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From Sturmian morphisms to the braid group B_4

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Sturmian sequences [Morse & Hedlund (1938)]

An infinite word w in two letters a and b is a *Sturmian sequence* if for all $n \geq 1$ the number of factors of w of length n is $n+1$. (w_1 is a *factor* of $w = w_0 w_1 w_2$.)

FIBONNACI SEQUENCE. If $\varphi(a) = ab$ and $\varphi(b) = a$, then $w_n = \varphi^n(a)$ converges to a Sturmian sequence w:

$$w_0 = a,\ w_1 = ab,\ w_2 = aba, \ldots, \quad w_{n+2} = w_{n+1} w_n.$$

The sequence $|w_n|$ of lengths is the Fibonnaci sequence.

SLOPE OF A STURMIAN SEQUENCE. If w is Sturmian and w_n is its prefix of length n, then $|w_n|_b / |w_n|$ tends to a limit, called the *slope* of w. The slope of a Sturmian sequence is an *irrational* number.

References

M. LOTHAIRE, *Algebraic combinatorics on words*, Cambridge University Press, Cambridge, 2002, Chapter 2, 45–110.

J.-P. ALLOUCHE, J. SHALLIT, *Automatic sequences. Theory, applications, generalizations*, Cambridge University Press, Cambridge, 2003.

The monoid of Sturmian morphisms

A substitution φ (an endomorphism of the free monoid $\{a,b\}^*$) is a *Sturmian morphism* if $\varphi(w)$ is a Sturmian sequence whenever w is a Sturmian sequence. Sturmian morphisms form a *monoid* St under composition.

FIVE STURMIAN MORPHISMS:

$$\begin{array}{cccc} & E(a)=b, & E(b)=a, & \\ D(a)=ba, & D(b)=b, & \widetilde{D}(a)=ab, & \widetilde{D}(b)=b, \\ G(a)=a, & G(b)=ab, & \widetilde{G}(a)=a, & \widetilde{G}(b)=ba. \end{array}$$

F. MIGNOSI, P. SÉÉBOLD (1993) : *The monoid* St *is generated by* E, D, $\widetilde{D}$ *(also by* E, G, $\widetilde{G}$*).*

$$\begin{array}{ccc} \mathrm{Aut}^+(F_2) & \longrightarrow & \mathrm{Aut}(F_2) \\ \downarrow & & \downarrow \\ \mathrm{End}(\{a,b\}^*) & \longrightarrow & \mathrm{End}(F_2) \end{array}$$

Z.-X. WEN, Z.-Y. WEN (1994) : *The monoid* $\mathrm{Aut}^+(F_2)$ *of positive automorphisms is generated by* E, D, $\widetilde{D}$.

Conclusion: $\mathrm{St} = \mathrm{Aut}^+(F_2)$.

The special Sturmian monoid

Let St_0 be the submonoid of St generated by

$$D, \widetilde{D}, G, \widetilde{G}.$$

The subgroup of $\mathrm{Aut}(F_2)$ generated by D, $\widetilde{D}$, G, $\widetilde{G}$ is

$$\mathrm{Aut}_0(F_2) = \mathrm{Ker}\big(\mathrm{Aut}(F_2) \xrightarrow{\mathrm{ab}} \mathrm{GL}_2(\mathbf{Z}) \xrightarrow{\det} \{\pm 1\}\big).$$

Theorem. *The monoid* St_0 *has a* monoid presentation *with generators*

$$D, \widetilde{D}, G, \widetilde{G},$$

and relations

$$G\widetilde{G} = \widetilde{G}G, \quad D\widetilde{D} = \widetilde{D}D,$$

$$GD\widetilde{G} = \widetilde{G}\widetilde{D}G, \quad DG\widetilde{D} = \widetilde{D}\widetilde{G}D,$$

$$GD^k\widetilde{G} = \widetilde{G}\widetilde{D}^kG, \quad DG^k\widetilde{D} = \widetilde{D}\widetilde{G}^kD$$

($k \geq 2$).

Braid relations for the generators of St_0

The generators G, $\widetilde{G}$ of St_0 map to $A \in \mathrm{GL}_2(\mathbf{Z})$, and D, $\widetilde{D}$ map to $B \in \mathrm{GL}_2(\mathbf{Z})$, where

$$A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}.$$

The matrices A and B satisfy the braid relation

$$AB^{-1}A = B^{-1}AB^{-1}.$$

Lemma. *We have* $G\widetilde{G} = \widetilde{G}G$, $D\widetilde{D} = \widetilde{D}D$,

$$GD^{-1}G = D^{-1}GD^{-1}, \quad D^{-1}\widetilde{G}D^{-1} = \widetilde{G}D^{-1}\widetilde{G},$$
$$\widetilde{G}\widetilde{D}^{-1}\widetilde{G} = \widetilde{D}^{-1}\widetilde{G}\widetilde{D}^{-1}, \quad \widetilde{D}^{-1}G\widetilde{D}^{-1} = G\widetilde{D}^{-1}G.$$

Symbolically,

G D

A B

$\widetilde{D}$ $\widetilde{G}$

An action of B_4 on F_2

The *braid group* B_4 with generators σ_1, σ_2, σ_3 and relations $\sigma_1\sigma_3 = \sigma_3\sigma_1$,

$$\sigma_1\sigma_2\sigma_1 = \sigma_2\sigma_1\sigma_2, \quad \sigma_2\sigma_3\sigma_2 = \sigma_3\sigma_2\sigma_3.$$

Lemma. *There is a group homomorphism*

$$f : B_4 \to \mathrm{Aut}(F_2)$$

defined by

$$f(\sigma_1) = G, \quad f(\sigma_2) = D^{-1}, \quad f(\sigma_3) = \widetilde{G}.$$

Moreover, if $\sigma_4 = \sigma_3^{-1}\sigma_1\,\sigma_2\,\sigma_3\sigma_1^{-1}$, *then* $f(\sigma_4) = \widetilde{D}^{-1}$.

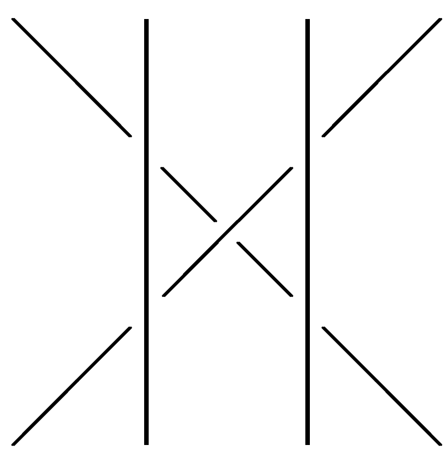


Figure 1. The braid σ_4 *in* B_4

An action of B_4 on F_2 (sequel)

THEOREM. *There is a commutative diagram of exact sequences*

$$\begin{array}{ccccccccccc}
 & & & & 1 & \longrightarrow & 1 & & & & \\
 & & & & \downarrow & & \downarrow & & & & \\
 & & 1 & \rightarrow & \mathrm{Ker}(q) & \xrightarrow{\cong} & F_2 & \rightarrow & 1 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
1 & \rightarrow & Z_4 & \rightarrow & B_4 & \xrightarrow{f} & \mathrm{Aut}(F_2) & \rightarrow & \mathbf{Z}/2 & \rightarrow & 1 \\
 & & \cong\downarrow & & q\downarrow & & \mathrm{ab}\downarrow & & =\downarrow & & \\
1 & \rightarrow & 2Z_3 & \rightarrow & B_3 & \longrightarrow & \mathrm{GL}_2(\mathbf{Z}) & \xrightarrow{\det} & \mathbf{Z}/2 & \rightarrow & 1 \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 1 & \rightarrow & 1 & \longrightarrow & 1 & \rightarrow & 1 & &
\end{array}$$

Centre of B_4 : $Z_4 = \langle\, (\sigma_1\sigma_2\sigma_3)^4 \,\rangle \cong \mathbf{Z}$
Centre of B_3 : $Z_3 = \langle\, (\sigma_1\sigma_2)^3 \,\rangle \cong \mathbf{Z}$

$$2Z_3 = \langle\, (\sigma_1\sigma_2)^6 \,\rangle$$

Homomorphism $q : B_4 \to B_3$:

$$q(\sigma_1) = q(\sigma_3) = \sigma_1 \ \text{ and } \ q(\sigma_2) = \sigma_2.$$

$\mathrm{Ker}(q)$ is free on $x = \sigma_1\sigma_3^{-1}$ and $y = \sigma_2\sigma_1\sigma_3^{-1}\sigma_2^{-1}$ [Gassner (1962), Gorin and Lin (1969)]

An action of B_4 on F_2 (end)

Involution ω of B_4:

$$\sigma_1 \leftrightarrow \sigma_2^{-1} \text{ and } \sigma_3 \leftrightarrow \sigma_4^{-1}.$$

- ω preserves the centre of B_4
- Compatibility of ω with $f : B_4 \to \mathrm{Aut}(F_2)$:

$$f(\omega(\sigma_i)) = Ef(\sigma_i)E \qquad (i = 1, 2, 3, 4)$$

- Karrass, Pietrowski, Solitar (1984):

$$\mathrm{Out}(B_4) \cong \mathbf{Z}/2 \text{ generated by } \omega$$

From the exact sequence (split by ω)

$$1 \to Z_4 \to B_4 \to \mathrm{Aut}(B_4) \to \mathrm{Out}(B_4) \to 1$$

it follows that $\mathrm{Aut}(B_4) \cong B_4/Z_4 \rtimes_\omega \mathbf{Z}/2$.

From previous theorem it then follows:

$$\mathrm{Aut}(F_2) \cong B_4/Z_4 \rtimes_\omega \mathbf{Z}/2 \cong \mathrm{Aut}(B_4).$$

Remarks:

- Dyer, Formanek, Grossman (1982): B_4/Z_4 is a subgroup of index two of $\mathrm{Aut}(F_2)$
- Karrass, Pietrowski, Solitar (1984): $\mathrm{Aut}(B_4) \cong \mathrm{Aut}(F_2)$

Lifting of St_0 to B_4

THEOREM. *The formulas*

$$i(G) = \sigma_1, \;\; i(\widetilde{G}) = \sigma_3, \;\; i(D) = \sigma_2^{-1}, \;\; i(\widetilde{D}) = \sigma_4^{-1}.$$

define a homomorphism of monoids $i : \mathrm{St}_0 \to B_4$. The composite map

$$\mathrm{St}_0 \xrightarrow{\;i\;} B_4 \xrightarrow{\;f\;} \mathrm{Aut}(F_2)$$

is the natural inclusion.

COROLLARY. *The submonoid of B_4 generated by*

$$\sigma_1, \;\sigma_2^{-1}, \;\sigma_3, \;\sigma_4^{-1}$$

is isomorphic to St_0. It has a monoid presentation *with generators σ_1, σ_2^{-1}, σ_3, σ_4^{-1}, and relations*

$$\sigma_1\sigma_2^{-k}\sigma_3 = \sigma_3\sigma_4^{-k}\sigma_1 \;\; \text{and} \;\; \sigma_2^{-1}\sigma_1^{k}\sigma_4^{-1} = \sigma_4^{-1}\sigma_3^{k}\sigma_2^{-1}$$

for all $k \geq 0$.

N. B. The submonoid of B_4 generated by σ_1 and σ_2^{-1} is a free monoid.

Bases of the free group F_2

$$\begin{array}{ccc} \mathrm{Aut}(F_2) & \cong & \{\text{bases of } F_2\} \\ \varphi & \mapsto & (\varphi(a), \varphi(b)) \end{array}$$

J. Nielsen (1918):

$$\begin{array}{ccc} \mathrm{Aut}(F_2)/\text{conjugacy} & \cong & \mathrm{GL}_2(\mathbf{Z}) \\ \cong\downarrow & & \cong\downarrow \\ \{\text{bases of } F_2\}/\text{conjugacy} & \cong & \{\text{bases of } \mathbf{Z}^2\} \end{array}$$

Problem: Given a basis of $\mathbf{Z}^2$, equivalently a matrix

$$M = \begin{pmatrix} p & r \\ q & s \end{pmatrix} \in \mathrm{GL}_2(\mathbf{Z}),$$

construct a basis of F_2 lifting M.

Solved by H. Cohn (1972), Osborne & Zieschang (1981), González-Acuña & Ramírez (1999).

Christoffel words:
a finitary version of Sturmian sequences

To $(p, q) \in \mathbf{Z}^2$ such that p and q are coprime, we attach an element $u(p, q) \in F_2$. When p, $q \in \mathbf{N}$:

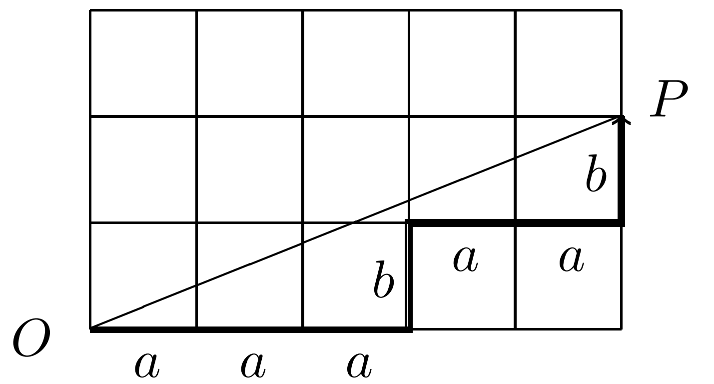


Figure 2. $u(5, 2) = a^3ba^2b$

Property. *If* $ps - qr = 1$ *with* p, q, r, $s \in \mathbf{N}$, *then*

$$u(p, q)u(r, s) = u(p + r, q + s).$$

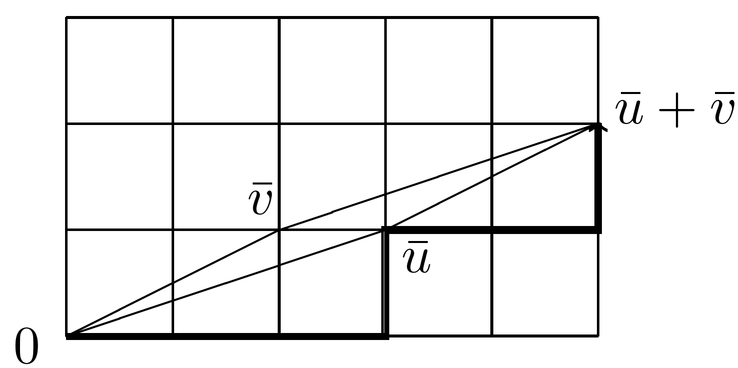


Figure 3. $u(3, 1)u(2, 1) = u(5, 2)$

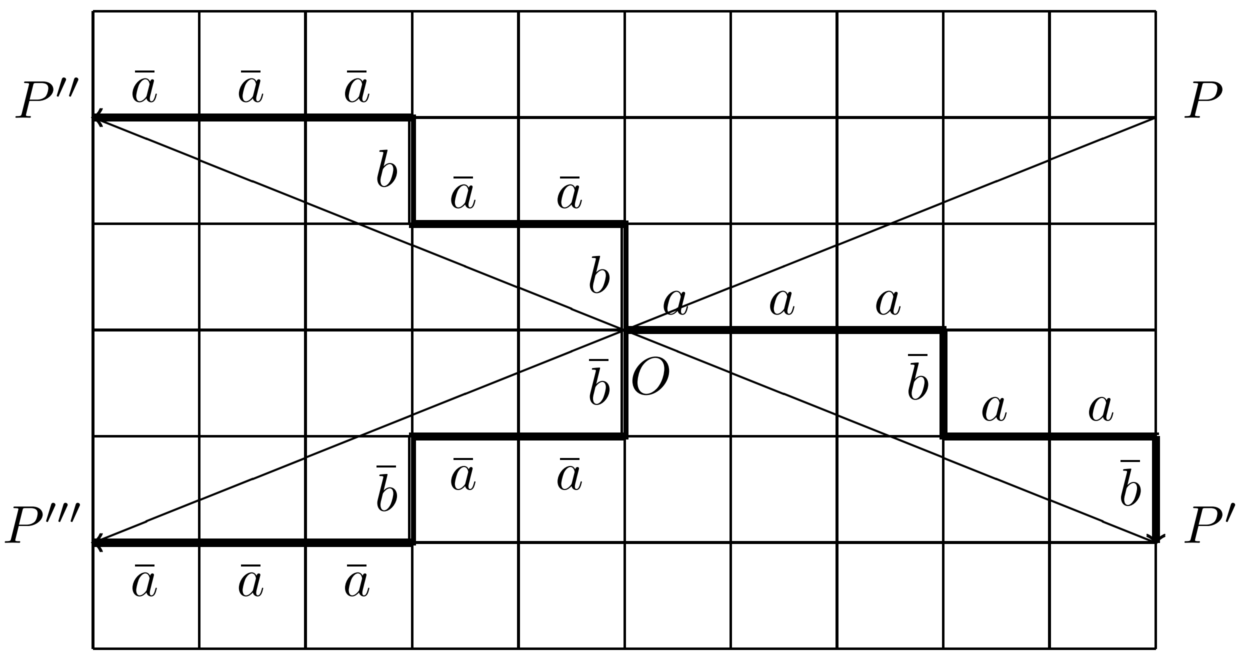
General Christoffel words

When p or q are negative, use the automorphism T: $a \mapsto a$, $b \mapsto b^{-1}$. For $p \geq 0$ and $q \geq 0$ set

$$u(p, -q) = Tu(p, q),$$

$$u(-p, q) = Tu(p, q)^{-1},$$

$$u(-p, -q) = u(p, q)^{-1}.$$



THEOREM. *If* $\{(p, q), (r, s)\}$ *is a basis of* $\mathbf{Z}^2$*, then* $\{u(p, q), u(r, s)\}$ *is a basis of* F_2*.*

Characterization of bases

M. Dehn: $\{u, v\}$ is a basis of F_2 if and only if $uvu^{-1}v^{-1}$ is conjugate to $(aba^{-1}b^{-1})^{\pm 1}$.

Rewriting system for $(u, v) \in \{a, b\}^* \times \{a, b\}^*$:

$$(au', av') \to (u'a, v'a) \text{ and } (bu', bv') \to (u'b, v'b)$$

Any couple is contained in a maximal chain of arrows. The length of the chain is the number of arrows.

Examples.

(a) (a, b^2) is contained in a chain of length 0.
(b) (b^2, b^3) is contained in a chain of infinite length

Theorem. *Let u, $v \in \{a, b\}^*$ of lengths $|u|$ and $|v|$.*

(a) The couple (u, v) is a basis of F_2 if and only if the maximal chain containing (u, v) is of length

$$|u| + |v| - 2.$$

(b) If the maximal chain containing (u, v) is of length $> |u| + |v| - 2$, then it is infinite and u, v are positive powers of a same word.

Corollary. *Any cyclically reduced basis (u, v) of F_2 is conjugate to exactly $|u| + |v| - 1$ cyclically reduced bases.*

Palindromes

Reverse: Let $u \mapsto \widetilde{u}$ be the anti-automorphism of F_2 such that $\widetilde{a} = a$ and $\widetilde{b} = b$.

Palindrome: $u \in F_2$ is a palindrome if $\widetilde{u} = u$.

Theorem *Any cyclically reduced basis $\{u, v\}$ of F_2 such that $|u|$ and $|v|$ are odd has exactly one cyclically reduced conjugate consisting of two palindromes.*

The maximal chain containing (aba^2b, aba):

$$\begin{array}{ccccc} (aba^2b, aba) & \longrightarrow & (ba^2ba, ba^2) & \longrightarrow & (a^2bab, a^2b) \\ & & & & \downarrow \\ & & & & (ababa, aba) \\ & & & & \downarrow \\ (ba^2ba, aba) & \longleftarrow & (aba^2b, a^2b) & \longleftarrow & (baba^2, ba^2) \end{array}$$

Length $6 = |aba^2b| + |aba| - 2$. Therefore (aba^2b, aba) is a basis.

Remark. A basis element of F_2 of even length cannot be conjugate to a palindrome.