Polynomial identities in noncommutative geometry

Christian Kassel

Institut de Recherche Mathématique Avancée
CNRS - Université Louis Pasteur
Strasbourg, France

Colloque d’algèbre non commutative
Université de Sherbrooke, Québec, Canada
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Introduction


We are interested in the concept of principal fiber bundles in noncommutative geometry

For these “quantum principal fiber bundles”, the structural group is a Hopf algebra

Motivations:

(a) Many interesting examples coming from quantum groups

(b) A well-known algebraic language to describe “quantum principal fiber bundles”, that of Hopf Galois extensions

(c) This leads to new questions on Hopf algebras
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Genesis of the results

- In August 2005 at the XVI° Coloquio Latinoamericano de Álgebra in Colonia del Sacramento, Uruguay, I reported results on the classification of Hopf Galois extensions (some joint with Hans-Jürgen Schneider)

In topology there are universal principal fiber bundles. In the context of noncommutative geometry I raised the question of the existence and the construction of universal Hopf Galois extensions.

- Eli Aljadeff suggested to use an appropriate theory of polynomial identities in order to answer this question. During his three-month stay in Strasbourg in Fall 2005, we made his idea work.

We obtained answers for a special class of Hopf Galois extensions, subsequently called twisted algebras, which are obtained from a Hopf algebra $H$ and a cocycle $\alpha$
What we actually do

To each pair \((H, \alpha)\), where \(H\) is a Hopf algebra and \(\alpha\) is a cocycle, and to the corresponding twisted algebra \(A\) we associate the following algebras:

- A commutative algebra \(B^\alpha_H\) constructed from what we call the generic cocycle cohomologous to the cocycle \(\alpha\).

  The algebra \(B^\alpha_H\) is a natural parameter space for a class of deformations of \(A\). The affine variety \(\text{Spec}(B^\alpha_H)\) is a natural geometrical object associated to the “noncommutative” pair \((H, \alpha)\)

- A noncommutative Hopf Galois extension \(A^\alpha_H\) over the commutative algebra \(B^\alpha_H\).

  The algebra \(A^\alpha_H\) is a flat deformation of \(A\) over the “parameter space” \(\text{Spec}(B^\alpha_H)\)

- An algebra \(U^\alpha_H\) built out of the polynomial identities satisfied by the twisted algebra \(A\)

We establish a connection between the noncommutative algebras \(A^\alpha_H\) and \(U^\alpha_H\)
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Plan

• **Part One:** Hopf Galois extensions, twisted algebras, the classification problem

• **Part Two:** The generic cocycle and the “parameter space”

• **Part Three:** An example: the Sweedler algebra

• **Part Four:** The generic Galois extension

• **Part Five:** Polynomial identities and the universal comodule algebra
Part One

HOPF GALOIS EXTENSIONS
Basic dictionary of noncommutative geometry

- Replacing spaces by associative algebras
  
  - space $X \leftrightarrow$ algebra $A(X)$ (\(=\) functions on $X$)
  - map $f : X \rightarrow Y \leftrightarrow$ algebra map $f^* : A(Y) \rightarrow A(X)$
  - product $f : X \times Y \leftrightarrow$ tensor product $A(X) \otimes A(Y)$
  - point $X = \{\ast\} \leftrightarrow$ ground field $A(\ast) = k$
Groups in noncommutative geometry

- **Groups.** Let $G$ be a group and $H = A(G)$

  product $G \times G \rightarrow G \quad \longleftrightarrow \quad$ coproduct $H \rightarrow H \otimes H$

  unit $\{\ast\} \rightarrow G \quad \longleftrightarrow \quad$ counit $H \rightarrow k$

  inverse $G \rightarrow G \quad \longleftrightarrow \quad$ antipode $S : H \rightarrow H$

  So $H$ is a Hopf algebra

- **Group actions.** We also need the concept of an $H$-comodule algebra

  action $X \times G \rightarrow X \quad \longleftrightarrow \quad$ coaction $\delta : A \rightarrow A \otimes H$

  orbit set $Y = X/G \quad \longleftrightarrow \quad$ coinvariants $B = \{a \in A | \delta(a) = a \otimes 1_H\}$
Definition of a Hopf algebra

Fix a field $k$. A Hopf algebra is an associative unital $k$-algebra $H$ together with

- **(Coproduct)** an algebra morphism $\Delta : H \to H \otimes H$
- **(Co\-unit)** an algebra morphism $\varepsilon : H \to k$
- **(Antipode)** an algebra antimorphism $S : H \to H$ such that

\[
(\Delta \otimes \text{id}_H) \circ \Delta = (\text{id}_H \otimes \Delta) \circ \Delta ,
\]

\[
\sum_{(x)} x_{(1)} \varepsilon(x_{(2)}) = \sum_{(x)} \varepsilon(x_{(1)}) x_{(2)} = x ,
\]

\[
\sum_{(x)} x_{(1)} S(x_{(2)}) = \sum_{(x)} S(x_{(1)}) x_{(2)} = \varepsilon(x) 1 ,
\]

where we use the Sweedler notation

\[
\Delta(x) = \sum_{(x)} x_{(1)} \otimes x_{(2)}
\]
Examples of Hopf algebras

*Example 1:* The algebra $H = k^G$ of $k$-valued functions on a finite group $G$ is a Hopf algebra with

$$\Delta(f)(g, h) = f(gh), \quad \varepsilon(f) = f(e), \quad S(f)(g) = f(g^{-1})$$

($f \in H$, $g, h \in G$; $e$ is the unit of the group).

*Example 2:* The algebra $H = k[G]$ of a group $G$ is a Hopf algebra with

$$\Delta(g) = g \otimes g, \quad \varepsilon(g) = 1, \quad S(g) = g^{-1}$$

($g \in G$).
Comodule algebras

Let $H$ be a Hopf algebra

• An $H$-comodule algebra is an associative unital algebra $A$ together with an algebra morphism

$$\delta : A \to A \otimes H$$

called the coaction and satisfying

$$(\delta \otimes \text{id}_H) \circ \delta = (\text{id}_H \otimes \Delta) \circ \delta \quad \text{and} \quad (\text{id}_H \otimes \varepsilon) \circ \delta = \text{id}_H$$

• Coinvariants:

$$A^H = \{ a \in A \mid \delta(a) = a \otimes 1_H \}$$

is a subalgebra and a subcomodule of $A$
Hopf Galois extensions

• The noncommutative analogue of a principal fiber bundle, or a $G$-torsor, is a Hopf Galois extension

• Let $H$ be a Hopf algebra and $B$ an associative algebra

**Definition.** An $H$-Galois extension $A$ of $B$ is an $H$-comodule algebra $A$ such that

(a) $B$ is the subalgebra of coinvariants of $A$, i.e., $B = A^H$

(b) the Galois condition is satisfied: the map $a \otimes b \mapsto (a \otimes 1_H) \delta(b)$ induces a bijection

$$A \otimes_B A \rightarrow A \otimes H$$

(c) as a left $B$-module, $A$ is faithfully flat
Central Hopf Galois extensions

- We consider only central Hopf Galois extensions, i.e., $H$-Galois extensions $A$ of $B$ such that $B$ is central in $A$

- *Functoriality property*: If $A$ is a central $H$-Galois extension of $B$ and $f : B \to B'$ is a morphism of commutative algebras, then

$$ f_\ast A = B' \otimes_B A $$

is a central $H$-Galois extension of $B'$
Twisted algebras

• Let $H$ be a Hopf algebra and $\alpha : H \times H \to k$ a bilinear form

• Let $\alpha H$ be a vector space isomorphic to $H$ via a linear isomorphism $x \in H \mapsto u_x \in \alpha H$. Equip $\alpha H$ with the product

\[
u_x \nu_y = \sum_{(x),(y)} \alpha(x_{(1)}, y_{(1)}) \nu_{x_{(2)}y_{(2)}} \tag{1}\]

This product is associative if $\alpha$ is a two-cocycle: for all $x, y \in H$,

\[
\sum_{(x),(y)} \alpha(x_{(1)}, y_{(1)}) \alpha(x_{(2)}y_{(2)}, z) = \sum_{(y),(z)} \alpha(y_{(1)}, z_{(1)}) \alpha(x, y_{(2)}z_{(2)}) \tag{2}\]

• The algebra $\alpha H$ has $u_1$ as unit if $\alpha$ is normalized: for all $x \in H$,

\[
\alpha(1, x) = \alpha(x, 1) = \varepsilon(x)
\]
Invertible cocycles

We shall consider only invertible two-cocycles

• A two-cocycle $\alpha : H \times H \to k$ is invertible if there is a bilinear form $\alpha^{-1} : H \times H \to k$ such that for all $x, y \in H$,

$$\sum_{(x),(y)} \alpha(x(1), y(1)) \alpha^{-1}(x(2), y(2)) = \varepsilon(x) \varepsilon(y)$$

and

$$\sum_{(x),(y)} \alpha^{-1}(x(1), y(1)) \alpha(x(2), y(2)) = \varepsilon(x) \varepsilon(y)$$

• If $G$ is a group, $H = k[G]$, and $\alpha$ is an invertible two-cocycle of $H$, then
  (a) the restriction of $\alpha$ to $G$ is a group-cocycle with values in $k^\times = k - \{0\}$
  (b) the algebra $^\alpha H$ is a so-called strongly $G$-graded algebra and the product is given for all $g, h \in G$ by

$$u_g \ u_h = \alpha(g, h) \ u_{gh}$$
Twisted algebras are Galois extensions

- For any two-cocycle $\alpha$, the twisted algebra $\alpha H$ is an $H$-comodule algebra with coaction

$$
\delta : \alpha H \rightarrow \alpha H \otimes H \\
\delta(u_x) = \sum_{(x)} u_{x(1)} \otimes x_{(2)}
$$

- The coinvariants of $\alpha H$ are trivial:

$$
(\alpha H)^H = k 1
$$

- The Galois condition is satisfied: for each cocycle $\alpha$, the twisted algebra $\alpha H$ is an $H$-Galois extension of $k$
The classification problem

• **Problem:** *Classify all* $H$-Galois *extensions of* $B$ *up to isomorphism*

*Recall:* Two $H$-Galois extensions $A, A'$ of $B$ are *isomorphic* if there is an algebra isomorphism $f : A \to A'$ preserving the coactions, i.e.,

\[(f \otimes \text{id}_H) \circ \delta = \delta' \circ f\]

• **Reformulation:** Determine the set $\text{CGal}_H(B)$ of isomorphism classes of central $H$-Galois extensions of $B$

This is a *difficult* problem in general
Universal classifying algebra

Let $H$ be a Hopf algebra

- **Definition.** A central $H$-Galois extension $A_{\mathcal{H}}$ of $B_{\mathcal{H}}$ is called (uni)versal if for any commutative algebra $B$, the map

  \[ \text{Alg}(B_{\mathcal{H}}, B) \rightarrow \text{CGal}_H(B) \]

  \[ f \mapsto f_\ast A_{\mathcal{H}} = B \otimes_{B_{\mathcal{H}}} A_{\mathcal{H}} \]

  is surjective (bijective)

- Universal central $H$-Galois extensions are unique up to isomorphism, but may not exist

- We shall construct versal Galois extensions for twisted algebras
Doi’s classification of twisted algebras

- **Lemma (Doi).** Two twisted algebras $\alpha H$ and $\beta H$ are isomorphic as $H$-comodule algebras if and only if $\alpha$ and $\beta$ are cohomologous in the following sense.

- **Definition.** Two-cocycles $\alpha$ and $\beta$ of $H$ are cohomologous, $\alpha \sim \beta$, if there is an invertible linear form $\lambda : H \to k$ with inverse $\lambda^{-1} : H \to k$ such that for all $x, y \in H$,

\[
\beta(x, y) = \sum_{(x), (y)} \lambda(x_{(1)}) \lambda(y_{(1)}) \alpha(x_{(2)}, y_{(2)}) \lambda^{-1}(x_{(3)} y_{(3)})
\]

- A linear form $\lambda : H \to k$ is invertible with inverse $\lambda^{-1} : H \to k$ if

\[
\sum_{(x)} \lambda(x_{(1)}) \lambda^{-1}(x_{(2)}) = \sum_{(x)} \lambda^{-1}(x_{(1)}) \lambda(x_{(2)}) = \varepsilon(x) 1
\]

for all $x \in H$ (An invertible linear form has a unique inverse)
Classifying twisted algebras

• By Doi’s lemma, there is a bijection

\[ \text{twisted algebras}/(\text{comodule algebra isomorphisms}) \]
\[ \cong \text{two-cocycles of } H/\sim \]

• In general, the set \( \text{two-cocycles of } H/\sim \) is not a group

• Only in special cases is \( \text{two-cocycles of } H/\sim \) a group, e.g., if \( H \) is cocommutative:

\[ \sum_{(x)} x_{(1)} \otimes x_{(2)} = \sum_{(x)} x_{(2)} \otimes x_{(1)} \]

The group Hopf algebra \( k[G] \) is cocommutative and

\[ \text{two-cocycles of } k[G]/\sim \cong H^2(G, k^\times) \]
THE GENERIC COCYCLE

(All results in the sequel are joint with Eli Aljadeff)
Towards a generic cocycle

- **Objective.** Let $H$ be a Hopf algebra and $\alpha$ a normalized invertible two-cocycle. We want to define a generic two-cocycle on $H$ that is cohomologous to $\alpha$.

- To this end, we “emulate” the equations

$$
\beta(x, y) = \sum_{(x), (y)} \lambda(x_1) \lambda(y_1) \alpha(x_2, y_2) \lambda^{-1}(x_3 y_3)
$$

and

$$
\sum_{(x)} \lambda(x_1) \lambda^{-1}(x_2) = \sum_{(x)} \lambda^{-1}(x_1) \lambda(x_2) = \varepsilon(x) 1
$$

(expressing that $\alpha, \beta$ are cohomologous)

by replacing the scalars $\lambda(x), \lambda^{-1}(x)$ by commuting symbols $t_x, t_x^{-1}$
The symbols $t_x$

Let $t_H$ be a copy of $H$ with linear isomorphism $x \in H \mapsto t_x \in t_H$

Consider the symmetric algebra

$$\text{Sym}(t_H) = \bigoplus_{r \geq 0} \text{Sym}^r(t_H)$$

on the vector space $t_H$. If $\{x_i\}_{i \in I}$ is a basis of $H$, then

$$\text{Sym}(t_H) \cong k[t_{x_i} \mid i \in I]$$

is the polynomial algebra in the (commuting) variables $t_{x_i}$
The symbols $t_{x}^{-1}$

- We also need variables $t_{x}^{-1}$ to deal with the equations
  \[
  \sum_{(x)} \lambda(x_{(1)}) \lambda^{-1}(x_{(2)}) = \sum_{(x)} \lambda^{-1}(x_{(1)}) \lambda(x_{(2)}) = \varepsilon(x) 1
  \]

- Let Frac Sym($t_{H}$) be the field of fractions of Sym($t_{H}$)

**Lemma.** *There is a unique linear map $x \mapsto t_{x}^{-1}$ from $H$ to Frac Sym($t_{H}$) such that for all $x \in H$,*

\[
\sum_{(x)} t_{x_{(1)}} t_{x_{(2)}}^{-1} = \sum_{(x)} t_{x_{(1)}}^{-1} t_{x_{(2)}} = \varepsilon(x) 1
\]
Computing $t^{-1}_x$

- If $x$ is **grouplike**, i.e., $\Delta(x) = x \otimes x$, then $\varepsilon(x) = 1$ and
  \[
t_xt_x^{-1} = 1, \quad \text{hence} \quad t_x^{-1} = \frac{1}{t_x}
  \]

- If $x$ is **skew-primitive**, i.e., $\Delta(x) = g \otimes x + x \otimes h$ for some grouplike elements $g, h$, then $\varepsilon(x) = 0$ and
  \[
t_g t_x^{-1} + t_x t_h^{-1} = 0, \quad \text{hence} \quad t_x^{-1} = -\frac{t_x}{t_g t_h}
  \]
The generic two-cocycle cohomologous to $\alpha$

- Let $H$ be a Hopf algebra and $\alpha : H \times H \to k$ a normalized invertible two-cocycle.

Define the generic two-cocycle $\sigma : H \times H \to \text{Frac Sym}(t_H)$ by

$$\sigma(x, y) = \sum_{(x), (y)} t_{x(1)} \ t_{y(1)} \ \alpha(x(2), y(2)) \ t_{x(3)y(3)}^{-1}$$

- The cocycle $\sigma$ is cohomologous to $\alpha$ over the fraction field $\text{Frac Sym}(t_H)$.

- In case $H = k[G]$ is a group algebra and $g, h \in G$, 

$$\sigma(g, h) = \alpha(g, h) \ \frac{t_g t_h}{t_{gh}}$$
The generic base algebra

• Let $H$ be a Hopf algebra and $\alpha : H \times H \to k$ a normalized invertible two-cocycle

• **Definition.** Let $\mathcal{B}_H^\alpha$ be the subalgebra of $\text{Frac Sym}(t_H)$ generated by the values of the generic two-cocycle $\sigma$ and of its inverse $\sigma^{-1}$

We call $\mathcal{B}_H^\alpha$ the generic base algebra

• **Immediate properties:**
  (a) $\mathcal{B}_H^\alpha$ is a domain
  (b) Transcendence degree of $\text{Frac } \mathcal{B}_H^\alpha \leq \dim H$
  (c) $\mathcal{B}_H^\alpha$ is finitely generated if $\dim H < \infty$
Computation of $\mathcal{B}_H^\alpha$ - the infinite cyclic case

- Consider the Hopf algebra of the group of integers $H = k[x, x^{-1}]$ with

$$\Delta(x) = x \otimes x, \quad \varepsilon(x) = 1, \quad S(x) = x^{-1}$$

together with the trivial two-cocycle: $\alpha(x^m, x^n) = 1$ \quad ($m, n \in \mathbb{Z}$)

- Here $\text{Sym}(t_H) = k[t_m \mid m \in \mathbb{Z}]$. For the generic cocycle,

$$\sigma(x^m, x^n) = \frac{t_m t_n}{t_{m+n}} = \frac{y_m y_n}{y_{m+n}} \quad (m, n \in \mathbb{Z})$$

where $y_m = t_m / t_1^m$ (note that $y_1 = 1$ and $y_0 = t_0$)

- The algebra $\mathcal{B}_H^\alpha$ is the Laurent polynomial algebra

$$\mathcal{B}_H^\alpha = k[y_m^{\pm 1} \mid m \in \mathbb{Z} - \{1\}]$$

inside the Laurent polynomial algebra $k[t_m^{\pm 1} \mid m \in \mathbb{Z}]$

- Is $k[t_m^{\pm 1} \mid m \in \mathbb{Z}]$ much bigger than $\mathcal{B}_H^\alpha$? Answer:

$$k[t_m^{\pm 1} \mid m \in \mathbb{Z}] = \mathcal{B}_H^\alpha[t_1^{\pm 1}]$$
Computation of $\mathcal{B}_H^\alpha$ - the finite cyclic case

- For the Hopf algebra $H = k[x]/(x^N - 1)$ of the cyclic group of order $N \geq 2$ with trivial two-cocycle,

$$\text{Sym}(t_H) = k[t_0, t_1, \ldots, t_{N-1}]$$

and the algebra $\mathcal{B}_H^\alpha$ is a Laurent polynomial algebra on $N$ variables:

$$\mathcal{B}_H^\alpha = k[y_0^\pm 1, y_2^\pm 1, \ldots, y_N^\pm 1] \subset k[t_0^\pm 1, t_1^\pm 1, \ldots, t_{N-1}^\pm 1]$$

where $y_N = t_0/t_1^N$

- Here $k[t_0^\pm 1, t_1^\pm 1, \ldots, t_{N-1}^\pm 1]$ is an integral extension of $\mathcal{B}_H^\alpha$:

$$k[t_0^\pm 1, t_1^\pm 1, \ldots, t_{N-1}^\pm 1] = \mathcal{B}_H^\alpha[t_1]/(t_1^N - y_0/y_N)$$
Part Three

THE SWEEDLER ALGEBRA
The Sweedler algebra: definition

- The Sweedler algebra is the smallest noncommutative noncocommutative Hopf algebra.

As an algebra,

\[ H = k\langle x, y \mid x^2 = 1, \ xy + yx = 0, \ y^2 = 0 \rangle \]

Its radical is the ideal \((y)\) and \(H/(y) \cong k[\mathbb{Z}/2]\)

- Hopf algebra structure:

  Coproduct: \( \Delta(x) = x \otimes x, \ \Delta(y) = 1 \otimes y + y \otimes x \)

  Co\(\epsilon\)nit: \( \varepsilon(x) = 1, \ \varepsilon(y) = 0 \)

  Antipode: \( S(x) = x, \ S(y) = xy \)

- The algebra \( H \) is four-dimensional with basis \( \{1, x, y, z\} \), where \( z = xy \)
The Sweedler algebra: the variables $t_x$ and $t_x^{-1}$

We need the variables $t_1, t_x, t_y, t_z$ and $t_1^{-1}, t_x^{-1}, t_y^{-1}, t_z^{-1}$.

They satisfy the equations

$$t_1 t_1^{-1} = 1, \quad t_x t_x^{-1} = 1,$$
$$t_1 t_y^{-1} + t_y t_x^{-1} = 0, \quad t_x t_z^{-1} + t_z t_1^{-1} = 0$$

Hence,

$$t_1^{-1} = \frac{1}{t_1}, \quad t_x^{-1} = \frac{1}{t_x}, \quad t_y^{-1} = -\frac{t_y}{t_1 t_x}, \quad t_z^{-1} = -\frac{t_z}{t_1 t_x}$$
Twisted Sweedler algebras

- **(Masuoka)** For any normalized invertible two-cocycle $\alpha$, the twisted algebra $\alpha H$ is up to isomorphism of the form

$$\alpha H = k\langle u_x, u_y \mid u_x^2 = a, \ u_x u_y + u_y u_x = b, \ u_y^2 = c \rangle$$

for some scalars $a, b, c$ with $a \neq 0$ ( $u_1$ is the unit of $\alpha H$)

- **Recall:** The generic two cocycle $\sigma$ is given by

$$\sigma(x, y) = \sum_{(x), (y)} t_{x(1)} t_{y(1)} \alpha(x(2), y(2)) t_{x(3)}^{-1} t_{y(3)}$$

Since $\dim H = 4$, we have to compute the 32 values of $\sigma$ and $\sigma^{-1}$ on the basis
Values of the generic cocycle

Values of \( \sigma \):

\[
\begin{align*}
\sigma(1,1) &= t_1, \\
\sigma(x,x) &= \frac{a t_x^2}{t_1}, \\
\sigma(y,y) &= \frac{a t_y^2 + b t_1 t_y + c t_1^2}{t_1}, \\
\sigma(x,y) &= \frac{a t_x t_y - t_1 t_z}{t_1}, \\
\sigma(y,x) &= \frac{b t_1 t_x + a t_x t_y + t_1 t_z}{t_1}, \\
\sigma(z,z) &= -\frac{t_z^2 + b t_x t_z + a c t_x^2}{t_1}
\end{align*}
\]

These rational fractions together with the values of \( \sigma^{-1} \) generate the algebra \( B_H^\alpha \).
Generators of $B^\alpha_H$ in the Sweedler case

If we set
\[ E = t_1, \quad R = a t_x^2, \quad S = a t_y^2 + b t_1 t_y + c t_1^2, \]
\[ T = t_x (2a t_y + b t_1), \quad U = a t_x^2 (2 t_z + b t_x), \]
then
\[ \sigma(1, 1) = E, \quad \sigma(x, y) = \frac{RT - EU}{2ER}, \]
\[ \sigma(x, x) = \frac{R}{E}, \quad \sigma(y, x) = \frac{RT + EU}{2ER}, \]
\[ \sigma(y, y) = \frac{S}{E}, \quad \sigma(z, z) = \frac{a U^2 - (b^2 - 4ac) R^3}{4a ER^2} \]

Consequence. The elements $E, E^{-1}, R, R^{-1}, S, T, U$ belong to $B^\alpha_H$ and generate it as an algebra. Moreover, $E, R, S, U$ are algebraically independent.
Theorem. *Presentation of $B^\alpha_H$ by generators and relations:*

$$B^\alpha_H \cong k[E^\pm 1, R^\pm 1, S, T, U]/(P_{a,b,c}),$$

where

$$P_{a,b,c} = T^2 - 4RS - \frac{b^2 - 4ac}{a} E^2 R$$
Part Four

THE GENERIC GALOIS EXTENSION
The generic Galois extension

- Since the generic cocycle takes values in $B^\alpha_H$, we can consider the twisted algebra
  \[ A^\alpha_H = B^\alpha_H \otimes \sigma H \]
  As a vector space, $A^\alpha_H = B^\alpha_H \otimes H$; it is equipped with the product
  \[
  (a \otimes u_x)(b \otimes u_y) = \sum_{(x),(y)} a b \sigma(x_1, y_1) \otimes u_{x_2} y_{2}
  \]
  \[
  (a, b \in B^\alpha_H, x, y \in H)
  \]
  The twisted algebra $A^\alpha_H$ is called the generic Galois extension

- **Proposition.** (a) The map $\delta = \text{id}_{B^\alpha_H} \otimes \Delta$ is a coaction, turning $A^\alpha_H$ into an $H$-comodule algebra whose subalgebra of coinvariants is $B^\alpha_H$:
  \[
  B^\alpha_H = \{ a \in A^\alpha_H \mid \delta(a) = a \otimes 1_H \}
  \]
  (b) $B^\alpha_H$ is a central subalgebra of $A^\alpha_H$
  Thus $A^\alpha_H$ is a central $H$-Galois extension of $B^\alpha_H$
The generic Galois extension

Since the generic cocycle takes values in $B^\alpha_H$, we can consider the twisted algebra

$$A^\alpha_H = B^\alpha_H \otimes \sigma H$$

As a vector space, $A^\alpha_H = B^\alpha_H \otimes H$; it is equipped with the product

$$(a \otimes u_x) (b \otimes u_y) = \sum_{(x),(y)} a b \sigma(x_1, y_1) \otimes u_{x_2} y_{(2)}$$

$(a, b \in B^\alpha_H, x, y \in H)$

The twisted algebra $A^\alpha_H$ is called the generic Galois extension

**Proposition.** (a) The map $\delta = \text{id}_{B^\alpha_H} \otimes \Delta$ is a coaction, turning $A^\alpha_H$ into an $H$-comodule algebra whose subalgebra of coinvariants is $B^\alpha_H$:

$$B^\alpha_H = \{ a \in A^\alpha_H \mid \delta(a) = a \otimes 1_H \}$$

(b) $B^\alpha_H$ is a central subalgebra of $A^\alpha_H$

Thus $A^\alpha_H$ is a central $H$-Galois extension of $B^\alpha_H$
A rigidity property

- $A^\alpha_H$ is a flat deformation of $\alpha H$ over the commutative algebra $B^\alpha_H$:

**Proposition.** There is a comodule algebra isomorphism

$$A^\alpha_H/m_0A^\alpha_H \cong \alpha H$$

for some maximal ideal $m_0$ of $B^\alpha_H$.

The ideal $m_0$ is the kernel of an algebra morphism $\chi_0 : B^\alpha_H \to k$ sending each element $\sigma(x, y)$ to $\alpha(x, y)$.

- If $\alpha H$ is (semi)simple, then $A^\alpha_H$ is generically (semi)simple. More precisely:

**Theorem.** Assume that $\text{char}(k) = 0$ and $\dim(H) < \infty$. If $\alpha H$ is (semi)simple, then so is

$$\text{Frac } B^\alpha_H \otimes_{B^\alpha_H} A^\alpha_H = \text{Frac } B^\alpha_H \otimes \sigma H$$
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**Theorem.** Assume that $\text{char}(k) = 0$ and $\dim(H) < \infty$.

If $\alpha H$ is (semi)simple, then so is

$$\text{Frac} \ B^\alpha_H \otimes_{B^\alpha_H} A^\alpha_H = \text{Frac} \ B^\alpha_H \otimes \sigma H$$
• Let $\beta : H \times H \to K \supset k$ be a normalized invertible two-cocycle. The twisted algebra $K \otimes \beta H$ is a $K$-form of $\alpha H$ if there is a field $L \supset K$ and an $L$-linear isomorphism of $H$-comodule algebras

$$L \otimes_K (K \otimes \beta H) \cong L \otimes_k \alpha H.$$ 

• **Theorem.** If $K \otimes \beta H$ is a $K$-form of $\alpha H$, then there is an algebra morphism $\chi : B_H^\alpha \to K$ such that

$$K \chi \otimes B_H^\alpha A_H^\alpha \cong \beta H$$

In other words, any form of $\alpha H$ is obtained from the generic Galois extension $A_H^\alpha$ by a central specialization.

There is a converse to the previous theorem; it requires an additional condition.
Theorem If the algebra \( \text{Frac} \, \text{Sym}(t_H) \) is integral over the subalgebra \( B_\alpha^H \), then for any algebra morphism \( \chi : B_\alpha^H \to K \supset k \), the algebra \( K_\chi \otimes_{B_\alpha^H} A_\alpha^H \) is a \( K \)-form of \( \alpha \_H \)

Theorem. Under the previous integrality condition, if the algebra \( \alpha \_H \) is simple, then the generic extension \( A_\alpha^H \) is an Azumaya algebra with center \( B_\alpha^H \)

An algebra \( A \) is Azumaya if \( A/\mathfrak{m} \) is simple for any maximal ideal \( \mathfrak{m} \) of its center. E.g. \( A = M_n(R) \), where \( R \) is a commutative ring
**Theorem.** If the algebra Frac Sym($t_H$) is integral over the subalgebra $B^\alpha_H$, then for any algebra morphism $\chi : B^\alpha_H \to K \supset k$, the algebra $K_\chi \otimes_{B^\alpha_H} A^\alpha_H$ is a $K$-form of $\alpha H$.

**Theorem.** Under the previous integrality condition, if the algebra $\alpha H$ is simple, then the generic extension $A^\alpha_H$ is an Azumaya algebra with center $B^\alpha_H$.

An algebra $A$ is Azumaya if $A/\mathfrak{m}$ is simple for any maximal ideal $\mathfrak{m}$ of its center. E.g. $A = M_n(R)$, where $R$ is a commutative ring.
Versal deformation space

• If $\text{Frac Sym}(t_H)$ is integral over $B_H^{\alpha}$, then the map

$$\text{Alg}(B_H^{\alpha}, K) \rightarrow K \text{- Forms}(H^{\alpha})$$

$$\chi \mapsto K_{\chi} \otimes B_H^{\alpha} A_H^{\alpha}$$

is a surjection from the set of algebra morphisms $B_H^{\alpha} \rightarrow K$ to the set of isomorphism classes of $K$-forms of $H^{\alpha}$.

• Thus the set $\text{Alg}(B_H^{\alpha}, K)$ parametrizes the $K$-forms of $H^{\alpha}$.

The extension $B_H^{\alpha} \subset A_H^{\alpha}$ is a versal deformation space for the forms of $H^{\alpha}$.

• Remark. To determine the set $\text{Alg}(B_H^{\alpha}, K)$, it is important to find a presentation by generators and relations of $B_H^{\alpha}$. 

The integrality condition

- **Question.** Under which condition on \((H, \alpha)\) is the algebra \(\text{Frac} \text{Sym}(t_H)\) integral over the subalgebra \(B^\alpha_H\)?

- **Proposition.** If \(H\) is a finite-dimensional Hopf algebra that is generated by grouplike and skew-primitive elements, and \(\alpha\) is any two-cocycle, then \(\text{Frac} \text{Sym}(t_H)\) is integral over the subalgebra \(B^\alpha_H\).

*Proof in the case \(H\) is the algebra of a finite group \(G\).* It suffices to show that each \(t_g \ (g \in G)\) is integral over \(B^\alpha_H\). Since

\[
\sigma(g, h) = \alpha(g, h) \frac{t_g t_h}{t_{gh}} \in B^\alpha_H,
\]

we have \(t_g t_h = b t_{gh}\) for some \(b \in B^\alpha_H\). Consequently for all \(n \geq 2\),

\[
t^n = b' t^n
\]

for some \(b' \in B^\alpha_H\). If \(n\) is the order of \(g\), then since \(\sigma(1, 1) = t_1\),

\[
t^n = b' t^n = b' t_1 = b' \sigma(1, 1),
\]

which shows that \(t^n\) belongs to \(B^\alpha_H\). QED
Theorem  The generic Galois extension $A^\alpha_H$ is given by

$$A^\alpha_H \cong B^\alpha_H \langle X, Y \rangle / (X^2 - R, \ XY + YX - T, \ Y^2 - S)$$

Compare with

$$\alpha H = k\langle u_x, u_y \rangle / (u_x^2 - a, \ u_x u_y + u_y u_x - b, \ u_y^2 - c)$$

Recall:

$$B^\alpha_H \cong k[E^{\pm 1}, R^{\pm 1}, S, T, U]/(P_{a,b,c}),$$

where

$$P_{a,b,c} = T^2 - 4RS - \frac{b^2 - 4ac}{a} E^2 R$$
Computation of $A_H^\alpha$ for a group algebra

(By D. Haile and M. Natapov)

Let $G = \langle g, h \mid g^9 = h^9 = 1, \ gh = h^4 g \rangle = \mathbb{Z}/9 \times \mathbb{Z}/9$

Let $H = k[G]$ and set $X = X_g$ and $Y = X_h$. Then

$$B_H^\alpha \cong k[(X^9)^{\pm 1}, (Y^9)^{\pm 1}, Z]/(Z^3 - \omega (X^9)^3 (Y^9)^2),$$

where $Z = X Y X^8 Y^5$ and $\omega$ is a primitive third root of 1

The generic Galois extension: $A_H^\alpha = B_H^\alpha \langle X, Y \rangle / I$, where $I$ is the two-sided ideal generated by

$$X^3 Y - \omega Y X^3, \quad Y^3 X - \omega^2 X Y^3,$$

$$X Y X Y - \omega^2 Y^2 X^2, \quad Y X Y X - X^2 Y^2, \quad X Y^2 X - \omega^2 Y X^2 Y$$
Computation of $A^\alpha_H$ for a group algebra

(By D. Haile and M. Natapov)

Let $G = \langle g, h | g^9 = h^9 = 1, \ gh = h^4 g \rangle = \mathbb{Z}/9 \times \mathbb{Z}/9$

Let $H = k[G]$ and set $X = X_g$ and $Y = X_h$. Then

$B^\alpha_H \cong k[(X^9)^{\pm 1}, (Y^9)^{\pm 1}, Z]/(Z^3 - \omega (X^9)^3 (Y^9)^2)$,

where $Z = XYX^8 Y^5$ and $\omega$ is a primitive third root of 1

The generic Galois extension: $A^\alpha_H = B^\alpha_H \langle X, Y \rangle / I$, where $I$ is the two-sided ideal generated by

$X^3 Y - \omega YX^3, \ Y^3 X - \omega^2 XY^3, \ XYXY - \omega^2 Y^2 X^2, \ YXXY - X^2 Y^2, \ XY^2 X - \omega^2 YX^2 Y$
Computation of $\mathcal{A}_{H}^\alpha$ for a group algebra

(By D. Haile and M. Natapov)

- Let $G = \langle g, h \mid g^9 = h^9 = 1, \ gh = h^4g \rangle = \mathbb{Z}/9 \times \mathbb{Z}/9$

- Let $H = k[G]$ and set $X = X_g$ and $Y = X_h$. Then

$$\mathcal{B}_{H}^\alpha \cong k[(X^9)^{\pm 1}, (Y^9)^{\pm 1}, Z]/(Z^3 - \omega (X^9)^3 (Y^9)^2),$$

where $Z = XYX^8Y^5$ and $\omega$ is a primitive third root of 1

- The generic Galois extension: $\mathcal{A}_{H}^\alpha = \mathcal{B}_{H}^\alpha \langle X, Y \rangle / I$, where $I$ is the two-sided ideal generated by

  \begin{align*}
  X^3Y - \omega YX^3, & \quad Y^3X - \omega^2 XY^3, \\
  XYXY - \omega^2 Y^2X^2, & \quad YXYX - X^2Y^2, \quad XY^2X - \omega^2 YX^2Y
  \end{align*}
Part Five

POLYNOMIAL IDENTITIES
How to find generators for $\mathcal{B}_H^\alpha$

• In the Sweedler case, how did we find the generators

$$E = t_1, \quad R = a t_x^2, \quad S = a t_y^2 + b t_1 t_y + c t_1^2,$$

$$T = t_x (2a t_y + b t_1), \quad U = a t_x^2 (2 t_z + b t_x),$$

of $\mathcal{B}_H^\alpha$?

• To explain this we need a new set of symbols
The symbols $X_x$ and the tensor algebra

- We now assume that the ground field $k$ is infinite. Let $H$ be a Hopf algebra.

- Let $X_H$ be a copy of $H$ with a linear isomorphism $x \mapsto X_x$.

- Consider the tensor algebra

  $$T(X_H) = \bigoplus_{r \geq 0} T^r(X_H)$$

  on the vector space $X_H$. If \( \{x_i\}_{i \in I} \) is a basis of $H$, then

  $$T(X_H) \cong k \langle X_{x_i} \mid i \in I \rangle$$

  is the algebra of noncommutative polynomials in $X_{x_i} \ (i \in I)$.

- The algebra $T(X_H)$ is an $H$-comodule algebra with coaction

  $\delta : T(X_H) \to T(X_H) \otimes H$ given by

  $$\delta(X_x) = \sum_{(x)} X_{x(1)} \otimes x_{(2)}$$
The universal evaluation map

- Let $H$ be a Hopf algebra, $\alpha : H \times H \to k$ a normalized invertible two-cocycle, and $\alpha H$ the corresponding twisted algebra

- Consider the algebra morphism

$$
\mu_\alpha : T(X_H) \longrightarrow \text{Sym}(t_H) \otimes ^\alpha H
$$

$$
X_x \longmapsto \sum_{(x)} t_{x(1)} \otimes u_{x(2)}
$$

We call $\mu_\alpha$ the universal evaluation map for $^\alpha H$
Properties of the map $\mu_\alpha$

**Lemma.** (a) The map $\mu_\alpha : T(X_H) \rightarrow \text{Sym}(t_H) \otimes ^\alpha H$ is an $H$-comodule algebra morphism

(b) For every $H$-comodule algebra morphism $\mu : T(X_H) \rightarrow ^\alpha H$, there is a unique algebra morphism $\chi : \text{Sym}(t_H) \rightarrow k$ such that

$$\mu = (\chi \otimes \text{id}) \circ \mu_\alpha$$

In other words, any comodule algebra morphism $T(X_H) \rightarrow ^\alpha H$ is a specialization of $\mu_\alpha$, and $\text{Sym}(t_H)$ parametrizes the set of such comodule algebra morphisms.
Constructing elements of $B^\alpha_H$ using $\mu^\alpha$

- **Proposition.** If $P \in T(X_H)$ is coinvariant, i.e., $\delta(P) = P \otimes 1_H$, then $\mu^\alpha(P)$ belongs to $B^\alpha_H$

- The following “universal” formulas provide coinvariant elements of $T(X_H)$ (where $x, y \in H$):

  \[ P_x = \sum_{(x)} X_{x(1)} X_{S(x(2))} \]

  \[ P_{x,y} = \sum_{(x),(y)} X_{x(1)} X_{y(1)} X_{S(x(2)y(2))} \]

- **Example:** For the Sweedler algebra:

  \[ R = \mu^\alpha(P_x), \quad T = \mu^\alpha(P_{y-z}), \]

  \[ U = \mu^\alpha(P_{x,z}), \quad ES = \mu^\alpha(P_{y,y}) \]
Constructing elements of $\mathcal{B}_H^\alpha$ using $\mu_\alpha$

- **Proposition.** If $P \in T(X_H)$ is coinvariant, i.e., $\delta(P) = P \otimes 1_H$, then $\mu_\alpha(P)$ belongs to $\mathcal{B}_H^\alpha$

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\[
U = \mu_\alpha(P_{x,z}), \quad ES = \mu_\alpha(P_{y,y})
\]
A theory of identities for comodule algebras

Let $A$ be an $H$-comodule algebra for some Hopf algebra $H$

**Definition.** An element $P \in T(X_H)$ is an $H$-identity of $A$ if $\mu(P) = 0$ for all comodule algebra morphisms $\mu : T(X_H) \to A$

Let $I_H(A)$ be the vector space of all $H$-identities of $A$

**Proposition.** (a) $I_H(A)$ is a two-sided ideal of $T(X_H)$ such that

$$\delta(I_H(A)) \subset I_H(A) \otimes H$$

(b) The ideal $I_H(A)$ is graded and

$$I_H(A) \subset \bigoplus_{r \geq 2} T^r(X_H)$$
A theory of identities for comodule algebras

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(b) The ideal $I_H(A)$ is graded and

$$I_H(A) \subset \bigoplus_{r \geq 2} T^r(X_H)$$
The universal comodule algebra

- **Definition.** The *universal comodule algebra* of identities of the $H$-comodule algebra $A$ is the quotient-algebra

\[ \mathcal{U}_H(A) = T(X_H)/I_H(A) \]

- **Properties.**
  
  (a) $\mathcal{U}_H(A)$ is a *graded* algebra coinciding with $T(X_H)$ in degrees 0 and 1
  (b) $\mathcal{U}_H(A)$ is an $H$-comodule algebra
  (c) All $H$-identities of $A$ vanish in $\mathcal{U}_H(A)$
Detecting the identities for twisted algebras

Let $\alpha H$ be a twisted algebra for some Hopf algebra $H$ and some normalized invertible two-cocycle $\alpha$.

Recall the universal evaluation map

$$\mu_\alpha : T(X_H) \to \text{Sym}(t_H) \otimes \alpha H$$

Theorem. We have $I_H(\alpha H) = \text{Ker} \ \mu_\alpha$.

In other words, the map $\mu_\alpha$ detects the $H$-identities of $\alpha H$. 
Let $\alpha H$ be a twisted algebra for some Hopf algebra $H$ and some normalized invertible two-cocycle $\alpha$.

Recall the universal evaluation map

$$\mu_\alpha : T(X_H) \rightarrow \text{Sym}(t_H) \otimes \alpha H$$

**Theorem.** We have $I_H(\alpha H) = \text{Ker } \mu_\alpha$.

In other words, the map $\mu_\alpha$ detects the $H$-identities of $\alpha H$.
Embedding $\mathcal{U}_H^\alpha$ into a controllable algebra

**Consequences of the previous theorem:** Set $\mathcal{U}_H^\alpha = T(X_H)/l_H(\alpha H)$

- Since $\text{Ker } \mu_\alpha = \text{Ker}(T(X_H) \to \mathcal{U}_H^\alpha)$, the map $\mu_\alpha$ induces an embedding
  \[ \mathcal{U}_H^\alpha \hookrightarrow \text{Sym}(t_H) \otimes \alpha H \]
  of the universal comodule algebra into a twisted product

- $u \in \mathcal{U}_H^\alpha$ is coinvariant if and only if $\mu_\alpha(u)$ belongs to $\text{Sym}(t_H) \otimes 1$

- $u \in \mathcal{U}_H^\alpha$ is central if and only if $\mu_\alpha(u)$ belongs to $\text{Sym}(t_H) \otimes Z(\alpha H)$, where $Z(\alpha H)$ is the center of $\alpha H$

- The center $Z_H^\alpha$ of $\mathcal{U}_H^\alpha$ is a domain if $Z(\alpha H)$ is a domain
Embedding $\mathcal{U}_H^\alpha$ into a controllable algebra

Consequences of the previous theorem: Set $\mathcal{U}_H^\alpha = T(X_H) / I_H(\alpha H)$

- Since $\text{Ker} \mu_\alpha = \text{Ker}(T(X_H) \to \mathcal{U}_H^\alpha)$, the map $\mu_\alpha$ induces an embedding
  \[ \mathcal{U}_H^\alpha \hookrightarrow \text{Sym}(t_H) \otimes \alpha H \]
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- The center $Z_H^\alpha$ of $\mathcal{U}_H^\alpha$ is a domain if $Z(\alpha H)$ is a domain
Embedding $\mathcal{U}_H^\alpha$ into a controllable algebra

Consequences of the previous theorem: Set $\mathcal{U}_H^\alpha = T(X_H)/I_H(\alpha H)$

- Since $\text{Ker} \; \mu_\alpha = \text{Ker}(T(X_H) \to \mathcal{U}_H^\alpha)$, the map $\mu_\alpha$ induces an embedding
  
  $$\mathcal{U}_H^\alpha \hookrightarrow \text{Sym}(t_H) \otimes \alpha H$$

  of the universal comodule algebra into a twisted product

- $u \in \mathcal{U}_H^\alpha$ is coinvariant if and only if $\mu_\alpha(u)$ belongs to $\text{Sym}(t_H) \otimes 1$

- $u \in \mathcal{U}_H^\alpha$ is central if and only if $\mu_\alpha(u)$ belongs to $\text{Sym}(t_H) \otimes Z(\alpha H)$, where $Z(\alpha H)$ is the center of $\alpha H$

- The center $Z_H^\alpha$ of $\mathcal{U}_H^\alpha$ is a domain if $Z(\alpha H)$ is a domain
Embedding $\mathcal{U}^\alpha_H$ into a controllable algebra

**Consequences of the previous theorem:** Set $\mathcal{U}^\alpha_H = T(X_H)/I_H(\alpha H)$

- Since $\text{Ker} \; \mu_\alpha = \text{Ker}(T(X_H) \rightarrow \mathcal{U}^\alpha_H)$, the map $\mu_\alpha$ induces an embedding

  $\mathcal{U}^\alpha_H \hookrightarrow \text{Sym}(t_H) \otimes \alpha H$

  of the universal comodule algebra into a twisted product

- $u \in \mathcal{U}^\alpha_H$ is coinvariant if and only if $\mu_\alpha(u)$ belongs to $\text{Sym}(t_H) \otimes 1$

- $u \in \mathcal{U}^\alpha_H$ is central if and only if $\mu_\alpha(u)$ belongs to $\text{Sym}(t_H) \otimes Z(\alpha H)$, where $Z(\alpha H)$ is the center of $\alpha H$

- The center $\mathcal{Z}^\alpha_H$ of $\mathcal{U}^\alpha_H$ is a domain if $Z(\alpha H)$ is a domain
Consider the Sweedler algebra $H$ and the twisted algebra

$$\alpha H = k\langle u_x, u_y | u_x^2 = a, \ u_x u_y + u_y u_x = b, \ u_y^2 = c \rangle$$

The following are examples of $H$-identities for $\alpha H$:

$$(X_x X_y + X_y X_x)^2 - 4X_x^2 X_y^2 - \frac{b^2 - 4ac}{a} X_1^2 X_x^2$$

$$2X_x^2(X_1 X_z - X_x X_y) - X_1 X_x(X_x X_z + X_z X_x) + X_x^2(X_x X_y + X_y X_x)$$

(Check using the universal evaluation map)
Relating $\mathcal{U}_H^\alpha$ and $\mathcal{A}_H^\alpha$

- To $(H, \alpha)$ we associated
  
  (a) the generic Galois extension
  
  $$\mathcal{A}_H^\alpha = \mathcal{B}_H^\alpha \otimes \sigma H$$
  
  built from the generic cocycle $\sigma$

  (b) the universal comodule algebra
  
  $$\mathcal{U}_H^\alpha = T(X_H)/I_H(\alpha H)$$
  
  built out of the $H$-identities of $\alpha H$; its center is $Z_H^\alpha$

- Theorem. (a) There is an embedding of $H$-comodule algebras
  
  $$\mathcal{U}_H^\alpha \hookrightarrow \mathcal{A}_H^\alpha$$

  (b) If $Z(\alpha H) = k$, then $Z_H^\alpha \hookrightarrow \mathcal{B}_H^\alpha$
Structure of $\mathcal{U}_H^\alpha$ after central localization

- We assume that $Z(\alpha H) = k$, so that $\mu_\alpha$ embeds $Z_H^\alpha$ into $B_H^\alpha$

**Theorem.** If in addition $B_H^\alpha$ is a localization of $Z_H^\alpha$, then there is an isomorphism of $H$-comodule algebras

$$B_H^\alpha \otimes_{Z_H^\alpha} \mathcal{U}_H^\alpha \cong A_H^\alpha$$

In other words, after localization of the center, the universal comodule algebra $\mathcal{U}_H^\alpha$ becomes the generic Galois extension $A_H^\alpha$


THANK YOU FOR YOUR ATTENTION
Strongly graded algebras

- Given a group $G$, consider a strongly $G$-graded algebra:

$$A = \bigoplus_{g \in G} A_g$$

with $A_g A_h = A_{gh}$ and $\dim A_g = 1$

- Let $u_g$ be a spanning vector of $A_g$. Then

$$u_g u_h = \alpha(g, h) u_{gh}$$

for some $\alpha(g, h) \in k^\times = k - \{0\}$

- The associativity of the product of $A$ implies that the map $\alpha : G \times G \to k^\times$ is a group-cocycle:

$$\alpha(g, h) \alpha(gh, k) = \alpha(h, k) \alpha(g, hk)$$
Isomorphism classes and group cohomology

• If $v_g = \lambda(g) u_g$ with $\lambda(g) \neq 0$, then

$$v_g v_h = \beta(g, h) v_{gh},$$

where

$$\beta(g, h) = \alpha(g, h) \frac{\lambda(g) \lambda(h)}{\lambda(gh)} \quad (5)$$

• Call $\alpha$, $\beta$ cohomologous, $\alpha \sim \beta$, if they are related by (5)

• The set $\{\text{group-cocycles of } G\}/\sim$ is the cohomology group $H^2(G, k^\times)$

• Isomorphism classes of strongly graded algebras:

$$\{\text{strongly } G\text{-graded algebras}\}/(\text{isomorphisms}) \cong H^2(G, k^\times)$$