Some universal constructions on Hopf algebras

Christian Kassel

Institut de Recherche Mathématique Avancée
CNRS - Université Louis Pasteur
Strasbourg, France

Conference “Shuffles, descents and representations”
Nice - 12 September 2007

Introductory quotation

F. Patras, C. Reutenauer, *On descent algebras and twisted algebras*, Moscow Math. J. 4 (2004), 199–216:

> *Many phenomena in algebraic combinatorics can be explained, and sometimes are discovered, using a suitable bialgebra structure.*

Work on graded (co)commutative bialgebras

• *Some references:*

- A. M. Garsia, C. Reutenauer, *A decomposition of Solomon's descent algebra*, Adv. Math. 77 (1989), 189–262.
- F. Patras, *L'algèbre des descentes d'une bigèbre graduée*, J. Algebra 170 (1994), 547–566.
- C. Reutenauer, *Free Lie algebras*, Clarendon, Oxford, 1993.

• Identities for graded (co)commutative bialgebras

• *Models:* Tensor bialgebras or their duals, "combinatorial Hopf algebras"

• Formulas often live in the algebra of the symmetric groups, related to descents,...

By contrast, my joint work with E. Aljadeff

- I now report on joint work with Eli Aljadeff (Technion, Haifa)
- We work with arbitrary Hopf algebras, e.g., finite dimensional Hopf algebras, *not combinatorial Hopf algebras*
- *Motivation:* From Noncommutative geometry and quantum groups
- *Models:* The Hopf algebra of a (finite) group or its dual
- We construct universal algebras associated to each Hopf algebra
- We get
  - identities
  - deformations

  of the given Hopf algebra
- So far no connection with symmetric groups

Part One

COMBINATORICS FROM A COALGEBRA

The free commutative Hopf algebra on a coalgebra

- **Recall work by Takeuchi:**
M. Takeuchi, *Free Hopf algebras generated by coalgebras*,
J. Math. Soc. Japan 23 (1971), 561–582.

- Given a coalgebra C, the free commutative Hopf algebra on C is a commutative Hopf algebra $H(C)$ together with a coalgebra morphism $i : C \to H(C)$ such that, for any commutative Hopf algebra H and any coalgebra morphism $f : C \to H$, there is a unique Hopf algebra morphism $\bar{f} : H(C) \to H$ with $f = \bar{f} \circ i$:

$$\begin{array}{ccc} C & \xrightarrow{\ i\ } & H(C) \\ & {\scriptstyle f}\searrow & \downarrow{\scriptstyle \bar{f}} \\ & & H \end{array}$$

- Construction of $H(C)$ comes next

The bialgebra $S(t_C)$

- Let $t : C \to t_C$ be a linear isomorphism
- Consider the symmetric algebra $S(t_C) = \bigoplus_{r \geq 0} S^r(t_c)$ on the vector space $t_C \cong C$
- The algebra $S(t_C)$ is a commutative bialgebra with coproduct Δ and coünit ε extended from C:

$$\text{if} \quad \Delta(x) = \sum_{(x)} x_{(1)} \otimes x_{(2)}\,, \qquad \text{then} \quad \Delta(t_x) = \sum_{(x)} t_{x_{(1)}} \otimes t_{x_{(2)}}$$

and $\varepsilon(t_x) = \varepsilon(x)$ for all $x \in C$.

- Any coalgebra morphism $f : C \to B$ into a commutative bialgebra B factors through $t : C \to S(t_C)$:

$$\begin{array}{ccc} C & \xrightarrow{\ t\ } & S(t_C) \\ & \searrow^{f} & \downarrow \bar{f} \\ & & B \end{array}$$

The Hopf algebra $H(C)$

- Let $\mathrm{Frac}\, S(t_C)$ be the field of fractions of $S(t_C)$

- **Lemma.** *The map* $t \in \mathrm{Hom}(C, S(t_C))$ *is convolution invertible in* $\mathrm{Hom}(C, \mathrm{Frac}\, S(t_C))$, *i.e., there is a map* $t^{-1} : C \to \mathrm{Frac}\, S(t_C)$ *such that*

$$\sum_{(x)} t_{x_{(1)}} t^{-1}_{x_{(2)}} = \sum_{(x)} t^{-1}_{x_{(1)}} t_{x_{(2)}} = \varepsilon(x)\, 1$$

- **Definition.** $H(C)$ is the subalgebra of $\mathrm{Frac}\, S(t_C)$ generated by $t(C)$ and $t^{-1}(C)$

- $H(C)$ is a commutative Hopf algebra with antipode S given by

$$S(t_x) = t_x^{-1} \quad \text{and} \quad S(t_x^{-1}) = t_x$$

- If $x \in C$ is grouplike, then $t_x^{-1} = 1/t_x$

- If $x \in C$ is primitive, then $t_x^{-1} = -t_x$

$H(C)$ as a localization of $S(t_C)$

- **Theorem (Takeuchi).** *(a) If* $\dim C < \infty$*, then there is* $\Theta_C \in S(t_C)$ *such that*

$$H(C) = S(t_C)\left[\frac{1}{\Theta_C}\right].$$

- *(b) If* $\dim C = \infty$*, in which case* C *is the union of a family* $(C_\kappa)_\kappa$ *of finite-dimensional subcoalgebras,*

$$H(C) = S(t_C)\left[\left(\frac{1}{\Theta_{C_\kappa}}\right)_\kappa\right].$$

- From now on, we write $S(t_C)_\Theta$ instead of $H(C)$
- Construction of Θ_C comes next

The polynomial Θ_C

• Let C be a coalgebra. Then $C^* = \mathrm{Hom}(C, k)$ is an algebra over the ground field k. Assume that C is a finite-dimensional.

• Definition of Θ_C: image of t under the maps

$$t \in \mathrm{Hom}(C, S(t_C)) \xleftarrow{\cong} C^* \otimes S(t_C) \xrightarrow{N} S(t_C) \ni \Theta_C\,,$$

where N is the norm map: $N(\omega) =$ determinant of the left multiplication by ω

• *Properties:* (a) If C is a simple coalgebra, then the polynomial Θ_C is irreducible

(b) If $C = C_1 \times C_2$, then

$$\Theta_C = \Theta_{C_1} \otimes \Theta_{C_2} \;\text{ in }\; S(t_{C_1}) \otimes S(t_{C_2}) = S(t_C)$$

(c) Θ_C is grouplike in the bialgebra $S(t_C)$

Computation of Θ_C: two basic examples

- If $C = M_n(k)^*$ (simple coalgebra), then

$$\Theta_C \in S(t_C) = \det\big(t_{i,j}\big)_{1\leq i,j\leq n}$$

[determinant of the “generic” matrix]

- Let $C = k[G]$ with a basis G of grouplike elements. Then

$$S(t_C)_\Theta = k[\,g, g^{-1}\,|\,g \in G\,]$$

is an algebra of Laurent polynomials. If G is finite, then

$$\Theta_C = \prod_{g\in G} t_g$$

Dedekind's "Gruppen-Determinanten"

- For a finite group G, let $C = k^G = k[G]^*$ be the coalgebra of functions on G with coproduct $\Delta(f)(g,h) = f(gh)$. Then Θ_C is equal to Dedekind's group determinant Θ_G defined by

$$\Theta_G = \det\left(t_{gh^{-1}}\right)_{g,h\in G}$$

- **Frobenius** (1896) factorized Θ_G into irreducible polynomials:

$$\Theta_G = \prod_{\rho\in \mathrm{Irrep}\, G} (\Theta_\rho)^{d_\rho},$$

where for each representation $\rho : G \to GL_{d_\rho}(\mathbb{C})$,

$$\Theta_\rho = \det\left(\sum_{g\in G} t_g\, \rho(g)\right)$$

Θ_G for two non-abelian groups

- If $G = \langle g, x \mid g^2 = x^3 = e,\ gxg = x^{-1} \rangle$ is the dihedral group (or the symmetric group) of order 6, then

$$\begin{aligned}\Theta_G &= -(a+b+c+A+B+C)\\ &\quad\times(a+b+c-A-B-C)\\ &\quad\times\big(a^2+b^2+c^2+ab+bc+ac\\ &\qquad\quad -A^2-B^2-C^2-AB-BC-AC\big)^2,\end{aligned}$$

where $a = t_1$, $b = t_x$, $c = t_{x^2}$, $A = t_g$, $B = t_{gx}$, $C = t_{gx^2}$.

- If $G = \{\pm 1, \pm i, \pm j, \pm k\}$ is the quaternionic group, then

$$\begin{aligned}\Theta_G &= -(t_1+t_i+t_j+t_k+t_{-1}+t_{-i}+t_{-j}+t_{-k})\\ &\quad\times(t_1+t_i-t_j-t_k+t_{-1}+t_{-i}-t_{-j}-t_{-k})\\ &\quad\times(t_1-t_i+t_j-t_k+t_{-1}-t_{-i}+t_{-j}-t_{-k}\\ &\quad\times(t_1-t_i-t_j+t_k+t_{-1}-t_{-i}-t_{-j}+t_{-k})\\ &\quad\times\big((t_1-t_{-1})^2+(t_i-t_{-i})^2+(t_j-t_{-j})^2+(t_k-t_{-k})^2\big)^2\end{aligned}$$

Part Two

HOPF ALGEBRAS AND COCYCLES

(All results joint with Eli Aljadeff)

Group-graded algebras

- Given a group G, consider strongly G-graded algebras:

$$A = \bigoplus_{g \in G} A_g$$

with $A_g A_h = A_{gh}$ and $\dim A_g = 1$

- Suppose u_g spans A_g. Then

$$u_g \, u_h = \alpha(g,h) \, u_{gh}$$

for some $\alpha(g,h) \in k^\times = k - \{0\}$

- The map $\alpha : G \times G \to k^\times$ is a two-cocycle
- $\{$strongly graded algebras$\}/($isomorphism$) \cong H^2(G, k^\times)$

Twisted algebras

- Let H be a Hopf algebra and $\alpha : H \times H \to k$ a bilinear map
- Let ${}^{\alpha}H$ be a vector space isomorphic to H. We fix a linear isomorphism $x \in H \mapsto u_x \in {}^{\alpha}H$. Equip ${}^{\alpha}H$ with product

$$u_x \cdot_\alpha u_y = \sum_{(x),(y)} \alpha(x_{(1)}, y_{(1)})\, u_{x_{(2)}y_{(2)}}$$

This product is associative iff α is a two-cocycle, i.e., satisfies

$$\sum_{(x),(y)} \alpha(x_{(1)}, y_{(1)})\, \alpha(x_{(2)}y_{(2)}, z) = \sum_{(y),(z)} \alpha(y_{(1)}, z_{(1)})\, \alpha(x, y_{(2)}z_{(2)})$$

It has u_1 as unit if α is normalized, i.e., $\alpha(1, x) = \alpha(x, 1) = \varepsilon(x)$

- If G is a group, $H = k[G]$, and α is an invertible two-cocycle, then ${}^{\alpha}H$ is a strongly G-graded algebra (as in previous slide)

The H-comodule algebra $\mathcal{U}_H^\alpha$

• Let H be a Hopf algebra and α a normalized two-cocycle. Consider the extended twisted algebra $S(t_H) \otimes {}^\alpha H$
(Now $S(t_H)$ plays the rôle of scalars)

• For $x \in H$, define (dropping tensor product signs)

$$X_x = \sum_{(x)} t_{x_{(1)}} u_{x_{(2)}} \in S(t_H) \otimes {}^\alpha H$$

In case G is a group and $H = k[G]$, then $X_g = t_g u_g$ $(g \in G)$

• Our first "universal algebra" attached to (H, α):

Definition. Let $\mathcal{U}_H^\alpha$ be the subalgebra of $S(t_H) \otimes {}^\alpha H$ generated by all elements X_x, where $x \in H$

Properties of $\mathcal{U}_H^\alpha$

- $\mathcal{U}_H^\alpha$ is a positively graded algebra with $\deg X_x = 1$
- $\mathcal{U}_H^\alpha$ is an H-comodule algebra with coaction $\delta : \mathcal{U}_H^\alpha \to \mathcal{U}_H^\alpha \otimes H$:

$$\delta(X_x) = \sum_{(x)} X_{x_{(1)}} \otimes x_{(2)}$$

- *(An important property not to be developed here)*

$\mathcal{U}_H^\alpha$ is the universal quotient of $T(H)$ in which all identities satisfied by ${}^\alpha H$ as an H-comodule algebra vanish

- Center of $\mathcal{U}_H^\alpha$: if $Z({}^\alpha H)$ is the center of ${}^\alpha H$, then

$$\mathcal{Z}_H^\alpha = \mathcal{U}_H^\alpha \bigcap \left(S(t_H) \otimes Z({}^\alpha H)\right)$$

In particular, if $\omega \in \mathcal{U}_H^\alpha \bigcap (S(t_H)\, u_1)$, then $\omega \in \mathcal{Z}_H^\alpha$

The Sweedler algebra

- The smallest non-commutative non-cocommutative Hopf algebra

$$H = k\langle x, y \mid x^2 = 1\,, \;\; xy + yx = 0\,, \;\; y^2 = 0\rangle$$

Comultiplication: $\Delta(x) = x \otimes x, \quad \Delta(y) = 1 \otimes y + y \otimes x$

Coünit: $\varepsilon(x) = 1, \quad \varepsilon(y) = 0$

It is four-dimensional with basis $\{1, x, y, z\}$, where $z = xy$

- We have $S(t_H) = k[t_1, t_x, t_y, t_z]$ and

$$t_1^{-1} = \frac{1}{t_1}\,, \quad t_x^{-1} = \frac{1}{t_x}\,, \quad t_y^{-1} = -\frac{t_y}{t_1 t_x}\,, \quad t_z^{-1} = -\frac{t_z}{t_1 t_x}\,.$$

Thus, $S(t_H)_\Theta = k[t_1^{\pm 1}, t_x^{\pm 1}, t_y, t_z]$. Note that $\Theta_H = (t_1 t_x)^2$.

Twisted Sweedler algebras

- For any normalized invertible two-cocycle α, the twisted algebra ${}^{\alpha}H$ is of the form

$$ {}^{\alpha}H = k\langle u_x, u_y \mid u_x^2 = a\,,\;\; u_x u_y + u_y u_x = b\,,\;\; u_y^2 = c\rangle $$

for some scalars a, b, c with $a \neq 0$ $\quad$ (u_1 is the unit of ${}^{\alpha}H$)

- Generators of $\mathcal{U}_H^{\alpha}$:

$$ E = X_1\,,\;\; X = X_x\,,\;\; Y = X_y\,,\;\; Z = X_z \in S(t_H) \otimes {}^{\alpha}H $$

defined by

$$ E = t_1\, u_1\,, \qquad X = t_x\, u_x\,, $$

$$ Y = t_1\, u_y + t_y\, u_x\,, \quad Z = t_x\, u_z + t_z\, u_1 $$

Thus, $E \in \mathcal{Z}_H^{\alpha}$ (the center of $\mathcal{U}_H^{\alpha}$)

Computations in $\mathcal{U}_H^\alpha$

- Degree two central elements of $\mathcal{U}_H^\alpha$:

$$X^2 = (t_x u_x)^2 = t_x^2 u_x^2 = a t_x^2 \in \mathcal{Z}_H^\alpha$$

$$Y^2 = (t_1 u_y + t_y u_x)^2 = a t_y^2 + b t_1 t_y + c t_1^2 \in \mathcal{Z}_H^\alpha$$

$$T = XY + YX = t_x(2a t_y + b t_1) \in \mathcal{Z}_H^\alpha$$

- Degree three central element:

$$U = X(XZ + ZX) = a t_x^2 (2 t_z + b t_x) \in \mathcal{Z}_H^\alpha$$

- Degree four relation in $\mathcal{Z}_H^\alpha$:

$$T^2 - 4X^2Y^2 = \frac{b^2 - 4ac}{a} E^2 X^2$$

- Degree four relation in $\mathcal{U}_H^\alpha$: [elements in red belong to $\mathcal{Z}_H^\alpha$]

$$2X^2(EZ - XY) = EU - X^2 T$$

The generic two-cocycle cohomologous to α

- Let H be a Hopf algebra H and $\alpha : H \times H \to k$ an invertible two-cocycle. Define new invertible cocycle $\sigma : H \times H \to S(t_H)_\Theta$ by

$$\sigma(x, y) = \sum_{(x),(y)} t_{x_{(1)}} t_{y_{(1)}} \, \alpha(x_{(2)}, y_{(2)}) \, t^{-1}_{x_{(3)} y_{(3)}}$$

The cocycles σ and α are cohomologous over $S(t_H)_\Theta$

- In case $H = k[G]$ is a group algebra and $g, h \in G$,

$$\sigma(g, h) = \alpha(g, h) \, \frac{t_g t_h}{t_{gh}}$$

- Our second "universal algebra" attached to (H, α):

Definition. Let $\mathcal{B}_H^\alpha$ be the subalgebra of $S(t_H)_\Theta$ generated by all $\sigma(x, y)$ and $\sigma^{-1}(x, y)$, where $x, y \in H$

The algebra $\mathcal{B}_H^\alpha$ is a domain

The algebra $\mathcal{B}_H^\alpha$ in the Sweedler case - Generators

- Generators of $\mathcal{B}_H^\alpha$:

$$
\begin{aligned}
\sigma(1,1) &= t_1 = E\,, \\
\sigma(x,x) &= \frac{at_x^2}{t_1} = \frac{X^2}{E}\,, \\
\sigma(y,y) &= \frac{at_y^2 + bt_1t_y + ct_1^2}{t_1} = \frac{Y^2}{E}\,, \\
\sigma(x,y) &= \frac{at_xt_y - t_1t_z}{t_1} = \frac{X^2T - EU}{2EX^2}\,, \\
\sigma(y,x) &= \frac{bt_1t_x + at_xt_y + t_1t_z}{t_1} = \frac{X^2T + EU}{2EX^2}\,, \\
\sigma(z,z) &= -\frac{t_z^2 + bt_xt_z + act_x^2}{t_1} = \frac{U^2}{4EX^4} - \frac{b^2 - 4ac}{4a}\frac{X^2}{E}
\end{aligned}
$$

The algebra $\mathcal{B}_H^\alpha$ in the Sweedler case - Presentation

Theorem. *(a) $\mathcal{B}_H^\alpha$ is a localization of $\mathcal{Z}_H^\alpha$:*

$$\mathcal{B}_H^\alpha = \mathcal{Z}_H^\alpha\,[E^{-1}, (X^2)^{-1}]\,.$$

(b) Presentation of $\mathcal{B}_H^\alpha$ by generators and relations:

$$\mathcal{B}_H^\alpha \cong k[E^{\pm 1}, (X^2)^{\pm 1}, Y^2, T, U]/(P_{a,b,c})\,,$$

where

$$P_{a,b,c} = T^2 - 4X^2Y^2 - \frac{b^2 - 4ac}{a}\,E^2X^2\,.$$

Relationship between $\mathcal{Z}_H^\alpha$ and $\mathcal{B}_H^\alpha$ in general case

- **Theorem.** *If $Z({}^\alpha H) = k$, then $\mathcal{Z}_H^\alpha \subset \mathcal{B}_H^\alpha$ ($\subset S(t_H)_\Theta$)*

- **Question.** Under which condition is $\mathcal{B}_H^\alpha$ a localization of $\mathcal{Z}_H^\alpha$?

If H is the Sweedler algebra or if $H = k[G]$, then $\mathcal{B}_H^\alpha$ is a localization of $\mathcal{Z}_H^\alpha$

- **Proof for $H = k[G]$:** Consider the following elements of $\mathcal{Z}_H^\alpha$:

$$Z_g = X_g X_{g^{-1}} = (t_g u_g)(t_{g^{-1}} u_{g^{-1}}) = \alpha(g, g^{-1}) t_g t_{g^{-1}}$$

$$Z_{g,h} = X_g X_h X_{(gh)^{-1}} = \alpha(g, g^{-1}) \alpha(h, (gh)^{-1}) t_g t_h t_{(gh)^{-1}}$$

Now, for the generators $\sigma(g, h)$ of $\mathcal{B}_H^\alpha$, we have

$$\sigma(g, h) = \frac{\alpha(g, h) \alpha(gh, (gh)^{-1})}{\alpha(g, g^{-1}) \alpha(h, (gh)^{-1})} \cdot \frac{Z_{g,h}}{Z_{gh}}$$

The universal algebra $\mathcal{A}_H^\alpha$

Assume that $\mathcal{Z}_H^\alpha \subset \mathcal{B}_H^\alpha$ and that $\mathcal{B}_H^\alpha$ is a localization of $\mathcal{Z}_H^\alpha$

- Our third "universal algebra" attached to (H, α):

Definition. Let $\mathcal{A}_H^\alpha = \mathcal{B}_H^\alpha \otimes_{\mathcal{Z}_H^\alpha} \mathcal{U}_H^\alpha$ (central localization of $\mathcal{U}_H^\alpha$)

- **Theorem.** *(a) The center of $\mathcal{A}_H^\alpha$ is $\mathcal{B}_H^\alpha$*

(b) If the algebra ${}^\alpha H$ is simple, then so is $\mathrm{Frac}\,\mathcal{B}_H^\alpha \otimes_{\mathcal{B}_H^\alpha} \mathcal{A}_H^\alpha$

(c) $\mathcal{A}_H^\alpha$ is a twisted algebra: $\mathcal{A}_H^\alpha \cong \mathcal{B}_H^\alpha \otimes {}^\sigma H$ (cleft H-Galois extension)

(d) If ${}^\beta H$ is a form of ${}^\alpha H$ (they are isomorphic as comodule algebras after extension of scalars) for some two-cocycle $\beta : H \times H \to K \supset k$, then there is an algebra map $\lambda : \mathcal{B}_H^\alpha \to K$ such that $K \otimes_{\mathcal{B}_H^\alpha} \mathcal{A}_H^\alpha \cong {}^\beta H$

Computation of $\mathcal{A}_H^\alpha$ for the Sweedler algebra

If H is the Sweedler algebra, then

$$\mathcal{A}_H^\alpha \cong \mathcal{B}_H^\alpha \langle \xi, \eta \rangle / (\xi^2 - X^2,\ \eta^2 - Y^2,\ \xi\eta + \eta\xi - T)\,.$$

Recall

$$\mathcal{B}_H^\alpha \cong k[E^{\pm 1}, (X^2)^{\pm 1}, Y^2, T, U]/(T^2 - 4X^2Y^2 - d\,E^2X^2)\,,$$

where

$$d = \frac{b^2 - 4ac}{a}\,.$$

Computation of $\mathcal{A}_H^\alpha$ for a group algebra

(By D. Haile and M. Natapov)

Let $G = \langle g, h \,|\, g^9 = h^9 = 1,\; gh = h^4 g \rangle = \mathbb{Z}/9 \rtimes \mathbb{Z}/9$ and $H = k[G]$

Set $X = X_g$ and $Y = X_h$. Then

$$\mathcal{B}_H^\alpha \cong k[(X^9)^{\pm 1}, (Y^9)^{\pm 1}, Z]/(Z^3 - \omega\,(X^9)^3\,(Y^9)^2)\,,$$

where $Z = XYX^8Y^5$ and ω is a primitive third root of 1, and

$$\mathcal{A}_H^\alpha = \mathcal{B}_H^\alpha \langle X, Y \rangle / I\,,$$

where I is the two-sided ideal generated by

$$X^3Y - \omega\,YX^3,\;\; Y^3X - \omega^2\,XY^3,$$

$$XYXY - \omega^2\,Y^2X^2,\;\; YXYX - X^2Y^2,\;\; XY^2X - \omega^2\,YX^2Y$$

Reference for results of Part Two

E. Aljadeff, C. Kassel,
Polynomial identities and noncommutative versal torsors,
Preprint arXiv:0708.4108