The Hilbert scheme of n points on a torus and modular forms

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Introduction

• Report on joint work with Christophe Reutenauer (UQAM)

References :

∗ C. Kassel, C. Reutenauer, *The zeta function of the Hilbert scheme of n points on a two-dimensional torus*, arXiv:1505.07229.

∗ C. Kassel, C. Reutenauer, *The Fourier expansion of* $\eta(z)\eta(2z)\eta(3z)/\eta(6z)$, arXiv:1603.06357.

• We are interested in the infinite product

$$\prod_{i\geq 1} \frac{(1-t^i)^2}{(1-qt^i)(1-q^{-1}t^i)}\,.$$

We wish to compute its expansion as a formal power series in t.

• Why we are interested in it will be explained later.

Notation

- Define $\mathbf{C}_1(q), \mathbf{C}_2(q), \mathbf{C}_3(q) \ldots$ by

$$\prod_{i\geq 1} \frac{(1-t^i)^2}{(1-qt^i)(1-q^{-1}t^i)} = 1 + \sum_{n\geq 1} \mathbf{C}_n(q)\, t^n.$$

- **Immediate observations.**

(a) Each $\mathbf{C}_n(q)$ is a Laurent polynomial with integer coefficients

$$\mathbf{C}_n(q) \in \mathbb{Z}[q, q^{-1}].$$

(b) *(Vanishing)* Setting $q = 1$, we obtain $\mathbf{C}_n(1) = 0$ for all $n \geq 1$.

(c) *(Palindromicity)* We have $\mathbf{C}_n(q) = \mathbf{C}_n(q^{-1})$ so that we can write

$$\mathbf{C}_n(q) = c_{n,0} + \sum_{i\geq 1} c_{n,i} \left(q^i + q^{-i}\right).$$

The coefficients $c_{n,i}$ are integers. We will compute them explicitly.

Another observation

- If ω is a root of unity of order 2, 3, 4, or 6, then $\omega + \omega^{-1} \in \mathbb{Z}$, which together with

$$1 + \sum_{n \geq 1} \mathbf{C}_n(q)\, t^n = \prod_{i \geq 1} \frac{(1 - t^i)^2}{1 - (q + q^{-1})t^i + t^{2i}}$$

shows that $\mathbf{C}_n(\omega) \in \mathbb{Z}$ is an integer. This yields four sequences of integers, which we will compute.

- **Remark.** Let ω be a root of unity of order $d = 2$, 3, 4 or 6. Then

$$1 + \sum_{n \geq 1} \mathbf{C}_n(\omega)\, t^n = \begin{cases} \dfrac{\eta(z)^4}{\eta(2z)^2} & \text{if } d = 2, \\ \dfrac{\eta(z)^3}{\eta(3z)} & \text{if } d = 3, \\ \dfrac{\eta(z)^2\, \eta(2z)}{\eta(4z)} & \text{if } d = 4, \\ \dfrac{\eta(z)\, \eta(2z)\, \eta(3z)}{\eta(6z)} & \text{if } d = 6, \end{cases}$$

where $\eta(z) = t^{1/24} \prod_{n \geq 1} (1 - t^n)$ (and $t = e^{2\pi i z}$) is Dedekind's eta function.

The right-hand sides are modular forms of weight 1 and of respective levels 2, 3, 4, 6.

The first twelve Laurent polynomials $\mathbf{C}_n(q)$

n	$\mathbf{C}_n(q)$
1	$q^{-1} - \mathbf{2} + q$
2	$q^{-2} - q^{-1} + \mathbf{0} - q + q^2$
3	$q^{-3} - q^{-2} - q^{-1} + \mathbf{2} - q - q^2 + q^3$
4	$q^{-4} - q^{-3} + \mathbf{0} - q^3 + q^4$
5	$q^{-5} - q^{-4} - q^{-2} + q^{-1} + \mathbf{0} + q - q^2 - q^4 + q^5$
6	$q^{-6} - q^{-5} + q^{-1} - \mathbf{2} + q - q^5 + q^6$
7	$q^{-7} - q^{-6} - q^{-3} + q^{-2} + \mathbf{0} + q^2 - q^3 - q^6 + q^7$
8	$q^{-8} - q^{-7} + \mathbf{0} - q^7 + q^8$
9	$q^{-9} - q^{-8} - q^{-4} + q^{-3} + q^{-2} - q^{-1} + \mathbf{0} - q + q^2 + q^3 - q^4 - q^8 + q^9$
10	$q^{-10} - q^{-9} - q^{-1} + \mathbf{2} - q - q^9 + q^{10}$
11	$q^{-11} - q^{-10} - q^{-5} + q^{-4} + \mathbf{0} + q^4 - q^5 - q^{10} + q^{11}$
12	$q^{-12} - q^{-11} + q^{-3} - q^{-2} + \mathbf{0} - q^2 + q^3 - q^{11} + q^{12}$

Observations.

* the central coefficients are 0 or ± 2 (for which n do we have $c_{n,0} = \pm 2$?)
* the non-central coefficients are 0 or ± 1.
* $\deg \mathbf{C}_n(q) = n$.

Plan

Theorem 1 - The coefficients of $\mathbf{C}_n(q)$

The previous observations are corroborated by the following theorem obtained together with Christophe Reutenauer.

Theorem 1. Let $\mathbf{C}_n(q) = c_{n,0} + \sum_{i\geq 1} c_{n,i}\ (q^i + q^{-i})$.

(a) We have $c_{n,i} = 0$ for all $i > n$ and $c_{n,n} = 1$. Thus, $\deg \mathbf{C}_n(q) = n$.

(b) For the central coefficients we have

$$c_{n,0} = \begin{cases} 2\,(-1)^k & \text{if } n = k(k+1)/2 \text{ for some integer } k \geq 1, \quad (n \text{ is triangular}) \\ 0 & \text{otherwise.} \end{cases}$$

(c) For the non-central coefficients ($i \geq 1$) we have

$$c_{n,i} = \begin{cases} (-1)^k & \text{if } n = k(k+2i+1)/2 \text{ for some integer } k \geq 1, \\ (-1)^{k-1} & \text{if } n = k(k+2i-1)/2 \text{ for some integer } k \geq 1, \\ 0 & \text{otherwise.} \end{cases}$$

In (c) the first two conditions are mutually exclusive.

Theorem 1 - Values of $\mathbf{C}_n(q)$ at roots of unity

Theorem 1 (sequel).

(d) (Value at $q = -1$) We have $\mathbf{C}_n(-1) = (-1)^n r(n)$, where

$$r(n) = \text{card}\ \left\{(x,y) \in \mathbb{Z}^2 \mid n = x^2 + y^2\right\}.$$

(e) (Value at $q = i = \sqrt{-1}$) We have $\mathbf{C}_n(i) = (-1)^{\lfloor (n+1)/2 \rfloor}\, r'(n)$, where

$$r'(n) = \text{card}\ \left\{(x,y) \in \mathbb{Z}^2 \mid n = x^2 + 2y^2\right\}.$$

(f) (Value at $q = j = e^{2\pi i/3}$) We have $\mathbf{C}_n(j) = -3\lambda(n)$, where $\lambda(n)$ is a certain multiplicative function; it is related to

$$r''(n) = \text{card}\ \left\{(x,y) \in \mathbb{Z}^2 \mid n = x^2 + xy + y^2\right\}.$$

(g) (Value at $q = -j = e^{2\pi i/6}$) We have

$$\mathbf{C}_n(-j) = \begin{cases} (-1)^n r(n) & \text{if } n \equiv 0, \\ (-1)^n \dfrac{r(n)}{4} & \text{if } n \equiv 1, \qquad \pmod 3 \\ (-1)^{n+1} \dfrac{r(n)}{2} & \text{if } n \equiv 2. \end{cases}$$

Values of $\mathbf{C}_n(q)$ at roots of unity - On the proofs

- *(Values at $q = -1, i, j$)* The Fourier expansions of the three eta-products

$$\frac{\eta(z)^4}{\eta(2z)^2}, \qquad \frac{\eta(z)^3}{\eta(3z)}, \qquad \frac{\eta(z)^2\,\eta(2z)}{\eta(4z)}$$

are well-known. For instance,

$$\frac{\eta(z)^4}{\eta(2z)^2} = \left(\prod_{i\geq 1} \frac{1-t^i}{1+t^i}\right)^2 \overset{\text{Gauss}}{=} \left(\sum_{k\in\mathbb{Z}} (-t)^{k^2}\right)^2 = 1 + \sum_{n\geq 1} r(n)\,(-t)^n.$$

- *(Value at $q = -j$)* The eta-product $\eta(z)\,\eta(2z)\,\eta(3z)/\eta(6z)$ does not appear in G. Köhler's monography *"Eta products and theta series identities"*.
We computed its Fourier expansion in arXiv:1603.06357.

Remark. Köhler suggested another computation from the product of two modular forms of weight 1/2:

$$\frac{\eta(z)\,\eta(2z)\,\eta(3z)}{\eta(6z)} \quad = \quad \frac{\eta(z)^2}{\eta(2z)} \cdot \frac{\eta(2z)^2\,\eta(3z)}{\eta(z)\,\eta(6z)}.$$

(Gauss) (V. Kac)

Reversing the computation, we obtain an elementary proof of Kac's identity, which Victor Kac had derived from his theory of contragredient Lie superalgebras (1978).

Theorem 2 - The polynomials $\mathbf{P}_n(q)$

There is another interesting family of Laurent polynomials.

We know that $\mathbf{C}_n(1) = 0$; therefore $\mathbf{C}_n(q)$ is divisible by $q - 1$.

Theorem 2. *(a) The polynomial $\mathbf{C}_n(q)$ is divisible by $(q-1)^2$. Set*

$$\mathbf{P}_n(q) = \frac{q}{(q-1)^2}\,\mathbf{C}_n(q) = \frac{\mathbf{C}_n(q)}{q^{-1} - 2 + q}$$

Then $\mathbf{P}_n(q)$ is a Laurent polynomial of degree $n - 1$, with integer coefficients and it is palindromic:

$$\mathbf{P}_n(q) = \mathbf{P}_n(q^{-1}) \in \mathbb{Z}[q, q^{-1}].$$

(b) (to be continued)

Let us first have a look at the first twelve polynomials $\mathbf{P}_n(q)$.

The first twelve polynomials $\mathbf{P}_n(q)$

n	$\mathbf{P}_n(q)$	$\mathbf{P}_n(1)$
1	1	1
2	$q^{-1}+1+q$	3
3	$q^{-2}+q^{-1}+q+q^2$	4
4	$q^{-3}+q^{-2}+q^{-1}+1+q+q^2+q^3$	7
5	$q^{-4}+q^{-3}+q^{-2}+q^2+q^3+q^4$	6
6	$q^{-5}+q^{-4}+q^{-3}+q^{-2}+q^{-1}$ $+\mathbf{2}+q+q^2+q^3+q^4+q^5$	12
7	$q^{-6}+q^{-5}+q^{-4}+q^{-3}+q^3+q^4+q^5+q^6$	8
8	$q^{-7}+q^{-6}+q^{-5}+q^{-4}+q^{-3}+q^{-2}+q^{-1}$ $+1+q+q^2+q^3+q^4+q^5+q^6+q^7$	15
9	$q^{-8}+q^{-7}+q^{-6}+q^{-5}+q^{-4}+q^{-1}$ $+1+q+q^4+q^5+q^6+q^7+q^8$	13
10	$q^{-9}+q^{-8}+q^{-7}+q^{-6}+q^{-5}+q^{-4}$ $+q^{-3}+q^{-2}+q^{-1}+q+q^2+q^3$ $+q^4+q^5+q^6+q^7+q^8+q^9$	18
11	$q^{-10}+q^{-9}+q^{-8}+q^{-7}+q^{-6}+q^{-5}$ $+q^5+q^6+q^7+q^8+q^9+q^{10}$	12
12	$q^{-11}+q^{-10}+q^{-9}+q^{-8}+q^{-7}+q^{-6}+q^{-5}+q^{-4}$ $+q^{-3}+\mathbf{2}\,q^{-2}+\mathbf{2}\,q^{-1}+\mathbf{2}+\mathbf{2}\,q+\mathbf{2}\,q^2+q^3$ $+q^4+q^5+q^6+q^7+q^8+q^9+q^{10}+q^{11}$	28

Observation. The coefficients are all non-negative and $\mathbf{P}_n(1) \neq 0$ (actually, $\mathbf{P}_n(1) > n$).

Theorem 2 - Value of $\mathbf{P}_n(q)$ at $q = 1$

Theorem 2 (sequel).
(b) (Value at $q = 1$) We have

$$\mathbf{P}_n(1) = \sigma(n) = \sum_{d|n} d.$$

(c) (Positivity) The coefficients of $\mathbf{P}_n(q)$ are non-negative integers.

We may expect the coefficients of $\mathbf{P}_n(q)$ to count something.

This is indeed so (see next slide).

Theorem 2 - Coefficients of $\mathbf{P}_n(q)$

Write

$$\mathbf{P}_n(q) = a_{n,0} + \sum_{i=1}^{n-1} a_{n,i}\,(q^i + q^{-i}).$$

Theorem 2 (end).

(d) The coefficient $a_{n,i}$ is the number of divisors d of n satisfying the inequalities

$$\frac{i + \sqrt{2n + i^2}}{2} < d \leq i + \sqrt{2n + i^2}\,.$$

Remarks.

(a) In particular, the central coefficient $a_{n,0}$ is the number of divisors d of n in the interval $(\sqrt{2n}/2, \sqrt{2n}]$. The sequence $a_{n,0}$ grows very slowly:

$$a_{72,0} = 3, \quad a_{120,0} = 4, \quad a_{1800,0} = 5, \quad a_{28800,0} = 9, \quad a_{259200,0} = 13.$$

Nevertheless it is unbounded (Vatne, 2016).

(b) The sum of the coefficients of $\mathbf{P}_n(q)$ equals the sum of divisors of n. Hence the above intervals overlap heavily.

Plan

Our results

Why we got interested

How we computed the number of ideals of codimension n

Concluding questions

Appendix

Previous work by my coauthor

• Let $\mathbb{F}_q\langle x, y, x^{-1}, y^{-1}\rangle$ be the algebra of non-commutative Laurent polynomials in two variables:

$$\mathbb{F}_q\langle x, y, x^{-1}, y^{-1}\rangle \cong \mathbb{F}_q[F_2]$$

where F_2 is the free group on two generators and $\mathbb{F}_q$ is the field with q elements.

• **R. Bacher and C. Reutenauer (2014):** The number $C_n(q)$ of right ideals of codimension n of $\mathbb{F}_q\langle x, y, x^{-1}, y^{-1}\rangle$ is a polynomial in q of the form

$$(q-1)^{n+1} P_n(q)$$

where $P_n(q)$ is a polynomial whose coefficients are non-negative integers.

• Moreover, $P_n(1)$ is equal to the number of subgroups of index n in the free group F_2.

Counting right ideals of codimension n of the group algebra $\mathbb{F}_q[F_2]$ yields a q-analogue of the number of subgroups of the group F_2 of index n.

• **Question.** What happens when one replaces F_2 by the free abelian group $\mathbb{Z}^2$?

In this case, $\mathbb{F}_q[\mathbb{Z}^2] = \mathbb{F}_q[x, y, x^{-1}, y^{-1}]$ is the algebra of Laurent polynomials in two variables.

Do we get a q-analogue of the number of finite index subgroups of $\mathbb{Z}^2$ by counting the finite codimension ideals of $\mathbb{F}_q[x, y, x^{-1}, y^{-1}]$?

Counting the finite codimension ideals of $\mathbb{F}_q[x, y, x^{-1}, y^{-1}]$

Let $C_n(q)$ be the number of ideals of codimension n of $\mathbb{F}_q[x, y, x^{-1}, y^{-1}] = \mathbb{F}_q[\mathbb{Z}^2]$.

The subsequent results have been obtained with Christophe Reutenauer (arXiv:1505.07229).

- **Theorem 3.** *We have*

$$C_n(q) = q^n \, \mathbf{C}_n(q) \in \mathbb{Z}[q].$$

So $C_n(q)$ is a palindromic polynomial of degree $2n$ with integer coefficients.

Its coefficients $c_{n,i}$ have been determined above (Theorem 1).

- **Corollary.** *The polynomial $C_n(q)$ is divisible by $(q-1)^2$. Set*

$$P_n(q) = \frac{C_n(q)}{(q-1)^2}.$$

Then $P_n(q)$ is a palindromic polynomial of degree $2n-2$ whose coefficients $a_{n,i}$ (determined in Theorem 2) are non-negative integers.

- **Value of $P_n(q)$ at $q = 1$.** By Theorem 2 we have

$$P_n(1) = \mathbf{P}_n(1) = \sigma(n) = \sum_{d \mid n} d,$$

which is... the number of subgroups of $\mathbb{Z}^2$ of index n.

Digression: zeta function

We now give an application of Theorem 3, namely we give an explicit formula for the zeta function of the Hilbert scheme of n points on the torus $\mathbb{G}_m \times \mathbb{G}_m$.

Recall:

- The zeta function of an algebraic variety X over $\mathbb{F}_q$ is the formal power series

$$Z_{X/\mathbb{F}_q}(t) = \exp\left(\sum_{m\geq 1} |X(\mathbb{F}_{q^m})| \frac{t^m}{m}\right)$$

  where $|X(\mathbb{F}_{q^m})|$ is the number of points of X over the finite extension $\mathbb{F}_{q^m}$ of $\mathbb{F}_q$.

- **Dwork (1960)** If X is quasi-projective, then $Z_{X/\mathbb{F}_q}(t)$ is a rational function.

- **Deligne (1974)** If X is projective, then Poincaré duality implies a functional equation of the form

$$Z_{X/\mathbb{F}_q}\left(\frac{1}{q^d t}\right) = \pm(q^{d/2}t)^{\chi}\, Z_{X/\mathbb{F}_q}(t)$$

  where d is the dimension of X and χ its Euler characteristic.

Hilbert scheme of n points on the torus: zeta function

- The ideals of codimension n of $\mathbb{F}_q[x, y, x^{-1}, y^{-1}]$ are the $\mathbb{F}_q$-points of the Hilbert scheme

$$H^n = \mathrm{Hilb}^n(\mathbb{G}_m \times \mathbb{G}_m)$$

of n points on the two-dimensional torus (i.e., of the affine plane minus two distinct straight lines). This scheme is smooth and quasi-projective.

- As a consequence of Theorem 1, the zeta function of H^n is given by

$$Z_{H^n/\mathbb{F}_q}(t) = \frac{1}{(1 - q^n t)^{c_{n,0}}} \prod_{i=1}^{n} \frac{1}{[(1 - q^{n+i} t)(1 - q^{n-i} t)]^{c_{n,i}}}$$

where the exponents $c_{n,i}$ are the (integer) coefficients of the polynomial $\mathbf{C}_n(q)$.

Functional equation for the zeta function

- **Example** $n = 6$. Since

$$C_6(q) = q^{12} - q^{11} + q^7 - 2q^6 + q^5 - q + 1,$$

we obtain

$$Z_{H^6/\mathbb{F}_q}(t) = \frac{(1-qt)(1-q^6t)^2(1-q^{11}t)}{(1-t)(1-q^5t)(1-q^7t)(1-q^{12}t)}.$$

- Palindromicity of $\mathbf{C}_n(q)$ and vanishing of $\mathbf{C}_n(1)$ imply the functional equation

$$Z_{H^n/\mathbb{F}_q}\left(\frac{1}{q^{2n}t}\right) = Z_{H^n/\mathbb{F}_q}(t).$$

(and yet H^n is not projective!)

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Ellingsrud & Strømme's cellular decomposition

- **Ellingsrud & Strømme (1987).** The Hilbert scheme of n points in the affine plane

$$\mathrm{Hilb}^n(\mathbb{A}^2)$$

has a decomposition into affine spaces, indexed by the partitions λ of n:

$$\{I \subset k[x,y] \mid \mathrm{codim}\, I = n\} \overset{\text{(Ellingsrud–Strømme)}}{=} \coprod_{\lambda \vdash n} \mathcal{C}_\lambda\,, \quad \text{where } \mathcal{C}_\lambda \cong \mathbb{A}^{n+\ell(\lambda)}$$

$$\cup$$

$$\{\text{monomial ideals of } k[x,y]\} \longleftrightarrow \{\text{partitions of } n\}$$

Therefore,

$$\mathrm{card}\ \mathrm{Hilb}^n(\mathbb{A}^2_{\mathbb{F}_q}) = \sum_{\lambda \vdash n} \mathrm{card}\ \mathcal{C}_\lambda = \sum_{\lambda \vdash n} q^{n+\ell(\lambda)}$$

where $\ell(\lambda)$ is the length of the partition.

Conca & Valla's parametrization

• We identify an ideal J of codimension n of $k[x, y, x^{-1}, y^{-1}]$ with the ideal

$$I = J \cap k[x, y]$$

of $k[x, y]$: it is also of codimension n.

$$\{I \subset k[x, y] \mid \operatorname{codim} I = n\} \overset{\text{(Ellingsrud–Strømme)}}{=} \coprod_{\lambda \vdash n} \mathcal{C}_\lambda$$

$$\cup$$

$$\{J \subset k[x, y, x^{-1}, y^{-1}] \mid \operatorname{codim} J = n\} = \{I \subset k[x, y] \mid \operatorname{codim} I = n \,\&\, x, y \in (k[x, y]/I)^\times\}$$

$$= \coprod_{\lambda \vdash n} \mathcal{C}_\lambda^{\mathrm{inv}} \subset \coprod_{\lambda \vdash n} \mathcal{C}_\lambda$$

• Now Conca–Valla (2008) produced an explicit parametrization of each affine cell $\mathcal{C}_\lambda$. We use it to obtain a parametrization for each $\mathcal{C}_\lambda^{\mathrm{inv}}$, which is an open subset of $\mathcal{C}_\lambda$, and then to count the elements of $\mathcal{C}_\lambda^{\mathrm{inv}}$.

• Our polynomial $C_n(q)$, counting the number of ideals of codimension n of $\mathbb{F}_q[x, y, x^{-1}, y^{-1}]$, is given by

$$C_n(q) = \sum_{\lambda \vdash n} \operatorname{card} \mathcal{C}_\lambda^{\mathrm{inv}}.$$

Expressing $C_n(q)$ in terms of partitions of n

Using our parametrization of $\mathcal{C}_\lambda^{\mathrm{inv}}$, we obtain the following expression for the number $C_n(q)$ of ideals of codimension n of the algebra $\mathbb{F}_q[x, y, x^{-1}, y^{-1}]$.

• **Proposition**. *We have* $C_n(q) = \sum_{\lambda \vdash n} \mathrm{card}\, \mathcal{C}_\lambda^{\mathrm{inv}}$, *where*

$$\mathrm{card}\, \mathcal{C}_\lambda^{\mathrm{inv}} = (q-1)^{2v(\lambda)}\, q^{n-\ell(\lambda)} \prod_{\substack{i=1,\dots,t \\ d_i \geq 1}} \frac{q^{2d_i} - 1}{q^2 - 1}.$$

• **Notation**. Let λ be a partition of n. Consider its Ferrers diagram: it has n boxes. Then

$*$ t is the number of columns of the diagram,

$*$ $m_1 \leq m_2 \leq \cdots \leq m_t$ is the sequence of heights of columns
and $d_i = m_i - m_{i-1}$ is the sequence of differences of heights
(by convention $m_0 = 0$),

$*$ $v(\lambda)$ is the number of distinct values of the sequence $m_1 \leq m_2 \leq \cdots \leq m_t$,

$*$ $\ell(\lambda) = m_t$ is the number of parts of the partition.

Straightforward consequences

The following can be derived from the formula

$$C_n(q) = \sum_{\lambda \vdash n} (q-1)^{2v(\lambda)}\, q^{n-\ell(\lambda)} \prod_{\substack{i=1,\dots,t \\ d_i \geq 1}} \frac{q^{2d_i}-1}{q^2-1}.$$

• Since $v(\lambda) \geq 1$, the polynomial $C_n(q)$ is divisible by $(q-1)^2$.

• If $v(\lambda) = 1$, then the diagram of the partition is rectangular, $d_2 = \cdots = d_t = 0$ and we have $n = dt$, where $d = d_1$. This means that d is a divisor of n. Then

$$\operatorname{card} \mathcal{C}_\lambda^{\mathrm{inv}} = (q-1)^2 q^{n-d}\, \frac{q^{2d}-1}{q^2-1}.$$

• If $v(\lambda) \geq 2$, then $\operatorname{card} \mathcal{C}_\lambda^{\mathrm{inv}}$ is divisible by $(q-1)^4$.

• Consequently, the value at $q = 1$ of $P_n(q) = C_n(q)/(q-1)^2$ is equal to

$$P_n(1) = \sum_{d|n} \left(q^{n-d}\, \frac{q^{2d}-1}{q^2-1} \right)_{|q=1} = \sum_{d|n} d = \sigma(n).$$

The generating function of the polynomials $C_n(q)$

Let us now consider the the generating function of the polynomials $C_n(q)$.

• **Corollary.** *We have*

$$1+\sum_{n\geq 1}\frac{C_n(q)}{q^n}\,t^n=\prod_{i\geq 1}\frac{(1-t^i)^2}{1-(q+q^{-1})t^i+t^{2i}}.\qquad \longleftarrow \text{our infinite product}$$

Hence $C_n(q)=q^n\,\mathbf{C}_n(q)$.

• **Proof.** The set of all partitions is the free abelian monoid on the set $\mathbb{N}\setminus\{0\}$ of positive integers: any partition can be written uniquely as $\lambda=\prod_{i\geq 1} i^{e_i}$. Therefore,

$$\mathrm{LHS} = 1+\sum_{\lambda\neq\emptyset}\mathrm{card}\,\mathcal{C}_\lambda^{\mathrm{inv}}\,\frac{t^{|\lambda|}}{q^{|\lambda|}} \overset{\text{(free ab. mon.)}}{=} \prod_{i\geq 1}\left(1+\sum_{e\geq 1}\mathrm{card}\;\mathcal{C}_{i^e}^{\mathrm{inv}}\,\frac{t^{ie}}{q^{ie}}\right)$$

$$\overset{\text{(rectangular)}}{=} \prod_{i\geq 1}\left(1+\sum_{e\geq 1}(q-1)^2q^{ie-e}\,\frac{q^{2e}-1}{q^2-1}\,\frac{t^{ie}}{q^{ie}}\right)$$

$$= \prod_{i\geq 1}\left(1+\frac{(q-1)^2}{q^2-1}\left[\frac{qt^i}{1-qt^i}-\frac{q^{-1}t^i}{1-q^{-1}t^i}\right]\right)=\prod_{i\geq 1}\frac{(1-t^i)^2}{1-(q+q^{-1})t^i+t^{2i}}\,.$$

QED

Plan

Two questions

To conclude we ask the following two questions:

- **Question 1. Why are the polynomials $C_n(q)$ and $P_n(q)$ palindromic?**

- **Question 2. Why are the coefficients of $P_n(q)$ non-negative?**

"Why" means: are there geometric explanations?

Question 1 - Palindromicity

• **Question 1. Why are the polynomials $C_n(q)$ and $P_n(q)$ palindromic?**

T. Hausel, E. Letellier et F. Rodriguez-Villegas observed the same palindromicity in their work on character varieties and they called it *"curious Poincaré duality"*.

• The character varieties they consider are $GL_n(\mathbb{C})$ character varieties of a Riemann surface of genus g with k punctures of the form

$$\mathcal{M} = \{(A_1, B_1, \ldots, A_g, B_g, X_1, \ldots, X_k) \in GL_n(\mathbb{C})^{2g} \times C_1 \times \cdots \times C_k \\ \text{such that } [A_1, B_1] \cdots [A_g, B_g] \, X_1 \cdots X_k = I_n\} /\!/ GL_n(\mathbb{C}).$$

Here $C_1, \ldots, C_k \subset GL_n(\mathbb{C})$ are generic semisimple conjugacy classes.

They computed card $\mathcal{M}(\mathbb{F}_q)$ and showed it to be a palindromic polynomial.

• De Cataldo, Hausel and Migliorini provided a sophisticated geometrical explanation for palindromicity (based on mixed Hodge structures).

Question 2 - Positivity

Question 2. Why are the coefficients of $P_n(q)$ non-negative?

No similar phenomenon was observed for character varieties.

So is this positivity serendipitous (glücklicher Zufall)

or is there a very subtle geometric reason?

We don't know.

Ich danke für Ihre Aufmerksamkeit

References

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Plan

Conca and Valla's parametrization

- To a partition λ we associate the $(t+1)\times t$-matrix

$$M_\lambda(x,y)=\begin{pmatrix} y^{d_1}+p_{1,1} & 0 & \cdots & 0 & 0 & \cdots & 0\\ p_{2,1}-x & y^{d_2}+p_{2,2} & \cdots & 0 & 0 & \cdots & 0\\ \vdots & \vdots & \ddots & \vdots & \vdots & & \vdots\\ p_{i,1} & p_{i,2} & \cdots & y^{d_i}+p_{i,i} & 0 & \cdots & 0\\ p_{i+1,1} & p_{i+1,2} & \cdots & p_{i+1,i}-x & y^{d_{i+1}}+p_{i+1,i+1} & \cdots & 0\\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots\\ p_{t,1} & p_{t,2} & \cdots & p_{t,i} & p_{t,i+1} & \cdots & y^{d_t}+p_{t,t}\\ p_{t+1,1} & p_{t+1,2} & \cdots & p_{t+1,i} & p_{t+1,i+1} & \cdots & p_{t+1,t}-x \end{pmatrix}$$

where $(p_{i,j})_{i\geq j}\in\mathbb{F}_q[y]$ is a family of polynomials in the variable y such that

* $\deg p_{i,j}<d_j$ if $d_j\geq 1$,
* $p_{i,j}=0$ if $d_j=0$.

The set of all polynomials $(p_{i,j})_{i,j}$ forms an affine space T_λ of dimension $n+\ell(\lambda)$.

- Let I_λ be the ideal generated by the maximal minors (of size $t\times t$) of the matrix M_λ.

Conca–Valla (2008) : The map $(p_{i,j})_{i,j}\mapsto I_\lambda$ induces a bijection $T_\lambda\cong C_\lambda$.

Mixed Hodge structures

• Göttsche and Soergel (1993) determined the mixed Hodge structure of the punctual Hilbert scheme for any smooth algebraic surface over $\mathbb{C}$
(using Beilinson–Bernstein–Deligne–Gabber's decomposition theorem and its mixed Hodge version due to Saito; intersection cohomology, perverse sheaves).

• Applying Göttsche and Soergel's results to the Hilbert scheme $H^n_{\mathbb{C}}$ of n points on $\mathbb{C}^\times \times \mathbb{C}^\times$, Hausel, Letellier and Rodriguez-Villegas (2013) obtained the following formula for the generating function of the mixed Hodge polynomials of $H^n_{\mathbb{C}}$:

$$1 + \sum_{n \geq 1} H_c(H^n_{\mathbb{C}}; q, u) \frac{t^n}{q^n} = \prod_{i \geq 1} \frac{(1 + u^{2i+1} t^i)^2}{(1 - u^{2i+2} q t^i)(1 - u^{2i} q^{-1} t^i)}. \quad (1)$$

Note that, for $q = 1$, the specialization $H_c(H^n_{\mathbb{C}}; 1, u)$ is the Poincaré polynomial of $H^n_{\mathbb{C}}$ (whose coefficients are the Betti numbers of $H^n_{\mathbb{C}}$).

• When we set $u = -1$,...

E-polynomials

- When setting $u = -1$ in Formula (1), we obtain

$$1 + \sum_{n \geq 1} E(H^n_{\mathbb{C}}; q) \, \frac{t^n}{q^n} = \prod_{i \geq 1} \frac{(1 - t^i)^2}{(1 - qt^i)(1 - q^{-1}t^i)} \qquad \longleftarrow \text{our infinite product}$$

where $E(H^n_{\mathbb{C}}; q) = H_c(H^n_{\mathbb{C}}; q, -1)$ is the E-polynomial of $H^n_{\mathbb{C}}$.

- Nick Katz (2008) : when the variety $X_{\mathbb{C}}$ is of polynomial-count, i.e. when the number of points of any reduction to a finite field $\mathbb{F}_q$ is given by a "universal" polynomial in q, then

$$E(X_{\mathbb{C}}; q) = \text{card } X(\mathbb{F}_q).$$

- By our computation we know that the Hilbert scheme $H^n_{\mathbb{C}}$ is of polynomial-count, hence

$$E(H^n_{\mathbb{C}}; q) = C_n(q),$$

which gives another proof for the infinite product expansion of the generating function of the polynomials $C_n(q)$.

Question 1 - A geometric explanation for palindromicity

• **Question 1. Why are the polynomials** $C_n(q) = E(H^n_{\mathbb{C}}; q)$ **palindromic?**

Hausel *et al.* observed the same palindromicity for the E-polynomials of their character varieties and they called it *"curious Poincaré duality"*.

• **De Cataldo, Hausel et Migliorini (2013)**: Consider the elliptic curve $\mathcal{E} = \mathbb{C}/\mathbb{Z}[i]$ and a diffeomorphism $\varphi : \mathbb{C}^\times \times \mathbb{C}^\times \to \mathcal{E} \times \mathbb{C}$ (analytic, not algebraic), for instance,

$$\varphi(z_1, z_2) = \left(\frac{z_1}{|z_1|}, \frac{z_2}{|z_2|}; \ln|z_1|, \ln|z_2| \right).$$

The diffeomorphism induce an isomorphism on the cohomology of the Hilbert schemes

$$\Phi : H^*(H^n_{\mathbb{C}}, \mathbb{Q}) \cong H^*(\mathrm{Hilb}^n(\mathcal{E} \times \mathbb{C}), \mathbb{Q}).$$

The isomorphism Φ does not preserve the mixed Hodge structures; it identifies the weight filtration of $H^*(H^n_{\mathbb{C}}, \mathbb{Q})$ with the perverse Leray filtration of $H^*(\mathrm{Hilb}^n(\mathcal{E} \times \mathbb{C}), \mathbb{Q})$.

The Poincaré duality on the cohomology of $\mathcal{E}$ (which is projective) induces a duality on the perverse Leray filtration, which explains why $C_n(q)$ is palindromic.

Question 2 - Positivity

• The group $\mathbb{C}^\times \times \mathbb{C}^\times$ acts naturally on itself, which induces an action of $\mathbb{C}^\times \times \mathbb{C}^\times$ on the Hilbert scheme $H^n_{\mathbb{C}}$.

Consider the geometric quotient $\widetilde{H}^n_{\mathbb{C}} = H^n_{\mathbb{C}} //(\mathbb{C}^\times \times \mathbb{C}^\times)$. Then

$$E(\widetilde{H}^n_{\mathbb{C}}; q) = \frac{E(H^n_{\mathbb{C}}; q)}{(q-1)^2} = \frac{C_n(q)}{(q-1)^2} = P_n(q).$$

• **Question 2. Why are the coefficients of $P_n(q) = E(\widetilde{H}^n_{\mathbb{C}}; q)$ non-negative?**

No similar phenomenon was observed for character varieties.

The fact that $\widetilde{H}^n_{\mathbb{C}}$ has odd cohomology and a counting polynomial with non-negative coefficients implies non-trivial cancellation for its mixed Hodge numbers.

So is this positivity serendipitous (glücklicher Zufall)
or is there a very subtle geometric reason?

We don't know.

References for the Appendix

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