Polynomial identities in noncommutative geometry

Christian Kassel

Institut de Recherche Mathématique Avancée
CNRS - Université Louis Pasteur
Strasbourg, France

Colloque d'algèbre non commutative
Université de Sherbrooke, Québec, Canada
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Introduction

- Joint work with Eli Aljadeff (Technion): arXiv:0708.4108, Adv. Math. (2008), doi:10.1016/j.aim.2008.03.014

- We are interested in the concept of principal fiber bundles in noncommutative geometry

  For these "quantum principal fiber bundles", the structural group is a Hopf algebra

- *Motivations:*

  (a) Many interesting examples coming from quantum groups

  (b) A well-known algebraic language to describe "quantum principal fiber bundles", that of Hopf Galois extensions

  (c) This leads to new questions on Hopf algebras

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Genesis of the results

• In August 2005 at the XVI^{o} Coloquio Latinoamericano de Álgebra in Colonia del Sacramento, Uruguay, I reported results on the classification of Hopf Galois extensions (some joint with Hans-Jürgen Schneider)

In topology there are universal principal fiber bundles. In the context of noncommutative geometry I raised the question of the existence and the construction of universal Hopf Galois extensions.

• Eli Aljadeff suggested to use an appropriate theory of polynomial identities in order to answer this question. During his three-month stay in Strasbourg in Fall 2005, we made his idea work.

We obtained answers for a special class of Hopf Galois extensions, subsequently called twisted algebras, which are obtained from a Hopf algebra H and a cocycle α

What we actually do

To each pair (H, α), where H is a Hopf algebra and α is a cocycle, and to the corresponding twisted algebra A we associate the following algebras:

- A commutative algebra $\mathcal{B}_H^\alpha$ constructed from what we call the generic cocycle cohomologous to the cocycle α.

  The algebra $\mathcal{B}_H^\alpha$ is a natural parameter space for a class of deformations of A. The affine variety $\mathrm{Spec}(\mathcal{B}_H^\alpha)$ is a natural geometrical object associated to the "noncommutative" pair (H, α)

- A noncommutative Hopf Galois extension $\mathcal{A}_H^\alpha$ over the commutative algebra $\mathcal{B}_H^\alpha$.

  The algebra $\mathcal{A}_H^\alpha$ is a flat deformation of A over the "parameter space" $\mathrm{Spec}(\mathcal{B}_H^\alpha)$

- An algebra $\mathcal{U}_H^\alpha$ built out of the polynomial identities satisfied by the twisted algebra A

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Plan

- Part One: Hopf Galois extensions, twisted algebras, the classification problem

- Part Two: The generic cocycle and the “parameter space”

- Part Three: An example: the Sweedler algebra

- Part Four: The generic Galois extension

- Part Five: Polynomial identities and the universal comodule algebra

Part One

HOPF GALOIS EXTENSIONS

Basic dictionary of noncommutative geometry

- **Replacing spaces by associative algebras**

$$
\begin{array}{rcl}
\text{space } X & \longleftrightarrow & \text{algebra } A(X) \ (= \text{ functions on } X) \\
\text{map } f : X \to Y & \longleftrightarrow & \text{algebra map } f^* : A(Y) \to A(X) \\
\text{product } f : X \times Y & \longleftrightarrow & \text{tensor product } A(X) \otimes A(Y) \\
\text{point } X = \{*\} & \longleftrightarrow & \text{ground field } A(*) = k
\end{array}
$$

Groups in noncommutative geometry

- **Groups**. Let G be a group and $H = A(G)$

$$\text{product } G \times G \to G \quad \longleftrightarrow \quad \text{coproduct } H \to H \otimes H$$

$$\text{unit } \{*\} \to G \quad \longleftrightarrow \quad \text{counit } H \to k$$

$$\text{inverse } G \to G \quad \longleftrightarrow \quad \text{antipode } S : H \to H$$

So H is a Hopf algebra

- **Group actions**. We also need the concept of an H-comodule algebra

$$\text{action } X \times G \to X \quad \longleftrightarrow \quad \text{coaction } \delta : A \to A \otimes H$$

$$\text{orbit set } Y = X/G \quad \longleftrightarrow \quad \text{coinvariants } B = \{a \in A \mid \delta(a) = a \otimes 1_H\}$$

Definition of a Hopf algebra

Fix a field k. A Hopf algebra is an associative unital k-algebra H together with

- *(Coproduct)* an algebra morphism $\Delta : H \to H \otimes H$
- *(Coünit)* an algebra morphism $\varepsilon : H \to k$
- *(Antipode)* an algebra antimorphism $S : H \to H$ such that

$$(\Delta \otimes \mathrm{id}_H) \circ \Delta = (\mathrm{id}_H \otimes \Delta) \circ \Delta\,,$$

$$\sum_{(x)} x_{(1)}\, \varepsilon(x_{(2)}) = \sum_{(x)} \varepsilon(x_{(1)})\, x_{(2)} = x\,,$$

$$\sum_{(x)} x_{(1)}\, S(x_{(2)}) = \sum_{(x)} S(x_{(1)})\, x_{(2)} = \varepsilon(x)\, 1\,,$$

where we use the Sweedler notation

$$\Delta(x) = \sum_{(x)} x_{(1)} \otimes x_{(2)}$$

Examples of Hopf algebras

Example 1: The algebra $H = k^G$ of k-valued functions on a finite group G is a Hopf algebra with

$$\Delta(f)(g,h) = f(gh)\,, \quad \varepsilon(f) = f(e)\,, \quad S(f)(g) = f(g^{-1})$$

($f \in H$, $g, h \in G$; e is the unit of the group).

Example 2: The algebra $H = k[G]$ of a group G is a Hopf algebra with

$$\Delta(g) = g \otimes g\,, \quad \varepsilon(g) = 1\,, \quad S(g) = g^{-1}$$

($g \in G$).

Comodule algebras

Let H be a Hopf algebra

• An H-comodule algebra is an associative unital algebra A together with an algebra morphism

$$\delta : A \to A \otimes H$$

called the coaction and satisfying

$$(\delta \otimes \mathrm{id}_H) \circ \delta = (\mathrm{id}_H \otimes \Delta) \circ \delta \quad \text{and} \quad (\mathrm{id}_H \otimes \varepsilon) \circ \delta = \mathrm{id}_H$$

• Coinvariants:

$$A^H = \{a \in A \mid \delta(a) = a \otimes 1_H\}$$

is a subalgebra and a subcomodule of A

Hopf Galois extensions

• The noncommutative analogue of a principal fiber bundle, or a G-torsor, is a *Hopf Galois extension*

• Let H be a Hopf algebra and B an associative algebra

Definition. *An H-Galois extension A of B is an H-comodule algebra A such that*

(a) B is the subalgebra of coinvariants of A, i.e., $B = A^H$

(b) the Galois condition is satisfied: the map $a \otimes b \mapsto (a \otimes 1_H)\,\delta(b)$ induces a bijection

$$A \otimes_B A \to A \otimes H$$

(c) as a left B-module, A is faithfully flat

Central Hopf Galois extensions

• We consider only central Hopf Galois extensions, i.e., H-Galois extensions A of B such that B is central in A

• *Functoriality property:* If A is a central H-Galois extension of B and $f : B \longrightarrow B'$ is a morphism of commutative algebras, then

$$f_* A = B' \otimes_B A$$

is a central H-Galois extension of B'

Twisted algebras

• Let H be a Hopf algebra and $\alpha : H \times H \to k$ a bilinear form

• Let ${}^{\alpha}H$ be a vector space isomorphic to H via a linear isomorphism $x \in H \mapsto u_x \in {}^{\alpha}H$. Equip ${}^{\alpha}H$ with the product

$$u_x\, u_y = \sum_{(x),(y)} \alpha\big(x_{(1)}, y_{(1)}\big)\, u_{x_{(2)}y_{(2)}} \tag{1}$$

This product is associative if α is a two-cocycle: for all $x, y \in H$,

$$\sum_{(x),(y)} \alpha\big(x_{(1)}, y_{(1)}\big)\, \alpha\big(x_{(2)}y_{(2)}, z\big) = \sum_{(y),(z)} \alpha\big(y_{(1)}, z_{(1)}\big)\, \alpha\big(x, y_{(2)}z_{(2)}\big) \tag{2}$$

• The algebra ${}^{\alpha}H$ has u_1 as unit if α is normalized: for all $x \in H$,

$$\alpha(1, x) = \alpha(x, 1) = \varepsilon(x)$$

Invertible cocycles

We shall consider only invertible two-cocycles

• A two-cocycle $\alpha : H \times H \to k$ is invertible if there is a bilinear form $\alpha^{-1} : H \times H \to k$ such that for all $x, y \in H$,

$$\sum_{(x),(y)} \alpha(x_{(1)}, y_{(1)})\, \alpha^{-1}(x_{(2)}, y_{(2)}) = \varepsilon(x)\, \varepsilon(y)$$

and

$$\sum_{(x),(y)} \alpha^{-1}(x_{(1)}, y_{(1)})\, \alpha(x_{(2)}, y_{(2)}) = \varepsilon(x)\, \varepsilon(y)$$

• If G is a group, $H = k[G]$, and α is an invertible two-cocycle of H, then

(a) the restriction of α to G is a group-cocycle with values in $k^\times = k - \{0\}$

(b) the algebra ${}^{\alpha}H$ is a so-called strongly G-graded algebra and the product is given for all $g, h \in G$ by

$$u_g\, u_h = \alpha(g, h)\, u_{gh}$$

Twisted algebras are Galois extensions

- For any two-cocycle α, the twisted algebra ${}^{\alpha}H$ is an H-comodule algebra with coaction

$$\begin{aligned} \delta : {}^{\alpha}H &\to {}^{\alpha}H \otimes H \\ \delta(u_x) &= \sum_{(x)} u_{x_{(1)}} \otimes x_{(2)} \end{aligned}$$

- The coinvariants of ${}^{\alpha}H$ are trivial:

$$({}^{\alpha}H)^{H} = k\,1$$

- The Galois condition is satisfied: for each cocycle α, the twisted algebra ${}^{\alpha}H$ is an H-Galois extension of k

The classification problem

• **Problem:** *Classify all H-Galois extensions of B up to isomorphism*

Recall: Two H-Galois extensions A, A' of B are isomorphic if there is an algebra isomorphism $f : A \to A'$ preserving the coactions, i.e.,

$$(f \otimes \mathrm{id}_H) \circ \delta = \delta' \circ f$$

• **Reformulation:** Determine the set $\mathrm{CGal}_H(B)$ of isomorphism classes of central H-Galois extensions of B

This is a difficult problem in general

Universal classifying algebra

Let H be a Hopf algebra

• **Definition**. *A central H-Galois extension $\mathcal{A}_H$ of $\mathcal{B}_H$ is called (uni)versal if for any commutative algebra B, the map*

$$\begin{aligned} \mathrm{Alg}(\mathcal{B}_H, B) &\longrightarrow \mathrm{CGal}_H(B) \\ f &\longmapsto f_* \mathcal{A}_H = B \otimes_{\mathcal{B}_H} \mathcal{A}_H \end{aligned}$$

is surjective (bijective)

• Universal central H-Galois extensions are unique up to isomorphism, but may not exist

• We shall construct versal Galois extensions for twisted algebras

Doi's classification of twisted algebras

- **Lemma (Doi)**. *Two twisted algebras ${}^{\alpha}H$ and ${}^{\beta}H$ are isomorphic as H-comodule algebras if and only if α and β are cohomologous in the following sense*

- **Definition.** *Two-cocycles α and β of H are cohomologous, $\alpha \sim \beta$, if there is an invertible linear form $\lambda : H \to k$ with inverse $\lambda^{-1} : H \to k$ such that for all $x, y \in H$,*

$$\beta(x,y) = \sum_{(x),(y)} \lambda(x_{(1)})\,\lambda(y_{(1)})\,\alpha(x_{(2)}, y_{(2)})\,\lambda^{-1}(x_{(3)}y_{(3)})$$

- A linear form $\lambda : H \to k$ is invertible with inverse $\lambda^{-1} : H \to k$ if

$$\sum_{(x)} \lambda(x_{(1)})\,\lambda^{-1}(x_{(2)}) = \sum_{(x)} \lambda^{-1}(x_{(1)})\,\lambda(x_{(2)}) = \varepsilon(x)\,1$$

for all $x \in H$ (An invertible linear form has a unique inverse)

Classifying twisted algebras

- By Doi's lemma, there is a bijection

$$\{\text{twisted algebras}\}/(\text{comodule algebra isomorphisms}) \cong \{\text{two-cocycles of } H\}/\sim$$

- In general, the set $\{\text{two-cocycles of } H\}/\sim$ is not a group

- Only in special cases is $\{\text{two-cocycles of } H\}/\sim$ a group, e.g., if H is cocommutative:

$$\sum_{(x)} x_{(1)} \otimes x_{(2)} = \sum_{(x)} x_{(2)} \otimes x_{(1)}$$

The group Hopf algebra $k[G]$ is cocommutative and

$$\{\text{two-cocycles of } k[G]\}/\sim \cong H^2(G, k^\times)$$

Part Two

THE GENERIC COCYCLE

(All results in the sequel are joint with Eli Aljadeff)

Towards a generic cocycle

• **Objective.** Let H be a Hopf algebra and α a normalized invertible two-cocycle. We want to define a generic two-cocycle on H that is cohomologous to α.

• To this end, we "emulate" the equations

$$\beta(x,y) = \sum_{(x),(y)} \lambda(x_{(1)})\, \lambda(y_{(1)})\, \alpha(x_{(2)}, y_{(2)})\, \lambda^{-1}(x_{(3)}y_{(3)})$$

and

$$\sum_{(x)} \lambda(x_{(1)})\, \lambda^{-1}(x_{(2)}) = \sum_{(x)} \lambda^{-1}(x_{(1)})\, \lambda(x_{(2)}) = \varepsilon(x)\, 1$$

(expressing that α, β are cohomologous)

by replacing the scalars $\lambda(x), \lambda^{-1}(x)$ by commuting symbols t_x, t_x^{-1}

The symbols t_x

Let t_H be a copy of H with linear isomorphism $x \in H \mapsto t_x \in t_H$

Consider the symmetric algebra

$$\mathrm{Sym}(t_H) = \bigoplus_{r \geq 0} \mathrm{Sym}^r(t_H)$$

on the vector space t_H. If $\{x_i\}_{i \in I}$ is a basis of H, then

$$\mathrm{Sym}(t_H) \cong k[t_{x_i} \mid i \in I]$$

is the polynomial algebra in the (commuting) variables t_{x_i}

The symbols t_x^{-1}

- We also need variables t_x^{-1} to deal with the equations

$$\sum_{(x)} \lambda(x_{(1)})\, \lambda^{-1}(x_{(2)}) = \sum_{(x)} \lambda^{-1}(x_{(1)})\, \lambda(x_{(2)}) = \varepsilon(x)\, 1$$

- Let $\operatorname{Frac}\operatorname{Sym}(t_H)$ be the field of fractions of $\operatorname{Sym}(t_H)$

Lemma. *There is a unique linear map* $x \mapsto t_x^{-1}$ *from* H *to* $\operatorname{Frac}\operatorname{Sym}(t_H)$ *such that for all* $x \in H$,

$$\sum_{(x)} t_{x_{(1)}}\, t_{x_{(2)}}^{-1} = \sum_{(x)} t_{x_{(1)}}^{-1}\, t_{x_{(2)}} = \varepsilon(x)\, 1$$

Computing t_x^{-1}

- If x is grouplike, i.e., $\Delta(x) = x \otimes x$, then $\varepsilon(x) = 1$ and

$$t_x t_x^{-1} = 1\,, \quad \text{hence} \quad t_x^{-1} = \frac{1}{t_x}$$

- If x is skew-primitive, i.e., $\Delta(x) = g \otimes x + x \otimes h$ for some grouplike elements g, h, then $\varepsilon(x) = 0$ and

$$t_g t_x^{-1} + t_x t_h^{-1} = 0\,, \quad \text{hence} \quad t_x^{-1} = -\frac{t_x}{t_g t_h}$$

The generic two-cocycle cohomologous to α

• Let H be a Hopf algebra and $\alpha : H \times H \to k$ a normalized invertible two-cocycle

Define the generic two-cocycle $\sigma : H \times H \to \operatorname{Frac}\operatorname{Sym}(t_H)$ by

$$\sigma(x,y) = \sum_{(x),(y)} t_{x_{(1)}}\, t_{y_{(1)}}\, \alpha(x_{(2)}, y_{(2)})\, t^{-1}_{x_{(3)}y_{(3)}}$$

• The cocycle σ is cohomologous to α over the fraction field $\operatorname{Frac}\operatorname{Sym}(t_H)$

• In case $H = k[G]$ is a group algebra and $g, h \in G$,

$$\sigma(g,h) = \alpha(g,h)\,\frac{t_g\, t_h}{t_{gh}}$$

The generic base algebra

• Let H be a Hopf algebra and $\alpha : H \times H \to k$ a normalized invertible two-cocycle

• **Definition.** *Let $\mathcal{B}_H^\alpha$ be the subalgebra of* $\operatorname{Frac}\operatorname{Sym}(t_H)$ *generated by the values of the generic two-cocycle σ and of its inverse σ^{-1}*

We call $\mathcal{B}_H^\alpha$ the generic base algebra

• **Immediate properties**:

(a) $\mathcal{B}_H^\alpha$ is a domain

(b) Transcendence degree of $\operatorname{Frac}\mathcal{B}_H^\alpha \leq \dim H$

(c) $\mathcal{B}_H^\alpha$ is finitely generated if $\dim H < \infty$

Computation of $\mathcal{B}_H^\alpha$ - the infinite cyclic case

- Consider the Hopf algebra of the group of integers $H = k[x, x^{-1}]$ with

$$\Delta(x) = x \otimes x\,, \quad \varepsilon(x) = 1\,, \quad S(x) = x^{-1}$$

together with the trivial two-cocycle: $\alpha(x^m, x^n) = 1 \quad (m, n \in \mathbb{Z})$

- Here $\mathrm{Sym}(t_H) = k[t_m \,|\, m \in \mathbb{Z}]$. For the generic cocycle,

$$\sigma(x^m, x^n) = \frac{t_m t_n}{t_{m+n}} = \frac{y_m y_n}{y_{m+n}} \quad (m, n \in \mathbb{Z})$$

where $y_m = t_m / t_1^m$ (note that $y_1 = 1$ and $y_0 = t_0$)

- The algebra $\mathcal{B}_H^\alpha$ is the Laurent polynomial algebra

$$\mathcal{B}_H^\alpha = k[\, y_m^{\pm 1} \,|\, m \in \mathbb{Z} - \{1\}\,]$$

inside the Laurent polynomial algebra $k[t_m^{\pm 1} \,|\, m \in \mathbb{Z}]$

- Is $k[t_m^{\pm 1} \,|\, m \in \mathbb{Z}]$ much bigger than $\mathcal{B}_H^\alpha$? *Answer:*

$$k[t_m^{\pm 1} \,|\, m \in \mathbb{Z}] = \mathcal{B}_H^\alpha[t_1^{\pm 1}]$$

Computation of $\mathcal{B}_H^\alpha$ - the finite cyclic case

• For the Hopf algebra $H = k[x]/(x^N - 1)$ of the cyclic group of order $N \geq 2$ with trivial two-cocycle,

$$\mathrm{Sym}(t_H) = k[t_0, t_1, \ldots, t_{N-1}]$$

and the algebra $\mathcal{B}_H^\alpha$ is a Laurent polynomial algebra on N variables:

$$\mathcal{B}_H^\alpha = k[y_0^{\pm 1}, y_2^{\pm 1}, \ldots, y_N^{\pm 1}] \subset k[t_0^{\pm 1}, t_1^{\pm 1}, \ldots, t_{N-1}^{\pm 1}]$$

where $y_N = t_0/t_1^N$

• Here $k[t_0^{\pm 1}, t_1^{\pm 1}, \ldots, t_{N-1}^{\pm 1}]$ is an integral extension of $\mathcal{B}_H^\alpha$:

$$k[t_0^{\pm 1}, t_1^{\pm 1}, \ldots, t_{N-1}^{\pm 1}] = \mathcal{B}_H^\alpha[t_1]/(t_1^N - y_0/y_N)$$

Part Three

THE SWEEDLER ALGEBRA

The Sweedler algebra: definition

• The Sweedler algebra is the smallest noncommutative noncocommutative Hopf algebra

As an algebra,

$$H = k\langle x, y \mid x^2 = 1\,,\;\; xy + yx = 0\,,\;\; y^2 = 0\,\rangle$$

Its radical is the ideal (y) and $H/(y) \cong k[\mathbb{Z}/2]$

• **Hopf algebra structure:**

Coproduct: $\Delta(x) = x \otimes x, \quad \Delta(y) = 1 \otimes y + y \otimes x$

Coünit: $\varepsilon(x) = 1, \quad \varepsilon(y) = 0$

Antipode: $S(x) = x, \quad S(y) = xy$

• The algebra H is four-dimensional with basis $\{1, x, y, z\}$, where $z = xy$

The Sweedler algebra: the variables t_x and t_x^{-1}

We need the variables t_1, t_x, t_y, t_z and $t_1^{-1}, t_x^{-1}, t_y^{-1}, t_z^{-1}$.

They satisfy the equations

$$t_1\, t_1^{-1} = 1\,, \qquad t_x\, t_x^{-1} = 1\,,$$

$$t_1\, t_y^{-1} + t_y\, t_x^{-1} = 0\,, \qquad t_x\, t_z^{-1} + t_z\, t_1^{-1} = 0$$

Hence,

$$t_1^{-1} = \frac{1}{t_1}\,, \quad t_x^{-1} = \frac{1}{t_x}\,, \quad t_y^{-1} = -\frac{t_y}{t_1 t_x}\,, \quad t_z^{-1} = -\frac{t_z}{t_1 t_x}$$

Twisted Sweedler algebras

- **(Masuoka)** For any normalized invertible two-cocycle α, the twisted algebra ${}^{\alpha}H$ is up to isomorphism of the form

$$ {}^{\alpha}H = k\langle u_x, u_y \,|\, u_x^2 = a\,,\;\; u_x u_y + u_y u_x = b\,,\;\; u_y^2 = c\rangle $$

for some scalars a, b, c with $a \neq 0$ $\quad$ (u_1 is the unit of ${}^{\alpha}H$)

- *Recall:* The generic two cocycle σ is given by

$$ \sigma(x, y) = \sum_{(x),(y)} t_{x_{(1)}}\, t_{y_{(1)}}\, \alpha(x_{(2)}, y_{(2)})\, t^{-1}_{x_{(3)}y_{(3)}} $$

Since $\dim H = 4$, we have to compute the 32 values of σ and σ^{-1} on the basis

Values of the generic cocycle

Values of σ :

$$
\begin{aligned}
\sigma(1,1) &= t_1 \,, \\
\sigma(x,x) &= \frac{a\, t_x^2}{t_1} \,, \\
\sigma(y,y) &= \frac{a\, t_y^2 + b\, t_1 t_y + c\, t_1^2}{t_1} \,, \\
\sigma(x,y) &= \frac{a\, t_x t_y - t_1 t_z}{t_1} \,, \\
\sigma(y,x) &= \frac{b\, t_1 t_x + a\, t_x t_y + t_1 t_z}{t_1} \,, \\
\sigma(z,z) &= -\frac{t_z^2 + b\, t_x t_z + ac\, t_x^2}{t_1}
\end{aligned}
$$

These rational fractions together with the values of σ^{-1} generate the algebra $\mathcal{B}_H^\alpha$

Generators of $\mathcal{B}_H^\alpha$ in the Sweedler case

If we set

$$E = t_1 \,, \quad R = a\, t_x^2 \,, \quad S = a\, t_y^2 + b\, t_1 t_y + c\, t_1^2 \,,$$
$$T = t_x\,(2a\, t_y + b\, t_1)\,, \quad U = a\, t_x^2\,(2\, t_z + b\, t_x)\,,$$

then

$$\begin{aligned} \sigma(1,1) &= E\,, & \sigma(x,y) &= \frac{RT - EU}{2ER}\,, \\ \sigma(x,x) &= \frac{R}{E}\,, & \sigma(y,x) &= \frac{RT + EU}{2ER}\,, \\ \sigma(y,y) &= \frac{S}{E}\,, & \sigma(z,z) &= \frac{a\,U^2 - (b^2 - 4ac)\,R^3}{4a\,ER^2} \end{aligned}$$

Consequence. *The elements E, E^{-1}, R, R^{-1}, S, T, U belong to $\mathcal{B}_H^\alpha$ and generate it as an algebra. Moreover, E, R, S, U are algebraically independent.*

The generic base algebra in the Sweedler case

Theorem. *Presentation of $\mathcal{B}_H^\alpha$ by generators and relations:*

$$\mathcal{B}_H^\alpha \cong k[E^{\pm 1}, R^{\pm 1}, S, T, U]/(P_{a,b,c}),$$

where

$$P_{a,b,c} = T^2 - 4RS - \frac{b^2 - 4ac}{a} E^2 R$$

Part Four

THE GENERIC GALOIS EXTENSION

The generic Galois extension

- Since the generic cocycle takes values in $\mathcal{B}_H^\alpha$, we can consider the twisted algebra

$$\mathcal{A}_H^\alpha = \mathcal{B}_H^\alpha \otimes {}^\sigma H$$

As a vector space, $\mathcal{A}_H^\alpha = \mathcal{B}_H^\alpha \otimes H$; it is equipped with the product

$$(a \otimes u_x)\,(b \otimes u_y) = \sum_{(x),(y)} a\,b\,\sigma(x_{(1)}, y_{(1)}) \otimes u_{x_{(2)}\,y_{(2)}}$$

$(a, b \in \mathcal{B}_H^\alpha,\ x, y \in H)$

The twisted algebra $\mathcal{A}_H^\alpha$ is called the generic Galois extension

- **Proposition.** *(a) The map $\delta = \mathrm{id}_{\mathcal{B}_H^\alpha} \otimes \Delta$ is a coaction, turning $\mathcal{A}_H^\alpha$ into an H-comodule algebra whose subalgebra of coinvariants is $\mathcal{B}_H^\alpha$:*

$$\mathcal{B}_H^\alpha = \left\{ a \in \mathcal{A}_H^\alpha \,|\, \delta(a) = a \otimes 1_H \right\}$$

(b) $\mathcal{B}_H^\alpha$ is a central subalgebra of $\mathcal{A}_H^\alpha$

Thus $\mathcal{A}_H^\alpha$ is a central H-Galois extension of $\mathcal{B}_H^\alpha$

The generic Galois extension

- Since the generic cocycle takes values in $\mathcal{B}_H^\alpha$, we can consider the twisted algebra
$$\mathcal{A}_H^\alpha = \mathcal{B}_H^\alpha \otimes {}^\sigma H$$
As a vector space, $\mathcal{A}_H^\alpha = \mathcal{B}_H^\alpha \otimes H$; it is equipped with the product
$$(a \otimes u_x)\,(b \otimes u_y) = \sum_{(x),(y)} a\, b\, \sigma(x_{(1)}, y_{(1)}) \otimes u_{x_{(2)}\, y_{(2)}}$$
$(a, b \in \mathcal{B}_H^\alpha, x, y \in H)$

  The twisted algebra $\mathcal{A}_H^\alpha$ is called the generic Galois extension

- **Proposition.** *(a) The map $\delta = \mathrm{id}_{\mathcal{B}_H^\alpha} \otimes \Delta$ is a coaction, turning $\mathcal{A}_H^\alpha$ into an H-comodule algebra whose subalgebra of coinvariants is $\mathcal{B}_H^\alpha$:*
$$\mathcal{B}_H^\alpha = \left\{ a \in \mathcal{A}_H^\alpha \,|\, \delta(a) = a \otimes 1_H \right\}$$
(b) $\mathcal{B}_H^\alpha$ is a central subalgebra of $\mathcal{A}_H^\alpha$

  Thus $\mathcal{A}_H^\alpha$ is a central H-Galois extension of $\mathcal{B}_H^\alpha$

A rigidity propery

- $\mathcal{A}_H^\alpha$ is a flat deformation of ${}^\alpha H$ over the commutative algebra $\mathcal{B}_H^\alpha$:

  Proposition. *There is a comodule algebra isomorphism*

  $$\mathcal{A}_H^\alpha / \mathfrak{m}_0 \mathcal{A}_H^\alpha \cong {}^\alpha H$$

  for some maximal ideal $\mathfrak{m}_0$ *of* $\mathcal{B}_H^\alpha$

  The ideal $\mathfrak{m}_0$ is the kernel of an algebra morphism $\chi_0 : \mathcal{B}_H^\alpha \to k$ sending each element $\sigma(x, y)$ to $\alpha(x, y)$

- If ${}^\alpha H$ is (semi)simple, then $\mathcal{A}_H^\alpha$ is generically (semi)simple. More precisely

  Theorem. *Assume that* $\mathrm{char}(k) = 0$ *and* $\dim(H) < \infty$.
  If ${}^\alpha H$ *is (semi)simple, then so is*

  $$\mathrm{Frac}\, \mathcal{B}_H^\alpha \otimes_{\mathcal{B}_H^\alpha} \mathcal{A}_H^\alpha = \mathrm{Frac}\, \mathcal{B}_H^\alpha \otimes {}^\sigma H$$

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Forms

• Let $\beta : H \times H \to K \supset k$ be a normalized invertible two-cocycle. The twisted algebra $K \otimes {}^{\beta}H$ is a K-form of ${}^{\alpha}H$ if there is a field $L \supset K$ and an L-linear isomorphism of H-comodule algebras

$$L \otimes_K (K \otimes {}^{\beta}H) \cong L \otimes_k {}^{\alpha}H \, .$$

• **Theorem.** *If $K \otimes {}^{\beta}H$ is a K-form of ${}^{\alpha}H$, then there is an algebra morphism $\chi : \mathcal{B}_H^{\alpha} \to K$ such that*

$$K_{\chi} \otimes_{\mathcal{B}_H^{\alpha}} \mathcal{A}_H^{\alpha} \cong {}^{\beta}H$$

In other words, any form of ${}^{\alpha}H$ is obtained from the generic Galois extension $\mathcal{A}_H^{\alpha}$ by a central specialization

There is a converse to the previous theorem; it requires an additional condition

Azumaya

- **Theorem** *If the algebra* $\mathrm{Frac\,Sym}(t_H)$ *is integral over the subalgebra* $\mathcal{B}_H^\alpha$*, then for any algebra morphism* $\chi : \mathcal{B}_H^\alpha \to K \supset k$*, the algebra* $K_\chi \otimes_{\mathcal{B}_H^\alpha} \mathcal{A}_H^\alpha$ *is a* K*-form of* ${}^\alpha H$

- **Theorem.** *Under the previous integrality condition, if the algebra* ${}^\alpha H$ *is simple, then the generic extension* $\mathcal{A}_H^\alpha$ *is an Azumaya algebra with center* $\mathcal{B}_H^\alpha$

An algebra A is Azumaya if $A/\mathfrak{m}$ is simple for any maximal ideal $\mathfrak{m}$ of its center. E.g. $A = M_n(R)$, where R is a commutative ring

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Versal deformation space

- If $\mathrm{Frac}\,\mathrm{Sym}(t_H)$ is integral over $\mathcal{B}_H^\alpha$, then the map

$$\begin{array}{rcl} \mathrm{Alg}(\mathcal{B}_H^\alpha, K) & \longrightarrow & K\text{-}\mathrm{Forms}({}^\alpha H) \\ \chi & \mapsto & K_\chi \otimes_{\mathcal{B}_H^\alpha} \mathcal{A}_H^\alpha \end{array}$$

is a surjection from the set of algebra morphisms $\mathcal{B}_H^\alpha \to K$ to the set of isomorphism classes of K-forms of ${}^\alpha H$

- Thus the set $\mathrm{Alg}(\mathcal{B}_H^\alpha, K)$ parametrizes the K-forms of ${}^\alpha H$.

The extension $\mathcal{B}_H^\alpha \subset \mathcal{A}_H^\alpha$ is a versal deformation space for the forms of ${}^\alpha H$

- *Remark.* To determine the set $\mathrm{Alg}(\mathcal{B}_H^\alpha, K)$, it is important to find a presentation by generators and relations of $\mathcal{B}_H^\alpha$

The integrality condition

- **Question.** *Under which condition on* (H, α) *is the algebra* $\mathrm{Frac\,Sym}(t_H)$ *integral over the subalgebra* $\mathcal{B}_H^\alpha$*?*

- **Proposition.** *If* H *is a finite-dimensional Hopf algebra that is generated by grouplike and skew-primitive elements, and* α *is any two-cocycle, then* $\mathrm{Frac\,Sym}(t_H)$ *is integral over the subalgebra* $\mathcal{B}_H^\alpha$

Proof in the case H *is the algebra of a finite group* G. It suffices to show that each t_g $(g \in G)$ is integral over $\mathcal{B}_H^\alpha$. Since

$$\sigma(g,h) = \alpha(g,h)\,\frac{t_g\,t_h}{t_{gh}} \in \mathcal{B}_H^\alpha\,,$$

we have $t_g\,t_h = b\,t_{gh}$ for some $b \in \mathcal{B}_H^\alpha$. Consequently for all $n \geq 2$,

$$t_g^n = b'\,t_{g^n}$$

for some $b' \in \mathcal{B}_H^\alpha$. If n is the order of g, then since $\sigma(1,1) = t_1$,

$$t_g^n = b'\,t_{g^n} = b'\,t_1 = b'\,\sigma(1,1)\,,$$

which shows that t_g^n belongs to $\mathcal{B}_H^\alpha$. QED

The generic Galois extension in the Sweedler case

Theorem *The generic Galois extension $\mathcal{A}_H^\alpha$ is given by*

$$\mathcal{A}_H^\alpha \cong \mathcal{B}_H^\alpha \langle X, Y \rangle / (X^2 - R\,,\; XY + YX - T\,,\; Y^2 - S)$$

Compare with

$$^\alpha H = k\langle u_x, u_y \rangle / (u_x^2 - a\,,\; u_x u_y + u_y u_x - b\,,\; u_y^2 - c)$$

Recall:

$$\mathcal{B}_H^\alpha \cong k[E^{\pm 1}, R^{\pm 1}, S, T, U]/(P_{a,b,c})\,,$$

where

$$P_{a,b,c} = T^2 - 4RS - \frac{b^2 - 4ac}{a} E^2 R$$

Computation of $\mathcal{A}_H^\alpha$ for a group algebra

(By D. Haile and M. Natapov)

- Let $G = \langle g, h \mid g^9 = h^9 = 1,\ gh = h^4 g\rangle = \mathbb{Z}/9 \rtimes \mathbb{Z}/9$
- Let $H = k[G]$ and set $X = X_g$ and $Y = X_h$. Then

$$\mathcal{B}_H^\alpha \cong k[(X^9)^{\pm 1}, (Y^9)^{\pm 1}, Z]/(Z^3 - \omega\,(X^9)^3\,(Y^9)^2)\,,$$

  where $Z = XYX^8Y^5$ and ω is a primitive third root of 1
- The generic Galois extension: $\mathcal{A}_H^\alpha = \mathcal{B}_H^\alpha\langle X, Y\rangle/I$, where I is the two-sided ideal generated by

$$X^3Y - \omega\,YX^3,\quad Y^3X - \omega^2\,XY^3,$$

$$XYXY - \omega^2\,Y^2X^2,\quad YXYX - X^2Y^2,\quad XY^2X - \omega^2\,YX^2Y$$

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Part Five

POLYNOMIAL IDENTITIES

How to find generators for $\mathcal{B}_H^\alpha$

- In the Sweedler case, how did we find the generators

$$E = t_1\,, \quad R = a\,t_x^2\,, \quad S = a\,t_y^2 + b\,t_1 t_y + c\,t_1^2\,,$$

$$T = t_x\,(2a\,t_y + b\,t_1)\,, \quad U = a\,t_x^2\,(2\,t_z + b\,t_x)\,,$$

of $\mathcal{B}_H^\alpha$?

- To explain this we need a new set of symbols

The symbols X_x and the tensor algebra

- We now assume that the ground field k is infinite. Let H be a Hopf algebra

- Let X_H be a copy of H with a linear isomorphism $x \mapsto X_x$

- Consider the tensor algebra

$$T(X_H) = \bigoplus_{r \geq 0} T^r(X_H)$$

on the vector space X_H. If $\{x_i\}_{i \in I}$ is a basis of H, then

$$T(X_H) \cong k\langle X_{x_i} \,|\, i \in I \rangle$$

is the algebra of noncommutative polynomials in X_{x_i} $(i \in I)$

- The algebra $T(X_H)$ is an H-comodule algebra with coaction $\delta : T(X_H) \to T(X_H) \otimes H$ given by

$$\delta(X_x) = \sum_{(x)} X_{x_{(1)}} \otimes x_{(2)}$$

The universal evaluation map

• Let H be a Hopf algebra, $\alpha : H \times H \to k$ a normalized invertible two-cocycle, and ${}^{\alpha}H$ the corresponding twisted algebra

• Consider the algebra morphism

$$\begin{aligned} \mu_\alpha : T(X_H) &\longrightarrow \mathrm{Sym}(t_H) \otimes {}^{\alpha}H \\ X_x &\longmapsto \sum_{(x)} t_{x_{(1)}} \otimes u_{x_{(2)}} \end{aligned}$$

We call μ_α the universal evaluation map for ${}^{\alpha}H$

Properties of the map μ_α

Lemma. *(a) The map* $\mu_\alpha : T(X_H) \longrightarrow \mathrm{Sym}(t_H) \otimes {}^\alpha H$ *is an* H*-comodule algebra morphism*

(b) For every H*-comodule algebra morphism* $\mu : T(X_H) \longrightarrow {}^\alpha H$*, there is a unique algebra morphism* $\chi : \mathrm{Sym}(t_H) \longrightarrow k$ *such that*

$$\mu = (\chi \otimes \mathrm{id}) \circ \mu_\alpha$$

In other words, any comodule algebra morphism $T(X_H) \longrightarrow {}^\alpha H$ is a specialization of μ_α, and $\mathrm{Sym}(t_H)$ parametrizes the set of such comodule algebra morphisms

Constructing elements of $\mathcal{B}_H^\alpha$ using μ_α

- **Proposition.** *If $P \in T(X_H)$ is coinvariant, i. e., $\delta(P) = P \otimes 1_H$, then $\mu_\alpha(P)$ belongs to $\mathcal{B}_H^\alpha$*

- The following "universal" formulas provide coinvariant elements of $T(X_H)$ (where $x, y \in H$):

$$P_x = \sum_{(x)} X_{x_{(1)}} X_{S(x_{(2)})}$$

$$P_{x,y} = \sum_{(x),(y)} X_{x_{(1)}} X_{y_{(1)}} X_{S(x_{(2)} y_{(2)})}$$

- **Example:** For the Sweedler algebra:

$$R = \mu_\alpha(P_x)\,, \qquad T = \mu_\alpha(P_{y-z})\,,$$

$$U = \mu_\alpha(P_{x,z})\,, \qquad ES = \mu_\alpha(P_{y,y})$$

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A theory of identities for comodule algebras

- Let A be an H-comodule algebra for some Hopf algebra H

Definition. *An element $P \in T(X_H)$ is an H-identity of A if $\mu(P) = 0$ for all comodule algebra morphisms $\mu : T(X_H) \to A$*

- Let $I_H(A)$ be the vector space of all H-identities of A

Proposition. *(a) $I_H(A)$ is a two-sided ideal of $T(X_H)$ such that*

$$\delta(I_H(A)) \subset I_H(A) \otimes H$$

(b) The ideal $I_H(A)$ is graded and

$$I_H(A) \subset \bigoplus_{r \geq 2} T^r(X_H)$$

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The universal comodule algebra

- **Definition.** *The universal comodule algebra of identities of the H-comodule algebra A is the quotient-algebra*

$$\mathcal{U}_H(A) = T(X_H)/I_H(A)$$

- **Properties.**

  (a) $\mathcal{U}_H(A)$ is a graded algebra coinciding with $T(X_H)$ in degrees 0 and 1

  (b) $\mathcal{U}_H(A)$ is an H-comodule algebra

  (c) All H-identities of A vanish in $\mathcal{U}_H(A)$

Detecting the identities for twisted algebras

- Let ${}^{\alpha}H$ be a twisted algebra for some Hopf algebra H and some normalized invertible two-cocycle α

  Recall the universal evaluation map

$$\mu_\alpha : T(X_H) \to \mathrm{Sym}(t_H) \otimes {}^{\alpha}H$$

- **Theorem.** *We have* $I_H({}^{\alpha}H) = \mathrm{Ker}\ \mu_\alpha$

  In other words, the map μ_α detects the H-identities of ${}^{\alpha}H$

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Embedding $\mathcal{U}_H^\alpha$ into a controllable algebra

Consequences of the previous theorem: Set $\mathcal{U}_H^\alpha = T(X_H)/I_H({}^\alpha H)$

- Since Ker $\mu_\alpha = \text{Ker}(T(X_H) \to \mathcal{U}_H^\alpha)$, the map μ_α induces an embedding

$$\mathcal{U}_H^\alpha \hookrightarrow \text{Sym}(t_H) \otimes {}^\alpha H$$

  of the universal comodule algebra into a twisted product

- $u \in \mathcal{U}_H^\alpha$ is coinvariant if and only if $\mu_\alpha(u)$ belongs to $\text{Sym}(t_H) \otimes 1$
- $u \in \mathcal{U}_H^\alpha$ is central if and only if $\mu_\alpha(u)$ belongs to $\text{Sym}(t_H) \otimes Z({}^\alpha H)$, where $Z({}^\alpha H)$ is the center of ${}^\alpha H$
- The center $\mathcal{Z}_H^\alpha$ of $\mathcal{U}_H^\alpha$ is a domain if $Z({}^\alpha H)$ is a domain

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Consequences of the previous theorem: Set $\mathcal{U}_H^\alpha = T(X_H)/I_H({}^\alpha H)$

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- The center $\mathcal{Z}_H^\alpha$ of $\mathcal{U}_H^\alpha$ is a domain if $Z({}^\alpha H)$ is a domain

Identities in the Sweedler case

Consider the Sweedler algebra H and the twisted algebra

$$^{\alpha}H = k\langle u_x, u_y \mid u_x^2 = a\,,\ \ u_x u_y + u_y u_x = b\,,\ \ u_y^2 = c\rangle$$

The following are examples of H-identities for $^{\alpha}H$:

$$(X_x X_y + X_y X_x)^2 - 4X_x^2 X_y^2 - \frac{b^2 - 4ac}{a} X_1^2 X_x^2$$

$$2X_x^2(X_1 X_z - X_x X_y) - X_1 X_x (X_x X_z + X_z X_x) + X_x^2 (X_x X_y + X_y X_x)$$

(Check using the universal evaluation map)

Relating $\mathcal{U}_H^\alpha$ and $\mathcal{A}_H^\alpha$

- To (H, α) we associated

  (a) the generic Galois extension

$$\mathcal{A}_H^\alpha = \mathcal{B}_H^\alpha \otimes {}^\sigma H$$

built from the generic cocycle σ

  (b) the universal comodule algebra

$$\mathcal{U}_H^\alpha = T(X_H)/I_H({}^\alpha H)$$

built out of the H-identities of ${}^\alpha H$; its center is $\mathcal{Z}_H^\alpha$

- **Theorem.** *(a) There is an embedding of H-comodule algebras*

$$\mathcal{U}_H^\alpha \hookrightarrow \mathcal{A}_H^\alpha$$

  (b) If $Z({}^\alpha H) = k$, then $\mathcal{Z}_H^\alpha \hookrightarrow \mathcal{B}_H^\alpha$

Structure of $\mathcal{U}_H^\alpha$ after central localization

- We assume that $Z({}^\alpha H) = k$, so that μ_α embeds $\mathcal{Z}_H^\alpha$ into $\mathcal{B}_H^\alpha$

Theorem. *If in addition $\mathcal{B}_H^\alpha$ is a localization of $\mathcal{Z}_H^\alpha$, then there is an isomorphism of H-comodule algebras*

$$\mathcal{B}_H^\alpha \otimes_{\mathcal{Z}_H^\alpha} \mathcal{U}_H^\alpha \cong \mathcal{A}_H^\alpha$$

In other words, after localization of the center, the universal comodule algebra $\mathcal{U}_H^\alpha$ becomes the generic Galois extension $\mathcal{A}_H^\alpha$

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THANK YOU FOR YOUR ATTENTION

Strongly graded algebras

- Given a group G, consider a strongly G-graded algebra:

$$A = \bigoplus_{g \in G} A_g$$

with $A_g\, A_h = A_{gh}$ and $\dim A_g = 1$

- Let u_g be a spanning vector of A_g. Then

$$u_g\, u_h = \alpha(g,h)\, u_{gh} \tag{3}$$

for some $\alpha(g,h) \in k^\times = k - \{0\}$

- The associativity of the product of A implies that the map $\alpha : G \times G \to k^\times$ is a group-cocycle:

$$\alpha(g,h)\,\alpha(gh,k) = \alpha(h,k)\,\alpha(g,hk) \tag{4}$$

Isomorphism classes and group cohomology

- If $v_g = \lambda(g)\, u_g$ with $\lambda(g) \neq 0$, then

$$v_g\, v_h = \beta(g,h)\, v_{gh}\ ,$$

where

$$\beta(g,h) = \alpha(g,h)\, \frac{\lambda(g)\,\lambda(h)}{\lambda(gh)} \tag{5}$$

- Call α, β cohomologous, $\alpha \sim \beta$, if they are related by (5)

- The set $\{$group-cocycles of $G\}/\!\sim$ is the cohomology group $H^2(G, k^\times)$

- Isomorphism classes of strongly graded algebras:

$$\{\text{strongly}\, G\text{-graded algebras}\}/(\text{isomorphisms}) \cong H^2(G, k^\times)$$