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Hopf Galois extensions up to homotopy

Christian Kassel

Institut de Recherche Mathématique Avancée
CNRS - Université Louis Pasteur
Strasbourg, France
http://www-irma.u-strasbg.fr/~kassel/

Joint work with Hans-Jürgen Schneider
(Universität München)

Hopf Galois extensions are noncommutative analogues of principal fibre G-bundles (G-pfb).

Aim: Transfer basic topological properties of principal fibre bundles to noncommutative geometry:

• **Functoriality of the base space**:

$$\begin{array}{ccc} f^*X & \longrightarrow & X \\ {\scriptstyle p}\big\downarrow & & \big\downarrow{\scriptstyle p} \\ Y' & \xrightarrow{f} & Y \end{array}$$

where $f^*X = \{(y', x) \in Y' \times X \mid f(y') = p(x)\}$.

• **Change of structural group**: For a group morphism $\rho : G \to H$ and a G-pfb $p : X \to Y$, we have a H-pfb

$$p' = X \times_G H \to Y.$$

• **Homotopy invariance**: If $f, g : Y' \to Y$ are homotopic maps, then f^*X and g^*X are isomorphic fibre bundles.

Definition of Hopf Galois extensions

- Fix a commutative base ring k; $\otimes = \otimes_k$
- H is a *Hopf k-algebra* (coproduct $\Delta : H \to H \otimes H$)

  (H = functions on a group G)
- A is an *associative unital k-algebra*

  (A = functions on a space X)
- H *co-acts* on A via algebra morphism $\delta : A \to A \otimes H$

  (G acts on X)
- Subalgebra $B = \{a \in A \mid \delta(a) = a \otimes 1_H\}$ of *co-invariants*

  (B = G-invariant functions on X)
- *Canonical map* $\beta : A \otimes_B A \to A \otimes H$:

$$\beta(a \otimes a') = (a \otimes 1)\delta(a')$$

Definition: *A is a H-Galois extension of B if β is an isomorphism and A is a faithfully flat B-module*

Main problem: Determine the set of isomorphism classes of H-Galois extensions of B

$$\mathrm{Gal}_B(H/k) = \{H\text{-Galois extensions of } B\}/(\text{iso's})$$

What is known about $\mathrm{Gal}_B(H/k)$**?**

• If H is a *cocommutative* k-flat Hopf algebra, then $\mathrm{Gal}_k(H/k)$ is a group with product $(A, A') \mapsto A \,\square_H A'$, which is the kernel of

$$\delta_A \otimes A' - A \otimes \delta_{A'} : A \otimes A' \to A \otimes H \otimes A'$$

• If $H = k[G]$ is a group algebra, then any H-Galois extension A of k is a strongly graded G algebra, i.e., $A = \bigoplus_{g \in G} A_g$, $A_e = k$ and the product $A_g \otimes A_h \to A_g A_h = A_{gh}$ is bijective.

Proposition 1. *(a) There is an exact sequence*

$$0 \to H^2(G, U(k)) \to \mathrm{Gal}_k(k[G]/k) \to \mathrm{Hom}(G, \mathrm{Pic}(k)).$$

(b) If $\mathrm{Pic}(k) = 0$*, then*

$$H^2(G, U(k)) \cong \mathrm{Gal}_k(k[G]/k).$$

(Here $U(k)$ is the group of invertible elements of k)

Taft algebras I

- For an integer $N \geq 2$ and $q \in U(k)$ of order N,

$$H = k\langle\, g, x \mid g^N = 1, x^N = 0,\, gx = qxg \,\rangle$$

is a N^2-dimensional Hopf algebra with coproduct Δ

$$\Delta(g) = g \otimes g, \quad \Delta(x) = x \otimes g + 1 \otimes x.$$

- Masuoka (1994) described all H-Galois extensions.

- *Special case when $B = k$:* For $\alpha \in U(k)$ and $\beta \in k$,

$$A_{\alpha,\beta} = k\langle\, a, b \mid a^N = \alpha, b^N = \beta,\, ab = qba \,\rangle$$

is a H-comodule-algebra with co-action δ

$$\delta(a) = a \otimes g, \quad \delta(b) = b \otimes g + 1 \otimes x.$$

Proposition (Masuoka) :

(a) $A_{\alpha,\beta} \in \mathrm{Gal}_k(H/k)$.

(b) If $A \in \mathrm{Gal}_k(H/k)$, *then* $A \cong A_{\alpha,\beta}$ *for some* α, β.

(c) $A_{\alpha,\beta} \cong A_{\alpha',\beta'} \iff$ *there is* $\gamma \in U(k)$ *such that*

$$\alpha' = \alpha\gamma^N \ \text{ and } \ \beta' = \beta.$$

Corollary. $\qquad \mathrm{Gal}_k(H/k) \cong k^*/{k^*}^N \times k.$

Functoriality I

• *(Change of scalars)* Let $\alpha : k \to R$ be a morphism of commutative rings. If $A \in \mathrm{Gal}_B(H/k)$, then

$$R \otimes A \in \mathrm{Gal}_{R\otimes B}(R \otimes H/R).$$

Hence

$$R \mapsto \mathrm{Gal}_{R\otimes B}(R \otimes H/R)$$

is a *covariant functor* $\mathcal{C}omm\mathcal{A}lg_k \to \mathcal{S}ets$

• *(Base change)* If $f : B \to B'$ and $A \in \mathrm{Gal}_B(H/k)$, then

$$f_* A = B' \otimes_B A$$

is *not* an algebra in general. Hence $\mathrm{Gal}_B(H/k)$ is not functorial in B.

Let $\mathrm{ZGal}_B(H/k) = \{A \in \mathrm{Gal}_B(H/k) \mid B \text{ central in } A\}$. Then

$$\mathrm{ZGal}_B(H/k) \cong \mathrm{Gal}_B(B \otimes H/B)$$

is a *covariant functor* $\mathcal{C}omm\mathcal{A}lg_k \to \mathcal{S}ets$

Functoriality II

• *(Cotensor product)* For a morphism $\varphi : H' \to H$ of Hopf algebras and a right H-comodule, define

$$\varphi^* A = A \,\square_H H'$$

as the kernel of

$$\delta \otimes H' - A \otimes (\varphi \otimes H')\Delta : A \otimes H' \to A \otimes H \otimes H'.$$

• *(Change of Hopf algebras)* Let $\varphi : H' \to H$ be a morphism of Hopf algebras such that H' is a flat k-module. If $A \in \mathrm{Gal}_B(H/k)$, then

$$\varphi^* A \in \mathrm{Gal}_B(H'/k).$$

Hence

$$H \mapsto \mathrm{Gal}_B(H/k)$$

is a *contravariant functor* $\mathcal{H}opf_k \to \mathcal{S}ets$

Homotopy equivalence

Definition. *A, $A' \in \mathrm{Gal}_B(H/k)$ are homotopy equivalent (we write $A \sim A'$) if there is $\widetilde{A} \in \mathrm{Gal}_{B[t]}(H/k)$ with t central in $\widetilde{A}$ such that*

$$\widetilde{A}/(t) \cong A \ \text{ and } \ \widetilde{A}/(t-1) \cong A'.$$

• Define the set of homotopy classes of H-Galois extensions of B

$$\mathcal{H}_B(H/k) = \mathrm{Gal}_B(H/k)/\sim .$$

• Isomorphic H-Galois extensions are homotopy equivalent

$$A \cong A' \Longrightarrow A \sim A'.$$

Proposition 2. *If $\mathrm{Pic}(k[t]) = 0$ and $U(k[t]) = U(k)$, then*

$$H^2(G, U(k)) \cong \mathrm{Gal}_k(k[G]/k) \cong \mathcal{H}_k(k[G]/k).$$

Theorem 1 (Kassel-Schneider). *If $H = \bigoplus_{n\geq 0} H(n)$ is a graded Hopf k-algebra and H is flat as a k-module, then*

$$A \mapsto A \,\square_H H(0) : \mathrm{Gal}_B(H/k) \to \mathrm{Gal}_B(H(0)/k)$$

induces a bijection

$$\mathcal{H}_B(H/k) \cong \mathcal{H}_B(H(0)/k).$$

Taft algebras II

• The Taft algebra

$$H = k\langle\, g, x \mid g^N = 1, x^N = 0,\, gx = qxg \,\rangle$$

is a positively graded Hopf algebra:

$$H = \oplus_{n\geq 0}\, H(n)$$

with $g \in H(0)$ and $x \in H(1)$.

• The degree 0 Hopf subalgebra is a group algebra

$$H(0) = k\langle\, g \mid g^N = 1 \,\rangle \cong k[\mathbf{Z}/N]$$

• By Theorem 1 and Proposition 2,

$$\begin{aligned} \mathcal{H}_k(H/k) &\cong \mathcal{H}_k(H(0)/k) \cong \mathcal{H}_k(k[\mathbf{Z}/N]/k) \\ &\cong \mathrm{Gal}_k(k[\mathbf{Z}/N]/k) \cong H^2(\mathbf{Z}/N, k^*) \\ &\cong k^*/{k^*}^N \end{aligned}$$

Hence,

$$\mathrm{Gal}_k(H/k) \cong k^*/{k^*}^N \times k \ncong \mathcal{H}_k(H/k)$$

Drinfeld-Jimbo quantum enveloping algebras over C

• Finite-type Cartan matrix $(a_{ij})_{1\leq i,j\leq r}$ with $d_1, \ldots, d_r \in \{1,2,3\}$ such that $d_i a_{ij} = d_j a_{ji}$

• $q \neq 0$ such that $q^{2d_i} \neq 1$ for all i

• Presentation of $U_q(\mathbf{g})$

Generators: E_i, F_i, $K^{\pm 1}$ $(i = 1, \ldots, r)$

Relations: $K_i K_j = K_j K_i$,

$$K_i E_j = q^{d_i a_{ij}}\, E_j K_i, \quad K_i F_j = q^{-d_i a_{ij}}\, F_j K_i,$$
$$E_i F_j - F_j E_i = \delta_{ij}\, \frac{K_i - K_i^{-1}}{q^{d_i} - q^{-d_i}},$$

+ quantum Serre relations.

• Coproduct of $U_q(\mathbf{g})$: $\Delta(K_i) = K_i \otimes K_i$,

$$\Delta(E_i) = K_i \otimes E_i + E_i \otimes 1 \text{ and } \Delta(F_i) = 1 \otimes F_i + F_i \otimes K_i^{-1}.$$

• Grouplike elements of $U_q(\mathbf{g})$:

$$G = G(U_q(\mathbf{g})) = \langle K_1, \ldots, K_r \rangle \cong \mathbf{Z}^r$$

Theorem 2 (Kassel-Schneider). *For any algebra B,*

$$\mathcal{H}_B(U_q(\mathbf{g})/\mathbf{C}) \cong \mathcal{H}_B(\mathbf{C}[G]/\mathbf{C}).$$

Corollary. For $B = \mathbf{C}$,

$$\begin{aligned}\mathcal{H}_{\mathbf{C}}(U_q(\mathbf{g})/\mathbf{C}) &\cong \mathcal{H}_{\mathbf{C}}(\mathbf{C}[\mathbf{Z}^r]/\mathbf{C}) \\ &\cong H^2(\mathbf{Z}^r, \mathbf{C}^*) \\ &\cong \mathrm{Hom}_{\mathbf{Z}}(\Lambda^2_{\mathbf{Z}}(\mathbf{Z}^r), \mathbf{C}^*) \\ &\cong (\mathbf{C}^*)^{r(r-1)/2}.\end{aligned}$$

Idea of proof of Theorem 2: Twisting

• From Hopf algebra H and convolution-invertible 2-cocycle $\sigma : H \otimes H \to k$ build *twisted* Hopf algebra $H^\sigma = H$ as coalgebra with new product

$$x \cdot_\sigma y = \sum_{(x),(y)} \sigma(x_{(1)}, y_{(1)})\, x_{(2)} y_{(2)}\, \sigma^{-1}(x_{(3)}, y_{(3)}).$$

• From H-comodule algebra A build *twisted* H-comodule algebra $A^\sigma = A$ as H-comodule with new product

$$a \cdot_\sigma b = \sum_{(a),(b)} a_{(0)})b_{(0)}\, \sigma^{-1}(a_{(1)}, b_{(1)}).$$

Proposition (Montgomery-Schneider). *If $A \in \mathrm{Gal}_B(H/k)$, then $A^\sigma \in \mathrm{Gal}_B(H^\sigma/k)$.*

Corollary.

$$\mathrm{Gal}_B(H^\sigma/k) \cong \mathrm{Gal}_B(H/k) \;\; \textit{and} \;\; \mathcal{H}_B(H^\sigma/k) \cong \mathcal{H}_B(H/k).$$

Twisting Drinfeld-Jimbo algebras

- $U_q(\mathbf{g})$ is a filtered Hopf algebra

$$U_q(\mathbf{g}) \supset \cdots \supset U^n \supset U^{n-1} \supset \cdots \supset U^1 \supset U^0 \supset 0$$

with $K_i \in U^0$ and E_i, $F_i \in U^1 - U^0$.

- Graded Hopf algebra

$$\text{gr } U_q(\mathbf{g}) = \bigoplus_{n \geq 0} U(n) \quad \text{with} \;\; U(n) = U^n / U^{n-1}$$

and $U(0) = U^0 = \mathbf{C}[K_1^{\pm 1}, \ldots, K_r^{\pm 1}] = \mathbf{C}[G] \cong \mathbf{C}[\mathbf{Z}^r]$.

Proposition 3. *There is an invertible 2-cocycle* σ *on* gr $U_q(\mathbf{g})$ *inducing an isomorphism of Hopf algebras*

$$\Big(\text{gr } U_q(\mathbf{g})\Big)^{\sigma} \cong U_q(\mathbf{g}).$$

Hence, $\mathcal{H}_B(U_q(\mathbf{g})/\mathbf{C}) \cong \mathcal{H}_B(\text{gr } U_q(\mathbf{g})/\mathbf{C}) \cong \mathcal{H}_B(\mathbf{C}[\mathbf{Z}^r]/\mathbf{C})$.

Question: Many pointed Hopf algebras are 2-cocycle twists of graded Hopf algebras (Masuoka, Didt). Is there any pointed Hopf algebra not of this type?

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