Quantum principal fiber bundles and polynomial identities

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Introduction

- • We are interested in the noncommutative analogues of principal fiber bundles. For these "quantum principal fiber bundles", the structural group is a Hopf algebra

  • Our ultimate goal is to classify them; we approach this goal by constructing certain universal objects

- • *Idea:* Use an adequate theory of polynomial identities

  • This is a report on joint results with Eli Aljadeff (Technion) published in *Polynomial identities and noncommutative versal torsors*, arXiv:0708.4108, Adv. Math. 218 (2008), 1453–1495.

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Basic dictionary of noncommutative geometry

- **Replacing spaces by associative algebras**

$$\begin{array}{rcl} \text{space } X & \longleftrightarrow & \text{algebra } A(X) \ \ (= \text{ functions on } X) \\ \text{map } f : X \to Y & \longleftrightarrow & \text{algebra map } f^* : A(Y) \to A(X) \\ \text{product } f : X \times Y & \longleftrightarrow & \text{tensor product } A(X) \otimes A(Y) \\ \text{point } X = \{*\} & \longleftrightarrow & \text{ground field } A(*) = k \end{array}$$

Groups in noncommutative geometry

- **Groups**. Let G be a group and $H = A(G)$

$$\begin{aligned} \text{product} \;\; G \times G \to G \quad &\longleftrightarrow \quad \text{coproduct} \;\; \Delta : H \to H \otimes H \\ \text{unit} \;\; \{*\} \to G \quad &\longleftrightarrow \quad \text{counit} \;\; \varepsilon : H \to k \\ \text{inverse} \;\; G \to G \quad &\longleftrightarrow \quad \text{antipode} \;\; S : H \to H \end{aligned}$$

So H is a Hopf algebra

- **Group actions.** We also need the concept of an H-comodule algebra

$$\begin{aligned} \text{action} \;\; X \times G \to X \quad &\longleftrightarrow \quad \text{coaction} \;\; \delta : A \to A \otimes H \\ \text{orbit set} \;\; Y = X/G \quad &\longleftrightarrow \quad \text{coinvariants} \;\; B = \{a \in A \,|\, \delta(a) = a \otimes 1_H\} \end{aligned}$$

Comodule algebras

- Given a Hopf algebra H, an H-comodule algebra is an associative unital algebra A together with an algebra morphism

$$\delta : A \to A \otimes H$$

called the coaction and satisfying

$$(\delta \otimes \mathrm{id}_H) \circ \delta = (\mathrm{id}_H \otimes \Delta) \circ \delta \quad \text{and} \quad (\mathrm{id}_H \otimes \varepsilon) \circ \delta = \mathrm{id}_H$$

- Coinvariants:

$$A^H = \{a \in A \mid \delta(a) = a \otimes 1_H\}$$

is a subalgebra and a subcomodule of A

- **Examples of comodule algebras**. Given a group G,

(a) a G-graded algebra $A = \bigoplus_{g \in G} A_g$ is the same as an H-comodule algebra with $H = k[G]$

(b) a G-algebra, i.e., an algebra on which G acts by algebra automorphisms, is the same as an H-comodule algebra with $H = \{\text{functions on } G\}$

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POLYNOMIAL IDENTITIES
FOR COMODULE ALGEBRAS

A place where polynomial identities will live

• From now on we assume that the ground field k is infinite

• Polynomial identities are noncommutative polynomials. We take variables indexed by a basis of a Hopf algebra

Let H be a Hopf algebra and X_H a copy of H together with a linear isomorphism $x \mapsto X_x$

• Consider the tensor algebra

$$T(X_H) = \bigoplus_{r \geq 0} X_H^{\otimes r}$$

on the vector space X_H. If $\{x_i\}_{i \in I}$ is a basis of H, then

$$T(X_H) \cong k\langle\, X_{x_i} \,|\, i \in I \,\rangle$$

is the algebra of noncommutative polynomials in X_{x_i} $(i \in I)$

Identities for comodule algebras

- The algebra $T(X_H)$ is an H-comodule algebra with coaction $\delta : T(X_H) \to T(X_H) \otimes H$ given by

$$\delta(X_x) = \sum_{(x)} X_{x_{(1)}} \otimes x_{(2)} \,,$$

  where $\Delta(x) = \sum_{(x)} x_{(1)} \otimes x_{(2)}$

- Let A be an H-comodule algebra

  Definition. *An element $P \in T(X_H)$ is an H-identity of A if $\mu(P) = 0$ for all comodule algebra morphisms $\mu : T(X_H) \to A$*

  Recall: An algebra morphism $\mu : T(X_H) \to A$ is a comodule algebra morphism if it preserves the coactions, i.e.,

$$(\mu \otimes \mathrm{id}_H) \circ \delta = \delta \circ \mu$$

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The ideal of identities

• Let A be an H-comodule algebra for some Hopf algebra H and consider the vector space $I_H(A)$ of all H-identities of A

• **Proposition.** *(a) $I_H(A)$ is a two-sided ideal of $T(X_H)$ such that*

$$\delta\big(I_H(A)\big) \subset I_H(A) \otimes H$$

(b) The ideal $I_H(A)$ is graded and

$$I_H(A) \subset \bigoplus_{r \geq 2} X_H^{\otimes r}$$

The universal comodule algebra

• **Definition.** *The universal comodule algebra for identities of the H-comodule algebra A is the quotient-algebra*

$$\mathcal{U}_H(A) = T(X_H)/I_H(A)$$

• **Properties.**

(a) $\mathcal{U}_H(A)$ is a graded algebra coinciding with $T(X_H)$ in degrees 0 and 1

(b) $\mathcal{U}_H(A)$ is an H-comodule algebra

(c) All H-identities of A vanish in $\mathcal{U}_H(A)$

Questions

• Can we determine the universal comodule algebra for identities $\mathcal{U}_H(A)$, or its center $\mathcal{Z}_H(A)$, or its subalgebra of coinvariants $\mathcal{U}_H(A)^H$, or say something about their structure?

• Is $\mathcal{U}_H(A)$ free as a module over $\mathcal{U}_H(A)^H$? Do we have a $\mathcal{U}_H(A)^H$-linear comodule isomorphism

$$\mathcal{U}_H(A) \cong \mathcal{U}_H(A)^H \otimes H?$$

• We give a positive answer to these questions for a special class of comodule algebras after having performed some central localization

THE SWEEDLER ALGEBRA

An example: the Sweedler algebra

• The Sweedler algebra is the smallest noncommutative noncocommutative Hopf algebra

As an algebra,

$$H_4 = k\langle\, x, y \mid x^2 = 1\,,\;\; xy + yx = 0\,,\;\; y^2 = 0\,\rangle$$

• **Hopf algebra structure:**

Coproduct: $\Delta(x) = x \otimes x, \quad \Delta(y) = 1 \otimes y + y \otimes x$

Coünit: $\varepsilon(x) = 1, \quad \varepsilon(y) = 0$

Antipode: $S(x) = x, \quad S(y) = xy$

• The algebra H_4 is four-dimensional with basis $\{1, x, y, z\}$, where $z = xy$

A three-parameter family of comodule algebras

- Given scalars a, b, c with $a \neq 0$, consider the algebra

$$A_{a,b,c} = k\langle\, u_x, u_y \mid u_x^2 = a\,, \;\; u_x u_y + u_y u_x = b\,, \;\; u_y^2 = c\,\rangle$$

- The algebra $A_{a,b,c}$ is a comodule algebra over the Sweedler algebra H_4 with coaction $\delta : A_{a,b,c} \to A_{a,b,c} \otimes H_4$ given by

$$\delta(u_x) = u_x \otimes x \quad \text{and} \quad \delta(u_y) = 1 \otimes y + u_y \otimes x$$

- The subalgebra of coinvariants is trivial: $(A_{a,b,c})^H = k\,1$

Identities for the comodule algebras $A_{a,b,c}$

- We have $T(X_{H_4}) = k\langle E, X, Y, Z \rangle$, where

$$E = X_1\,,\quad X = X_x\,,\quad Y = X_y\,,\quad Z = X_z$$

We keep the same notation for their images in $\mathcal{U}_{a,b,c} = \mathcal{U}_{H_4}(A_{a,b,c})$

- **Theorem**.

1. *The center $\mathcal{Z}_{a,b,c}$ of $\mathcal{U}_{a,b,c}$ coincides with the subalgebra of coinvariants:*

$$\mathcal{Z}_{a,b,c} = (\mathcal{U}_{a,b,c})^H$$

2. *The following polynomials are central elements of $\mathcal{U}_{a,b,c}$:*

$$E\,,\quad R = X^2\,,\quad S = Y^2\,,\quad T = XY + YX\,,\quad U = X(XZ + ZX) \in \mathcal{Z}_{a,b,c}$$

3. *After inverting E and R, there is an algebra isomorphism*

$$\mathcal{Z}_{a,b,c}[E^{-1}, R^{-1}] \cong k[E^{\pm 1}, R^{\pm 1}, S, U][T]/(P_{a,b,c})\,,$$

where

$$P_{a,b,c} = T^2 - 4RS - \frac{b^2 - 4ac}{a}\,E^2 R$$

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The universal comodule algebra of $A_{a,b,c}$

- Recall $R = X^2,\ S = Y^2,\ T = XY + YX,\ U = X(XZ + ZX)$

Theorem. *After inverting E and R,*

1. *there is a $\mathcal{Z}_{a,b,c}[E^{-1}, R^{-1}]$-linear comodule isomorphism*

$$\mathcal{U}_{a,b,c}[E^{-1}, R^{-1}] \cong \mathcal{Z}_{a,b,c}[E^{-1}, R^{-1}] \otimes H_4$$

2. *and $\mathcal{U}_{a,b,c}[E^{-1}, R^{-1}]$ is isomorphic as an algebra to*

$$\mathcal{Z}_{a,b,c}[E^{-1}, R^{-1}]\langle X, Y \mid X^2 = R,\ XY + YX = T,\ Y^2 = S \rangle$$

- In $\mathcal{U}_{a,b,c}$ we have

$$ERZ = RXY + \frac{EU - RT}{2}$$

(The elements in red are central)

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How did we determine the identities in the Sweedler case?

• The comodule algebras $A_{a,b,c}$ belong to a special class of comodule algebras, called twisted algebras

• In the case of a twisted algebra A we can detect the H-identities of A and embed $\mathcal{U}_H(A)$ into an easily controllable algebra

• **Plan of the sequel**

(a) Define the twisted algebras

(b) Define the controllable algebra and embed $\mathcal{U}_H(A)$ into it

(c) What localization is needed?

(d) Results on $\mathcal{U}_H(A)$ after localization

TWISTED ALGEBRAS

Twisting the product with a cocycle

• Let H be a Hopf algebra and $\alpha : H \times H \to k$ be a two-cocycle, i.e., a bilinear form such that for all $x, y \in H$,

$$\sum_{(x),(y)} \alpha(x_{(1)}, y_{(1)})\, \alpha(x_{(2)}y_{(2)}, z) = \sum_{(y),(z)} \alpha(y_{(1)}, z_{(1)})\, \alpha(x, y_{(2)}z_{(2)})$$

where $\Delta(x) = \sum_{(x)} x_{(1)} \otimes x_{(2)}, \ldots$

Also assume α normalized: $\alpha(1, x) = \alpha(x, 1) = \varepsilon(x)$ for all $x \in H$

• Let ${}^{\alpha}H$ be a vector space isomorphic to H via a linear isomorphism $x \in H \mapsto u_x \in {}^{\alpha}H$. Equip ${}^{\alpha}H$ with the product

$$u_x\, u_y = \sum_{(x),(y)} \alpha(x_{(1)}, y_{(1)})\, u_{x_{(2)}y_{(2)}}$$

This product is associative with u_1 as unit

• The algebra ${}^{\alpha}H$ is called a twisted algebra

Twisted algebras are comodule algebras

- ${}^{\alpha}H$ is an H-comodule algebra with coaction $\delta : {}^{\alpha}H \to {}^{\alpha}H \otimes H$ given by

$$\delta(u_x) = \sum_{(x)} u_{x_{(1)}} \otimes x_{(2)}$$

where $\Delta(x) = \sum_{(x)} x_{(1)} \otimes x_{(2)}$

- The subalgebra of coinvariants of ${}^{\alpha}H$ is trivial: $({}^{\alpha}H)^H = k\,1$

The symbols t_x

- Let t_H be a copy of H with linear isomorphism $x \in H \mapsto t_x \in t_H$
- Consider the symmetric algebra

$$\mathrm{Sym}(t_H) = \bigoplus_{r \geq 0} \mathrm{Sym}^r(t_H)$$

on the vector space t_H. If $\{x_i\}_{i \in I}$ is a basis of H, then

$$\mathrm{Sym}(t_H) \cong k[t_{x_i} \mid i \in I]$$

is the polynomial algebra in the (commuting) variables t_{x_i}

The symbols t_x^{-1}

- We also need "inverse variables" t_x^{-1}. Let $\operatorname{Frac}\operatorname{Sym}(t_H)$ be the field of fractions of $\operatorname{Sym}(t_H)$

Lemma. *There is a unique linear map* $x \mapsto t_x^{-1}$ *from* H *to* $\operatorname{Frac}\operatorname{Sym}(t_H)$ *such that for all* $x \in H$,

$$\sum_{(x)} t_{x_{(1)}}\, t_{x_{(2)}}^{-1} = \sum_{(x)} t_{x_{(1)}}^{-1}\, t_{x_{(2)}} = \varepsilon(x)\, 1$$

- If x is grouplike, i.e., $\Delta(x) = x \otimes x$, then $\varepsilon(x) = 1$ and

$$t_x t_x^{-1} = 1\,, \quad \text{hence} \quad t_x^{-1} = \frac{1}{t_x}$$

- If x is skew-primitive, i.e., $\Delta(x) = g \otimes x + x \otimes h$ for some grouplike elements g, h, then $\varepsilon(x) = 0$ and

$$t_g t_x^{-1} + t_x t_h^{-1} = 0\,, \quad \text{hence} \quad t_x^{-1} = -\frac{t_x}{t_g t_h}$$

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The universal evaluation map

- Let ${}^{\alpha}H$ be a twisted algebra and consider the algebra morphism

$$\begin{aligned} \mu_\alpha : T(X_H) &\longrightarrow \mathrm{Sym}(t_H) \otimes {}^{\alpha}H \\ X_x &\longmapsto \sum_{(x)} t_{x_{(1)}} \otimes u_{x_{(2)}} \end{aligned}$$

- **Lemma.**
  1. *The map μ_α is an H-comodule algebra morphism*
  2. *For every H-comodule algebra morphism $\mu : T(X_H) \to {}^{\alpha}H$, there is a unique algebra morphism $\chi : \mathrm{Sym}(t_H) \to k$ such that*

  $$\mu = (\chi \otimes \mathrm{id}) \circ \mu_\alpha$$

  In other words, any comodule algebra morphism $T(X_H) \to {}^{\alpha}H$ is a specialization of μ_α

We call μ_α the universal evaluation map for ${}^{\alpha}H$

The universal evaluation map

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Detecting the identities for twisted algebras

- **Theorem.** *We have* $I_H({}^\alpha H) = \text{Ker } \mu_\alpha$

  In other words, the map μ_α detects the H-identities of ${}^\alpha H$

- Set $\mathcal{U}_H^\alpha = T(X_H)/I_H({}^\alpha H)$. Then the map μ_α induces an embedding

  $$\mathcal{U}_H^\alpha \hookrightarrow \text{Sym}(t_H) \otimes {}^\alpha H$$

  of the universal comodule algebra into a twisted algebra with extended scalars

- $u \in \mathcal{U}_H^\alpha$ is coinvariant if and only if $\mu_\alpha(u)$ belongs to $\text{Sym}(t_H) \otimes 1$

- $u \in \mathcal{U}_H^\alpha$ is central if and only if $\mu_\alpha(u)$ belongs to $\text{Sym}(t_H) \otimes Z({}^\alpha H)$, where $Z({}^\alpha H)$ is the center of ${}^\alpha H$

- The center $\mathcal{Z}_H^\alpha$ of $\mathcal{U}_H^\alpha$ is a domain if $Z({}^\alpha H)$ is a domain

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LOCALIZATION

The generic base algebra

- Let ${}^\alpha H$ be a twisted algebra. The bilinear map $\sigma : H \times H \to \operatorname{Frac}\operatorname{Sym}(t_H)$

$$\sigma(x,y) = \sum_{(x),(y)} t_{x_{(1)}}\, t_{y_{(1)}}\, \alpha(x_{(2)}, y_{(2)})\, t^{-1}_{x_{(3)}y_{(3)}}$$

  is a two-cocycle

- **Definition.** *The generic base algebra is the subalgebra $\mathcal{B}_H^\alpha$ of* $\operatorname{Frac}\operatorname{Sym}(t_H)$ *generated by the values of the generic two-cocycle σ and of its convolution inverse σ^{-1}*

- **Immediate properties**:

  (a) $\mathcal{B}_H^\alpha$ is a domain

  (b) Transcendence degree of $\operatorname{Frac}\mathcal{B}_H^\alpha \leq \dim H$

  (c) $\mathcal{B}_H^\alpha$ is finitely generated if $\dim H < \infty$

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The generic base algebra in the Sweedler case

- Each comodule algebra $A_{a,b,c}$ is a twisted algebra for some two-cocycle α
- Recall $R = X^2,\ S = Y^2,\ T = XY + YX,\ U = X(XZ + ZX) \in \mathcal{Z}_{H_4}^{\alpha}$
- **Presentation of $\mathcal{B}_{H_4}^{\alpha}$ by generators and relations**:

$$\mathcal{B}_{H_4}^{\alpha} \cong k[\widehat{E}^{\pm 1}, \widehat{R}^{\pm 1}, \widehat{S}, \widehat{T}, \widehat{U}]/(P_{a,b,c})$$

with

$$P_{a,b,c} = \widehat{T}^2 - 4\widehat{R}\widehat{S} - \frac{b^2 - 4ac}{a}\,\widehat{E}^2\widehat{R}$$

Here

$$\begin{aligned}
\widehat{E} &= \mu_\alpha(E) = t_1 \otimes 1 \\
\widehat{R} &= \mu_\alpha(R) = a\,t_x^2 \otimes 1 \\
\widehat{S} &= \mu_\alpha(S) = a\,t_y^2 + b\,t_1 t_y + c\,t_1^2 \otimes 1 \\
\widehat{T} &= \mu_\alpha(T) = t_x\,(2a\,t_y + b\,t_1) \otimes 1 \\
\widehat{U} &= \mu_\alpha(U) = a\,t_x^2\,(2\,t_z + b\,t_x) \otimes 1
\end{aligned}$$

Relating the center of the universal comodule algebra and the generic base algebra

• Let ${}^\alpha H$ be a twisted algebra and $\mathcal{Z}_H^\alpha$ be the center of the universal comodule algebra $\mathcal{U}_H^\alpha$

Theorem. *If* $Z({}^\alpha H) = k$, *then* $\mathcal{Z}_H^\alpha \hookrightarrow \mathcal{B}_H^\alpha$

• In the sequel we assume that $Z({}^\alpha H) = k$ and that $\mathcal{B}_H^\alpha$ is a localization of $\mathcal{Z}_H^\alpha$

The universal comodule algebra after localization

- **Definition.** *The generic twisted algebra is the comodule algebra*

$$\mathcal{A}_H^\alpha = \mathcal{B}_H^\alpha \otimes_{\mathcal{Z}_H^\alpha} \mathcal{U}_H^\alpha$$

with center $\mathcal{B}_H^\alpha = (\mathcal{A}_H^\alpha)^H$

- **Theorem.** (a) *There is an H-comodule algebra isomorphism*

$$\mathcal{A}_H^\alpha \cong \mathcal{B}_H^\alpha \otimes {}^\sigma H$$

(b) There is a maximal ideal $\mathfrak{m}_0$ of $\mathcal{B}_H^\alpha$ such that

$$\mathcal{A}_H^\alpha / \mathfrak{m}_0 \mathcal{A}_H^\alpha \cong {}^\alpha H$$

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Forms

- A twisted algebra ${}^{\beta}H$ is a form of ${}^{\alpha}H$ if there is a field extension $K \supset k$ and a K-linear isomorphism of H-comodule algebras

$$K \otimes {}^{\beta}H \cong K \otimes {}^{\alpha}H \,.$$

- **Theorem.** *If ${}^{\beta}H$ is a form of ${}^{\alpha}H$, then there is an algebra morphism $\chi : \mathcal{B}_H^{\alpha} \to k$ such that*

$$k_{\chi} \otimes_{\mathcal{B}_H^{\alpha}} \mathcal{A}_H^{\alpha} \cong {}^{\beta}H$$

  In other words, any form of ${}^{\alpha}H$ is obtained from the generic Galois extension $\mathcal{A}_H^{\alpha}$ by a central specialization.

- There is a converse to the previous theorem; it requires an additional condition

  Theorem *If the algebra* $\operatorname{Frac}\operatorname{Sym}(t_H)$ *is integral over the subalgebra $\mathcal{B}_H^{\alpha}$, then for any algebra morphism $\chi : \mathcal{B}_H^{\alpha} \to k$, the algebra $k_{\chi} \otimes_{\mathcal{B}_H^{\alpha}} \mathcal{A}_H^{\alpha}$ is a form of ${}^{\alpha}H$*

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Versal deformation space

- If $\operatorname{Frac}\operatorname{Sym}(t_H)$ is integral over $\mathcal{B}_H^\alpha$, then the map

$$\begin{array}{ccc} \operatorname{Alg}(\mathcal{B}_H^\alpha, k) & \longrightarrow & \operatorname{Forms}({}^\alpha H) \\ \chi & \longmapsto & k_\chi \otimes_{\mathcal{B}_H^\alpha} \mathcal{A}_H^\alpha \end{array}$$

is a surjection from the set of algebra morphisms $\mathcal{B}_H^\alpha \to k$ to the set of isomorphism classes of forms of ${}^\alpha H$

Thus the set $\operatorname{Alg}(\mathcal{B}_H^\alpha, k)$ parametrizes the forms of ${}^\alpha H$.

The extension $\mathcal{B}_H^\alpha \subset \mathcal{A}_H^\alpha$ is a versal deformation space for the forms of ${}^\alpha H$

- *Remark.* To determine the set $\operatorname{Alg}(\mathcal{B}_H^\alpha, k)$, it is important to find a presentation by generators and relations of $\mathcal{B}_H^\alpha$

The integrality condition

- **Question.** *Under which condition on* (H, α) *is the algebra* $\operatorname{Frac}\operatorname{Sym}(t_H)$ *integral over the subalgebra* $\mathcal{B}_H^\alpha$*?*
- **Proposition.** *If H is a finite-dimensional Hopf algebra generated by grouplike and skew-primitive elements, and α is any two-cocycle, then* $\operatorname{Frac}\operatorname{Sym}(t_H)$ *is integral over the subalgebra* $\mathcal{B}_H^\alpha$
- **Negative answer.** For $H = k[\mathbb{Z}]$ and α trivial, $\operatorname{Frac}\operatorname{Sym}(t_H)$ is transcendental (of degree 1) over $\operatorname{Frac}\mathcal{B}_H^\alpha$

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Rigidity properties of $\mathcal{A}_H^\alpha$

• **Theorem.** *Assume that* $\mathrm{char}(k) = 0$ *and* $\dim(H) < \infty$.
If ${}^\alpha H$ *is simple, then so is*

$$\mathrm{Frac}\,\mathcal{B}_H^\alpha \otimes_{\mathcal{B}_H^\alpha} \mathcal{A}_H^\alpha$$

• **Theorem.** *Under the previous integrality condition, if* ${}^\alpha H$ *is simple, then* $\mathcal{A}_H^\alpha$ *is an Azumaya algebra*

An algebra A is Azumaya if $A/\mathfrak{m}$ is simple for any maximal ideal $\mathfrak{m}$ of its center. E.g. $A = M_n(R)$, where R is a commutative ring

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THANK YOU FOR YOUR ATTENTION