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Surjectivité de la norme : problèmes effectifs en cohomologie des groupes

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Références

[AG] E. Aljadeff, Y. Ginosar, *Induction from elementary abelian subgroups*, J. Algebra 179 (1996), 599–606.

[AK1] E. Aljadeff, C. Kassel, *Explicit norm one elements for ring actions of finite abelian groups*, Israel J. Math. 129 (2002), 99–108.

[AK2] E. Aljadeff, C. Kassel, *Norm formulas for finite groups and induction from elementary abelian subgroups*, arXiv:math.RA/0402145 (9 Feb 2004).

The cyclic group $\mathbf{Z}/p^2$ [AK1]

- $G = \mathbf{Z}/p^2$ with generator σ (p prime number)
- $U \cong \mathbf{Z}/p \subset G$ generated by $t = \sigma^p$
- $x \in R$ such that

$$N_U(x) = (1 + t + t^2 + \cdots + t^{p-1})(x) = 1.$$

Define z, a, $y \in R$ by

$$z = \big(1 + \sigma + \sigma^2 + \cdots + \sigma^{p-1}\big)(x) - 1,$$

$$a = x + (1 - \sigma) \sum_{i=1}^{p-1} \Big(1 + t + t^2 + \cdots + t^{i-1}\Big)\big(xt^{-i}(z)\big),$$

and $y = ax$ or $y = xa$. Then $N_G(y) = 1$.

For $p = 2$, we obtain

$$y = \sigma(x)x - \sigma(x)x^2 + x\sigma^2(x)x + x\sigma^3(x)x - \sigma(x)\sigma^3(x)x.$$

A formula for $G = \mathbf{Z}/9$

For a ring automorphism σ of order 9 of the ring R and an element $x \in R$ such that

$$x + \sigma^3(x) + \sigma^6(x) = 1,$$

we have

$$\sum_{i=0}^{8} \sigma^i(y) = 1$$

for

$$\begin{aligned} y = & - x^2 + 2\sigma(x)x - \sigma^3(x)x + \sigma^4(x)x \\ & + x\sigma^3(x)x + x\sigma^4(x)x + x\sigma^5(x)x \\ & + x\sigma^6(x)x + x\sigma^7(x)x + x\sigma^8(x)x \\ & - \sigma(x)\sigma^4(x)x - \sigma(x)\sigma^5(x)x - \sigma(x)\sigma^6(x)x \\ & - \sigma(x)\sigma^7(x)x - \sigma(x)\sigma^8(x)x - \sigma(x)x^2 \\ & + \sigma^3(x)\sigma^6(x)x + \sigma^3(x)\sigma^7(x)x + \sigma^3(x)\sigma^8(x)x \\ & - \sigma^4(x)\sigma^7(x)x - \sigma^4(x)\sigma^8(x)x - \sigma^4(x)x^2. \end{aligned}$$

(RHS has 22 monomials)

The quaternion group Q_8 [AK2]

- $G = Q_8 = \langle \sigma, \tau \mid \sigma^4 = 1, \tau^2 = \sigma^2, \tau\sigma = \sigma^{-1}\tau \rangle$
- $U = \langle \sigma \rangle$, cyclic subgroup of order 4
- $x \in R$ such that $N_U(x) = (1 + \sigma + \sigma^2 + \sigma^3)(x) = 1$

The system (Σ) *with unknowns* $b(\sigma)$ *and* $b(\tau)$*:*

$$\begin{cases} (1 + \sigma + \sigma^2 + \sigma^3) b(\sigma) & = 0, \\ (1 + \tau) b(\tau) - (1 + \sigma) b(\sigma) & = 1, \\ (\sigma - 1) b(\tau) + (1 + \sigma\tau) b(\sigma) & = 0. \end{cases}$$

A solution of (Σ)*:*

$$b(\sigma) = (1 - \sigma\tau)(x) \ \text{ and } \ b(\tau) = (1 + \sigma)(x).$$

The apocalypse of Q_8

- $U = \langle \sigma \rangle$, cyclic subgroup of order 4 of $G = Q_8$
- $E = \langle \sigma^2 \rangle$, elementary abelian subgroup of order 2

If $x \in R$ such that $N_U(x) = 1$, then $N_G(y) = 1$ for

$$
\begin{aligned}
y = {} & x^2 + \sigma(x)x \\
& + x\sigma(x)x + x\sigma^2(x)x + x\sigma^3(x)x \\
& - x\tau(x)x - x(\sigma^2\tau)(x)x - x(\sigma^3\tau)(x)x \\
& + \sigma(x)\sigma^2(x)x + \sigma(x)\sigma^3(x)x \\
& - \sigma(x)\tau(x)x - \sigma(x)(\sigma^3\tau)(x)x \\
& + \sigma^2(x)\sigma^3(x)x - \sigma^2(x)\tau(x)x \\
& + \tau(x)x^2 + \tau(x)\sigma^2(x)x + \tau(x)\sigma^3(x)x \\
& - \tau(x)(\sigma\tau)(x)x - \tau(x)(\sigma^2\tau)(x)x - \tau(x)(\sigma^3\tau)(x)x \\
& + (\sigma^2\tau)(x)\sigma^2(x)x - (\sigma^2\tau)(x)(\sigma\tau)(x)x \\
& + (\sigma^3\tau)(x)\sigma^2(x)x + (\sigma^3\tau)(x)\sigma^3(x)x \\
& - (\sigma^3\tau)(x)(\sigma\tau)(x)x - (\sigma^3\tau)(x)(\sigma^2\tau)(x)x.
\end{aligned}
$$

(26 monomials of degree ≤ 3)

Replacing x by $x_E\sigma(x_E) + x_E\sigma(x_E)x_E - x_E^2\sigma(x_E)$ with $N_E(x_E) = 1$, then y becomes a polynomial in x_E with **666** monomials of degree ≤ 9.

The dihedral group D_8

- $G = D_8 = \langle \sigma, \tau \mid \sigma^4 = 1, \tau^2 = 1, \tau\sigma = \sigma^{-1}\tau \rangle$
- $U = \langle \sigma^2, \tau \rangle \cong \mathbf{Z}/2 \times \mathbf{Z}/2$
- $U' = \langle \sigma^2, \sigma\tau \rangle \cong \mathbf{Z}/2 \times \mathbf{Z}/2$
- $x' \in R$ with $N_{U'}(x') = (1 + \sigma\tau + \sigma^2 + \sigma^3\tau)(x') = 1$

The system (Σ) *with unknowns* $b(\sigma)$ *and* $b(\tau)$*:*

$$\left\{ \begin{array}{ccc} (1 + \sigma + \sigma^2 + \sigma^3)b(\sigma) & = & 2, \\ (1 + \tau)b(\tau) & = & 0, \\ (\sigma - 1)b(\tau) + (1 + \sigma\tau)b(\sigma) & = & 1. \end{array} \right.$$

A solution of (Σ)*:*

$b(\sigma) = \sigma(1 + \sigma\tau)(x')$ and $b(\tau) = (\tau - 1)(1 + \sigma^2)(x')$.

A nonabelian group of order 27

- $G = \langle \sigma, \tau \mid \sigma^9 = 1, \tau^3 = 1, \tau\sigma = \sigma^4\tau \rangle$
- $U = \langle \sigma^3, \tau \rangle \cong \mathbf{Z}/3 \times \mathbf{Z}/3$
- $Z = \langle \sigma^3 \rangle \cong \mathbf{Z}/3$ (center)
- $H = \langle \sigma\tau \rangle \cong \mathbf{Z}/9$ contains Z
- Let $x \in R$ such that $N_U(x) = 1$
- If $x_0 = (1 + \tau + \tau^2)(x) \in R$, then $N_Z(x_0) = 1$
- From x_0 derive $x' \in R$ such that $N_H(x') = 1$

The system (Σ) *with unknowns* $b(\sigma)$ *and* $b(\tau)$*:*

$$\begin{cases} (1 + \sigma + \sigma^2 + \cdots + \sigma^8)b(\sigma) & = & 3, \\ (1 + \tau + \tau^2)b(\tau) & = & 0, \\ (\sigma^4 - 1)b(\tau) + (1 + \sigma + \sigma^2 + \sigma^3 - \tau)b(\sigma) & = & 1. \end{cases}$$

A solution of (Σ)*:*

$$b(\sigma) = (1 + \sigma\tau + (\sigma\tau)^2)(x'),$$

$$b(\tau) = (\tau - 1)\big(\sigma^6 - \sigma(1 + \sigma + \sigma^2 + \sigma^4)\tau\big)(x').$$