The free group F_2, the braid group B_3, and palindromes

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Introduction

- Joint work with Christophe Reutenauer (UQAM): arXiv:0802.4359
- *Starting point:* Work of Aldo de Luca in combinatorics of words

  We work with words on two letters a, b, in particular with a class of words related to Christoffel words
- *What we do:*

  We extend de Luca's construction to the free group on a, b

  Some results of de Luca's still hold for our extension, some not
- *New features:*

  (a) Continuity properties

  (b) The braid group B_3 plays a role

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Palindromic closure

• The words on the alphabet $\{a, b\}$, including the empty word, form under concatenation the monoid $\{a, b\}^*$

• A word w is a palindrome if $w = \widetilde{w}$ is equal to its mirror image (the word $\widetilde{w}$ is what you obtain when you read w from right to left)

• **Right palindromic closure:** For a word w, we denote by w^+ the shortest palindrome having w as a prefix

Construction of w^+: if $w = uv$ is the unique decomposition, where $v = \widetilde{v}$ is the longest palindromic suffix of v, then

$$w^+ = uv\widetilde{u}$$

De Luca's iterated palindromic closure

- De Luca's defined the right iterated palindromic closure

$$P : \{a,b\}^* \longrightarrow \{a,b\}^*$$

recursively by $P(1) = 1$ and

$$P(wa) = \left(P(w)a\right)^+ \quad \text{and} \quad P(wb) = \left(P(w)b\right)^+$$

By definition, each $P(w)$ is a palindrome

- **Some values:**

$$\begin{aligned} P(a) &= a\,, & P(b) &= b\,, \\ P(aa) &= aa\,, & P(bb) &= bb\,, \\ P(ab) &= aba\,, & P(ba) &= bab\,, \\ P(aba) &= abaaba\,, & P(bab) &= babbab \end{aligned}$$

Properties of de Luca's map

Aldo de Luca proved the following result

Theorem. *The map P is a bijection from $\{a, b\}^*$ onto the subset of central words*

This means that

(a) $P : \{a, b\}^* \to \{a, b\}^*$ is injective

(b) Each $P(w)$ is a central word

(c) Any central word is $P(w)$ for some (unique) word w

Central words will be defined on the next slide

Central words

- A word is central if it has two coprime periods p, q and its length is $p + q - 2$

  Example: aba^2ba is central with periods 3, 5 and length 6:

$$aba^2ba = \underbrace{aba}\,\underbrace{aba} = \underbrace{aba^2b}\,\underbrace{a}$$

- All central words are palindromes

- There is a geometric definition of central words *via* Christoffel words, which label discrete approximations of segments in the plane (see Appendix 1)

  If w is a central word, then awb is a Christoffel word. Conversely, any Christoffel word starting with a and ending with b is of the form awb for some central word w

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Justin's formula

- **An action of the free monoid on itself.**
  There is a unique monoid homomorphism

$$\begin{array}{ccc} \{a,b\}^* & \longrightarrow & \mathrm{End}(\{a,b\}^*) \\ w & \longmapsto & R_w \end{array}$$

  such that R_a and R_b are the substitutions (endomorphisms)

$$R_a(a) = a\,, \qquad R_a(b) = ba\,,$$

$$R_b(a) = ab\,, \qquad R_b(b) = b\,.$$

- Justin proved the following multiplicative formula for de Luca's map P:

$$P(uv) = P(u)\,R_u\big(P(v)\big) \qquad (u, v \in \{a,b\}^*) \qquad (1)$$

  Consequently, de Luca's map P is determined by (1) and its values on a and b

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Trivial solutions of (1)

• If u, v were elements of a group G and P took values in a group E on which G acts by $w \mapsto R_w$ by group automorphisms, then the functional equation

$$P(uv) = P(u)\,R_u\big(P(v)\big)$$

would have as trivial solutions maps of the form

$$w \mapsto X^{-1} R_w(X) \qquad (w \in G) \tag{2}$$

for some fixed element X of E

• **Drawback.** In the case of de Luca's map, a trivial solution of the form (2) forces us to work in the free group F_2 on the letters a, b

• **Bonus.** If de Luca's map is of the form (2) for some $X \in F_2$, then Formula (2) makes sense in F_2, and thus provides an extension of P to F_2

Extending de Luca's map to the free group

For each $w \in F_2$, we (= CK & CR) set

$$\mathrm{Pal}(w) = (ab)^{-1} R_w(ab)$$

Our first observation was the following.

Proposition. *For all* $w \in \{a,b\}^*$,

$$\mathrm{Pal}(w) = P(w)$$

In other words, the map $\mathrm{Pal} : F_2 \to F_2$ extends de Luca's map $P : \{a,b\}^* \to \{a,b\}^*$

Proof. Since Pal is of the form (2), it suffices to check that $\mathrm{Pal}(a) = a$ and $\mathrm{Pal}(b) = b$. For instance,

$$\mathrm{Pal}(a) = (ab)^{-1} R_a(ab) = (ab)^{-1} R_a(a) R_a(b) = (ab)^{-1} aba = a$$

First properties of Pal

- **Proposition.** *Each* Pal(w) *is a palindrome:*

$$\mathrm{Pal}(w) = \widetilde{\mathrm{Pal}(w)}$$

Here $w \mapsto \widetilde{w}$ is the unique anti-automorphism of F_2 fixing a and b

- *(Counting with signs the numbers of occurrences of a and b in* Pal(w)*)*
Let $\pi : F_2 \to \mathbb{Z}^2$ be the abelianization map

Proposition. *For each $w \in F_2$,*

$$\pi(\mathrm{Pal}(w)) = (M_w - I_2)\begin{pmatrix}1\\1\end{pmatrix} \qquad (3)$$

where $w \mapsto M_w$ is the homomorphism $F_2 \to SL_2(\mathbb{Z})$ such that

$$M_a = \begin{pmatrix}1 & 1\\0 & 1\end{pmatrix} \quad \text{and} \quad M_b = \begin{pmatrix}1 & 0\\1 & 1\end{pmatrix}.$$

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Continuity of Pal

• The profinite topology on F_2 is the coarsest topology such that any homomorphism from F_2 to a finite group is continuous

The subgroups of F_2 of finite index form a system of neighborhoods of 1 for this topology

• **Theorem.** *The map* Pal : $F_2 \to F_2$ *is continuous for the profinite topology*

• Since $\{a, b\}^*$ is dense in F_2 for this topology, we obtain

Corollary. *The map* Pal : $F_2 \to F_2$ *is the unique continuous extension of de Luca's map to* F_2

Non-continuity results

• *Recall:* De Luca constructed the map P using the right palindromic closure

$$w \mapsto w^+$$

Lemma. *The map $w \mapsto w^+$ is not continuous*

Proof. When $n \to \infty$,

$$w_n = ab^{n!} \to a \quad \text{and} \quad (w_n)^+ = (ab^{n!})^+ = ab^{n!}a \to aa \neq a^+$$

• For a given prime p, the pro-p-finite topology on F_2 is defined as the coarsest topology such that any homomorphism from F_2 to a finite p-group is continuous

Lemma. *The map* Pal *is not continuous for the pro-p-finite topology*

Pal is not injective

- De Luca's map $P : \{a,b\}^* \to \{a,b\}^*$ is injective, but

$$\mathrm{Pal}(ba^{-1}) = a^{-1} = \mathrm{Pal}(a^{-1}) \quad \text{and} \quad \mathrm{Pal}(aba^{-1}) = 1 = \mathrm{Pal}(1)$$

- **A good reason why** Pal **cannot be injective.** In the automorphism group $\mathrm{Aut}(F_2)$,

$$R_{ab^{-1}a} = R_a R_b^{-1} R_a = R_b^{-1} R_a R_b^{-1} = R_{b^{-1}ab^{-1}} \tag{4}$$

- By contrast, when considered as substitutions, R_a and R_b generate a free monoid in $\mathrm{End}(\{a,b\}^*)$

Link with the braid group

- The braid group of braids with three strands:

$$B_3 = \langle \sigma_1, \sigma_2 \mid \sigma_1\sigma_2\sigma_1 = \sigma_2\sigma_1\sigma_2 \rangle$$

Let $\varphi : F_2 \longrightarrow B_3$ be the homomorphism such that

$$\varphi(a) = \sigma_1 \quad \text{and} \quad \varphi(b) = \sigma_2^{-1}$$

- Since $\mathrm{Pal}(w) = (ab)^{-1} R_w(ab)$ and $R_{ab^{-1}a} = R_{b^{-1}ab^{-1}}$,

$$\varphi(w) = \varphi(w') \implies \mathrm{Pal}(w) = \mathrm{Pal}(w')$$

The "kernel" of Pal

- **Theorem.** $\mathrm{Pal}(w) = \mathrm{Pal}(w') \iff \varphi(w^{-1}w') \in \langle\, \sigma_1\sigma_2^{-1}\sigma_1^{-1} \,\rangle \subset B_3$

- The proof of the theorem relies on the following facts:

  (a) The subgroup of $\mathrm{Aut}(F_2)$ generated by R_a and R_b is isomorphic to B_3

  (b) If $M_w \in SL_2(\mathbb{Z})$ fixes the vector $(1, 1)$, then w is a power of aba^{-1}

Note that $\varphi(aba^{-1}) = \sigma_1\sigma_2^{-1}\sigma_1^{-1}$

Characterization of the image of Pal

• For $w, w' \in F_2$, we write $w \sim w'$ if they are conjugate in the group

Theorem. *An element $u \in F_2$ belongs to the image of* Pal *if and only if*

$$abu \sim bau$$

• We reduce the proof to

(a) the following result by Pirillo:

If u is a word in an alphabet A
such that $abu \sim bau$ for some distinct $a, b \in A$,
then u is a central word in a and b

(b) and de Luca's characterization of central words as the images of his map P

• **Corollary.** *The image of* Pal *is a closed subset of F_2 w. r. t. the profinite topology*

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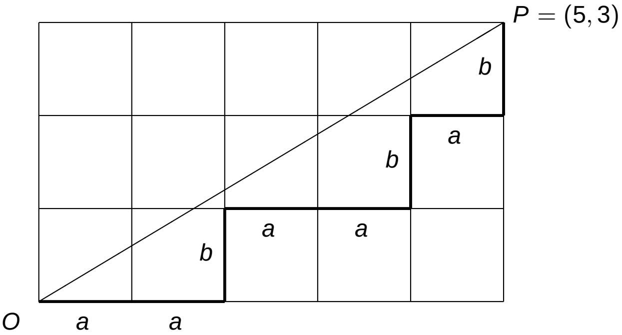
Appendix 1: Christoffel words

- The Christoffel word attached to the primitive vector $(5,3)$ is *aabaabab*: it labels the closest stair-case path approximating *OP* from below
- The corresponding central word is the palindrome *abaaba*



Appendix 2: Inverting de Luca's map

- By de Luca's result, if u is a central word, then $u = P(w)$ for a unique w.

How to recover w from u?

- *De Luca's algorithm:* If $u_0 = 1, u_1, \ldots, u_r$ are the proper palindromic prefixes of u in increasing length, and a_i is the letter following u_i in u, then

$$w = a_0 a_1 \cdots a_r$$

- *Example:* For the central word $u = abaaba$,

$$u_0 = 1\,, \qquad a_0 = a$$

$$u_1 = a\,, \qquad a_1 = b$$

$$u_2 = aba\,, \qquad a_2 = a$$

Hence,

$$w = aba$$