Polynomial identities

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Overview

• We give a short introduction to the classical theory of polynomial identities. . .

• . . . followed by a description of some recent developments involving group gradings

• Our main objects of study are matrix algebras

The classical theory of polynomial identities

Definition of polynomial identities

• We fix a field k of characteristic zero (e.g., $k = \mathbb{C}$)

All algebras whe shall consider are associative unital k-algebras

• **Definition.** *Let A be an algebra. A polynomial identity (PI) for A is a non-zero polynomial $f(X_1, \ldots, X_n)$ in a finite number of non-commuting variables $X_1, \ldots, X_n$ with coefficients in k such that*

$$f(a_1, \ldots, a_n) = 0$$

for all $a_1, \ldots, a_n \in A$.

• We shall only consider homogeneous PIs

(Recall that a polynomial is homogeneous if it is a linear combination of monomials all having the same degree in each variable)

There is no loss of generality in considering only homogeneous PIs since each homogeneous summand of a PI is a PI

PI-algebras

- An algebra for which there is a PI is called a PI-algebra
- Any commutative algebra A is a PI-algebra since

$$f(X, Y) = [X, Y] \stackrel{\text{def}}{=} XY - YX$$

is a PI for A.

- As we shall see, all finite-dimensional algebras are PI-algebras

In particular, all matrix algebras are PI-algebras

- **Not all algebras are PI-algebras**:

E.g., free algebras $k\langle X_1, \ldots, X_n\rangle$ with $n \geq 2$ are not PI-algebras

A polynomial identity for 2×2-matrices

- **A degree 5 polynomial identity:** The polynomial

$$\begin{aligned} f(X,Y,Z) &= [[X,Y]^2, Z] \\ &= XYXYZ - XY^2XZ - YX^2YZ + YXYXZ \\ &\quad - ZXYXY + ZXY^2X + ZYX^2Y - ZYXYX \end{aligned}$$

  is a PI for the algebra $M_2(k)$ of 2×2-matrices with entries in k.

  Consequently, $M_2(k)$ is a PI-algebra

- *Proof.* By Cayley-Hamilton any 2×2-matrix M satisfies the identity

$$M^2 = \mathrm{tr}(M)\, M - \det(M)\, I$$

  If $M = [X,Y]$, then $\mathrm{tr}(M) = 0$. Hence its square

$$M^2 = -\det(M)\, I$$

  is a scalar matrix, which then commutes with any matrix Z. $\square$

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Subalgebras of PI-algebras are PI-algebras

Any subalgebra of a PI-algebra is clearly a PI-algebra

Let us draw two simple interesting consequences:

- **Quaternions form a PI-algebra:** This follows from the well-known fact that the algebra $\mathbb{H}$ of quaternions is a subalgebra of $M_2(\mathbb{C})$
- **Non-linearity of free algebras:** Since $M_2(K)$ is a PI-algebra for any field extension K of k and a non-commutative free algebra is not a PI-algebra, one cannot embed $k\langle X_1, \ldots, X_n\rangle$ (with $n \geq 2$) as a subalgebra into a matrix algebra of the form $M_2(K)$

  Remark. By contrast, any free group F_n can be embedded as a subgroup into $SL_2(\mathbb{Z})$, hence into the general linear group $GL_2(\mathbb{Q})$

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The matrix algebra $M_3(k)$ is a PI-algebra

We construct a PI for the algebra $M_3(k)$ of 3×3-matrices as follows:

By Cayley-Hamilton any matrix $X \in M_3(k)$satisfies a relation of the form

$$X^3 = \lambda_2 X^2 + \lambda_1 X + \lambda_0 I$$

where $\lambda_0, \lambda_1, \lambda_2$ are scalars (depending on X). Taking the commutator with another matrix Y, we obtain

$$[X^3, Y] = \lambda_2 [X^2, Y] + \lambda_1 [X, Y]$$

Now we take the commutator with $[X, Y]$, thus obtaining

$$[[X^3, Y], [X, Y]] = \lambda_2 [[X^2, Y], [X, Y]]$$

Finally, taking the commutator with $[[X^2, Y], [X, Y]]$, we obtain for $M_3(k)$ the degree 11 polynomial identity

$$[[[X^3, Y], [X, Y]], [[X^2, Y], [X, Y]]]$$

All matrix algebras are PI-algebras

• Proceeding in a similar fashion, we obtain a polynomial identity for any algebra for which there is an integer $n \geq 1$ such that any element $a \in A$ satisfies an identity of the form

$$a^n + \lambda_{n-1} a^{n-1} + \cdots + \lambda_1 a + \lambda_0 = 0$$

where $\lambda_0, \lambda_1, \ldots, \lambda_{n-1}$ are scalars (depending on a)

• In this way we obtain for the algebra $M_n(k)$ of $n \times n$-matrices a polynomial identity of degree $2^{n+1} - 2^{n-1} - 1$

• Are there polynomial identities for $M_n(k)$ of (much) smaller degree?

For an answer, see the next slides

Standard polynomials

- The standard polynomial S_d of degree d is given by

$$S_d(X_1, \ldots, X_d) = \sum_{\sigma \in \mathrm{Sym}_d} \varepsilon(\sigma)\, X_{\sigma(1)} X_{\sigma(2)} \cdots X_{\sigma(d)}$$

Clearly, $S_d(a_1, \ldots, a_d) = 0$ if $a_i = a_j$ for some $i \neq j$

Proposition. *If* $\dim(A) < d$*, then* $S_d(X_1, \ldots, X_d)$ *is a PI for* A

- *Proof.* Let $(e_i)_i$ be a basis of A. Then for any $a_1, \ldots, a_d \in A$ the element $S_d(a_1, \ldots, a_d)$ is a linear combination of elements of the form $S_d(e_{i_1}, \ldots, e_{i_d})$. Each of the latter must be zero since for dimension reasons there must be a repetition in each sequence $e_{i_1}, \ldots, e_{i_d}$. □

- Consequently, any finite-dimensional algebra is a PI-algebra

• It follows from the above proposition that there is a PI of degree $n^2 + 1$ for the matrix algebra $M_n(k)$

• We can still do better…

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The Amitsur-Levitzki theorem

• The following deep result is due to Amitsur and Levitzki (1950)

Theorem. *The standard polynomial $S_{2n}(X_1, \ldots, X_{2n})$ of degree $2n$ is a polynomial identity for the matrix algebra $M_n(k)$*

• The degree $2n$ is optimal because there are no PIs of degree $< 2n$ for $M_n(k)$

• The proof of the theorem is difficult, but the latter assertion is easy to prove

So let us give a proof. . . (see next slide)

Multilinear polynomials

• A homogeneous polynomial is called multilinear if it is a linear combination of monomials all of degree one in each variable

• A multilinear polynomial of degree d is of the form

$$\sum_{\sigma \in \mathrm{Sym}_d} \lambda_\sigma X_{\sigma(1)} X_{\sigma(2)} \cdots X_{\sigma(d)} \tag{1}$$

where λ_σ are scalars

• By a standard multilinearization procedure, one shows that if an algebra has a homogeneous PI of some degree, then it has a multilinear PI of the same degree

There are no PIs of degree $< 2n$ for $M_n(k)$

• Indeed, suppose that $M_n(k)$ has a PI of degree $d \leq 2n - 1$, then it would have a multilinear PI of the form (1)

• To prove that this is impossible, it suffices to exhibit d matrices $M_1, M_2, \ldots, M_d$ such that the product $M_1 M_2 \cdots M_d$ is non-zero and the product

$$M_{\sigma(1)} M_{\sigma(2)} \cdots M_{\sigma(d)}$$

of any other permutation σ of these matrices is zero

• Consider the elementary matrices $E_{i,j}$ with $1 \leq i, j \leq n$; they form a basis of $M_n(k)$ and they multiply as

$$E_{i,j}\, E_{m,n} = \delta_{j,m}\, E_{i,n}$$

The product of the $2n - 1$ matrices

$$E_{1,1}\,,\ E_{1,2}\,,\ E_{2,2}\,,\ E_{2,3}\,,\ E_{3,3}\,, \ldots, E_{n-1,n}\,,\ E_{n,n}$$

is non-zero whereas the product of any other permutation of this sequence of matrices is zero. This shows that $M_n(k)$ has no PI of degree $2n - 1$

• The same proof can be made to work for any degree $d < 2n - 1$

T-ideals

• Let A be a PI-algebra. The set $\mathrm{Id}(A)$ of polynomial identities for A (together with 0) forms a two-sided ideal of the free algebra $k\langle X_1, X_2, \ldots\rangle$

• A T-ideal is a two-sided ideal of $k\langle X_1, X_2, \ldots\rangle$ that is preserved under all substitution of variables, equivalently, under all algebra endomorphisms of $k\langle X_1, X_2, \ldots\rangle$

The ideal $\mathrm{Id}(A)$ of polynomial identities is a T-ideal

• **Problem.** Determine the ideal $\mathrm{Id}(A)$ for a given PI-algebra A

This is a difficult problem, solved only for a handful of algebras:

(a) If A is commutative, then $\mathrm{Id}(A)$ is generated by the degree 2 standard polynomial $S_2 = [X_1, X_2]$

(b) If $A = M_2(k)$, then $\mathrm{Id}(A)$ is generated by the degree 4 standard polynomial S_4 and the degree 5 polynomial $[[X, Y]^2, Z]$

The Specht problem

• **The problem.** Is $\mathrm{Id}(A)$ generated by a finite number of polynomial identities as a T-ideal?

• In 1987 Kemer gave a positive answer to the Specht problem for any PI-algebra (whether it is finitely generated or not)

The proof of Kemer's theorem is difficult and does not yield a bound on the number of generators or their degrees

• We now turn to a class of algebras with extra structure...

Graded polynomial identities

Graded algebras

Fix a finite group G

• An algebra A is G-graded if it has a decomposition into a family of vector subspaces indexed by the elements of the group

$$A = \bigoplus_{g \in G} A_g$$

and compatible with the product on A in the sense that for all $g, h \in G$,

$$A_g \cdot A_h \subset A_{gh}$$

The subspace A_g is called the homogeneous component of degree g

The component A_e corresponding to the unit e of G is a subalgebra of A

• $\mathbb{Z}/2$-graded algebras are also called superalgebras; they come up in physics

Weyl algebras and Clifford algebras are superalgebras

Elementary gradings of matrices

- Fix a group G
- Let us describe a special type of G-grading of $M_n(k)$ called an elementary grading (a description of all gradings of $M_n(k)$ will be given in the Appendix)
- An elementary grading of $M_n(k)$ is obtained for each sequence $g_1, \ldots, g_n$ of length n of elements of G: we grade $M_n(k)$ by specifying that each elementary matrix $E_{i,j}$ belongs to the homogeneous component of degree $g_i^{-1} g_j$
- **Example of an elementary $G = \mathbb{Z}/2$-grading.** We have $M_3(k) = A_0 \oplus A_1$ where A_0 is the 5-dimensional vector space consisting of the matrices

$$\begin{pmatrix} * & 0 & * \\ 0 & * & 0 \\ * & 0 & * \end{pmatrix}$$

and A_1 is the 4-dimensional vector space consisting of the matrices

$$\begin{pmatrix} 0 & * & 0 \\ * & 0 & * \\ 0 & * & 0 \end{pmatrix}$$

(This grading corresponds to the triple $(0, 1, 0) \in (\mathbb{Z}/2)^3$)

Graded polynomial identities

• To define polynomial identities for ordinary algebras, we used an arbitrary finite number of non-commuting variables

To define identities for G-graded algebras, we need a numerable set of non-commuting variables for each element of G

Let $\{X_1(g), X_2(g), \ldots\}$ be the set of variables corresponding to the element $g \in G$

• **Definition.** *Let A be a G-graded algebra. A graded polynomial identity (G-PI) for A is a non-zero polynomial in the variables $X_i(g)$ that vanishes in A whenever we replace each variable $X_i(g)$ by an element of the homogeneous component A_g of degree g*

Examples of graded polynomial identities

• It is easier to find G-PIs than ordinary PIs, as show the following stupid examples:

(a) If A is a G-graded algebra and f is an ordinary PI for the subalgebra A_e (where e is the unit of G), then f is a G-PI for A (Replace each variable X_i by the variable $X_i(e)$)

(b) Suppose that A is trivially graded, i.e., $A_g = 0$ for each $g \neq e$. Then $X(g)$ is a G-PI for each $g \neq e$

• **Non-trivial examples of graded polynomial identities**. Consider the elementary $\mathbb{Z}/2$-grading of $M_2(k)$ corresponding to the couple $(0,1) \in (\mathbb{Z}/2)^2$. We have $M_2(k) = A_0 \oplus A_1$, where

$$A_0 = \left\{ \begin{pmatrix} * & 0 \\ 0 & * \end{pmatrix} \right\} \quad \text{and} \quad A_1 = \left\{ \begin{pmatrix} 0 & * \\ * & 0 \end{pmatrix} \right\}$$

Check that $X_1(0)\, X_1(1) - X_1(1)\, X_1(0)$ and

$$X_1(0)\,\big(X_1(1)X_2(1) + X_2(1)X_1(1)\big) - \big(X_1(1)X_2(1) + X_2(1)X_1(1)\big)\, X_1(0)$$

are G-PIs for $M_2(k)$

The Specht problem in the graded case

- Let f be an algebra endomorphism of the free algebra on the variables $X_i(g)$ where $g \in G$ and $i = 1, 2, \ldots$

  We say that f is admissible if each $f(X_i(g))$ is a linear combination of monomials $X_{i_1}(g_1)X_{i_2}(g_2)\cdots X_{i_r}(g_r)$ such that $g_1 g_2 \cdots g_r = g$

  • A two-sided ideal of the free algebra on the variables $X_i(g)$ is a G-T-ideal if it is preserved under all admissible algebra endomorphisms

  • The set $\mathrm{Id}_G(A)$ of G-PIs for A (together with 0) is a G-T-ideal

- In 2008 Aljadeff and Kanel-Belov proved the following

  Theorem. *If A is a PI-algebra, then $\mathrm{Id}_G(A)$ is generated by a finite number of graded polynomial identities as a G-T-ideal*

  • This provides an example of a (hard) result for ordinary polynomial identities that extends to graded polynomial identities

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A result about nilpotency

We now give an example of a result on ordinary polynomial identities that does not extend to graded polynomial identities

Proposition. *If f^r is a PI for $M_n(\mathbb{C})$ for some $r \geq 2$, then f is a PI*

This means that if any evaluation of f is a nilpotent matrix, then this matrix is zero

Proof. It is based on the existence of a division ring D inside $M_n(\mathbb{C})$ such that there is an algebra isomorphism

$$M_n(\mathbb{C}) \cong D \otimes_K \mathbb{C}$$

where K is the center of D (for instance, $M_2(\mathbb{C}) \cong \mathbb{H} \otimes_{\mathbb{R}} \mathbb{C}$)

Now if $f(X_1, \ldots, X_d)^r \in \mathrm{Id}(M_n(\mathbb{C}))$, then $f(a_1, \ldots, a_d)^r = 0$ for all $a_1, \ldots, a_d \in M_n(\mathbb{C})$, hence for all $a_1, \ldots, a_d \in D$. Since a division ring has no non-zero nilpotent elements, $f(a_1, \ldots, a_d) = 0$ for all $a_1, \ldots, a_d \in D$. This shows that $f(X_1, \ldots, X_d)$ is a PI for D, hence for $M_n(\mathbb{C})$. □

In this proof we have used the fact that if an algebra A is obtained from an algebra B by an extension of scalars, then they have the same polynomial identities:

$$\mathrm{Id}(A) = \mathrm{Id}(B)$$

A different behavior in the graded case

By contrast with what happens in the ungraded case, there are examples of gradings on matrix algebras and of graded polynomials f such that f^r is a G-PI for some $r \geq 2$, but not f

Example. Let $G = \mathbb{Z}/6 \rtimes \mathrm{Sym}_3$, where the symmetric group Sym_3 acts on $\mathbb{Z}/6$ by the sign of permutations; this is a group of order 36 with the following presentation:

$$G = \langle s, t, z \,|\, s^3 = t^2 = z^6 = 1\,,\ ts = s^{-1}t\,,\ sz = zs\,,\ tz = z^{-1}t \rangle$$

There is a G-graded algebra structure on $M_6(\mathbb{C})$ (its homogeneous components are all one-dimensional) for which the following holds:

Aljadeff, Haile & Natapov: The polynomial

$$f = X(s)\,X(t)^2\,X(z^2)^3 - \omega\,X(t)\,X(z^2)\,X(t)\,X(z^2)^2\,X(s)$$

is not a G-PI for $M_6(\mathbb{C})$, but f^3 is a G-PI for $M_6(\mathbb{C})$

(here $\omega = e^{2\pi i/3}$ is a primitive third root of unity)

When Hopf algebras come in…

Towards more general polynomial identities

• It is possible to extend the theory of polynomial identities to a bigger class of algebras, namely to comodule algebras over Hopf algebras

Graded algebras are instances of comodule algebras

• Let H be a Hopf algebra; this is essentially an algebra together with an algebra map $\Delta : H \to H \otimes H$, called the coproduct, satisfying certain properties (coassociativity, counitality) dual to the properties satisfied by the product in an associative unital algebra

Hopf algebras are essential tools in the theory of quantum groups

• An H-comodule algebra is an algebra A together with an algebra map $\delta : A \to A \otimes H$, called the coaction, satisfying certain standard properties

Comodule algebras

• A G-graded algebra is the same as an H-comodule algebra, where $H = kG$ is the group algebra with coproduct given by

$$\Delta(g) = g \otimes g \qquad (g \in G)$$

• There is an important class of comodule algebras called Hopf Galois extensions

Classical Galois extensions of fields are instances of Hopf Galois extensions with H being the dual Hopf algebra of the algebra of the Galois group

Hopf Galois extensions can be viewed as analogues of principal fiber bundles in "non-commutative geometry"; now the rôle of the structural group is played by a Hopf algebra. There are numerous instances of such "quantum principal fiber bundles" in quantum group theory

Polynomial identities for comodule algebras

• To define a theory of polynomial identities for comodule algebras over a given Hopf algebra H, we fix a basis $(h_i)_i$ of H, and for each basis element h_i we pick a numerable set $\{X_1(h_i), X_2(h_i), \ldots\}$ of non-commuting variables

• Let T_H be the free non-commutative algebra generated by these variables. There is a structure of H-comodule algebra on T_H induced by the coproduct of H

Definition. *Let A be an H-comodule algebra. An H-polynomial identity for A is an element $f \in T_H$ such that $\mu(f) = 0$ for every algebra map $\mu : T_H \to A$ preserving the coactions, i.e., such that the following square commutes:*

$$\begin{array}{ccc} T_H & \xrightarrow{\mu} & A \\ \delta\downarrow & & \delta\downarrow \\ T_H \otimes H & \xrightarrow{\mu \otimes \mathrm{id}_H} & A \otimes H \end{array}$$

• An ordinary polynomial identity is an H-polynomial identity for the one-dimensional Hopf algebra $H = k$

A graded polynomial identity is an H-polynomial identity for a group algebra $H = kG$

A universal algebra

• Given an H-comodule algebra A, let $\mathrm{Id}_H(A)$ be the two-sided ideal of T_H consisting of all H-polynomial identities for A

• The Specht problem for $\mathrm{Id}_H(A)$ is open for a general Hopf algebra

• The quotient-algebra $U_H(A) = T_H / \mathrm{Id}_H(A)$ is the universal algebra in which all H-polynomial identities for A vanish

• The algebra $U_H(A)$ turns out be useful:

After a suitable localization of the (big) center of $U_H(A)$ we obtain an algebra parametrizing an important class of Hopf Galois extensions

See details in joint work with Eli Aljadeff (Adv. Math., 2008)

APPENDIX

Gradings on matrix algebras

Fine grading and twisted group algebras

Let $A = \bigoplus_{g \in G} A_g$ be a G-graded algebra for a finite group G

• Such a grading is called fine if for all $g, h \in G$,

$$\dim(A_g) = 1 \quad \text{and} \quad A_g \cdot A_h = A_{gh}$$

Then necessarily $\dim(A) = |G|$

• Pick a non-zero element u_g in each A_g. Then

$$u_g \, u_h = \alpha(g, h) \, u_{gh} \tag{2}$$

for some non-zero scalar $\alpha(g, h)$. Because of the associativity of the product, the scalars $\alpha(g, h)$ satisfy the cocycle condition:

$$\alpha(g, h) \, \alpha(gh, k) = \alpha(g, hk) \, \alpha(h, k) \qquad (g, h, k \in G) \tag{3}$$

• Conversely, given a cocycle on G, i.e., a map $\alpha : G \times G \to k - \{0\}$ satisfying (3), we can define an associative algebra whose underlying vector space has a basis $(u_g)_g$ indexed by the elements of G and whose product is given by (2)

Such an algebra is called a twisted group algebra and is denoted by $k^\alpha G$

Groups of central type and non-degenerate cocycles

- Let $k^\alpha G$ be a twisted group algebra for some cocycle α

If there is an algebra isomorphism

$$k^\alpha G \cong M_n(k)$$

for some $n \geq 1$, then we say that G is of central type and α is non-degenerate

- If α is non-degenerate, then $|G| = n^2$ and $M_n(k)$ has a fine G-grading

- *(Howlett, Isaacs,1982)* Any group of central type is solvable

- It is easy to check that an abelian group G is of central type if and only if

$$G \cong H \times H$$

for some abelian group H

- The group $\mathbb{Z}/6 \rtimes \mathrm{Sym}_3$ considered above is of central type

Gradings on $M_n(\mathbb{C})$

Let G be a finite group

- Suppose we are given the following data:
  - (a) a subgroup H of G of central type
  - (b) a non-degenerate cocycle α on H
  - (c) a sequence $g_1, \ldots, g_d$ of elements of G

Then we can define a G-grading on $A = \mathbb{C}^\alpha H \otimes M_d(\mathbb{C})$ by setting

$$A_g = \text{Span}\{u_h \otimes E_{i,j} \mid g^{-1}hg_j = g\}$$

- Bahturin and Zaicev (2002) proved that each G-grading of $M_n(\mathbb{C})$ is of this form

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THANK YOU FOR YOUR ATTENTION