a detailed study of the chain maps and homotopies connecting the three complexes $C_*^{\mathrm{cyc}}(A)$, $CC_*(A)$, $\mathcal{B}_*(A)$; it uses a very useful tool in homological algebra, namely the so-called *perturbation lemma*).

Using Proposition 6.10, we can freely switch between the cyclic bicomplex $CC_*(A)$ and the $b-B$-bicomplex $\mathcal{B}_*(A)$.

6.11. Mixed Complexes and Connes's Long Exact Sequence. A *mixed complex* is a triple $C = (C_*, b, B)$, where $C_* = (C_q)_{q\geq 0}$ is a positively graded vector space, $b : C_* \to C_*$ is a linear endomorphism of degree -1, $B : C_* \to C_*$ is a linear endomorphism of degree $+1$ satisfying

$$b^2 = B^2 = Bb + bB = 0. \tag{6.20}$$

The chain complex (C_*, b) is called the underlying Hochschild complex of the mixed complex C and its homology $H_*(C_*, b)$ will be denoted by $HH_*(C)$.

A *morphism* $f : (C_*, b, B) \to (C'_*, b, B)$ *of mixed complexes* is a degree 0 linear map $f : C_* \to C'_*$ such that $bf = fb$ and $Bf = fB$. As Lemma 6.8 shows, any algebra A gives rise to a mixed complex $C(A) = (C_*(A), b, B)$ and any morphism of algebras gives rise to a morphism of mixed complexes. We shall see more examples of mixed complexes in Section 7.

To a mixed complex we can associate the bicomplex $\mathcal{B}_{**}(C)$ described by (6.17). We denote by $(\mathcal{B}_*(C), \nabla)$ the associated complex. We define the cyclic homology of the mixed complex $C = (C_*, b, B)$ by

$$HC_*(C) = H_*(\mathcal{B}_*(C), \nabla). \tag{6.21}$$

The underlying Hochschild complex of a mixed complex C, viewed as the leftmost column of the bicomplex $\mathcal{B}_{**}(C)$, is a subcomplex of $(\mathcal{B}_*(C), \nabla)$. The quotient chain complex is the chain complex associated to the bicomplex $\mathcal{B}_{**}(C)$ from which we have removed the leftmost column. Therefore we have the short exact sequence of chain complexes

$$0 \to C_* \xrightarrow{I} \mathcal{B}_*(C) \xrightarrow{S} s^2\mathcal{B}_*(C) \to 0, \tag{6.22}$$

where $s^2\mathcal{B}_*(C)$ is the complex defined by $(s^2\mathcal{B}_*(C))_q = s^2\mathcal{B}_*(C)_{q-2}$ and the same differential ∇. The exact sequence (6.22) gives rise to a long exact sequence in homology, which is called *Connes's long exact sequence*:

$$\cdots \xrightarrow{B} HH_q(C) \xrightarrow{I} HC_q(C) \xrightarrow{S} HC_{q-2}(C) \xrightarrow{B} HH_{q-1}(C) \xrightarrow{I} HC_{q-1}(C) \xrightarrow{S} \cdots \tag{6.23}$$

This sequence ends in low degrees with

$$\cdots \xrightarrow{S} HC_0(C) \xrightarrow{B} HH_1(C) \xrightarrow{I} HC_1(C) \to 0 \to HH_0(C) \xrightarrow{I} HC_0(C) \to 0. \tag{6.24}$$

The degree -2 map $S : HC_q(C) \to HC_{q-2}(C)$ is called the *periodicity map.*

Connes's long exact sequence has the following immediate consequence.

6.12. PROPOSITION.— *For any mixed complex C we have $HC_0(C) \cong HH_0(C)$. Moreover, if there is q such that $HH_i(C) = 0$ for all $i > q$, then the periodicity map $S : HC_{i+2}(C) \to HC_i(C)$ is an isomorphism when $i \geq q$ and a monomorphism (injection) when $i = q - 1$*

Observe that the periodicity map S also exists on the level of the cyclic complex $CC_*(A)$; it is induced by killing the two leftmost columns in the bicomplex (6.11).

6.13. EXERCISES. (a) Show that the cyclic homology groups of the ground field k are given by

$$HC_q(k) = \begin{cases} k & \text{if } q \text{ is even,} \\ 0 & \text{if } q \text{ is odd,} \end{cases} \tag{6.25}$$

(b) For the Weyl algebra $A_1(k)$ over a field of characteristic zero show that

$$HC_q(A_1(k)) = \begin{cases} k & \text{if } q \text{ is even} \geq 2, \\ 0 & \text{otherwise,} \end{cases} \tag{6.26}$$

(c) Compute the cyclic homology of the tensor algebra $T(V)$, of the truncated polynomial algebra of Section 5.9.

7. Cyclic homology and differential forms

In this section we assume that all algebras are *commutative.*

7.1. DIFFERENTIAL FORMS. For a commutative algebra A let Ω^1_A be the left A-module generated by all symbols of the form da, where a runs over all elements of A, subject to the relations

$$d(a_1 a_2) = a_1 da_2 + a_2 da_1 \tag{7.1}$$

for all a_1, $a_2 \in A$. Relation (7.1) states that the k-linear map $d : A \to \Omega^1_A$ is a *derivation.* The A-module Ω^1_A is called the module of *1-differential forms* (or *Kähler differentials*) on A. It satisfies the following universal property: for any derivation $D : A \to M$ on A with values in a left A-module M, there is a unique A-linear map $f : \Omega^1_A \to M$ such that $D(a) = f(da)$ for all $a \in A$.

The k-linear map $d : A \to \Omega^1_A$ can be extended into a cochain complex

$$0 \to A \xrightarrow{d} \Omega^1_A \xrightarrow{d} \Omega^2_A \xrightarrow{d} \Omega^3_A \xrightarrow{d} \cdots \tag{7.2}$$

called the *de Rham complex* of A; the differential d is called the *de Rham differential*. For $q \geq 2$, the vector space Ω^q_A is defined as the q-th exterior power of the A-module Ω^1_A:

$$\Omega^q_A = \Lambda^q_A \Omega^1_A. \tag{7.3}$$

Its elements are linear combinations of expressions of the form $a_0 da_1 \cdots da_q$, where $a_0, a_1, \ldots, a_q \in A$. These expressions are antisymmetric in the variables $a_1, \ldots, a_q$. By convention we set $\Omega^0_A = A$. The de Rham differential $d : \Omega^q_A \to \Omega^{q+1}_A$ is defined by

$$d(a_0 da_1 \cdots da_q) = da_0 da_1 \cdots da_q. \tag{7.4}$$

It satisfies $d^2 = 0$. The cohomology $H^*_{\mathrm{DR}}(A)$ of the cochain complex (7.2) is called the *de Rham cohomology* of the (commutative) algebra A.

As an exercise, compute the de Rham cohomology of k and of $k[X]$ when k is a field of characteristic zero.

Let us denote by Ω_A the mixed complex $(\Omega^q_A, 0, d)$ in the sense of Section 6.11. Its Hochschild and cyclic homology are clearly given for all q by

$$HH_q(\Omega_A) = \Omega^q_A \tag{7.5a}$$

and

$$HC_q(\Omega_A) = \Omega^q_A / d\Omega^{q-1}_A \oplus H^{q-2}_{\mathrm{DR}}(A) \oplus H^{q-4}_{\mathrm{DR}}(A) \oplus \cdots \tag{7.5b}$$

7.2. Relating Cyclic Homology and de Rham Cohomology. Assume that k is of characteristic zero. For $q \geq 0$ let $\mu : C_q(A) = A^{\otimes(q+1)} \to \Omega^q_A$ be defined by

$$\mu(a_0 \otimes a_1 \otimes \cdots \otimes a_q) = a_0 da_1 \cdots da_q. \tag{7.6}$$

The following relations are easy to check:

$$\mu b = 0 \quad \text{and} \quad \mu B = d\mu. \tag{7.7}$$

This implies that μ defines a morphism of mixed complexes $(C_*(A), b, B) \to (\Omega^q_A, 0, d)$, hence there are maps

$$HH_q(A) \to \Omega^q_A \tag{7.8a}$$

and

$$HC_q(A) \to \Omega^q_A / d\Omega^{q-1}_A \oplus H^{q-2}_{\mathrm{DR}}(A) \oplus H^{q-4}_{\mathrm{DR}}(A) \oplus \cdots \tag{7.8b}$$

For a commutative algebra A the Hochschild complex turns out to be

$$\cdots \xrightarrow{b} A \otimes A \otimes A \xrightarrow{b} A \otimes A \xrightarrow{0} A \to 0$$

in low degrees. Therefore, as we already know, $HH_0(A) = A = \Omega^0_A$, and in degree one we obtain $HH_1(A) = A \otimes A/b(A \otimes A \otimes A)$, where $b(A \otimes A \otimes A)$ is spanned by all expressions of the form

$$a_0a_1 \otimes a_2 - a_0 \otimes a_1a_2 + a_2a_0 \otimes a_1.$$

From this it follows immediately that μ induces an isomorphism

$$HH_1(A) \cong \Omega^1_A. \tag{7.9}$$

Together with the exact sequence (6.24), this implies

$$HC_1(A) \cong \Omega^1_A/dA. \tag{7.10}$$

In higher degree Hochschild homology is in general different from differential forms. Nevertheless for an important class of algebras, we still have isomorphisms. This is the class of so-called *smooth algebras*. I will not give a definition of a smooth algebra. Let me just mention that the algebras of functions on a smooth affine variety are smooth. For instance, the polynomial algebras are smooth, the algebra $k[x,y]/(x^2+y^2-1)$ is smooth, but the truncated polynomial algebra $k[x]/(x^2)$ is not smooth (it has a singularity at 0).

7.3. Theorem.— *If A is a smooth commutative algebra over a field of characteristic zero, then μ induces isomorphisms*

$$HH_q(A) \cong \Omega^q_A \quad \textit{and} \quad HC_q(A) \cong \Omega^q_A/d\Omega^{q-1}_A \oplus H^{q-2}_{\mathrm{DR}}(A) \oplus H^{q-4}_{\mathrm{DR}}(A) \oplus \cdots$$

for all $q \geq 0$.

For Hochschild homology this theorem is due to Hochschild, Kostant, Rosenberg [1962]. The implication for cyclic homology is an immediate consequence; it is due to Loday & Quillen [1984].

The Hochschild and cyclic homology of a *singular* (i.e., non-smooth) commutative algebra A can also be expressed in terms of differential forms, but we must replace the differential forms on A by differential forms on a resolution of A in the category of commutative differential graded algebras, see Burghelea & Vigué-Poirrier [1988], Vigué-Poirrier [1991].

We may retain from this section that Connes's operator B is the noncommutative analogue of the de Rham differential and that cyclic homology is the noncommutative version of de Rham cohomology. It was introduced by Connes precisely to have a version of de Rham cohomology for noncommutative algebras.

7.4. Brylinski's Mixed Complex. Before we close this section, we give another interesting example of a mixed complex, also related to differential forms.

Let A be a *filtered* algebra, i.e., equipped with an increasing filtration

$$0 = A^{-1} \subset A^0 \subset A^1 \subset \cdots \subset A^q \subset \cdots$$

of subspaces of A such that $\bigcup_{q\geq 0} = A$, the multiplication of A induces a map $A^p \otimes A^q \to A^{p+q}$ for all $p, q \geq 0$, and the unit of A belongs to A^0.

To any filtered algebra A we associate a graded algebra $S = \oplus_{q\geq 0} S_q$, where $S_q = A^q/A^{q-1}$ for all $q \geq 0$. If S happens to be commutative, then it is a *Poisson algebra*, i.e., S possesses a bilinear map $\{\, , \,\} : S \times S \to S$, called the *Poisson bracket*, satisfying the relations

$$\{g,f\} = -\{f,g\}, \quad \{f,\{g,h\}\} + \{g,\{h,f\}\} + \{h,\{f,g\}\} = 0, \tag{7.11a}$$

$$\{f,gh\} = \{f,g\}h + g\{f,h\} \tag{7.11b}$$

for all f, g, $h \in S$. The Poisson bracket $\{f,g\}$ is defined as follows for $f \in S^p$ and $g \in S^q$: lift f to $a \in A^p$ and g to $b \in A^q$. Since S is assumed to be commutative, the commutator $[a,b] = ab - ba \in A$, which belongs to A^{p+q}, actually sits in the smaller subspace A^{p+q-1}. We define $\{f,g\}$ to be the image of $[a,b]$ in S^{p+q-1}. It is an exercise to check that this bracket is well-defined and is a Poisson bracket.

Brylinski [1988] showed that, when A is a filtered algebra over a field of characteristic zero such that the associated graded algebra S is *commutative and smooth*, then there exists a mixed complex (Ω^*_S, δ, d), where δ is the degree -1 differential on the differential forms on S defined for all $f_0, f_1, \ldots, f_q \in S$ by

$$\begin{aligned}\delta(f_0 df_1 \cdots df_q) = & \sum_{i=1}^{q+1} (-1)^{i+1} \{f_0, f_i\} df_1 \cdots \widehat{df_i} \cdots df_q \\ & + \sum_{1\leq i<j\leq q} (-1)^{i+j}\, f_0 d\{f_i, f_j\} df_1 \cdots \widehat{df_i} \cdots \widehat{df_j} \cdots df_q,\end{aligned} \tag{7.12}$$

where we have removed the expressions under the hats in the previous sums.

The following result was proved in Kassel [1988a].

7.5. Theorem.— *If A is a filtered algebra over a field of characteristic zero such that the associated graded algebra S is isomorphic to a polynomial algebra $k[x_1, \ldots, x_n]$, then the Hochschild and cyclic homology of A are isomorphic to the Hochschild and cyclic homology of the mixed complex (Ω^*_S, δ, d):*

$$HH_q(A) \cong H_q(\Omega^*_S, \delta) \quad \textit{and} \quad HC_q(A) \cong HC_q(\Omega^*_S, \delta, d)$$

for all $q \geq 0$.

This applies in particular to the Weyl algebra $A_1(k)$ and to the universal enveloping algebra of a Lie algebra. See an application of this theorem to the computation of the cyclic homology of the universal enveloping algebras of all Lie algebras of dimension three in Nuss [1991].

8. The algebraic Chern character

In this section we give a *bivariant* version of cyclic (co)homology and of algebraic K-theory. We connect these bivariant theories through a map that is a noncommutative analogue of the Chern character in differential geometry.

8.1. Bivariant Cyclic Cohomology. John Jones and the author (see Jones & Kassel [1989], Kassel [1989]) introduced *bivariant Hochschild cohomology* groups $HH^q(A_1, A_2)$ and *bivariant cyclic cohomology* groups $HC^q(A_1, A_2)$, depending on a rational integer $q \in \mathbf{Z}$ and on two algebras A_1, A_2. These groups satisfy the following properties:

(a) they are contravariant functors in A_1 and covariant functors in A_2,

(b) they are equipped with natural associative composition products

$$HH^q(A_1, A_2) \otimes HH^r(A_2, A_3) \xrightarrow{\cup} HH^{q+r}(A_1, A_3) \qquad (8.1a)$$

and

$$HC^q(A_1, A_2) \otimes HC^r(A_2, A_3) \xrightarrow{\cup} HC^{q+r}(A_1, A_3), \qquad (8.1b)$$

(c) there is a Connes-type long exact sequence

$$\cdots \xrightarrow{B} HC^{q-2}(A_1, A_2) \xrightarrow{S} HC^q(A_1, A_2) \xrightarrow{I} HH^q(A_1, A_2) \xrightarrow{B} HC^{q-1}(A_1, A_2) \xrightarrow{S} \cdots \qquad (8.2)$$

(d) When $A_2 = k$, then we recover Hochschild and cyclic cohomology:

$$HH^q(A_1, k) \cong HH^q(A_1) \quad \text{and} \quad HC^q(A_1, k) \cong HC^q(A_1). \qquad (8.3)$$

When $A_1 = k$, we recover Hochschild homology:

$$HH^q(k, A_2) \cong HH_{-q}(A_2). \qquad (8.3)$$

The groups $HC^q(k, A_2)$ are not isomorphic to the cyclic homology groups $HC_{-q}(A_2)$, but to Goodwillie's *negative cyclic homology* groups $HC^-_{-q}(A_2)$. We will not define HC^-_* here; the reader may take

$$HC^-_q(A) = HC^{-q}(k, A) \qquad (8.4)$$

as a definition.

The bivariant Hochschild cohomology groups $HH^*(A_1, A_2)$ are defined as the cohomology groups of the Hom-complex

$$\mathrm{Hom}^*(C_*(A_1), C_*(A_2))$$

whose elements in degree q are families of degree $-q$ maps from the Hochschild complex $C_*(A_1)$ to the Hochschild complex $C_*(A_2)$.

Similarly, the bivariant cyclic cohomology groups $HC^*(A_1, A_2)$ are defined as the cohomology groups of the Hom-complex

$$\mathrm{Hom}^*_S(CC_*(A_1), CC_*(A_2))$$

whose elements in degree q are families of degree $-q$ maps from the cyclic complex $CC_*(A_1)$ to the cyclic complex $CC_*(A_2)$, with the additional requirement that such families commute with the periodicity map S. The composition products (8.1) are induced by the composition in the Hom-complexes.

Let us concentrate on the group $HC^0(A_1, A_2)$: each of its elements can be represented by a (degree 0) morphism of complexes

$$u : CC_*(A_1) \to CC_*(A_2)$$

such that $Su = uS$. Let us give a few examples of such morphisms.

(a) A *morphism of algebras* $f : A_1 \to A_2$ clearly induces a morphism of complexes $CC_*(A_1) \to CC_*(A_2)$ commuting with S. Let us denote the corresponding element of $HC^0(A_1, A_2)$ by $[f]$. In particular, for any algebra A we have the canonical element $[\mathrm{id}_A] \in HC^0(A, A)$.

If $g : A_2 \to A_3$ is another morphism of algebras, then we immediately have

$$[g \circ f] = [f] \cup [g] \in HC^0(A_1, A_3). \tag{8.5}$$

From the definition of the complex CC_* we see that, in order for $[f]$ to be defined, we do not need the morphism of algebras f to preserve the units. This observation is very useful when we deal with matrix algebras. For instance, the inclusion i of an algebra A into the matrix algebra $M_p(A)$ given by

$$i(a) = \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix} \tag{8.6}$$

is not unit-preserving, but nevertheless it induces an element $[i] \in HC^0(A, M_p(A))$.

(b) The *trace of matrices* $\mathrm{tr} : M_p(A) \to A$ can be extended to a morphism of complexes $\mathrm{tr}_* : CC_*(M_p(A)) \to CC_*(A)$ commuting with S; it thus defines an element $[\mathrm{tr}] \in HC^0(M_p(A), A)$. The morphism tr_* is given on $C_q(M_p(A)) = M_p(A)^{\otimes(q+1)}$ by

$$a(0) \otimes a(1) \otimes \cdots \otimes a(q) \mapsto \sum_{1 \le i_0, i_1, \ldots, i_q \le p} a(0)_{i_0 i_1} \otimes a(1)_{i_1 i_2} \otimes \cdots \otimes a(q)_{i_q i_0} \tag{8.7}$$

$(a(0), a(1), \ldots, a(q) \in M_p(A))$. When we take the composition product of $[i] \in HC^0(A, M_p(A))$ with $[\mathrm{tr}] \in HC^0(M_p(A), A)$, we obviously obtain

$$[i] \cup [\mathrm{tr}] = [\mathrm{id}_A] \in HC^0(A, A). \tag{8.8}$$

The following was proved in Jones & Kassel [1988] and Kassel [1990].

8.2. PROPOSITION.— *We have*

$$[\mathrm{tr}] \cup [i] = [\mathrm{id}_{M_p(A)}] \in HC^0(M_p(A), M_p(A)).$$

Let us say that two algebras A_1 and A_2 are *HC-equivalent* of there are elements $\alpha \in HC^0(A_1, A_2)$ and $\beta \in HC^0(A_2, A_1)$ such that $\alpha \cup \beta = [\mathrm{id}_{A_1}]$ and $\beta \cup \alpha = [\mathrm{id}_{A_2}]$. It is clear that if, A_1 and A_2 are HC-equivalent, then there are natural isomorphisms

$$HC^*(A, A_1) \cong HC^*(A, A_2), \quad HH^*(A, A_1) \cong HH^*(A, A_2) \tag{8.9a}$$

$$HC^*(A_1, A) \cong HC^*(A_2, A), \quad HH^*(A_1, A) \cong HH^*(A_2, A) \tag{8.9b}$$

for all algebras A. This means that HC-equivalent algebras are undistinguishable from the point of view of cyclic and Hochschild (co)homology.

As a consequence of (8.8) and of Proposition 8.2, any algebra A is HC-equivalent to any of its matrix algebras $M_p(A)$.

8.3. BIVARIANT ALGEBRAIC K-THEORY. Given two algebras A_1, A_2, let $\mathcal{R}ep(A_1, A_2)$ be the category of all A_1-A_2-bimodules that are finitely generated projective as right A_2-modules. This category, equipped with all exact sequences, gives rise to algebraic K-groups following QUILLEN [1973]. In particular, we have a K_0-group defined as in Section 4.3, which we denote by $K^0(A_1, A_2)$.

The group $K^0(A_1, A_2)$ is clearly a contravariant functor in A_1 and a covariant functor in A_2.

If P is an object of $\mathcal{R}ep(A_1, A_2)$ and Q is an object of $\mathcal{R}ep(A_2, A_3)$, then it is easy to check that $P \otimes_{A_2} Q$ is an object of $\mathcal{R}ep(A_1, A_3)$. This induces a composition product

$$K^0(A_1, A_2) \otimes_{\mathbf{Z}} K^0(A_2, A_3) \longrightarrow K^0(A_1, A_3). \tag{8.10}$$

Observe that $\mathcal{R}ep(k, A_2)$ is the category of finitely generated projective right A_2-modules and that $K^0(k, A_2)$ is the group $K_0(A_2)$ of Section 4.3.

The category $\mathcal{R}ep(A_1, k)$ is the category of all finite-dimensional representations $A_1 \to M_p(k)$ of A_1.

8.4. THE BIVARIANT CHERN CHARACTER. To any object P of $\mathcal{R}ep(A_1, A_2)$ we now attach an element of $HC^0(A_1, A_2)$. As we observed in Exercise 2.5 (b), the bimodule P gives rise to a morphism of algebras $\rho : A_1 \to \mathrm{End}_{A_2}(P)$. It thus defines an element $[\rho] \in HC^0(A_1, \mathrm{End}_{A_2}(P))$.

The bimodule P being finitely generated projective over A_2, there are an A_2-module P', an integer $p \geq 1$ and an isomorphism $P \oplus P' \cong A_2^p$ of A_2-modules. Let ι be the composite map

$$\iota : \mathrm{End}_{A_2}(P) \xrightarrow{\mathrm{id}_P \oplus 0_{P'}} \mathrm{End}_{A_2}(P \oplus P') \xrightarrow{\cong} M_p(A_2);$$

this map is a morphism of algebras that does not preserve the units. It defines an element $[\iota] \in HC^0(\mathrm{End}_{A_2}(P), M_p(A))$.

Define $\mathrm{ch}(P)$ to be the element

$$\mathrm{ch}(P) = [\rho] \cup [\iota] \cup [\mathrm{tr}] \in HC^0(A_1, A_2). \tag{8.11}$$

The following theorem was proved in KASSEL [1989] (see also KASSEL [1988b]).

8.5. THEOREM.— *(a) The element* $\mathrm{ch}(P) \in HC^0(A_1, A_2)$ *depends only of the class of* P *in* $K^0(A_1, A_2)$ *and it defines a homomorphism of abelian groups*

$$\mathrm{ch} : K^0(A_1, A_2) \to HC^0(A_1, A_2).$$

(b) If P *is an object of* $\mathcal{R}ep(A_1, A_2)$ *and* Q *is an object of* $\mathcal{R}ep(A_2, A_3)$*, then we have*

$$\mathrm{ch}(P \otimes_{A_2} Q) = \mathrm{ch}(P) \cup \mathrm{ch}(Q).$$

Theorem 8.5 has the following immediate consequence (which was proved in a different way by MCCARTHY [1988]).

8.6. COROLLARY.— *Two Morita equivalent algebras are HC-equivalent.*

PROOF.— Let $(A, B, P, Q, \alpha, \beta)$ be a Morita context as in Section 2.8. Then P is an object in $\mathcal{R}ep(A, B)$ and Q is an object in $\mathcal{R}ep(B, A)$ such that $P \otimes_A Q \cong {}_{\mathrm{id}_A}A$ and $Q \otimes_B P \cong {}_{\mathrm{id}_B}B$. Applying Theorem 8.5, we obtain

$$\mathrm{ch}(P) \cup \mathrm{ch}(Q) = \mathrm{ch}({}_{\mathrm{id}_A}A) \quad \text{and} \quad \mathrm{ch}(Q) \cup \mathrm{ch}(P) = \mathrm{ch}({}_{\mathrm{id}_B}B).$$

It is easy to see that $\mathrm{ch}({}_{\mathrm{id}_A}A) = [\mathrm{id}_A] \in HC^0(A, A)$ and $\mathrm{ch}({}_{\mathrm{id}_B}B) = [\mathrm{id}_B] \in HC^0(B, B)$. Therefore, A and B are HC-equivalent. □

8.7. REMARKS. (a) When $A_1 = k$, we obtain a Chern character

$$\mathrm{ch} : K_0(A_2) = K^0(k, A_2) \to HC^0(k, A_2) = HC_0^-(A_2). \tag{8.12}$$

This is the right formalization for the characteristic classes defined by CONNES [1985] and KAROUBI [1982] on the level of K_0. When composing the map (8.12) with the map $I : HC^0(k, A_2) = HC_0^-(A_2) \to HH^0(k, A_2) = HH_0(A_2)$ of (8.2), we recover the Hattori-Stallings trace of Section 4.6.

(b) CONNES [1985] (for the case $i = 1$) and KAROUBI [1983b] (for all $i \geq 1$) defined Chern characters for higher algebraic K-groups

$$K_i(A_2) \to HC^{-i}(k, A_2) = HC_i^-(A_2). \tag{8.13}$$

One can extend these to higher bivariant Chern characters

$$K^i(A_1, A_2) \to HC^i(A_1, A_2).$$

(c) There are other objects having an image in bivariant cyclic cohomology, such as the quasi-homomorphisms introduced by Cuntz in order to simplify the definition of Kasparov's KK-theory (see KASSEL [1989], Section III.6).

(d) Using the periodicity map S in the long exact sequence (8.2), we may define the *bivariant periodic cyclic cohomology* groups $HP^*(A_1, A_2)$ as the limit of the inductive system

$$\{\cdots \xrightarrow{S} HC^{*+2p}(A_1, A_2) \xrightarrow{S} HC^{*+2p+2}(A_1, A_2) \xrightarrow{S} \cdots\}. \tag{8.14}$$

These groups are periodic of period two. They have the important additional property that they are *homotopy-invariant*, i.e., when one evaluates an element $\alpha(t) \in HP^*(A_1, A_2[t])$ at $t = 0$ and at $t = 1$, the elements $\alpha(0)$ and $\alpha(1)$ we obtain coincide in $HP^*(A_1, A_2)$.

Notes

Hochschild cohomology was introduced by HOCHSCHILD [1945], [1946]. The first presentation as derived functor was given in CARTAN & EILENBERG [1956]. Hochschild homology was related to differential forms for the first time by HOCHSCHILD, KOSTANT, ROSENBERG [1962]. The operator B, as named by CONNES [1985], had actually been defined by RINEHART [1963], who also defined on the Hochschild complex the Lie derivative and the interior produit with respect to a derivation.

The bicomplex $CC_{**}(A)$ of Section 6, implicit in TSYGAN [1983], was made explicit by LODAY & QUILLEN [1984]. The cyclic complex $C^*_{\mathrm{cyc}}(A)$ and the $b-B$-bicomplex are due to CONNES [1985], as well as Connes's long exact sequence. The computation of the cyclic homology of smooth algebras is due to LODAY & QUILLEN [1984] in the algebraic case and to CONNES [1985] in the C^∞-case. Mixed complexes (whose name was suggested by A. Borel) were introduced and extensively studied in KASSEL [1987].

For Künneth formulas allowing to compute the cyclic homology of the tensor products of two algebras, see BURGHELEA & OGLE [1986] and KASSEL [1987]. An important property of cyclic homology, namely *excision*, was proved by WODZICKI [1989], with applications to K-theory (see SUSLIN & WODZICKI [1992]).

Below we have given a number of other references relevant to the subject. In addition to the above-mentioned references, we recommend reading CUNTZ & QUILLEN [1995a], [1995b], FEIGIN & TSYGAN [1987], GOODWILLIE [1985a], [1985b], [1986], JONES [1987], KASSEL [1988a], [1990], [1992], LODAY [1989], WAMBST [1993], [1997].

A standard textbook on cyclic homology is LODAY [1992a], see also HUSEMÖLLER [1991], WEIBEL [1994].

References

Monographies

H. Bass [1968]: *Algebraic K-theory*, W. A. Benjamin, Inc., New York-Amsterdam 1968.

H. Cartan, S. Eilenberg [1956]: *Homological algebra*, Princeton Univ. Press, Princeton, NJ, 1956.

A. Connes [1994]: *Noncommutative geometry*, Academic Press, 1994.

D. Husemöller [1991]: *Lectures on cyclic homology*, Notes by R. Sujatha, Lectures on Mathematics and Physics, Tata Institute of Fundamental Research, 83. Tata Institute of Fundamental Research, Berlin: Springer-Verlag, (ISBN 3-540-54667-7). viii, 101 p. (1991).

N. Jacobson [1989]: *Basic algebra, II*, Second edition, W. H. Freeman and Company, New York, 1989.

M. Karoubi [1987]: *Homologie cyclique et K-théorie*, Astérisque 149, Soc. Math. France, Paris, 1987.

C. Kassel [1994]: *Quantum groups*, Grad. Texts in Math., vol. 155, Springer-Verlag, New York, 1994.

J.-L. Loday [1992a]: *Cyclic homology*, Springer-Verlag, Heidelberg, 1 992.

J. Rosenberg [1994]: *Algebraic K-theory and its applications*, Grad. Texts in Math., vol. 147, Springer-Verlag, New York, 1994.

C. Weibel [1994]: *An introduction to homological algebra*, Cambridge Studies in Advanced Mathematics, 38, Cambridge University Press, Cambridge, 1994.

C. Weibel [2004]: *Introduction to algebraic K-theory*, a graduate textbook in progress, http://math.rutgers.edu/~weibel/Kbook.html

Articles and proceedings

J.-L. Brylinski [1988]: *A differential complex for Poisson manifolds*, J. Diff. Geom. 28 (1988), 93–114.

D. Burghelea, C. Ogle [1986]: *Künneth formula in cyclic homology*, Math. Z. 193 (1986), 527–536.

D. Burghelea, M. Vigué-Poirrier [1988]: *Cyclic homology of commutative algebras, I*, in Algebraic topology—rational homotopy (Louvain-la-Neuve, 1986), 51–72, Lecture Notes in Math., 1318, Springer, Berlin, 1988.

A. Connes [1985]: *Non-commutative differential geometry*, Publ. IHES 62 (1985), 41–144.

J. Cuntz, D. Quillen [1995a]: *Algebra extensions and nonsingularity*, Journal Amer. Math. Soc. 8 (1995), 251–289.
— [1995b]: *Cyclic homology and nonsingularity*, Journal Amer. Math. Soc. 8 (1995), 373–442.

B. L. Feigin, B. L. Tsygan [1983]: *Cohomology of Lie algebras of generalized Jacobi matrices*, Funct. Anal. Appl. 17 (1983), 153–155.

— [1985]: *Additive K-theory and crystalline cohomology*, Funct. Anal. Appl. 19 (1985), 124–132.

— [1987]: *Additive K-theory*, in *K-theory, arithmetic and geometry (Manin Seminar, Moscow 1984-86)*, Lect. Notes in Math. 1289, 67–209, Springer-Verlag, Heidelberg, 1987.

M. Gerstenhaber, S. D. Schack [1987]: *A Hodge-type decomposition for commutative algebra cohomology*, J. Pure Appl. Algebra 48 (1987), 229–247.

T. G. Goodwillie [1985a]: *Cyclic homology, derivations and the free loopspace*, Topology 24 (1985), 187–215.

— [1985b]: *On the general linear group and Hochschild homology*, Annals Math. 121 (1985), 383–407. Corrections: Annals Math. 124 (1986), 627–628.

— [1986]: *Relative algebraic K-theory and cyclic homology*, Annals Math. 124 (1986), 347–402.

J. A. Guccione, J. J. Guccione [1990]: *Cyclic homology of monogenic extensions*, J. Pure Appl. Algebra 66 (1990), 251–269.

— [1996]: *The theorem of excision for Hochschild and cyclic homology*, J. Pure Appl. Algebra 106 (1996), 57–60.

— [1997]: *Hochschild and cyclic homology of Ore extensions and some examples of quantum algebras*, K-Theory 12 (1997), 259–276.

— [1998]: *Hochschild homology of some quantum algebras*, J. Pure Appl. Algebra 132 (1998), 129–147.

A. Hattori [1965]: *Rank element of a projective module*, Nagoya Math. J. 25 (1965), 113–120.

G. Hochschild [1945]: *On the cohomology groups of an associative algebra*, Annals Math. 46 (1945), 58–67.

— [1946]: *On the cohomology theory for associative algebras*, Annals Math. 47 (1946), 568–579.

G. Hochschild, B. Kostant, A. Rosenberg [1962]: *Differential forms on regular affine algebras*, Trans. Amer. Math. Soc. 102(1962), 383–408.

J. D. S. Jones [1987]: *Cyclic homology and equivariant homology*, Invent. Math. 87 (1987), 403–423.

J. D. S. Jones, C. Kassel [1989]: *Bivariant cyclic theory*, K-Theory 3 (1989), 339–365.

M. Karoubi [1982]: *Connexions, courbures et classes caractéristiques en K-théorie algébrique*, Canad. Math. Soc. Proc., vol. 2, Part 2, (1982), 1–19.

— [1983a]: *Homologie cyclique des groupes et des algèbres*, C. R. Acad. Sci. Paris 297 (1983), 381–384.

— [1983b]: *Homologie cyclique et K-théorie algébrique, I and II*, C. R. Acad. Sci. Paris 297 (1983), 447–450 and 513–516.

C. Kassel [1987]: *Cyclic homology, comodules and mixed complexes*, J. Algebra 107 (1987), 195–216.

— [1988a]: *L'homologie cyclique des algèbres enveloppantes*, Invent. Math. 91 (1988), 221–251.

— [1988b]: *K-théorie algébrique et cohomologie cyclique bivariantes*, C. R. Acad. Sci. Paris 306 (1988), 799–802.

— [1989]: *Caractère de Chern bivariant*, K-Theory 3 (1989), 367–400.

— [1990]: *Homologie cyclique, caractère de Chern et lemme de perturbation*, J. reine angew. Math. 408 (1990), 159–180.

— [1992]: *Cyclic homology of differential operators, the Virasoro algebra and a q-analogue*, Commun. Math. Phys. 146 (1992), 343–356.

C. Kassel, J.-L. Loday [1982]: *Extensions centrales d'algèbres de Lie*, Ann. Inst. Fourier (Grenoble) 32 (1982), 119–142.

C. Kassel, M. Vigué-Poirrier [1992]: *Homologie des quotients primitifs de l'algèbre enveloppante de* $\mathbf{sl}(2)$, Math. Ann. 294 (1992), 483–502.

J.-L. Loday [1989]: *Opérations sur l'homologie cyclique des algèbres commutatives*, Invent. math. 96 (1989), 205–230.

— [1992b]: *Introduction to algebraic K-theory and cyclic homology*, Lect. Notes in Math. 1491, 31–54, Springer-Verlag, Heidelberg, 1992.

J.-L. Loday, D. Quillen [1984]: *Cyclic homology and the Lie algebra homology of matrices*, Comment. Math. Helvetici 59 (1984), 565–591.

T. Masuda [1986]: *Cyclic cohomology of $A[X]$ and $A[X, X^{-1}]$*, Publ. RIMS, Kyoto Univ. 22 (1986), 365–393.

R. McCarthy [1988]: *L'équivalence de Morita et homologie cyclique*, C. R. Acad. Sci. Paris 307 (1988), 211–215.

P. Nuss [1991]: *L'homologie cyclique des algèbres enveloppantes des algèbres de Lie de dimension trois*, J. Pure Appl. Algebra 73 (1991), 39–71.

D. Quillen [1973]: *Higher algebraic K-theory, I*, Lect. Notes in Math. 341, 85–147, Springer-Verlag, Heidelberg, 1973.

G. Rinehart [1963]: *Differential forms on general commutative algebras*, Trans. Amer. Math. Soc. 108 (1963), 195–222.

J. Stallings [1965]: *Centerless groups—an algebraic formulation of Gottlieb's theorem*, Topology 4 (1965), 129–134.

A. A. Suslin, M. Wodzicki [1992]: *Excision in algebraic K-theory*, Annals Math. 136 (1992), 51–122.

B. L. Tsygan [1983]: *Homology of matrix algebras over rings and Hochschild homology*, Russian Math. Surveys 38:2 (1983), 198–199.

M. Vigué-Poirrier [1991]: *Cyclic homology of algebraic hypersurfaces*, J. Pure Appl. Algebra 72 (1991), 95–108.

M. WAMBST [1993]: *Complexes de Koszul quantiques*, Ann. Inst. Fourier (Grenoble) 43 (1993), 1089–1156.
— [1996]: *Homologie de l'algèbre quantique des symboles pseudo-différentiels sur le cercle*, Bull. Soc. Math. France 124 (1996), 523–544.
— [1997]: *Hochschild and cyclic homology of the quantum multiparametric torus*, J. Pure Appl. Algebra 114 (1997), 321–329.

M. WODZICKI [1987]: *Cyclic homology of differential operators*, Duke Math. J. 54 (1987), 641–647.
— [1989]: *Excision in cyclic homology and in rational algebraic K-theory*, Annals Math. 129 (1989), 591–639.

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