Non-commutative torsors

Christian Kassel

Institut de Recherche Mathématique Avancée
CNRS - Université de Strasbourg
Strasbourg, France

Oslo University
2 September 2011

Introduction

• Report on joint work with Pierre Guillot (Strasbourg)

about the classification of non-commutative torsors over a finite group G

Reference: P. Guillot, C. Kassel,

Cohomology of invariant Drinfeld twists on group algebras,

Internat. Math. Res. Notices 2010 no. 10, 1894–1939; arXiv:0903.2807

• The idea of considering non-commutative G-torsors comes from quantum group theory as well as the proofs of the main results

We obtain a new (quantum) invariant for finite groups

Introduction

• Report on joint work with Pierre Guillot (Strasbourg)

about the classification of non-commutative torsors over a finite group G

Reference: P. Guillot, C. Kassel,

Cohomology of invariant Drinfeld twists on group algebras,

Internat. Math. Res. Notices 2010 no. 10, 1894–1939; arXiv:0903.2807

• The idea of considering non-commutative G-torsors comes from quantum group theory as well as the proofs of the main results

We obtain a new (quantum) invariant for finite groups

Plan

I. Torsors and bitorsors

II. Non-commutative torsors

III. Main results

IV. A proof using quantum group theory

Plan

Classical G-torsors

• **Recall:** Let G be an algebraic group defined over a field k. A right G-torsor is a right G-variety T (over k) such that the map

$$\begin{array}{ccc} T \times G & \longrightarrow & T \times T \\ (t, g) & \longmapsto & (t, tg) \end{array}$$

is an isomorphism

This means that for any $t, t' \in T$ there is a unique $g \in G$ such that $t' = tg$

• If $k = \bar{k}$ is algebraically closed, then any torsor is isomorphic to $T = G$ with G acting by right translations

Classical G-torsors

• **Recall:** Let G be an algebraic group defined over a field k. A right G-torsor is a right G-variety T (over k) such that the map

$$\begin{array}{ccc} T \times G & \longrightarrow & T \times T \\ (t, g) & \longmapsto & (t, tg) \end{array}$$

is an isomorphism

This means that for any $t, t' \in T$ there is a unique $g \in G$ such that $t' = tg$

• If $k = \bar{k}$ is algebraically closed, then any torsor is isomorphic to $T = G$ with G acting by right translations

Galois cohomology

• When $k \neq \bar{k}$, torsors are classified by Serre's non-abelian Galois cohomology set:

$$\{G\text{-torsors}\}/(\text{isomorphism}) \cong H^1(\mathrm{Gal}(\bar{k}/k), G)$$

This is a pointed set, not a group (unless G is abelian)

• If G is a finite group, then $H^1(\mathrm{Gal}(\bar{k}/k), G)$

$$\cong \{\text{continuous homomorphisms } \varphi : \mathrm{Gal}(\bar{k}/k) \to G\}/(\text{conjugacy})$$

Recall: a homomorphism $\mathrm{Gal}(\bar{k}/k) \to G$ is continuous if it factors through $\mathrm{Gal}(k'/k)$ for some finite Galois extension k' of k

Remark. The set $H^1(\mathrm{Gal}(\bar{k}/k), G)$ is an arithmetical analogue of the topologists' classifying set for flat G-connections

$$\mathrm{Hom}(\pi_1(S), G)/(\text{conjugacy})$$

Galois cohomology

• When $k \neq \bar{k}$, torsors are classified by Serre's non-abelian Galois cohomology set:

$$\{G\text{-torsors}\}/(\text{isomorphism}) \cong H^1(\mathrm{Gal}(\bar{k}/k), G)$$

This is a pointed set, not a group (unless G is abelian)

• If G is a finite group, then $H^1(\mathrm{Gal}(\bar{k}/k), G)$

$$\cong \{\text{continuous homomorphisms } \varphi : \mathrm{Gal}(\bar{k}/k) \to G\}/(\text{conjugacy})$$

Recall: a homomorphism $\mathrm{Gal}(\bar{k}/k) \to G$ is continuous if it factors through $\mathrm{Gal}(k'/k)$ for some finite Galois extension k' of k

Remark. The set $H^1(\mathrm{Gal}(\bar{k}/k), G)$ is an arithmetical analogue of the topologists' classifying set for flat G-connections

$$\mathrm{Hom}(\pi_1(S), G)/(\text{conjugacy})$$

Bitorsors

• A *G*-bitorsor is both a left and right *G*-torsor such the left and right actions commute

• Bitorsors can be multiplied: if T, T' are bitorsors, then so is

$$T * T' = T \times_G T' = (T \times T')/\langle (tg, t') = (t, gt') \rangle$$

• Under this product, the set of isomorphism classes of bitorsors becomes a group, which is isomorphic to the Galois cohomology group

$$\{G\text{-bitorsors}\}/(\text{isomorphism}) \cong H^1(\mathrm{Gal}(\bar{k}/k), Z(G))$$

where $Z(G)$ is the center of G

Bitorsors

- A *G*-bitorsor is both a left and right *G*-torsor such the left and right actions commute

- Bitorsors can be multiplied: if T, T' are bitorsors, then so is

$$T * T' = T \times_G T' = (T \times T')/\langle (tg, t') = (t, gt') \rangle$$

- Under this product, the set of isomorphism classes of bitorsors becomes a group, which is isomorphic to the Galois cohomology group

$$\{G\text{-bitorsors}\}/(\text{isomorphism}) \cong H^1(\mathrm{Gal}(\bar{k}/k), Z(G))$$

where $Z(G)$ is the center of G

Bitorsors

- A G-bitorsor is both a left and right G-torsor such the left and right actions commute

- Bitorsors can be multiplied: if T, T' are bitorsors, then so is

$$T * T' = T \times_G T' = (T \times T')/\langle (tg, t') = (t, gt') \rangle$$

- Under this product, the set of isomorphism classes of bitorsors becomes a group, which is isomorphic to the Galois cohomology group

$$\{G\text{-bitorsors}\}/(\text{isomorphism}) \cong H^1(\text{Gal}(\bar{k}/k), Z(G))$$

where $Z(G)$ is the center of G

Plan

Non-commutative geometry

- In the spirit of non-commutative geometry, replace

$$
\begin{aligned}
k\text{-variety} &\longleftrightarrow k\text{-algebra}\\
\text{group } G &\longleftrightarrow \text{Hopf algebra}\\
\text{group action } G &\longleftrightarrow \text{Hopf algebra coaction}\\
\text{torsor condition } G &\longleftrightarrow \text{Galois condition}\\
\text{left and right action } G &\longleftrightarrow \text{left and right coaction}
\end{aligned}
$$

- Using this dictionary, one can define non-commutative bitorsors for any Hopf algebra

- As in the classical case, the set of isomorphism classes of non-commutative bitorsors forms a group

Non-commutative H-torsors

• Given a Hopf algebra H, a right non-commutative H-torsor (also called a right H-Galois object) is an associative unital algebra A together with an algebra morphism

$$\delta : A \to A \otimes H \qquad \text{(coaction)}$$

satisfying the coassociativity and the counitarity conditions:

$$(\delta \otimes \mathrm{id}_H) \circ \delta = (\mathrm{id}_H \otimes \Delta) \circ \delta \quad \text{and} \quad (\mathrm{id}_H \otimes \varepsilon) \circ \delta = \mathrm{id}_H$$

and the Galois condition: the map

$$A \otimes A \to A \otimes H \, ; \quad a \otimes a' \mapsto (a \otimes 1_H)\, \delta(a')$$

is an isomorphism

• **Why it is called "Galois condition"**:
If K/k is a Galois extension of fields with finite Galois group G, then K is a non-commutative H-torsor for $H = (kG)^*$

• Similarly one defines left non-commutative H-torsors (with left coaction $\delta : A \to H \otimes A$) and non-commutative H-bitorsors

Non-commutative H-torsors

• Given a Hopf algebra H, a right non-commutative H-torsor (also called a right H-Galois object) is an associative unital algebra A together with an algebra morphism

$$\delta : A \to A \otimes H \qquad \text{(coaction)}$$

satisfying the coassociativity and the counitarity conditions:

$$(\delta \otimes \mathrm{id}_H) \circ \delta = (\mathrm{id}_H \otimes \Delta) \circ \delta \quad \text{and} \quad (\mathrm{id}_H \otimes \varepsilon) \circ \delta = \mathrm{id}_H$$

and the Galois condition: the map

$$A \otimes A \to A \otimes H \, ; \quad a \otimes a' \mapsto (a \otimes 1_H)\, \delta(a')$$

is an isomorphism

• **Why it is called "Galois condition"**:
If K/k is a Galois extension of fields with finite Galois group G, then K is a non-commutative H-torsor for $H = (kG)^*$

• Similarly one defines left non-commutative H-torsors (with left coaction $\delta : A \to H \otimes A$) and non-commutative H-bitorsors

Non-commutative H-torsors

• Given a Hopf algebra H, a right non-commutative H-torsor (also called a right H-Galois object) is an associative unital algebra A together with an algebra morphism

$$\delta : A \to A \otimes H \qquad \text{(coaction)}$$

satisfying the coassociativity and the counitarity conditions:

$$(\delta \otimes \mathrm{id}_H) \circ \delta = (\mathrm{id}_H \otimes \Delta) \circ \delta \quad \text{and} \quad (\mathrm{id}_H \otimes \varepsilon) \circ \delta = \mathrm{id}_H$$

and the Galois condition: the map

$$A \otimes A \to A \otimes H \,; \quad a \otimes a' \mapsto (a \otimes 1_H)\, \delta(a')$$

is an isomorphism

• **Why it is called "Galois condition"**:
If K/k is a Galois extension of fields with finite Galois group G, then K is a non-commutative H-torsor for $H = (kG)^*$

• Similarly one defines left non-commutative H-torsors (with left coaction $\delta : A \to H \otimes A$) and non-commutative H-bitorsors

Non-commutative H-torsors

• Given a Hopf algebra H, a right non-commutative H-torsor (also called a right H-Galois object) is an associative unital algebra A together with an algebra morphism

$$\delta : A \to A \otimes H \qquad \text{(coaction)}$$

satisfying the coassociativity and the counitarity conditions:

$$(\delta \otimes \mathrm{id}_H) \circ \delta = (\mathrm{id}_H \otimes \Delta) \circ \delta \quad \text{and} \quad (\mathrm{id}_H \otimes \varepsilon) \circ \delta = \mathrm{id}_H$$

and the Galois condition: the map

$$A \otimes A \to A \otimes H \,; \quad a \otimes a' \mapsto (a \otimes 1_H)\, \delta(a')$$

is an isomorphism

• **Why it is called "Galois condition"**:
If K/k is a Galois extension of fields with finite Galois group G, then K is a non-commutative H-torsor for $H = (kG)^*$

• Similarly one defines left non-commutative H-torsors (with left coaction $\delta : A \to H \otimes A$) and non-commutative H-bitorsors

Non-commutative G-bitorsors

Let G be a finite group

• By definition, a non-commutative G-bitorsor is a non-commutative H-bitorsor for the Hopf algebra $(kG)^*$ of k-valued functions on G

• The set of isomorphism classes of non-commutative G-bitorsors forms a group we denote by

$$\mathcal{H}^2(G/k)$$

• **Our aim**: To compute the group $\mathcal{H}^2(G/k)$ of non-commutative G-bitorsors and to compare it with the group $H^1(\mathrm{Gal}(\bar{k}/k), Z(G))$ of classical G-bitorsors

Non-commutative G-bitorsors

Let G be a finite group

• By definition, a non-commutative G-bitorsor is a non-commutative H-bitorsor for the Hopf algebra $(kG)^*$ of k-valued functions on G

• The set of isomorphism classes of non-commutative G-bitorsors forms a group we denote by

$$\mathcal{H}^2(G/k)$$

• **Our aim**: To compute the group $\mathcal{H}^2(G/k)$ of non-commutative G-bitorsors and to compare it with the group $H^1(\mathrm{Gal}(\bar{k}/k), Z(G))$ of classical G-bitorsors

Invariant Drinfeld twists

The following facts can be proved.

• **Fact 1**. Any non-commutative G-bitorsor is an algebra A_F indexed by an invariant Drinfeld twist F on the group algebra kG (the definition of A_F is given in the Appendix)

Definition. *(a) A Drinfeld twist on kG is an invertible element $F \in kG \otimes kG$ satisfying the condition*

$$(F \otimes 1)\,(\Delta \otimes \mathrm{id})(F) = (1 \otimes F)\,(\mathrm{id} \otimes \Delta)(F) \in kG \otimes kG \otimes kG$$

(b) An element $F \in kG \otimes kG$ is invariant if

$$\Delta(a)F = F\Delta(a) \qquad (a \in kG)$$

Recall that the coproduct Δ of kG is defined by $\Delta(g) = g \otimes g$ for all $g \in G$

• **Fact 2**. Two non-commutative bitorsors $A_F, A_{F'}$ are isomorphic as bitorsors if and only if there exists a central invertible element $a \in kG$ such that

$$F' = F(a \otimes a)\Delta(a)^{-1} = (a \otimes a)\Delta(a)^{-1}F$$

Invariant Drinfeld twists

The following facts can be proved.

• **Fact 1**. Any non-commutative G-bitorsor is an algebra A_F indexed by an invariant Drinfeld twist F on the group algebra kG (the definition of A_F is given in the Appendix)

Definition. *(a) A Drinfeld twist on kG is an invertible element $F \in kG \otimes kG$ satisfying the condition*

$$(F \otimes 1)\,(\Delta \otimes \mathrm{id})(F) = (1 \otimes F)\,(\mathrm{id} \otimes \Delta)(F) \in kG \otimes kG \otimes kG$$

(b) An element $F \in kG \otimes kG$ is invariant if

$$\Delta(a)F = F\Delta(a) \qquad (a \in kG)$$

Recall that the coproduct Δ of kG is defined by $\Delta(g) = g \otimes g$ for all $g \in G$

• **Fact 2**. Two non-commutative bitorsors $A_F, A_{F'}$ are isomorphic as bitorsors if and only if there exists a central invertible element $a \in kG$ such that

$$F' = F(a \otimes a)\Delta(a)^{-1} = (a \otimes a)\Delta(a)^{-1}F$$

Invariant Drinfeld twists

The following facts can be proved.

• **Fact 1**. Any non-commutative G-bitorsor is an algebra A_F indexed by an invariant Drinfeld twist F on the group algebra kG (the definition of A_F is given in the Appendix)

Definition. *(a) A Drinfeld twist on kG is an invertible element $F \in kG \otimes kG$ satisfying the condition*

$$(F \otimes 1)\,(\Delta \otimes \mathrm{id})(F) = (1 \otimes F)\,(\mathrm{id} \otimes \Delta)(F) \in kG \otimes kG \otimes kG$$

(b) An element $F \in kG \otimes kG$ is invariant if

$$\Delta(a)F = F\Delta(a) \qquad (a \in kG)$$

Recall that the coproduct Δ of kG is defined by $\Delta(g) = g \otimes g$ for all $g \in G$

• **Fact 2**. Two non-commutative bitorsors $A_F, A_{F'}$ are isomorphic as bitorsors if and only if there exists a central invertible element $a \in kG$ such that

$$F' = F(a \otimes a)\Delta(a)^{-1} = (a \otimes a)\Delta(a)^{-1}F$$

Invariant Drinfeld twists

The following facts can be proved.

• **Fact 1**. Any non-commutative G-bitorsor is an algebra A_F indexed by an invariant Drinfeld twist F on the group algebra kG (the definition of A_F is given in the Appendix)

Definition. *(a) A Drinfeld twist on kG is an invertible element $F \in kG \otimes kG$ satisfying the condition*

$$(F \otimes 1)\,(\Delta \otimes \mathrm{id})(F) = (1 \otimes F)\,(\mathrm{id} \otimes \Delta)(F) \in kG \otimes kG \otimes kG$$

(b) An element $F \in kG \otimes kG$ is invariant if

$$\Delta(a)F = F\Delta(a) \qquad (a \in kG)$$

Recall that the coproduct Δ of kG is defined by $\Delta(g) = g \otimes g$ for all $g \in G$

• **Fact 2**. Two non-commutative bitorsors $A_F, A_{F'}$ are isomorphic as bitorsors if and only if there exists a central invertible element $a \in kG$ such that

$$F' = F(a \otimes a)\Delta(a)^{-1} = (a \otimes a)\Delta(a)^{-1}F$$

A description of $\mathcal{H}^2(G/k)$

• Consequently,

the group $\mathcal{H}^2(G/k)$ of isomorphism classes of non-commutative G-bitorsors is isomorphic to the group of invariant Drinfeld twists on kG modulo the trivial twists, i.e., those of the form $(a \otimes a)\Delta(a)^{-1}$, where $a \in kG$ is central and invertible

$$\mathcal{H}^2(G/k) \cong \{\text{invariant Drinfeld twists on } kG\} \,/\, \{\text{trivial twists}\}$$

• Using this interpretation in terms of Drinfeld twists, the group $\mathcal{H}^2(G/k)$ can be defined for any group G, not only finite groups

S. Neshveyev and L. Tuset (Oslo) computed $\mathcal{H}^2(G/k)$ for any connected compact group G and $k = \mathbb{C}$

• We next show how to compute $\mathcal{H}^2(G/k)$ for any finite group G

A description of $\mathcal{H}^2(G/k)$

• Consequently,

the group $\mathcal{H}^2(G/k)$ of isomorphism classes of non-commutative G-bitorsors is isomorphic to the group of invariant Drinfeld twists on kG modulo the trivial twists, i.e., those of the form $(a \otimes a)\Delta(a)^{-1}$, where $a \in kG$ is central and invertible

$$\mathcal{H}^2(G/k) \cong \{\text{invariant Drinfeld twists on } kG\} \,/\, \{\text{trivial twists}\}$$

• Using this interpretation in terms of Drinfeld twists, the group $\mathcal{H}^2(G/k)$ can be defined for any group G, not only finite groups

S. Neshveyev and L. Tuset (Oslo) computed $\mathcal{H}^2(G/k)$ for any connected compact group G and $k = \mathbb{C}$

• We next show how to compute $\mathcal{H}^2(G/k)$ for any finite group G

A description of $\mathcal{H}^2(G/k)$

• Consequently,

the group $\mathcal{H}^2(G/k)$ of isomorphism classes of non-commutative G-bitorsors is isomorphic to the group of invariant Drinfeld twists on kG modulo the trivial twists, i.e., those of the form $(a \otimes a)\Delta(a)^{-1}$, where $a \in kG$ is central and invertible

$$\mathcal{H}^2(G/k) \cong \{\text{invariant Drinfeld twists on } kG\} \,/\, \{\text{trivial twists}\}$$

• Using this interpretation in terms of Drinfeld twists, the group $\mathcal{H}^2(G/k)$ can be defined for any group G, not only finite groups

S. Neshveyev and L. Tuset (Oslo) computed $\mathcal{H}^2(G/k)$ for any connected compact group G and $k = \mathbb{C}$

• We next show how to compute $\mathcal{H}^2(G/k)$ for any finite group G

Plan

The rationality exact sequence

Assume that k is of characteristic zero with algebraic closure $\bar{k}$

Theorem 1. *Let G be a finite group. If all irreducible $\bar{k}$-representations of G can be realized over k, then there is an exact sequence of groups*

$$1 \longrightarrow H^1(\mathrm{Gal}(\bar{k}/k), Z(G)) \longrightarrow \mathcal{H}^2(G/k) \longrightarrow \mathcal{H}^2(G/\bar{k}) \longrightarrow 1$$

(classical bitorsors) (non-comm. bitorsors)

(ARITHMETIC) (GEOMETRIC)

In particular, if G is centerless, then $\mathcal{H}^2(G/k) \cong \mathcal{H}^2(G/\bar{k})$

The rationality exact sequence

Assume that k is of characteristic zero with algebraic closure $\bar{k}$

Theorem 1. *Let G be a finite group. If all irreducible $\bar{k}$-representations of G can be realized over k, then there is an exact sequence of groups*

$$1 \longrightarrow H^1(\mathrm{Gal}(\bar{k}/k), Z(G)) \longrightarrow \mathcal{H}^2(G/k) \longrightarrow \mathcal{H}^2(G/\bar{k}) \longrightarrow 1$$

(classical bitorsors) (non-comm. bitorsors)

(ARITHMETIC) (GEOMETRIC)

In particular, if G is centerless, then $\mathcal{H}^2(G/k) \cong \mathcal{H}^2(G/\bar{k})$

On the proof of Theorem 1

- $\mathcal{H}^2(G/k)$ appears as a "cohomology group" in the short cochain complex:

$$1 \to [(kG)^\times]^G \xrightarrow{\delta^1} [(kG \otimes kG)^\times]^G \overset{\delta^2_L, \delta^2_R}{\Longrightarrow} [(kG \otimes kG \otimes kG)^\times]^G$$

where $\delta^1(a) = (a \otimes a)\,\Delta(a^{-1})$ and

$$\delta^2_L(F) = (F \otimes 1)\,(\Delta \otimes \mathrm{id})(F) \quad \text{and} \quad \delta^2_R(F) = (1 \otimes F)\,(\mathrm{id} \otimes \Delta)(F)$$

Then

$$\mathcal{H}^2(G/k) \cong \mathrm{Equalizer}(\delta^2_L, \delta^2_R) / \mathrm{Im}(\delta^1)$$

- The groups in the cochain complex are algebraic groups:

$$[(\bar{k}G)^\times]^G \cong \prod_{\rho \in \mathrm{Irrep}(G)} \mathrm{Aut}_G(V_\rho) \cong \prod_{\rho \in \mathrm{Irrep}(G)} \bar{k}^\times \qquad \text{(split torus)}$$

$$((\bar{k}G \otimes \bar{k}G)^\times)^G \cong \prod_{\rho, \sigma \in \mathrm{Irrep}(G)} \mathrm{Aut}_G(V_\rho \otimes V_\sigma) \qquad \text{(product of } GL_n(\bar{k})\text{'s)}$$

- Apply the functor $H^0(\mathrm{Gal}(\bar{k}/k), -)$ and use Hilbert's Theorem 90

On the proof of Theorem 1

- $\mathcal{H}^2(G/k)$ appears as a "cohomology group" in the short cochain complex:

$$1 \to [(kG)^\times]^G \xrightarrow{\delta^1} [(kG \otimes kG)^\times]^G \overset{\delta_L^2, \delta_R^2}{\Longrightarrow} [(kG \otimes kG \otimes kG)^\times]^G$$

where $\delta^1(a) = (a \otimes a)\,\Delta(a^{-1})$ and

$$\delta_L^2(F) = (F \otimes 1)\,(\Delta \otimes \mathrm{id})(F) \quad \text{and} \quad \delta_R^2(F) = (1 \otimes F)\,(\mathrm{id} \otimes \Delta)(F)$$

Then

$$\mathcal{H}^2(G/k) \cong \mathrm{Equalizer}(\delta_L^2, \delta_R^2)/\,\mathrm{Im}(\delta^1)$$

- The groups in the cochain complex are algebraic groups:

$$[(\bar{k}G)^\times]^G \cong \prod_{\rho \in \mathrm{Irrep}(G)} \mathrm{Aut}_G(V_\rho) \cong \prod_{\rho \in \mathrm{Irrep}(G)} \bar{k}^\times \qquad \text{(split torus)}$$

$$((\bar{k}G \otimes \bar{k}G)^\times)^G \cong \prod_{\rho,\sigma \in \mathrm{Irrep}(G)} \mathrm{Aut}_G(V_\rho \otimes V_\sigma) \qquad \text{(product of } GL_n(\bar{k})\text{'s)}$$

- Apply the functor $H^0(\mathrm{Gal}(\bar{k}/k), -)$ and use Hilbert's Theorem 90

On the proof of Theorem 1

- $\mathcal{H}^2(G/k)$ appears as a "cohomology group" in the short cochain complex:

$$1 \to [(kG)^\times]^G \xrightarrow{\delta^1} [(kG \otimes kG)^\times]^G \overset{\delta_L^2,\delta_R^2}{\Longrightarrow} [(kG \otimes kG \otimes kG)^\times]^G$$

where $\delta^1(a) = (a \otimes a)\,\Delta(a^{-1})$ and

$$\delta_L^2(F) = (F \otimes 1)\,(\Delta \otimes \mathrm{id})(F) \quad \text{and} \quad \delta_R^2(F) = (1 \otimes F)\,(\mathrm{id} \otimes \Delta)(F)$$

Then

$$\mathcal{H}^2(G/k) \cong \mathrm{Equalizer}(\delta_L^2, \delta_R^2)/\operatorname{Im}(\delta^1)$$

- The groups in the cochain complex are algebraic groups:

$$[(\bar{k}G)^\times]^G \cong \prod_{\rho \in \mathrm{Irrep}(G)} \mathrm{Aut}_G(V_\rho) \cong \prod_{\rho \in \mathrm{Irrep}(G)} \bar{k}^\times \qquad \text{(split torus)}$$

$$((\bar{k}G \otimes \bar{k}G)^\times)^G \cong \prod_{\rho,\sigma \in \mathrm{Irrep}(G)} \mathrm{Aut}_G(V_\rho \otimes V_\sigma) \qquad \text{(product of } GL_n(\bar{k})\text{'s)}$$

- Apply the functor $H^0(\mathrm{Gal}(\bar{k}/k), -)$ and use Hilbert's Theorem 90

The algebraically closed case

• We are now reduced to computing $\mathcal{H}^2(G/k)$ over an algebraically closed ground field k

• We need two ingredients :

  * a set $\mathcal{B}(G)$ obtained as a colimit over normal abelian subgroups of G

  * a group $\mathrm{Out}_c(G)$ of outer automorphisms of G

The algebraically closed case

- We are now reduced to computing $\mathcal{H}^2(G/k)$ over an algebraically closed ground field k

- We need two ingredients :

  - $*$ a set $\mathcal{B}(G)$ obtained as a colimit over normal abelian subgroups of G

  - $*$ a group $\mathrm{Out}_c(G)$ of outer automorphisms of G

The algebraically closed case

- We are now reduced to computing $\mathcal{H}^2(G/k)$ over an algebraically closed ground field k

- We need two ingredients :
  - $*$ a set $\mathcal{B}(G)$ obtained as a colimit over normal abelian subgroups of G
  - $*$ a group $\mathrm{Out}_c(G)$ of outer automorphisms of G

First ingredient: the pointed set $\mathcal{B}(G)$

- **Definition.** *Let $\mathcal{A}$ be the category whose objects are the normal abelian subgroups A of G and whose arrows are the inclusions. Define*

$$\mathcal{B}(G) = \bigcup_{A \in \mathcal{A}} H^2(\widehat{A}, k^\times)^G \qquad \textit{(a colimit of pointed sets)}$$

Here $\widehat{A} = \mathrm{Hom}(A, k^\times)$ is the group of characters of A and $H^2(\widehat{A}, k^\times)$ is the second cohomology group of $\widehat{A}$

- The set $\mathcal{B}(G)$ is
  - ∗ non-empty: it is pointed by the zero element 0 lying in all $H^2(\widehat{A}, k^\times)^G$
  - ∗ it is finite

- If G has a unique maximal normal abelian subgroup A, then

$$\mathcal{B}(G) = H^2(\widehat{A}, k^\times)^G$$

First ingredient: the pointed set $\mathcal{B}(G)$

- **Definition.** *Let $\mathcal{A}$ be the category whose objects are the normal abelian subgroups A of G and whose arrows are the inclusions. Define*

$$\mathcal{B}(G) = \bigcup_{A \in \mathcal{A}} H^2(\widehat{A}, k^\times)^G \qquad \text{(a colimit of pointed sets)}$$

Here $\widehat{A} = \mathrm{Hom}(A, k^\times)$ is the group of characters of A and $H^2(\widehat{A}, k^\times)$ is the second cohomology group of $\widehat{A}$

- The set $\mathcal{B}(G)$ is
  - $*$ non-empty: it is pointed by the zero element 0 lying in all $H^2(\widehat{A}, k^\times)^G$
  - $*$ it is finite

- If G has a unique maximal normal abelian subgroup A, then

$$\mathcal{B}(G) = H^2(\widehat{A}, k^\times)^G$$

First ingredient: the pointed set $\mathcal{B}(G)$

- **Definition.** *Let $\mathcal{A}$ be the category whose objects are the normal abelian subgroups A of G and whose arrows are the inclusions. Define*

$$\mathcal{B}(G) = \bigcup_{A \in \mathcal{A}} H^2(\widehat{A}, k^\times)^G \qquad \text{(a colimit of pointed sets)}$$

Here $\widehat{A} = \mathrm{Hom}(A, k^\times)$ is the group of characters of A and $H^2(\widehat{A}, k^\times)$ is the second cohomology group of $\widehat{A}$

- The set $\mathcal{B}(G)$ is
  - ∗ non-empty: it is pointed by the zero element 0 lying in all $H^2(\widehat{A}, k^\times)^G$
  - ∗ it is finite

- If G has a unique maximal normal abelian subgroup A, then

$$\mathcal{B}(G) = H^2(\widehat{A}, k^\times)^G$$

Second ingredient: class-preserving automorphisms

- Let $\mathrm{Aut}_c(G)$ be the group of automorphisms φ of G such that for all $g \in G$

$$\varphi(g) = hgh^{-1} \qquad \text{for some } h \in G$$

The group $\mathrm{Inn}(G)$ of inner automorphism is normal in $\mathrm{Aut}_c(G)$

- **Definition.**

$$\mathrm{Out}_c(G) = \mathrm{Aut}_c(G)/\,\mathrm{Inn}(G)$$

- **Examples.** $\mathrm{Out}_c(G) = 1$ if

  * $G = S_n$ (symmetric group)
  * $G = SL_n(\mathbb{F}_q)$
  * G is simple (Feit-Seitz, 1989)

- Finding finite groups with $\mathrm{Out}_c(G) \neq 1$ is not straightforward

Nevertheless, there are groups G with non-trivial $\mathrm{Out}_c(G)$, even with non-abelian $\mathrm{Out}_c(G)$

Second ingredient: class-preserving automorphisms

- Let $\mathrm{Aut}_c(G)$ be the group of automorphisms φ of G such that for all $g \in G$

$$\varphi(g) = hgh^{-1} \qquad \text{for some } h \in G$$

The group $\mathrm{Inn}(G)$ of inner automorphism is normal in $\mathrm{Aut}_c(G)$

- **Definition.**

$$\mathrm{Out}_c(G) = \mathrm{Aut}_c(G)/\,\mathrm{Inn}(G)$$

- **Examples**. $\mathrm{Out}_c(G) = 1$ if
  - $G = S_n$ (symmetric group)
  - $G = SL_n(\mathbb{F}_q)$
  - G is simple (Feit-Seitz, 1989)

- Finding finite groups with $\mathrm{Out}_c(G) \neq 1$ is not straightforward

Nevertheless, there are groups G with non-trivial $\mathrm{Out}_c(G)$, even with non-abelian $\mathrm{Out}_c(G)$

Second ingredient: class-preserving automorphisms

- Let $\mathrm{Aut}_c(G)$ be the group of automorphisms φ of G such that for all $g \in G$

$$\varphi(g) = hgh^{-1} \qquad \text{for some } h \in G$$

The group $\mathrm{Inn}(G)$ of inner automorphism is normal in $\mathrm{Aut}_c(G)$

- **Definition.**

$$\mathrm{Out}_c(G) = \mathrm{Aut}_c(G)/\,\mathrm{Inn}(G)$$

- **Examples**. $\mathrm{Out}_c(G) = 1$ if
  - $G = S_n$ (symmetric group)
  - $G = SL_n(\mathbb{F}_q)$
  - G is simple (Feit-Seitz, 1989)

- Finding finite groups with $\mathrm{Out}_c(G) \neq 1$ is not straightforward

Nevertheless, there are groups G with non-trivial $\mathrm{Out}_c(G)$,
even with non-abelian $\mathrm{Out}_c(G)$

Non-trivial $\mathrm{Out}_c(G)$

• **Burnside (1912)** was the first one to construct groups with $\mathrm{Out}_c(G) \neq 1$ (his smallest example is of order $729 = 3^6$)

• **G. E. Wall (1947)** proved that $\mathrm{Out}_c(G) = \mathbb{Z}/2$ for

$$G = \mathbb{Z}/8 \rtimes \mathrm{Aut}(\mathbb{Z}/8)$$

which is of order 32

• Burnside stated that $\mathrm{Out}_c(G)$ is always abelian, but. . .

• **C.-H. Sah (1968):** There are groups such that $\mathrm{Out}_c(G)$ is non-abelian (his smallest example is of order 2^{15})

Non-trivial $\mathrm{Out}_c(G)$

- **Burnside (1912)** was the first one to construct groups with $\mathrm{Out}_c(G) \neq 1$ (his smallest example is of order $729 = 3^6$)

- **G. E. Wall (1947)** proved that $\mathrm{Out}_c(G) = \mathbb{Z}/2$ for

$$G = \mathbb{Z}/8 \rtimes \mathrm{Aut}(\mathbb{Z}/8)$$

which is of order 32

- Burnside stated that $\mathrm{Out}_c(G)$ is always abelian, but. . .

- **C.-H. Sah (1968):** There are groups such that $\mathrm{Out}_c(G)$ is non-abelian (his smallest example is of order 2^{15})

Non-trivial $\mathrm{Out}_c(G)$

- **Burnside (1912)** was the first one to construct groups with $\mathrm{Out}_c(G) \neq 1$ (his smallest example is of order $729 = 3^6$)

- **G. E. Wall (1947)** proved that $\mathrm{Out}_c(G) = \mathbb{Z}/2$ for

$$G = \mathbb{Z}/8 \rtimes \mathrm{Aut}(\mathbb{Z}/8)$$

which is of order 32

- Burnside stated that $\mathrm{Out}_c(G)$ is always abelian, but. . .

- **C.-H. Sah (1968):** There are groups such that $\mathrm{Out}_c(G)$ is non-abelian (his smallest example is of order 2^{15})

Determining $\mathcal{H}^2(G/k)$

Now our main result in the algebraically closed case

Assume that the field k is algebraically closed of characteristic prime to $|G|$

Theorem 2. *There is a set-theoretic map $\Theta : \mathcal{H}^2(G/k) \to \mathcal{B}(G)$ such that*

(a) $\mathcal{H}_0 = \Theta^{-1}(\{0\})$ is a subgroup of $\mathcal{H}^2(G/k)$ with

$$\mathcal{H}_0 \cong \mathrm{Out}_c(G)$$

(b) All fibers of Θ are in bijection with $\mathcal{H}_0$; more precisely,

$$\Theta(\alpha) = \Theta(\beta) \Longleftrightarrow \beta \in \alpha\,\mathcal{H}_0 \qquad (\alpha, \beta \in \mathcal{H}^2(G/k))$$

(c) If $|G|$ is odd, then Θ is surjective

Determining $\mathcal{H}^2(G/k)$

Now our main result in the algebraically closed case

Assume that the field k is algebraically closed of characteristic prime to $|G|$

Theorem 2. *There is a set-theoretic map* $\Theta : \mathcal{H}^2(G/k) \to \mathcal{B}(G)$ *such that*

(a) $\mathcal{H}_0 = \Theta^{-1}(\{0\})$ *is a subgroup of* $\mathcal{H}^2(G/k)$ *with*

$$\mathcal{H}_0 \cong \mathrm{Out}_c(G)$$

(b) All fibers of Θ *are in bijection with* $\mathcal{H}_0$*; more precisely,*

$$\Theta(\alpha) = \Theta(\beta) \iff \beta \in \alpha\,\mathcal{H}_0 \qquad (\alpha, \beta \in \mathcal{H}^2(G/k))$$

(c) If $|G|$ *is odd, then* Θ *is surjective*

Determining $\mathcal{H}^2(G/k)$

Now our main result in the algebraically closed case

Assume that the field k is algebraically closed of characteristic prime to $|G|$

Theorem 2. *There is a set-theoretic map* $\Theta : \mathcal{H}^2(G/k) \to \mathcal{B}(G)$ *such that*

(a) $\mathcal{H}_0 = \Theta^{-1}(\{0\})$ *is a subgroup of* $\mathcal{H}^2(G/k)$ *with*

$$\mathcal{H}_0 \cong \mathrm{Out}_c(G)$$

(b) All fibers of Θ *are in bijection with* $\mathcal{H}_0$*; more precisely,*

$$\Theta(\alpha) = \Theta(\beta) \Longleftrightarrow \beta \in \alpha\,\mathcal{H}_0 \qquad (\alpha, \beta \in \mathcal{H}^2(G/k))$$

(c) If $|G|$ *is odd, then* Θ *is surjective*

Determining $\mathcal{H}^2(G/k)$

Now our main result in the algebraically closed case

Assume that the field k is algebraically closed of characteristic prime to $|G|$

Theorem 2. *There is a set-theoretic map* $\Theta : \mathcal{H}^2(G/k) \to \mathcal{B}(G)$ *such that*

(a) $\mathcal{H}_0 = \Theta^{-1}(\{0\})$ *is a subgroup of* $\mathcal{H}^2(G/k)$ *with*

$$\mathcal{H}_0 \cong \mathrm{Out}_c(G)$$

(b) All fibers of Θ *are in bijection with* $\mathcal{H}_0$*; more precisely,*

$$\Theta(\alpha) = \Theta(\beta) \Longleftrightarrow \beta \in \alpha\,\mathcal{H}_0 \qquad (\alpha, \beta \in \mathcal{H}^2(G/k))$$

(c) If $|G|$ *is odd, then* Θ *is surjective*

Immediate consequences of Theorem 2

- The group $\mathcal{H}^2(G/k)$ is finite: it sits between $\mathrm{Out}_c(G)$ and $\mathcal{B}(G)$
- The group $\mathcal{H}^2(G/k)$ can be non-abelian since $\mathcal{H}^2(G/k) \supset \mathcal{H}_0 \cong \mathrm{Out}_c(G)$
- If $\mathrm{Out}_c(G) = 1$ and $\mathcal{B}(G)$ is trivial, then $\mathcal{H}^2(G/k) = 1$
- **Examples of groups with trivial** $\mathcal{H}^2(G/k)$:
  - $*$ $G = S_n$ or G is simple (no normal abelian subgroup)
  - $*$ $G = SL_n(\mathbb{F}_q)$

(all normal abelian subgroups are cyclic and $H^2(\text{cyclic}, k^\times) = 0$)

Immediate consequences of Theorem 2

- The group $\mathcal{H}^2(G/k)$ is finite: it sits between $\mathrm{Out}_c(G)$ and $\mathcal{B}(G)$
- The group $\mathcal{H}^2(G/k)$ can be non-abelian since $\mathcal{H}^2(G/k) \supset \mathcal{H}_0 \cong \mathrm{Out}_c(G)$
- If $\mathrm{Out}_c(G) = 1$ and $\mathcal{B}(G)$ is trivial, then $\mathcal{H}^2(G/k) = 1$
- **Examples of groups with trivial $\mathcal{H}^2(G/k)$:**
  - $G = S_n$ or G is simple (no normal abelian subgroup)
  - $G = SL_n(\mathbb{F}_q)$

(all normal abelian subgroups are cyclic and $H^2(\text{cyclic}, k^\times) = 0$)

Immediate consequences of Theorem 2

- The group $\mathcal{H}^2(G/k)$ is finite: it sits between $\mathrm{Out}_c(G)$ and $\mathcal{B}(G)$
- The group $\mathcal{H}^2(G/k)$ can be non-abelian since $\mathcal{H}^2(G/k) \supset \mathcal{H}_0 \cong \mathrm{Out}_c(G)$
- If $\mathrm{Out}_c(G) = 1$ and $\mathcal{B}(G)$ is trivial, then $\mathcal{H}^2(G/k) = 1$
- **Examples of groups with trivial $\mathcal{H}^2(G/k)$:**
  - $G = S_n$ or G is simple (no normal abelian subgroup)
  - $G = SL_n(\mathbb{F}_q)$

(all normal abelian subgroups are cyclic and $H^2(\text{cyclic}, k^\times) = 0$)

Immediate consequences of Theorem 2

- The group $\mathcal{H}^2(G/k)$ is finite: it sits between $\mathrm{Out}_c(G)$ and $\mathcal{B}(G)$
- The group $\mathcal{H}^2(G/k)$ can be non-abelian since $\mathcal{H}^2(G/k) \supset \mathcal{H}_0 \cong \mathrm{Out}_c(G)$
- If $\mathrm{Out}_c(G) = 1$ and $\mathcal{B}(G)$ is trivial, then $\mathcal{H}^2(G/k) = 1$
- **Examples of groups with trivial $\mathcal{H}^2(G/k)$:**
  - $*$ $G = S_n$ or G is simple (no normal abelian subgroup)
  - $*$ $G = SL_n(\mathbb{F}_q)$

(all normal abelian subgroups are cyclic and $H^2(\text{cyclic}, k^\times) = 0$)

Immediate consequences of Theorem 2

- The group $\mathcal{H}^2(G/k)$ is finite: it sits between $\mathrm{Out}_c(G)$ and $\mathcal{B}(G)$
- The group $\mathcal{H}^2(G/k)$ can be non-abelian since $\mathcal{H}^2(G/k) \supset \mathcal{H}_0 \cong \mathrm{Out}_c(G)$
- If $\mathrm{Out}_c(G) = 1$ and $\mathcal{B}(G)$ is trivial, then $\mathcal{H}^2(G/k) = 1$
- **Examples of groups with trivial** $\mathcal{H}^2(G/k)$:
  - $*$ $G = S_n$ or G is simple (no normal abelian subgroup)
  - $*$ $G = SL_n(\mathbb{F}_q)$

(all normal abelian subgroups are cyclic and $H^2(\text{cyclic}, k^\times) = 0$)

An example with non-trivial $\mathcal{H}^2(G/k)$

• Let p be an odd prime and let G be the wreath product

$$G = \mathbb{Z}/p \wr \mathbb{Z}/p = A \rtimes \mathbb{Z}/p$$

with $\mathbb{Z}/p$ acting cyclically on $A = (\mathbb{Z}/p)^p$

The subgroup A is the unique maximal normal abelian subgroup of G

• For this group $\mathrm{Out}_c(G) = 1$ and $|G|$ is odd:

$$\Theta : \mathcal{H}^2(G/k) \longrightarrow \mathcal{B}(G) = H^2(\widehat{A}, k^\times)^G$$

is bijective by Theorem 2

• **Proposition.** $\mathcal{H}^2(G/k) = H^2(\widehat{A}, k^\times)^G \;\cong\; (\mathbb{Z}/p)^{(p-1)/2}$

The second isomorphism follows from a standard cohomological calculation

An example with non-trivial $\mathcal{H}^2(G/k)$

- Let p be an odd prime and let G be the wreath product

$$G = \mathbb{Z}/p \wr \mathbb{Z}/p = A \rtimes \mathbb{Z}/p$$

with $\mathbb{Z}/p$ acting cyclically on $A = (\mathbb{Z}/p)^p$

The subgroup A is the unique maximal normal abelian subgroup of G

- For this group $\mathrm{Out}_c(G) = 1$ and $|G|$ is odd:

$$\Theta : \mathcal{H}^2(G/k) \longrightarrow \mathcal{B}(G) = H^2(\widehat{A}, k^\times)^G$$

is bijective by Theorem 2

- **Proposition.** $\mathcal{H}^2(G/k) = H^2(\widehat{A}, k^\times)^G \;\cong\; (\mathbb{Z}/p)^{(p-1)/2}$

The second isomorphism follows from a standard cohomological calculation

An example with non-trivial $\mathcal{H}^2(G/k)$

- Let p be an odd prime and let G be the wreath product

$$G = \mathbb{Z}/p \wr \mathbb{Z}/p = A \rtimes \mathbb{Z}/p$$

with $\mathbb{Z}/p$ acting cyclically on $A = (\mathbb{Z}/p)^p$

The subgroup A is the unique maximal normal abelian subgroup of G

- For this group $\mathrm{Out}_c(G) = 1$ and $|G|$ is odd:

$$\Theta : \mathcal{H}^2(G/k) \longrightarrow \mathcal{B}(G) = H^2(\widehat{A}, k^\times)^G$$

is bijective by Theorem 2

- **Proposition.** $\mathcal{H}^2(G/k) = H^2(\widehat{A}, k^\times)^G \;\cong\; (\mathbb{Z}/p)^{(p-1)/2}$

The second isomorphism follows from a standard cohomological calculation

To be or not to be surjective in the even case

If $|G|$ is even, then Θ may or may not be surjective

• **Example 1**. For $p = 2$ the wreath product $G = \mathbb{Z}/p \wr \mathbb{Z}/p$ is the dihedral group of order 8: it has two maximal normal abelian subgroups. In this case,

$$\mathcal{H}^2(G/k) = 1 \quad \text{although} \quad |\mathcal{B}(G)| = 3$$

and the map $\Theta : \mathcal{H}^2(G/k) \to \mathcal{B}(G)$ is not surjective

• **Example 2**. If $G = A_4$ is the alternating group, then $\mathrm{Out}_c(A_4) = 1$ so that the map $\Theta : \mathcal{H}^2(A_4/k) \to \mathcal{B}(A_4)$ is injective

Proposition. *The map Θ is bijective and $\mathcal{H}^2(A_4/k) \cong \mathcal{B}(A_4) \cong \mathbb{Z}/2$*

To prove the surjectivity of Θ, we exhibit an element $\alpha \in \mathcal{H}^2(A_4/k)$ such that $\Theta(\alpha)$ is the non-trivial element of $\mathcal{B}(A_4)$

To be or not to be surjective in the even case

If $|G|$ is even, then Θ may or may not be surjective

• **Example 1**. For $p = 2$ the wreath product $G = \mathbb{Z}/p \wr \mathbb{Z}/p$ is the dihedral group of order 8: it has two maximal normal abelian subgroups. In this case,

$$\mathcal{H}^2(G/k) = 1 \quad \text{although} \quad |\mathcal{B}(G)| = 3$$

and the map $\Theta : \mathcal{H}^2(G/k) \to \mathcal{B}(G)$ is not surjective

• **Example 2**. If $G = A_4$ is the alternating group, then $\mathrm{Out}_c(A_4) = 1$ so that the map $\Theta : \mathcal{H}^2(A_4/k) \to \mathcal{B}(A_4)$ is injective

Proposition. *The map Θ is bijective and $\mathcal{H}^2(A_4/k) \cong \mathcal{B}(A_4) \cong \mathbb{Z}/2$*

To prove the surjectivity of Θ, we exhibit an element $\alpha \in \mathcal{H}^2(A_4/k)$ such that $\Theta(\alpha)$ is the non-trivial element of $\mathcal{B}(A_4)$

To be or not to be surjective in the even case

If $|G|$ is even, then Θ may or may not be surjective

• **Example 1**. For $p = 2$ the wreath product $G = \mathbb{Z}/p \wr \mathbb{Z}/p$ is the dihedral group of order 8: it has two maximal normal abelian subgroups. In this case,

$$\mathcal{H}^2(G/k) = 1 \quad \text{although} \quad |\mathcal{B}(G)| = 3$$

and the map $\Theta : \mathcal{H}^2(G/k) \to \mathcal{B}(G)$ is not surjective

• **Example 2**. If $G = A_4$ is the alternating group, then $\mathrm{Out}_c(A_4) = 1$ so that the map $\Theta : \mathcal{H}^2(A_4/k) \to \mathcal{B}(A_4)$ is injective

Proposition. *The map Θ is bijective and $\mathcal{H}^2(A_4/k) \cong \mathcal{B}(A_4) \cong \mathbb{Z}/2$*

To prove the surjectivity of Θ, we exhibit an element $\alpha \in \mathcal{H}^2(A_4/k)$ such that $\Theta(\alpha)$ is the non-trivial element of $\mathcal{B}(A_4)$

Plan

On the proof of Theorem 2

- **Theorem 2 (Extract).** *There is a set-theoretic map*

$$\Theta : \mathcal{H}^2(G/k) \to \mathcal{B}(G/k)$$

such that $\mathcal{H}_0 = \Theta^{-1}(\{0\})$ *is a subgroup of* $\mathcal{H}^2(G/k)$ *with*

$$\mathcal{H}_0 \cong \mathrm{Out}_c(G)$$

- The proof uses quantum group theory

Let us first explain the construction of the map Θ

The universal R-matrix attached to a Drinfeld twist

- Represent an element of $\mathcal{H}^2(G/k)$ by an invariant Drinfeld twist

$$F \in kG \otimes kG$$

- Consider

$$R_F = F_{21}\, F^{-1} \in kG \otimes kG$$

This is a universal R-matrix for kG, i.e., an invertible element satisfying

$$\Delta(a)\, R_F = R_F\, \Delta(a) \qquad \text{for all } a \in kG$$

and

$$(\Delta \otimes \mathrm{id})(R_F) = (R_F)_{13}\,(R_F)_{23} \quad \text{and} \quad (\mathrm{id} \otimes \Delta)(R_F) = (R_F)_{13}\,(R_F)_{12}$$

- The universal R-matrix R_F induces a braiding γ on the tensor category of G-modules

$$\begin{aligned} \gamma_{V,W} : V \otimes W &\xrightarrow{\cong} W \otimes V \\ v \otimes w &\longmapsto \big(R_F(v \otimes w)\big)_{21} \end{aligned}$$

The universal R-matrix attached to a Drinfeld twist

- Represent an element of $\mathcal{H}^2(G/k)$ by an invariant Drinfeld twist

$$F \in kG \otimes kG$$

- Consider

$$R_F = F_{21}\, F^{-1} \in kG \otimes kG$$

This is a universal R-matrix for kG, i.e., an invertible element satisfying

$$\Delta(a)\, R_F = R_F\, \Delta(a) \qquad \text{for all } a \in kG$$

and

$$(\Delta \otimes \mathrm{id})(R_F) = (R_F)_{13}\,(R_F)_{23} \quad \text{and} \quad (\mathrm{id} \otimes \Delta)(R_F) = (R_F)_{13}\,(R_F)_{12}$$

- The universal R-matrix R_F induces a braiding γ on the tensor category of G-modules

$$\begin{aligned} \gamma_{V,W} : V \otimes W &\xrightarrow{\cong} W \otimes V \\ v \otimes w &\longmapsto \big(R_F(v \otimes w)\big)_{21} \end{aligned}$$

The universal R-matrix attached to a Drinfeld twist

- Represent an element of $\mathcal{H}^2(G/k)$ by an invariant Drinfeld twist

$$F \in kG \otimes kG$$

- Consider

$$R_F = F_{21}\, F^{-1} \in kG \otimes kG$$

This is a universal R-matrix for kG, i.e., an invertible element satisfying

$$\Delta(a)\, R_F = R_F\, \Delta(a) \qquad \text{for all } a \in kG$$

and

$$(\Delta \otimes \mathrm{id})(R_F) = (R_F)_{13}\,(R_F)_{23} \quad \text{and} \quad (\mathrm{id} \otimes \Delta)(R_F) = (R_F)_{13}\,(R_F)_{12}$$

- The universal R-matrix R_F induces a braiding γ on the tensor category of G-modules

$$\begin{aligned} \gamma_{V,W} : V \otimes W &\xrightarrow{\cong} W \otimes V \\ v \otimes w &\longmapsto \big(R_F(v \otimes w)\big)_{21} \end{aligned}$$

Constructing the map Θ: Part 1

• By work of Radford, there is a minimal Hopf subalgebra $H \subset kG$ such that

$$R_F \in H \otimes H$$

This Hopf algebra is self-dual: $H^* \cong H$

• Since kG is cocommutative, so is H. From self-duality, H is bicommutative

One deduces that $H = kA$ for the abelian subgroup A of G given by

$$A = \{a \in H \mid \Delta(a) = a \otimes a \text{ and } \varepsilon(a) = 1\}$$

• Since F is G-invariant, so is R_F, and A is normal in G

Constructing the map Θ: Part 1

• By work of Radford, there is a minimal Hopf subalgebra $H \subset kG$ such that

$$R_F \in H \otimes H$$

This Hopf algebra is self-dual: $H^* \cong H$

• Since kG is cocommutative, so is H. From self-duality, H is bicommutative

One deduces that $H = kA$ for the abelian subgroup A of G given by

$$A = \{a \in H \mid \Delta(a) = a \otimes a \;\text{ and }\; \varepsilon(a) = 1\}$$

• Since F is G-invariant, so is R_F, and A is normal in G

Constructing the map Θ: Part 1

- By work of Radford, there is a minimal Hopf subalgebra $H \subset kG$ such that

$$R_F \in H \otimes H$$

This Hopf algebra is self-dual: $H^* \cong H$

- Since kG is cocommutative, so is H. From self-duality, H is bicommutative

One deduces that $H = kA$ for the abelian subgroup A of G given by

$$A = \{a \in H \mid \Delta(a) = a \otimes a \text{ and } \varepsilon(a) = 1\}$$

- Since F is G-invariant, so is R_F, and A is normal in G

Constructing the map Θ: Part 2

- By duality, $R_F \in kA \otimes kA$ corresponds to a bilinear form

$$(kA)^* \times (kA)^* \to k$$

- Now the discrete Fourier transform induces a Hopf algebra isomorphism

$$(kA)^* \cong k\widehat{A}$$

(where $\widehat{A} = \mathrm{Hom}(A, k^\times)$) so that we obtain a bilinear form $k\widehat{A} \times k\widehat{A} \to k$

- Restricting to $\widehat{A} \times \widehat{A}$, we obtain a G-invariant 2-cocycle

$$b_F : \widehat{A} \times \widehat{A} \to k^\times$$

which represents an element $[b_F] \in H^2(\widehat{A}, k^\times)^G$

- We define

$$\Theta(F) = [b_F] \in \mathcal{B}(G) = \bigcup_{A \in \mathcal{A}} H^2(\widehat{A}, k^\times)^G$$

One checks that $\Theta(F)$ depends only on the class of F in $\mathcal{H}^2(G/k)$

Constructing the map Θ: Part 2

- By duality, $R_F \in kA \otimes kA$ corresponds to a bilinear form

$$(kA)^* \times (kA)^* \to k$$

- Now the discrete Fourier transform induces a Hopf algebra isomorphism

$$(kA)^* \cong k\widehat{A}$$

(where $\widehat{A} = \mathrm{Hom}(A, k^\times)$) so that we obtain a bilinear form $k\widehat{A} \times k\widehat{A} \to k$

- Restricting to $\widehat{A} \times \widehat{A}$, we obtain a G-invariant 2-cocycle

$$b_F : \widehat{A} \times \widehat{A} \to k^\times$$

which represents an element $[b_F] \in H^2(\widehat{A}, k^\times)^G$

- We define

$$\Theta(F) = [b_F] \in \mathcal{B}(G) = \bigcup_{A \in \mathcal{A}} H^2(\widehat{A}, k^\times)^G$$

One checks that $\Theta(F)$ depends only on the class of F in $\mathcal{H}^2(G/k)$

Constructing the map Θ: Part 2

- By duality, $R_F \in kA \otimes kA$ corresponds to a bilinear form

$$(kA)^* \times (kA)^* \to k$$

- Now the discrete Fourier transform induces a Hopf algebra isomorphism

$$(kA)^* \cong k\widehat{A}$$

(where $\widehat{A} = \mathrm{Hom}(A, k^\times)$) so that we obtain a bilinear form $k\widehat{A} \times k\widehat{A} \to k$

- Restricting to $\widehat{A} \times \widehat{A}$, we obtain a G-invariant 2-cocycle

$$b_F : \widehat{A} \times \widehat{A} \to k^\times$$

which represents an element $[b_F] \in H^2(\widehat{A}, k^\times)^G$

- We define

$$\Theta(F) = [b_F] \in \mathcal{B}(G) = \bigcup_{A \in \mathcal{A}} H^2(\widehat{A}, k^\times)^G$$

One checks that $\Theta(F)$ depends only on the class of F in $\mathcal{H}^2(G/k)$

Constructing the map Θ: Part 2

- By duality, $R_F \in kA \otimes kA$ corresponds to a bilinear form

$$(kA)^* \times (kA)^* \to k$$

- Now the discrete Fourier transform induces a Hopf algebra isomorphism

$$(kA)^* \cong k\widehat{A}$$

(where $\widehat{A} = \mathrm{Hom}(A, k^\times)$) so that we obtain a bilinear form $k\widehat{A} \times k\widehat{A} \to k$

- Restricting to $\widehat{A} \times \widehat{A}$, we obtain a G-invariant 2-cocycle

$$b_F : \widehat{A} \times \widehat{A} \to k^\times$$

which represents an element $[b_F] \in H^2(\widehat{A}, k^\times)^G$

- We define

$$\Theta(F) = [b_F] \in \mathcal{B}(G) = \bigcup_{A \in \mathcal{A}} H^2(\widehat{A}, k^\times)^G$$

One checks that $\Theta(F)$ depends only on the class of F in $\mathcal{H}^2(G/k)$

The "kernel" of Θ

To determine $\mathcal{H}_0 = \Theta^{-1}((\{1\}, b_0)) \cong \mathrm{Out}_c(G)$, we use the following results:

• **Etingof and Gelaki (2000):** *If F is a Drinfeld twist such that $R_F = 1 \otimes 1$, or equivalently F is symmetric: $F = F_{21}$, then*

$$F = (a \otimes a)\, \Delta(a^{-1})$$

for some invertible element $a \in kG$

Their result follows from classical Tannakian theory:

Deligne and Milne (1982): *Any exact and fully faithful symmetric tensor functor from the category of kG-modules to the category of k-vector spaces is isomorphic to the forgetful functor*

A symmetric Drinfeld twist gives rise to a symmetric tensor functor, to which Etingof and Gelaki apply Deligne and Milne's result

• A symmetric Drinfeld twist $F = (a \otimes a)\, \Delta(a^{-1})$ is invariant if and only if the automorphism $x \mapsto axa^{-1}$ preserves G and is an element of $\mathrm{Aut}_c(G)$

From these remarks it is easy to deduce that $\mathcal{H}_0 \cong \mathrm{Out}_c(G)$

The “kernel” of Θ

To determine $\mathcal{H}_0 = \Theta^{-1}((\{1\}, b_0)) \cong \mathrm{Out}_c(G)$, we use the following results:

• **Etingof and Gelaki (2000):** *If F is a Drinfeld twist such that $R_F = 1 \otimes 1$, or equivalently F is symmetric: $F = F_{21}$, then*

$$F = (a \otimes a)\,\Delta(a^{-1})$$

for some invertible element $a \in kG$

Their result follows from classical Tannakian theory:

Deligne and Milne (1982): *Any exact and fully faithful symmetric tensor functor from the category of kG-modules to the category of k-vector spaces is isomorphic to the forgetful functor*

A symmetric Drinfeld twist gives rise to a symmetric tensor functor, to which Etingof and Gelaki apply Deligne and Milne's result

• A symmetric Drinfeld twist $F = (a \otimes a)\,\Delta(a^{-1})$ is invariant if and only if the automorphism $x \mapsto axa^{-1}$ preserves G and is an element of $\mathrm{Aut}_c(G)$

From these remarks it is easy to deduce that $\mathcal{H}_0 \cong \mathrm{Out}_c(G)$

The “kernel” of Θ

To determine $\mathcal{H}_0 = \Theta^{-1}((\{1\}, b_0)) \cong \mathrm{Out}_c(G)$, we use the following results:

• **Etingof and Gelaki (2000):** *If F is a Drinfeld twist such that $R_F = 1 \otimes 1$, or equivalently F is symmetric: $F = F_{21}$, then*

$$F = (a \otimes a)\, \Delta(a^{-1})$$

for some invertible element $a \in kG$

Their result follows from classical Tannakian theory:

Deligne and Milne (1982): *Any exact and fully faithful symmetric tensor functor from the category of kG-modules to the category of k-vector spaces is isomorphic to the forgetful functor*

A symmetric Drinfeld twist gives rise to a symmetric tensor functor, to which Etingof and Gelaki apply Deligne and Milne’s result

• A symmetric Drinfeld twist $F = (a \otimes a)\, \Delta(a^{-1})$ is invariant if and only if the automorphism $x \mapsto axa^{-1}$ preserves G and is an element of $\mathrm{Aut}_c(G)$

From these remarks it is easy to deduce that $\mathcal{H}_0 \cong \mathrm{Out}_c(G)$

Appendix. The definition of A_F

• The Hopf algebra $(kG)^*$, dual of the group algebra kG, has a basis $(e_g)_{g\in G}$ of orthogonal idempotents defined by $\langle e_g, h\rangle = \delta_{g,h}$ $(g, h \in G)$

• The non-commutative G-bitorsor A_F associated to an invariant Drinfeld twist $F \in kG \otimes kG$ is a deformation of $(kG)^*$: it has a basis $(u_g)_{g\in G}$ in which the product of A_F is given by

$$u_g u_h = \sum_{s\in G} \alpha(gs^{-1}, hs^{-1})\, u_s$$

where the scalars $\alpha(g, h)$ are the coefficients of F:

$$F = \sum_{g,h\in G} \alpha(g, h)\, g \otimes h$$

• The coaction $\delta : A_F \to A_F \otimes (kG)^*$ is given by

$$\delta(u_g) = \sum_{s\in G} u_{gs^{-1}} \otimes e_s$$

References on Hopf algebras

J. Bichon, G. Carnovale, *Lazy cohomology: an analogue of the Schur multiplier for arbitrary Hopf algebras*, J. Pure Appl. Algebra 204 (2006), 627–665.

V. G. Drinfeld, *Quasi-Hopf algebras*, Algebra i Analiz 1 (1989), 114–148; English translation: Leningrad Math. J. 1 (1990), 1419–1457.

P. Guillot, C. Kassel, *Cohomology of invariant Drinfeld twists on group algebras*, Internat. Math. Res. Notices 2010, 1894–1939; arXiv:0903.2807.

D. E. Radford, *Minimal quasitriangular Hopf algebras* J. Algebra 157 (1993), no. 2, 285–315.

P. Schauenburg, *Hopf bimodules, coquasibialgebras, and an exact sequence of Kac*, Adv. Math. 165 (2002), 194–263.

M. E. Sweedler, *Cohomology of algebras over Hopf algebras*, Trans. Amer. Math. Soc. 133 (1968), 205–239.

References on groups

W. Burnside, *On the outer automorphisms of a group*, Proc. London Math. Soc. (2) 11 (1912), 225–245.

W. Feit, G. M. Seitz, *On finite rational groups and related topics*, Illinois J. Math. 33 (1989), no. 1, 103–131.

C.-H. Sah, *Automorphisms of finite groups*, J. Algebra 10 (1968), 47–68.

G. E. Wall, *Finite groups with class-preserving outer automorphisms*, J. London Math. Soc. 22 (1947), 315–320.

Takk for oppmerksomheten!

Thank you for your attention!