Tate-Vogel cohomology

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Report on work by Vogel

- This is a short report on unpublished work done by Pierre Vogel in the early 1980's

- In this work Vogel extended the Tate cohomology of finite groups to any group, and even to any ring

- The definition by Vogel is very simple and elegant, using unbounded chain complexes

- At that time, Vogel was working on a strong version of Novikov's conjecture

- In an email dated 28 September 2010, Pierre wrote to me the following:

<< C’est au cours de mes nombreuses tentatives pour montrer la conjecture que j’ai manipulé beaucoup de modules différentiels gradués et que j’ai pensé à cette algèbre homologique à la Tate. Le fait que je n’ai rien écrit sur ces sujets est que je n’ai rien obtenu de significatif sauf des conjectures et des jolies constructions. >>

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What is Tate cohomology?

Generalities on chain complexes

Vogel's extension of Tate cohomology

References

Group cohomology

- Let G be a group and $R = \mathbb{Z}G$ its group ring
- The cohomology of G with coefficients in a left R-module M is defined as

$$H^*(G, M) = \mathrm{Ext}^*_R(\mathbb{Z}, M)$$

- It can be computed as follows: if

$$\ldots \longrightarrow F_2 \longrightarrow F_1 \longrightarrow F_0 \longrightarrow \mathbb{Z} \tag{1}$$

is a resolution of the trivial R-module $\mathbb{Z}$ by projective left R-modules, then

$$H^*(G, M) = H^*\left(\mathrm{Hom}_R(F, M)\right)$$

The case of finite groups

- **Notation.** If M is a left R-module, then the dual module

$$M^{\vee} = \mathrm{Hom}_R(M, R)$$

is a right R-module (which can be turned into a left module)

- Now suppose that the group G is finite

There exist resolutions of the form (1) where the projective modules F_i are all finitely generated

- Dualizing such a resolution, one gets an acyclic complex

$$0 \longrightarrow \mathbb{Z} \longrightarrow F_0 \longrightarrow F_1 \longrightarrow F_2 \longrightarrow \cdots \tag{2}$$

of finitely generated projective modules

Tate cohomology

• Splicing the complexes (1) and (2) together and setting $F_{-i} = F_{i-1}^{\vee}$ for $i > 0$, we obtain a complete resolution for G, that is, an acyclic complex of finitely generated projective modules

$$\ldots \longrightarrow F_2 \longrightarrow F_1 \longrightarrow F_0 \longrightarrow F_{-1} \longrightarrow F_{-2} \longrightarrow \cdots \tag{3}$$

together with an R-linear map $F_0 \longrightarrow \mathbb{Z}$ such that

$$\ldots \longrightarrow F_2 \longrightarrow F_1 \longrightarrow F_0 \longrightarrow \mathbb{Z} \longrightarrow 0$$

is acyclic

• The Tate cohomology of G with coefficients in M is defined as

$$\widehat{H}^*(G, M) = H^*\left(\mathrm{Hom}_R(F, M)\right)$$

where F is a complete resolution of the form (3)

These groups are independent of the chosen complete resolution

Properties of Tate cohomology

Tate cohomology enjoys standard properties of ordinary group cohomology such as:

• If $0 \to M' \to M \to M'' \to 0$ is a short exact sequence of R-modules, then there is a long exact sequence of cohomology groups

$$\cdots \to \widehat{H}^i(G, M') \to \widehat{H}^i(G, M) \to \widehat{H}^i(G, M'') \to \widehat{H}^{i+1}(G, M') \to \cdots$$

• There are restriction and transfer maps with respect to subgroups of G

• There are associative cup-products

$$\widehat{H}^i(G, M) \times \widehat{H}^j(G, N) \xrightarrow{\cup} \widehat{H}^{i+j}(G, M \otimes N)$$

These are useful to express periodicity in group cohomology (see next slide)

Periodic cohomology

- **Definition.** *A finite group has periodic cohomology if there is an integer* $d \neq 0$ *and an element* $\alpha \in \widehat{H}^d(G, \mathbb{Z})$ *that is invertible with respect to the cup-product*

Such an element induces natural isomorphisms

$$\widehat{H}^i(G, M) \xrightarrow{\cup \alpha} \widehat{H}^{i+d}(G, M)$$

for all $i \in \mathbb{Z}$ and all G-modules M

- **Examples.** The following finite groups have periodic cohomology:

  (a) the cyclic groups ($d = 2$)

  (b) the order 8 quaternionic group Q_8 ($d = 4$)

See Chapter XII of Cartan and Eilenberg's book for a complete classification of finite groups with periodic cohomology

Farrell's extension

• **Definition.** *A group has finite cohomological dimension (cd) if the trivial G-module $\mathbb{Z}$ has a projective resolution of finite length*

Any group of finite cd is torsion-free; thus a finite group has infinite cd

• **Definition.** *A group has finite virtual cohomological dimension (vcd) if it admits a finite index subgroup that has finite cd*

For instance, a finite group G has finite vcd: the trivial subgroup of G is of finite index and has finite cd

• Farrell (1977) extended Tate cohomology to all groups with finite vcd

• Any group of finite vcd has a complete resolution in the following general sense: there exist an acyclic complex of projective modules

$$\ldots \longrightarrow F_2 \longrightarrow F_1 \longrightarrow F_0 \longrightarrow F_{-1} \longrightarrow F_{-2} \longrightarrow \cdots$$

and a projective resolution

$$\ldots \longrightarrow P_2 \longrightarrow P_1 \longrightarrow P_0 \longrightarrow \mathbb{Z} \longrightarrow 0$$

such that the complexes F and P coincide in large enough degrees

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The standard Hom-complex

• Let C, d and C', d' be chain complexes of left R-modules (graded by $\mathbb{Z}$ and not necessarily bounded)

• Recall that that there is an internal Hom in the category of chain complexes of left R-modules:

Define $\mathrm{Hom}(C, C')$ as the chain complex such that

$$\mathrm{Hom}(C, C')_n = \prod_{i \in \mathbb{Z}} \mathrm{Hom}_R(C_i, C'_{i+n}) \qquad (n \in \mathbb{Z})$$

with differential given for $f \in \mathrm{Hom}_R(C_i, C'_{i+n})$ by

$$\partial(f) = d' \circ f - (-1)^n f \circ d$$

The bounded Hom-complex

- The complex $\mathrm{Hom}(C, C')$ has a subcomplex $\mathrm{Hom}_b(C, C')$ such that

$$\mathrm{Hom}_b(C, C')_n = \bigoplus_{i \in \mathbb{Z}} \mathrm{Hom}_R(C_i, C'_{i+n}) \qquad (n \in \mathbb{Z})$$

We call $\mathrm{Hom}_b(C, C')$ the bounded Hom-complex

- Observe that $\mathrm{Hom}_b(C, C') = \mathrm{Hom}(C, C')$ if

  (a) one of the complexes C, C' is bounded above and below, or

  (b) one of them is bounded above and the other one is bounded below

- **Definition.** The complete Hom-complex is the quotient complex

$$\widehat{\mathrm{Hom}}(C, C') = \mathrm{Hom}(C, C') / \mathrm{Hom}_b(C, C')$$

- Observe that $\widehat{\mathrm{Hom}}(C, C') = 0$ if one of Conditions (a) or (b) above holds

The tensor product of complexes

• Let C'', d'' be a chain complex of right R-modules and C', d' be a chain complex of left R-modules

• Recall that their tensor product is the chain complex $C'' \otimes C'$ such that

$$(C'' \otimes C')_n = \bigoplus_{i+j=n} C''_i \otimes_R C'_j \qquad (n \in \mathbb{Z})$$

with differential given for $x \in C''_i$ and $y \in C'_j$ by

$$d(x \otimes y) = d''(x) \otimes y + (-1)^i x \otimes d'(y)$$

Relating $\otimes$ and Hom for modules

- Let us connect the Hom-complexes with the tensor product of complexes

- Recall that if M and N are left R-modules, then there is a natural map

$$M^{v} \otimes_R N \longrightarrow \mathrm{Hom}_R(M, N)$$

sending $f \otimes y \in M^{v} \otimes_R N$ to the element $x \mapsto f(x)y$ of $\mathrm{Hom}_R(M, N)$

This map is an isomorphism if M is a finitely generated projective R-module

Relating $\otimes$ and Hom for complexes

- Let C and C' be chain complexes of left R-modules
- The dual complex C^v is a chain complex of right R-modules with

$$(C^v)_n = (C_{-n})^v \qquad (i \in \mathbb{Z})$$

- **Lemma.** *If C_n is a finitely generated projective R-module for all $n \in \mathbb{Z}$, then the natural chain map*

$$C^v \otimes C' \longrightarrow \mathrm{Hom}_b(C, C')$$

is an isomorphism of chain complexes

Proof of Lemma

Proof. On the chain level, we have the following natural identifications

$$\begin{aligned}
\mathrm{Hom}_\mathrm{b}(C, C')_n &= \bigoplus_{i\in\mathbb{Z}} \mathrm{Hom}_R(C_i, C'_{i+n}) \\
&= \bigoplus_{i\in\mathbb{Z}} (C_i)^v \otimes_R C'_{i+n} \\
&= \bigoplus_{j\in\mathbb{Z}} (C^v)_j \otimes_R C'_{n-j} \\
&= (C^v \otimes C')_n
\end{aligned}$$

∗ The first and last equality hold by definition

∗ The second equality holds because of the assumption on the modules C_n

∗ For the third equality set $j = -i$

Q.E.D.

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Definition of Tate-Vogel cohomology

• Given left R-modules M, N, and a projective resolution P (resp. Q) of M (resp. of N), Pierre Vogel defined

$$\widehat{\operatorname{Ext}}_R^*(M, N) = H^*\left(\widehat{\operatorname{Hom}}(P, Q)\right)$$

One checks that $\widehat{\operatorname{Ext}}_R^*(M, N)$ is independent of the chosen resolutions

• **Definition.** The Tate-Vogel cohomology of a group G with coefficients in a left R-module M is given by

$$\widehat{H}^*(G, M) = \widehat{\operatorname{Ext}}_R^*(\mathbb{Z}, M)$$

where $R = \mathbb{Z}G$

• Tate-Vogel cohomology enjoys the standard properties of Tate cohomology

Tate-Vogel cohomology extends Tate cohomology

- **Proposition.** *If G is a finite group, then the Tate-Vogel cohomology of G coincides with the Tate cohomology of G*

- **Proof.** Let F be a complete resolution for G such that

$$F^+ : \quad \cdots \longrightarrow F_2 \longrightarrow F_1 \longrightarrow F_0 \longrightarrow 0$$

is a resolution of $\mathbb{Z}$; all F_n are assumed to be finitely generated projective

$*$ Let M be a left G-module and Q be a projective resolution of M

$*$ The Tate-Vogel cohomology of G with coefficients in M is the cohomology of the chain complex $\widehat{\mathrm{Hom}}(F^+, Q)$

$*$ The Tate cohomology of G with coefficients in M is the cohomology of the chain complex $\mathrm{Hom}(F, M)$

$*$ Enough to show that $\widehat{\mathrm{Hom}}(F^+, Q)$ and $\mathrm{Hom}(F, M)$ are quasi-isomorphic

(to be continued)

Sequel of proof of Proposition

$*$ Let F' be the kernel of the natural surjective map of complexes $F \to F^+$

The exact sequence of complexes $0 \to F' \to F \to F^+ \to 0$ induces an exact sequence

$$0 \to \widehat{\mathrm{Hom}}(F^+, Q) \to \widehat{\mathrm{Hom}}(F, Q) \to \widehat{\mathrm{Hom}}(F', Q) \to 0$$

between the complete Hom-complexes

$*$ Since F' is bounded above and Q is bounded below,

$$\widehat{\mathrm{Hom}}(F', Q) = 0$$

It follows from this and from the previous exact sequence that

$$\widehat{\mathrm{Hom}}(F^+, Q) = \widehat{\mathrm{Hom}}(F, Q)$$

(to be continued)

End of proof of Proposition

$*$ Consider now the exact sequence defining $\widehat{\mathrm{Hom}}(F, Q)$:

$$0 \to \mathrm{Hom}_b(F, Q) \to \mathrm{Hom}(F, Q) \to \widehat{\mathrm{Hom}}(F, Q) \to 0$$

$*$ Since F consists of finitely generated projective R-modules, we have

$$\mathrm{Hom}_b(F, Q) \cong F^v \otimes Q$$

For the same reason, the acyclicity of F implies the acyclicity of F^v, of $F^v \otimes Q$, hence of $\mathrm{Hom}_b(F, Q)$

It follows that $\mathrm{Hom}(F, Q) \to \widehat{\mathrm{Hom}}(F, Q)$ is a quasi-isomorphism

$*$ Observe that $\mathrm{Hom}(F, Q) \to \mathrm{Hom}(F, M)$ is a quasi-isomorphism as well

$*$ By putting everything together, we obtain the chain of quasi-isomorphisms

$$\widehat{\mathrm{Hom}}(F^+, Q) \xrightarrow{=} \widehat{\mathrm{Hom}}(F, Q) \longleftarrow \mathrm{Hom}(F, Q) \longrightarrow \mathrm{Hom}(F, M)$$

Q.E.D.

Some history

• Pierre Vogel gave lectures on this subject around 1983–84, but never published his results

• Accounts of his results appeared, one in a paper by François Goichot (1992), another one in a paper by Daniel Conduché, and Hvedri and Nick Inassaridze (2004)

• Benson and Carlson rediscovered Vogel's construction in a paper published in 1992

• Mislin (1994) gave a definition of Tate-Vogel cohomology using satellites

• Goichot extended Tate-Vogel cohomology to equivariant cohomology, and Conduché *et al*. to mod q cohomology

• Tate-Vogel cohomology for rings has been used recently by Mori to prove a Riemann-Roch theorem in non-commutative algebraic geometry and by Martínez Villa and Martsinkovsky to establish some non-commutative Serre duality

(see references below)

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Basic references

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