DIALGEBRAS

Jean-Louis LODAY

There is a notion of "non-commutative Lie algebra" called *Leibniz algebra*, which is characterized by the following property. The bracketing $[-, z]$ is a derivation for the bracket operation, that is, it satisfies the Leibniz identity:

$$[[x, y], z] = [[x, z], y] + [x, [y, z]].$$

cf. [L1]. When it happens that the bracket is skew-symmetric, we get a Lie algebra since the Leibniz identity becomes equivalent to the Jacobi identity.

Any associative algebra gives rise to a Lie algebra by $[x, y] = xy - yx$. The purpose of this article is to introduce and study a new notion of algebra which gives, by a similar procedure, a Leibniz algebra. The idea is to start with two distinct operations for the product xy and the product yx, so that the bracket is not necessarily skew-symmetric any more. Explicitly, we define an *associative dialgebra* (or simply dialgebra for short) as a vector space D equipped with two associative operations $\dashv$ and $\vdash$, called respectively left and right product, satisfying 3 more axioms:

$$\begin{cases} x \dashv (y \dashv z) = x \dashv (y \vdash z), \\ (x \vdash y) \dashv z = x \vdash (y \dashv z), \\ (x \dashv y) \vdash z = (x \vdash y) \vdash z. \end{cases}$$

It is immediate to check that $[x, y] := x \dashv y - y \vdash x$ defines a Leibniz bracket. Hence any dialgebra gives rise to a Leibniz algebra.

A typical example of dialgebra is constructed as follows. Let (A, d) be a differential associative algebra, and put

$$x \dashv y := x\ dy \quad \text{and} \quad x \vdash y := dx\ y.$$

One easily checks that $(A, \dashv, \vdash)$ is a dialgebra. For instance there is a natural dialgebra structure on the de Rham complex of a manifold.

Observe that, since the relations defining a dialgebra do not involve sums, there is a well-defined notion of *dimonoid*.

In this article we construct and study a (co)homology theory for dialgebras. Since an associative algebra is a particular case of dialgebra, we get a new (co)homology theory for associative algebras as well. The surprizing fact, in the construction of the chain complex, is the appearance of the combinatorics of *planar binary trees*. The principal result about this homology theory HY is its vanishing on free dialgebras. In order to state some of the properties of the theory HY, we introduce another type of algebras with two operations:

the *dendriform algebras* (sometimes called dendriform dialgebras). This notion dichotomizes the notion of associative algebra in the following sense: there are two operations $\prec$ and $\succ$, such that the product $*$ made of the sum of them

$$x * y := x \prec y + x \succ y,$$

is associative. The axioms relating these two products are

$$\begin{aligned}
\text{(i)} \quad & (a \prec b) \prec c = a \prec (b \prec c) + a \prec (b \succ c), \\
\text{(ii)} \quad & (a \succ b) \prec c = a \succ (b \prec c), \\
\text{(iii)} \quad & (a \prec b) \succ c + (a \succ b) \succ c = a \succ (b \succ c).
\end{aligned}$$

The free dendriform algebra can be constructed by means of the planar binary trees, whence the terminology.

The results intertwining associative dialgebras and dendriform algebras are best expressed in the framework of *algebraic operads.* The notion of dialgebra defines an algebraic operad *Dias*, which is binary and quadratic. By the theory of Ginzburg and Kapranov (cf. [GK]), there is a well-defined "dual operad" $Dias^!$. We show that this is precisely the operad *Dend* of the "dendriform algebras", in other words a dual associative dialgebra is nothing but a dendriform algebra. The vanishing of HY of a free dialgebra implies that those two operads are of a special kind: they are "Koszul operads". As a consequence the cohomology of a dialgebra is a graded dendriform algebra and, a fortiori, a graded associative algebra. The explicit description of the free dendriform algebra in terms of trees permits us to describe the notion of *strong homotopy associative dialgebra.*

The categories of algebras over these operads assemble into a commutative diagram of functors which reflects the Koszul duality.

$$\begin{array}{ccccccccc}
 & & \mathbf{Dend} & & & & \mathbf{Dias} & & \\
 & \nearrow & & \searrow & & \nearrow & & \searrow & \\
\mathbf{Zinb} & & & & \mathbf{As} & & & & \mathbf{Leib} \\
 & \searrow & & \nearrow & & \searrow & & \nearrow & \\
 & & \mathbf{Com} & & & & \mathbf{Lie} & &
\end{array}$$

In this diagram **Zinb** denotes the categories of Zinbiel algebras, which are Koszul dual to the Leibniz algebras.

This paper is part of a long-standing project whose ultimate aim is to study periodicity phenomenons in algebraic K-theory. This project is described in

[L4]. The next step would consist in computing the dialgebra homology of the augmentation ideal of $K[GL(A)]$, for an associative algebra A.

Here is the content of this article. In the first section we introduce the notion of associative dimonoid, or dimonoid for short, and develop the calculus in a dimonoid. In particular we describe the free dimonoid on a given set. In the second section we introduce the notion of dialgebra and give several examples. We explicitly describe the free dialgebra over a vector space. In the third section we construct the chain complex of a dialgebra D, which gives rise to homology and cohomology groups denoted $HY(D)$. The main tool is made of the planar binary trees and operations on them. We prove that HY of a free dialgebra vanishes (hence the operad associated to dialgebras is a Koszul operad). We also introduce a variation of the chain complex by replacing the trees by increasing trees, or, equivalently, by permutations. This variation appears naturally in the computation of the Leibniz homology of dialgebras of matrices (cf. [F1]). (Co)homology of dialgebras with non-trivial coefficients is treated by Alessandra Frabetti in [F4].

Section 4 is devoted to the relationship between Leibniz algebras and dialgebras. The functor which assigns to any dialgebra $(D, \vdash, \dashv)$ the Leibniz algebra $(D, [x,y] := x \dashv y - y \vdash x)$ has a left adjoint which is the *universal enveloping dialgebra* of a Leibniz algebra. Then we compare the diverse types of free algebras and we propose a definition for a *Poisson dialgebra*. The Hopf-type properties of the universal enveloping dialgebra are studied by François Goichot in [Go].

In the fifth section we introduce the notion of *dendriform algebra*, which is closely connected to the notion of associative dialgebra. For instance the tensor product of a dialgebra and of a dendriform algebra is naturally equipped with a structure of Lie algebra. The main result of this section is to make explicit the free dendriform algebra. It turns out that it is best expressed in terms of planar binary trees. The dendriform algebra structure on the vector space generated by the planar binary trees is the core of this section. It uses the grafting operation and the nesting operation on trees and it induces a graded associative algebra structure on the same vector space. In a sense associative algebras are closely connected with the integers (including addition and multiplication). Similarly dendriform algebras are closely connected with planar binary trees and a calculus on them. This arithmetic aspect of the theory will be treated elsewhere.

In section 6 we construct (co)homology groups for dendriform algebras. They vanish on free dendriform algebras.

In section 7 we relate dendriform algebras with Zinbiel algebras (i.e. dual-Leibniz algebras) and associative algebras. It is based on the relationship between binary trees and permutations as described in Appendix A.

The aim of the eighth section is to interpret the preceding results in the context of algebraic operads. The basics on algebraic operads and Koszul duality are recalled in Appendix B. We show that the operads associated to dialgebras and to dendriform algebras are dual in the operad sense. Then we show that the (co)homology groups HY for dialgebras (resp. H^{Dend} for dendriform algebras) constructed in section 3 (resp. 4) are the ones predicted by the operad theory. Hence, by the vanishing of HY of a free dialgebra, both operads $Dias$ and $Dend$ are Koszul. It implies, among several consequences, the vanishing of the homology of a free dendriform algebra. Some of the theorems in sections 2 to 6 can be proved either directly or by appealing to the operad theory. In general we write down the most elementary one.

The last section describes the notion of *strong homotopy associative dialgebras*. For any Koszul operad the notion of algebra up to homotopy is theoretically well-defined from the bar construction over the dual operad. Since, in our case, we know explicitly the structure of a free dendriform algebra, we can make the notion of dialgebra up to homotopy completely explicit.

Part of the results of this article has been announced in a "Note aux Comptes Rendus" [L2]. I thank Ale Frabetti, Benoit Fresse, Victor Gnedbaye, François Goichot, Phil Hanlon, Muriel Livernet, Teimuraz Pirashvili and Maria Ronco for fruitful conversations on this subject.

Contents.

1. DIMONOIDS

1.1. Definition. An *associative dimonoid*, or dimonoid for short, is a set X equipped with two maps called respectively *left product* and *right product*:

$$\text{(left)} \qquad \dashv\ : X \times X \to X,$$
$$\text{(right)} \qquad \vdash\ : X \times X \to X,$$

satisfying the following axioms

$$\begin{cases} x \dashv (y \dashv z) \overset{1}{=} (x \dashv y) \dashv z \overset{2}{=} x \dashv (y \vdash z), \\ (x \vdash y) \dashv z \overset{3}{=} x \vdash (y \dashv z), \\ (x \dashv y) \vdash z \overset{4}{=} x \vdash (y \vdash z) \overset{5}{=} (x \vdash y) \vdash z, \end{cases}$$

for all x, y and $z \in X$.

In the notation $x \dashv y$, $y \vdash x$, the element x is said to be on the *pointer* side and the element y is said to be on the *bar* side.

The numbers 1 to 5 of the relations are for future reference.

Observe that relations 1 and 5 are the "associativity" of the products $\dashv$ and $\vdash$ respectively. Relation 3 will be referred to as "inside associativity", since the products point inside. Relations 2 and 4 can be replaced by the relations 12 and 45:

$$x \dashv (y \dashv z) \overset{12}{=} x \dashv (y \vdash z) \quad \text{and} \quad (x \dashv y) \vdash z \overset{45}{=} (x \vdash y) \vdash z,$$

which can be summarized as "on the bar side, does not matter which product". All these relations are referred to as "diassociativity".

A *morphism* of dimonoids is a map $f : X \to Y$ between two dimonoids X and Y such that $f(x \dashv x') = f(x) \dashv f(x')$ and $f(x \vdash x') = f(x) \vdash f(x')$ for any $x, x' \in X$.

Observe that one can define a di-object in any monoidal category. One does not need the monoidal category to be symmetric since in each relation the variables stay in the same order.

1.2. Bar-unit. An element $e \in X$ is said to be a *bar-unit* of the dimonoid X if

$$x \dashv e = x = e \vdash x, \quad \text{for any } x \in X.$$

So it is only assumed that e acts trivially from the bar side. There is no reason for a bar-unit to be unique. The set of bar-units is called the *halo*.

A morphism of dimonoids is said to be *unital* if the image of a bar-unit is a bar-unit.

1.3. Examples.

a) Let M be a monoid (without unit), that is a set M with an associative product $(m, m') \mapsto mm'$. Putting $m \dashv m' = mm' = m \vdash m'$ gives a dimonoid structure on M. Indeed each relation 1 to 5 is the associativity property. A unit of the monoid is a bar-unit of the associated dimonoid.

Conversely, if in a dimonoid D there is a unit, that is an element $1 \in D$ which satisfies either $1 \dashv x = x$ or $x = x \vdash 1$ for all $x \in D$, then, by axiom 3 or 5, one has $\dashv = \vdash$ and D is simply the dimonoid associated to a unital monoid.

b) Let X be a set and define

$$x \dashv y = x = y \vdash x, \quad \text{for any} \quad x, y \in X.$$

Then, obviously, X is a (not so interesting) dimonoid and it coincides with its halo.

c) Let M be a monoid. Put $D = M \times M$ and define the products by

$$\begin{cases} (m,n) \dashv (m',n') := (m, nm'n') \\ (m,n) \vdash (m',n') := (mnm', n'). \end{cases}$$

With these definitions $D = (D, \dashv, \vdash)$ is a dimonoid. Let us check relation 3 for instance:

$$((m,n) \vdash (m',n')) \dashv (m'',n'') = (mnm',n') \dashv (m'',n'') = (mnm', n'm''n'')$$
$$(m,n) \vdash ((m',n') \dashv (m'',n'')) = (m,n) \vdash (m', n'm''n'') = (mnm', n'm''n'').$$

Let $1 \in M$ be a unit for M. Then $e = (1,1)$ is a bar-unit for D, but one has $e \dashv x \neq x$ and $x \vdash e \neq x$ in D in general. For any invertible element m the element $(m, m^{-1}) \in D$ is a bar-unit.

d) Let G be a group and X a G-set. The following formulas define a dimonoid structure on $X \times G$ (cf. 7.5):

$$\begin{cases} (x,g) \dashv (y,h) := (x, gh), \\ (x,g) \vdash (y,h) := (g \cdot y, gh). \end{cases}$$

1.4. Opposite dimonoid. Let D be a dimonoid. Define new operations $\dashv'$ and $\vdash'$ on D by

$$x \dashv' y := y \vdash x,$$
$$x \vdash' y := y \dashv x.$$

It is immediate to check that $(D, \dashv', \vdash')$ is a new dimonoid which we call the *opposite* dimonoid that we denote by D^{op}.

Observe that if we put

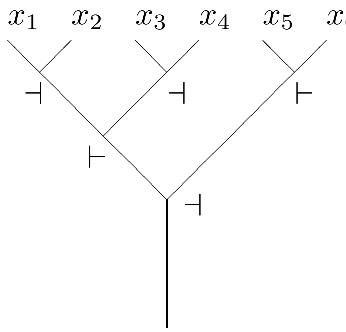
$$\begin{cases} x \dashv'' y := y \dashv x, \\ x \vdash'' y := y \vdash x, \end{cases}$$

then $(D, \vdash'', \dashv'')$ is *not* a dimonoid.

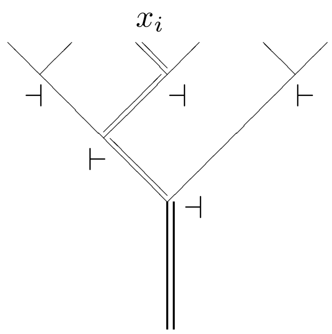
1.5. Monomials in a dimonoid. Let $x_1, \dots, x_n$ be elements in the dimonoid D. A *monomial* in D is a parenthesizing together with product signs, for instance

$$((x_1 \dashv x_2) \vdash (x_3 \dashv x_4)) \dashv (x_5 \vdash x_6),$$

giving rise to an element in D. Such a monomial is completely determined by a binary tree, where each vertex is labelled by $\dashv$ or $\vdash$:



1.6. The middle of a monomial. Given a monomial as above we define the *middle* of the monomial as being the entry x_i determined by the following algorithm. Starting at the root of the tree one goes up by choosing the route indicated by the pointer. The middle of the monomial is the abutment of the path. In this example x_3 is the middle.



1.7. Theorem (Dimonoid calculus). *Let x_i, $i \in \mathbf{Z}$, be elements in a dimonoid D.*

a) Any parenthesizing of

$$x_{-n} \vdash x_{-n+1} \vdash \dots \vdash x_{-1} \vdash x_0 \dashv x_1 \dashv \dots \dashv x_{m-1} \dashv x_m$$

gives the same element in D, which we denote by

$$x_{-n} \dots x_{-1} \check{x}_0 x_1 \dots x_m.$$

b) Let $m = x_1 \dots x_k$ be a monomial in D. Let x_i be its middle entry. Then $m = x_1 \dots \check{x}_i \dots x_k$.

c) One has the following formulas in D:

$$(x_1 \dots \check{x}_i \dots x_k) \dashv (x_{k+1} \dots \check{x}_j \dots x_\ell) = x_1 \dots \check{x}_i \dots x_k x_{k+1} \dots x_j \dots x_\ell$$
$$(x_1 \dots \check{x}_i \dots x_k) \vdash (x_{k+1} \dots \check{x}_j \dots x_\ell) = x_1 \dots x_i \dots x_k x_{k+1} \dots \check{x}_j \dots x_\ell.$$

For instance, in the above example, one has

$$((x_1 \dashv x_2) \vdash (x_3 \dashv x_4)) \dashv (x_5 \vdash x_6) = x_1 x_2 \check{x}_3 x_4 x_5 x_6.$$

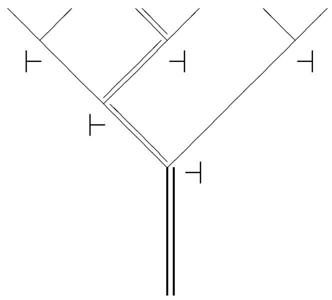
Proof. By axiom 1 (associativity of $\dashv$) any parenthesizing of $x_1 \dashv \dots \dashv x_m$ gives the same element. So, in such a monomial we can ignore the parentheses (and analogously for $\vdash$ thanks to axiom 5).

Consider a generic monomial with first entry x_{-n}, last entry x_m and middle entry x_0 (where $-n \leq 0 \leq m$). By axioms $1-3-5$ it is clear that the element

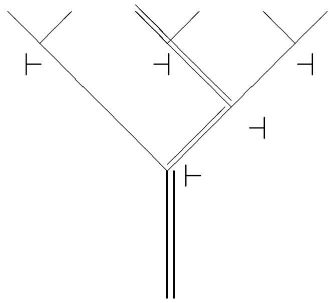
$$(*) \qquad (x_{-n} \vdash \dots \vdash x_{-1}) \vdash x_0 \dashv (x_1 \dashv \dots \dashv x_m)$$

is well-defined. We denote it by $x_{-n} \dots \check{x}_0 \dots x_m$.

Consider the labelled tree of our generic monomial. Let v be a vertex which is on the route from the root to the middle entry x_0. Thanks to axioms 12 and 45 all the vertices on the bar side of v can be labelled with the same label as v. In our example



Then by axiom 3 we can modify the tree so that all labels $\vdash$ come first:



This new tree corresponds to a monomial of the form $(*)$ and therefore we have proved that our starting monomial has value $x_{-n} \dots \check{x}_0 \dots x_m$. So parts a) and b) are proved.

By a) and b) it follows that in order to compute

$$(x_1 \dots \check{x}_i \dots x_k) \dashv (x_{k+1} \dots \check{x}_j \dots x_\ell) \quad \text{and} \quad (x_1 \dots \check{x}_i \dots x_k) \vdash (x_{k+1} \dots \check{x}_j \dots x_\ell)$$

it suffices to determine which entry is the middle of these monomials. By the algorithm described in 1.6, the middle entry is x_i in the first case and x_j in the second case. □

1.8. Corollary. *The free dimonoid on the set X is the disjoint union*

$$\mathcal{D}(X) = \coprod_{n \geq 1} (\underbrace{X^n \cup \dots \cup X^n}_{n \text{ copies}}).$$

Denoting by $x_1 \dots \check{x}_i \dots x_n$ an element in the i-th summand, the products are given by

$$(x_1 \dots \check{x}_i \dots x_k) \dashv (x_{k+1} \dots \check{x}_j \dots x_\ell) = x_1 \dots \check{x}_i \ \dots x_\ell \ ,$$
$$(x_1 \dots \check{x}_i \dots x_k) \vdash (x_{k+1} \dots \check{x}_j \dots x_\ell) = x_1 \dots \ \check{x}_j \dots x_\ell \ .$$

□

2. ASSOCIATIVE DIALGEBRAS

In the sequel K denotes a field referred to as the *ground field*. Later on it will be supposed to be of characteristic zero. The tensor product over K is denoted by $\otimes_K$ or, more often, by $\otimes$.

After introducing the notion of dialgebra, we give some examples, including free dialgebras, which we describe explicitly, and define modules and representations over a dialgebra.

2.1. Definition. An *associative dialgebra*, or *dialgebra* for short, over K is a K-module D equipped with two K-linear maps

$$\begin{aligned} \dashv &: D \otimes D \longrightarrow D, \\ \vdash &: D \otimes D \longrightarrow D, \end{aligned}$$

satisfying the *di-associativity* axioms

$$\begin{cases} (1) & (x \dashv y) \dashv z = x \dashv (y \vdash z), \\ (2) & (x \dashv y) \dashv z = x \dashv (y \dashv z), \\ (3) & (x \vdash y) \dashv z = x \vdash (y \dashv z), \\ (4) & (x \dashv y) \vdash z = x \vdash (y \vdash z), \\ (5) & (x \vdash y) \vdash z = x \vdash (y \vdash z). \end{cases}$$

The maps $\dashv$ and $\vdash$ are called respectively the *left product* and the *right product.*

Here is an equivalent formulation of these axioms: the products $\dashv$ and $\vdash$ are associative and satisfy:

$$\begin{cases} (12) & x \dashv (y \dashv z) = x \dashv (y \vdash z), \\ (3) & (x \vdash y) \dashv z = x \vdash (y \dashv z), \\ (45) & (x \dashv y) \vdash z = (x \vdash y) \vdash z. \end{cases}$$

Observe that the analogue of formula (3), but with the product symbols pointing outward, is not valid in general: $(x \dashv y) \vdash z \neq x \dashv (y \vdash z)$.

A *morphism* of dialgebras from D to D' is a K-linear map $f : D \to D'$ such that

$$f(x \dashv y) = f(x) \dashv f(y) \quad \text{and} \quad f(x \vdash y) = f(x) \vdash f(y) \quad \text{for all} \quad x, y \in D.$$

We denote by **Dias** the category of dialgebras.

A *bar-unit* in D is an element $e \in D$ such that

$$x \dashv e = x = e \vdash x \quad \text{for all} \quad x \in D.$$

A bar-unit need not be unique. The subset of bar-units of D is called its *halo.*

A *unital dialgebra* is a dialgebra with a specified bar-unit e. This choice gives rise to a preferred K-linear map $K \hookrightarrow D, \lambda \mapsto \lambda e$.

A morphism of dialgebras is said to be *unital* if the image of any bar-unit is a bar-unit.

Observe that if a dialgebra has a unit ϵ, that is an element which satisfies $\epsilon \dashv x = x$ for any x, then $\dashv = \vdash$ by axiom 12, and D is an associative algebra with unit.

An *ideal* I in a dialgebra D is a submodule of D such that $x \dashv y$ and $x \vdash y$ are in I whenever one of the variables is in I. Clearly the quotient D/I is a dialgebra. Conversely, the kernel of a dialgebra morphism is an ideal.

2.2. Examples.

a) Associative algebra. If A is an associative algebra over K, then the formulas $a \dashv b = ab = a \vdash b$ define a structure of dialgebra on A. If 1 is a unit of the associative algebra, then $e = 1$ is a unit of the dialgebra and the halo is just $\{1\}$.

b) Differential associative algebra. Let (A, d) be a differential associative algebra. So, by hypothesis, $d(ab) = da\, b + a\, db$ (here we work in the non-graded setting) and $d^2 = 0$. Define left and right products on A by the formulas

$$x \dashv y := x\ dy \quad \text{and} \quad x \vdash y := dx\ y.$$

It is immediate to check that A equipped with these two products is a dialgebra. A similar construction holds in the graded (or more accurately super) algebra framework.

c) Dimonoid algebra. Let X be a dimonoid, and denote by $K[X]$ the free K-module on X. Then obviously $K[X]$ is a dialgebra.

d) Bimodule map. Let A be an associative algebra and let M be an A-bimodule. Let $f : M \to A$ be an A-bimodule map. Then one can put a dialgebra structure on M as follows:

$$m \dashv m' := mf(m'),$$
$$m \vdash m' := f(m)m',$$

The verification is left to the reader. One can systematize this procedure by considering the tensor category of linear maps as follows (cf. [LP2], [Ku] for details). The category of linear maps over K is made of the K-linear maps $f : V \to W$ as objects. It can be equipped with a tensor product by

$$(V \xrightarrow{f} W) \otimes (V' \xrightarrow{f'} W') = V \otimes W' \ \oplus\ W \otimes V' \xrightarrow{f\otimes 1 + 1 \otimes f'} W \otimes W'.$$

An associative algebra in this tensor category defines a dialgebra structure on the source object.

The particular case of the projection $M \oplus A \to A$ shows that there is a dialgebra structure on $M \oplus A$ (cf. P. Higgins [Hi]).

e) Tensor product, matrices. If D and D' are two dialgebras, then the tensor product $D \otimes D'$ is also a dialgebra by $(a \otimes a') \star (b \otimes b') = (a \star b) \otimes (a' \star b')$ for

$\star = \dashv$ and $\vdash$. For instance the module of $n\times n-$matrices $\mathcal{M}_n(D) = \mathcal{M}_n(K)\otimes D$ is a dialgebra. The left and right products are given by

$$(\alpha \dashv \beta)_{ij} = \sum_k \alpha_{ik} \dashv \beta_{kj} \quad \text{and} \quad (\alpha \vdash \beta)_{ij} = \sum_k \alpha_{ik} \vdash \beta_{kj}.$$

f) Opposite dialgebra. As for dimonoids, the opposite dialgebra of D is the dialgebra D^{op} with the same underlying K-module and with products given by

$$x \dashv' y = y \vdash x,\ x \vdash' y = y \dashv x.$$

g) Let A be an associative algebra over K. Put $D = A \otimes A$ and define

$$\begin{aligned} a\otimes b \dashv a'\otimes b' &:= a\otimes ba'b', \\ a\otimes b \vdash a'\otimes b' &:= aba'\otimes b'. \end{aligned}$$

Extending these formulas by linearity on $A\otimes A$ gives well-defined product maps $\dashv$ and $\vdash$ on D which satisfy the diassociativity axioms. If $1\in A$ is a unit of the associative algebra, then $1\otimes 1$ is a bar-unit for the dialgebra. More generally, for any invertible element x in A, the element $x\otimes x^{-1}$ is a bar-unit. If I is a left ideal and J is a right ideal, then the same formulas define a diassociative algebra structure on $I\otimes_K J$.

h) Let A be an associative algebra and n be a positive integer. On the module of n-vectors $D = A^n$ one puts:

$$\begin{aligned} (x\dashv y)_i &= x_i\Big(\sum_{j=1}^n y_j\Big) \quad \text{for} \quad 1\le i\le n \quad \text{and} \\ (x\vdash y)_i &= \Big(\sum_{j=1}^n x_j\Big)y_i \quad \text{for} \quad 1\le i\le n. \end{aligned}$$

One easily checks that D is a dialgebra. For $n=1$, this is example (a). In fact this construction can be extended to any dialgebra A.

2.3. Module, bimodule, extension. A *left module over a dialgebra D* is a K-module M equipped with two linear maps

(right structure) $\quad \dashv : D\otimes M \to M,$

(left structure) $\quad \vdash : D\otimes M \to M,$

satisfying the axioms (1)-(5) whenever they make sense. There is, of course, a similar definition for right modules.

A *bimodule* over a dialgebra D, also called a *representation*, is a K-module M equipped with four linear maps

(right structures) $\quad \dashv, \vdash : M\otimes D \to M,$

(left structures) $\quad \dashv, \vdash : D\otimes M \to M,$

satisfying the axioms (1) to (5), whenever one of the entries x, y or z is in M and the two others are in D.

Obviously a bimodule over D is, a fortiori, a left module and also a right module over D ; and D is a bimodule over itself.

Let

$$0 \to M \to \overline{D} \to D \to 0$$

be an abelian extension of dialgebras, that is an exact sequence of dialgebras such that any product of two elements in M is trivial. Then, it is immediate to check that M is a representation of D in the above sense.

2.4. Free associative dialgebra. Let V be a K-module. By definition the *free dialgebra* on V is the dialgebra $Dias(V)$ equipped with a K-linear map $i : V \to Dias(V)$ such that for any K-module map $f : V \to D$, where D is a dialgebra over K, there is a unique factorization

$$f : V \xrightarrow{i} Dias(V) \xrightarrow{\phi} D,$$

where ϕ is a dialgebra morphism.

Equivalently the functor $Dias : (K\text{-}\mathrm{Mod}) \to \mathbf{Dias}$ is left adjoint to the forgetful functor. The following proposition proves the existence of the free dialgebra $Dias(V)$ and gives an explicit description of it in terms of the tensor module

$$T(V) := K \oplus V \oplus V^{\otimes 2} \oplus \cdots \oplus V^{\otimes n} \oplus \cdots.$$

2.5. Theorem. *The free dialgebra on V is the K-module*

$$Dias(V) = T(V) \otimes V \otimes T(V)$$

equipped with the two products induced by:

$$\begin{aligned}(v_{-n} \cdots v_{-1} \otimes v_0 \otimes v_1 \cdots v_m) &\dashv (w_{-p} \cdots w_{-1} \otimes w_0 \otimes w_1 \cdots w_q) \\ &= v_{-n} \cdots v_{-1} \otimes v_0 \otimes v_1 \cdots v_m w_{-p} \cdots w_q, \\ (v_{-n} \cdots v_{-1} \otimes v_0 \otimes v_1 \cdots v_m) &\vdash (w_{-p} \cdots w_{-1} \otimes w_0 \otimes w_1 \cdots w_q) \\ &= v_{-n} \cdots v_m w_{-p} \cdots w_{-1} \otimes w_0 \otimes w_1 \cdots w_q,\end{aligned}$$

where v_i, $w_j \in V$.

With our notation (cf. 1.7) any additive generator of $Dias(V)$ can be written

$$v_{-n} \cdots v_{-1} \otimes v_0 \otimes v_1 \cdots v_m = v_{-n} \cdots v_{-1} \check{v}_0 v_1 \cdots v_m.$$

Proof. It is immediate to check that $Dias(V) = (T(V) \otimes V \otimes T(V), \dashv, \vdash)$ is a dialgebra (cf. 1.7). The map $i : V \to Dias(V)$ is the composite $V \simeq 1 \cdot K \otimes V \otimes 1 \cdot K \hookrightarrow T(V) \otimes V \otimes T(V)$. Starting with $f : V \to D$ the map $\phi : Dias(V) \to D$ is given by

$$\phi(v_{-n} \cdots v_{-1} \check{v}_0 v_1 \cdots v_m) = f(v_{-n}) \cdots f(v_{-1}) \check{f(v_0)} f(v_1) \cdots f(v_m).$$

It is obviously a dialgebra morphism. Moreover, by theorem 1.7, it is uniquely determined since it should coincide with f on $V \cong 1\cdot K \otimes V \otimes 1\cdot K$ and it should be a morphism of dialgebras. Hence the inclusion $V \to Dias(V)$ is universal. $\square$

Remark. A free dialgebra is a particular case of example 2.2.d, with $M = T(V) \otimes V \otimes T(V)$, $A = T(V)$ (the associative tensor algebra) and $f : M \to A$ being the concatenation.

Let V be finite dimensional over K generated by $x_1, \cdots, x_n$. Let us describe the degree n part of $T(V) \otimes V \otimes T(V)$ which is generated by all the monomials containing x_i once and only once, $1 \le i \le n$. We denote it by $Dias(n)$. These monomials are the elements

$$(\sigma, i)(x_1, \cdots, x_n) := x_{\sigma^{-1}(1)} \cdots \check{x}_{\sigma^{-1}(i)} \cdots x_{\sigma^{-1}(n)}, \quad \sigma \in S_n,\ 1 \le i \le n,$$

where S_n is the symmetric group. Therefore, as a left S_n-module, the multilinear part of this space is isomorphic to n copies of the regular representation of S_n:

$$Dias(n) \cong nK[S_n].$$

The element σ in the i-th copy corresponds to the operation (σ, i) described above (cf. Corollary 1.8).

Examples:

- $n = 1$, one generator: $\check{x}_1$.
- $n = 2$, four generators: $\check{x}_1 x_2, \check{x}_2 x_1, x_1 \check{x}_2, x_2 \check{x}_1$.
- $n = 3$, eighteen generators: $\check{x}_i x_j x_k, x_i \check{x}_j x_k, x_i x_j \check{x}_k$

for all permutations i, j, k of 1,2,3.

2.6. Associative algebra associated to a dialgebra. For any dialgebra D let D_{As} be the quotient of D by the ideal generated by the elements $x \dashv y - x \vdash y$, for all $x, y \in D$. It is clear that $\dashv = \vdash$ in D_{As}, hence D_{As} is an associative algebra (non-unital in general). The quotient map $\mu : D \twoheadrightarrow D_{As}$ is universal among the maps from D to associative algebras. In other words the associativization functor $(-)_{As} : \mathbf{Dias} \to \mathbf{As}$ is left adjoint to $inc : \mathbf{As} \to \mathbf{Dias}$.

Axioms 12 and 45 imply that the element $x \vdash y \dashv z$ in D depends only on the values of x and z in D_{As}. Hence D is a D_{As}-bimodule and the projection map μ is a D_{As}-bimodule map. On the other hand the dialgebra structure of D is completely determined by μ and the D_{As}-bimodule structure on the space D since

$$x \dashv y = x\,\mu(y) \quad \text{and} \quad x \vdash y = \mu(x)\,y\,,$$

cf. example 2.2.d. It is useful to write the element $x \vdash y \dashv z$ as $x\check{y}z$. Under this notation the dialgebra calculus rules are

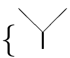
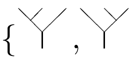
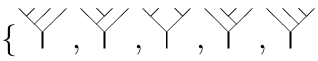
$$x\check{y}z \dashv s\check{t}u = x\check{y}zstu,$$
$$x\check{y}z \vdash s\check{t}u = xyzs\check{t}u.$$

3. (CO)HOMOLOGY OF ASSOCIATIVE DIALGEBRAS

In this section we introduce a chain complex which permits us to define homology groups $HY_*(D)$ and cohomology groups $HY^*(D)$ of a dialgebra D. The main ingredient is the set of *planar binary trees.* The main result of this section is the vanishing of the dialgebra homology of a free dialgebra.

An extension of this theory to a theory with coefficients is to be found in [F4].

3.1. Planar binary trees. A planar tree is *binary* if any vertex is trivalent. We denote by Y_n the set of planar binary trees with $n+1$ leaves. Since we only use planar binary trees in this section we abbreviate it into tree (or n-tree to specify that it has $n+1$ leaves, or, equivalently, n interior vertices).

$Y_0 = \{ | \}$, $Y_1 = \{$  $\}$, $Y_2 = \{$  $\}$, $Y_3 = \{$  $\}$.

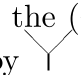
We will use the permutation-like notation of trees (cf. Appendix A):

[0] , [1] , [12], [21] , [123], [213], [131], [312], [321]

The number of elements in Y_n is the Catalan number $c_n = \frac{(2n)!}{n!(n+1)!}$. For any $y \in Y_n$ we label the $n+1$ leaves by $\{0, 1, \cdots, n\}$ from left to right.

3.2. Face and degeneracy maps. For any i, $0 \le i \le n$, there is a map, called a *face map*, $d_i : Y_n \to Y_{n-1}$ which assigns to the tree y the tree $d_i y$ obtained from y by deleting the i-th leaf. For instance:

$$d_0[213] = [12],\ d_1[213] = [12],\ d_2[213] = [12],\ d_3[213] = [21].$$

For any i, $0 \le i \le n$, there is a map, called a *degeneracy map*, s_i : $Y_n \to Y_{n+1}$ which assigns to the tree y the $(n+1)$-tree $s_i y$ obtained by bifurcating the i-th leaf, that is replace it by  . For instance

$$s_0[0] = [1], s_0[1] = [12]\ , s_1[1] = [21].$$

The face and degeneracy maps satisfy all the classical simplicial relations, *except* for the relation $s_i s_i = s_{i+1} s_i$. Indeed, this relation is not fulfilled on trees, because

$$s_0 s_0([0]) = [12] \text{ and } s_1 s_0([0]) = [21]\ .$$

So $Y_.$ is not a simplicial set, but only an *almost simplicial set*, (cf. [F3]).

For any i, $1 \le i \le n-1$, there is a map

$$\circ_i : Y_n \ \to \{\dashv, \vdash\}$$

defined as follows. The image of $y \in Y_n$ is $\circ_i^y = \dashv$ (resp. $\vdash$) if the i-th leaf points from the vertex to the left (resp. to the right). For instance:

$$\circ_1^{[131]} = \vdash \quad \text{and} \quad \circ_2^{[131]} = \dashv \quad .$$

More generally one has

$$\circ_i^{[j_1,\cdots,j_n]} = \dashv \text{ if } j_i > j_{i+1} \text{ and } \circ_i^{[j_1,\cdots,j_n]} = \vdash \text{ if } j_i < j_{i+1}, \text{ for } 1 \le i \le n-1.$$

Here is the table in low dimension:

y	d_1	$\circ_1$	d_2	$\circ_2$
[12]	[1]	$\vdash$		
[21]	[1]	$\dashv$		
[123]	[12]	$\vdash$	[12]	$\vdash$
[213]	[12]	$\dashv$	[12]	$\vdash$
[131]	[21]	$\vdash$	[12]	$\dashv$
[312]	[21]	$\dashv$	[21]	$\vdash$
[321]	[21]	$\dashv$	[21]	$\dashv$

3.3. The chain complex of a dialgebra. Let D be a dialgebra over K. Define the module of n-chains by

$$CY_n(D) := K[Y_n] \otimes D^{\otimes n},$$

in particular $CY_1(D) \cong D$, $CY_2(D) \cong D^{\otimes 2} \oplus D^{\otimes 2}$, more generally $CY_n(D)$ is isomorphic to the direct sum of c_n copies of $D^{\otimes n}$ (indexed by Y_n).

Define a linear map $d : CY_n(D) \to CY_{n-1}(D)$ by the following formula:

$$d(y; a_1, \cdots, a_n) := -\sum_{i=1}^{n-1} (-1)^i (d_i(y); a_1, \cdots, a_{i-1}, a_i \circ_i^y a_{i+1}, \cdots, a_n),$$

where $y \in Y_n$ and $a_i \in D$. This formula has a meaning since $\circ_i^y = \dashv$ or $\vdash$ and D is a dialgebra. It is convenient to define

$$d_i(y; a_1, \cdots, a_n) := (d_i(y); a_1, \cdots, a_{i-1}, a_i \circ_i^y a_{i+1}, \cdots, a_n)$$

so that $d = -\sum_{i=1}^{n-1} (-1)^i d_i$.

3.4. Lemma. *The face maps $d_i : CY_n(D) \to CY_{n-1}(D)$ satisfy the simplicial relations*

$$d_i d_j = d_{j-1} d_i, \textit{ for any } 1 \le i < j \le n-1.$$

Proof. We first prove this identity in the lowest dimension, that is

$$(*) \qquad d_1 d_2 = d_1 d_1 : CY_3(D) \to CY_2(D)$$

The computation of $d_i d_j(y; a, b, c)$ splits into 5 cases corresponding to the five trees with 4 leaves (cf. 3.1).

- *Case* [123] :

$$d_1 d_2([123]; a, b, c) = d_1([12]; a, b \vdash c) = ([1]; a \vdash (b \vdash c)),$$
$$d_1 d_1([123]; a, b, c) = d_1([12]; (a \vdash b, c) = ([1]; (a \vdash b) \vdash c).$$

So relation $(*)$ follows from axiom 5.

- *Case* [213] :

$$d_1 d_2([213]; a, b, c) = d_1([12]; a, b \vdash c) = ([1]; a \vdash (b \vdash c)),$$
$$d_1 d_1([213]; a, b, c) = d_1([12]; (a \dashv b, c) = ([1]; (a \dashv b) \vdash c).$$

So relation $(*)$ follows from axiom 4.

- *Case* [131] :

$$d_1 d_2([131]; a, b, c) = d_1([12]; a, b \dashv c) = ([1]; a \vdash (b \dashv c)),$$
$$d_1 d_1([131]; a, b, c) = d_1([21]; (a \vdash b, c) = ([1]; (a \vdash b) \dashv c).$$

So relation $(*)$ follows from axiom 3.

- *Case* [312] :

$$d_1 d_2([312]; a, b, c) = d_1([21]; a, b \vdash c) = ([1]; a \dashv (b \vdash c)),$$
$$d_1 d_1([312]; a, b, c) = d_1([21]; (a \dashv b, c) = ([1]; (a \dashv b) \dashv c).$$

So relation $(*)$ follows from axiom 2.

- *Case* [321] :

$$d_1 d_2([321]; a, b, c) = d_1([21]; a, b \dashv c) = ([1]; a \dashv (b \dashv c)),$$
$$d_1 d_1([321]; a, b, c) = d_1([21]; (a \dashv b, c) = ([1]; (a \dashv b) \dashv c).$$

So relation $(*)$ follows from axiom 1.

The proof of the general case $d_i d_j = d_{j-1} d_i$ for $i < j$ splits into two different cases.

First, if $j = i + 1$, then the proof is exactly as in low dimension and so follows from the axioms of a dialgebra. Second, if $j > i+1$, then both operations $d_i d_j$ and $d_{j-1} d_i$ amount to perform the same modification: removing the leaves number j and i of the tree y, and replace $(a_1, \cdots, a_n)$ by

$$(a_1, \cdots, a_i \circ_i^y a_{i+1}, \cdots, a_j \circ_j^y a_{j+1}, \cdots, a_n).$$

The point is that the leaf number j of y is the leaf number $j - 1$ of $d_i(y)$.

So we have proved that $d_i d_j = d_{j-1} d_i$ for $i < j$. □

3.5. Proposition. *One has* $d \circ d = 0$ *and so* $(CY_*(D), d)$ *is a chain-complex.*

Proof. This is an immediate consequence of the previous lemma, like for a pre-simplicial module. □

Observe that in the chain complex

$$CY_*(D): \quad \cdots \to K[Y_n] \otimes D^{\otimes n} \to \cdots \to K[Y_3] \otimes D^{\otimes 3} \to K[Y_2] \otimes D^{\otimes 2} \xrightarrow{(\dashv,\vdash)} D$$

the module of n-chains is the direct sum of c_n copies of $D^{\otimes n}$ (indexed by the set of trees Y_n), the first differential is induced by the two products $\dashv, \vdash$, and the first relation $d^2 = 0$ coincides precisely with the 5 axioms of a dialgebra.

3.6. Homology and cohomology of a dialgebra. By definition the *homology of the dialgebra* D is the homology of the chain-complex $CY_*(D)$:

$$HY_n(D) := H_n(CY_*(D), d), \ n \geq 1.$$

For $n = 1$ it is immediate that $HY_1(D)$ is the quotient of D by the submodule generated by all the elements $x \dashv y$ and $x \vdash y$,

$$HY_1(D) = D/\{x \dashv y, x \vdash y \mid x, y \in D\},$$

which we denote, sometimes, by D/D^2.

By definition the *cohomology of the dialgebra* D is

$$HY^n(D) := H^n(\mathrm{Hom}(CY_*(D), K)), \ n \geq 1.$$

3.7. The chain bicomplex of a dialgebra. The chain complex of a dialgebra is in fact the total chain complex associated to a bicomplex. Indeed, let $Y_{p,q}$ be the subset of Y_n made of the trees which are obtained by grafting a p-tree with a q-tree (cf. Appendix A), where $p + q + 1 = n$. For instance

$$Y_{0,2} = \{[321], [312]\}, \quad Y_{1,1} = \{[131]\}, \quad Y_{2,0} = \{[213], [123]\}.$$

Let $CY_{p,q} := K[Y_{p,q}] \otimes D^{\otimes n}$. Since for any $y \in Y_{p,q}$ the element $d_i(y)$ is either in $Y_{p-1,q}$ or in $Y_{p,q-1}$, the face map d_i takes value either in $CY_{p-1,q}$ or in $CY_{p,q-1}$. So the chain bicomplex $CY_{**}(D)$ is well-defined and its associated total complex is $CY_*(D)$. Remark that, with our choice of notation, one has $CY_n = \bigoplus_{p+q+1=n} CY_{p,q}$. This bicomplex gives rise to two spectral sequences abutting to $HY_*(D)$.

3.8. Theorem. *Let* V *be a vector space over* K *and* $Dias(V) = TV \otimes V \otimes TV$ *be the free dialgebra over* V *(cf.2.5). Then, one has*

$$HY_1(Dias(V)) \cong V,$$
$$HY_n(Dias(V)) = 0, \ for \ n > 1.$$

Proof. The first statement is obvious since $V \cong K \otimes V \otimes K$ is the quotient of $CY_1 = TV \otimes V \otimes TV$ by the submodule generated by all the products of elements of V.

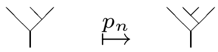
To show that $HY_n = 0$ for $n > 1$ we construct a homotopy

$$h = h_n : K[Y_n] \otimes D^{\otimes n} \to K[Y_{n+1}] \otimes D^{\otimes n+1}$$

such that, for $n > 1$,

$$dh_n + h_{n-1}d = \mathrm{id}_n \, .$$

In order to write down h explicitly we use the degeneracy maps introduced in 3.2 and the following construction. Given an n-tree y, we denote by $p_n(y)$ the $(n+1)$-tree obtained from y by adding a new leaf at the left of the last one and parallel to it:



There are five different formulas for $h_n(x)$ depending on the form of $x \in K[Y_n] \otimes D^{\otimes n}$.

• Case (a): $x = (y; \check{\omega}_1, ..., \check{\omega}_{n-1}, \check{\omega}_n u)$. One puts

$$h_n(y; \check{\omega}_1, ..., \check{\omega}_{n-1}, \check{\omega}_n u) := (-1)^n (s_n(y); \check{\omega}_1, ..., \check{\omega}_{n-1}, \check{\omega}_n, \check{u}).$$

First, one has $d_i h_n(x) + h_{n-1} d_i(x) = 0$ for $1 \le i \le n-1$ because the modifications on x performed by h and d are disjoint. Second, one has

$$(-1)^n d_n h_n(x) = d_n(s_n(y); \check{\omega}_1, ..., \check{\omega}_{n-1}, \check{\omega}_n, \check{u}) = (y; \check{\omega}_1, ..., \check{\omega}_{n-1}, \check{\omega}_n u) = x$$

since $d_n s_n(y) = y$ and $\circ_n^{s_n(y)} = \dashv$. So we have proved relation (*) in this case.

• Case (b): $x = (y; \check{\omega}_1, ..., \check{\omega}_{n-1}, \omega_n v \check{u})$. One puts

$$h_n(y; \check{\omega}_1, ..., \check{\omega}_{n-1}, \omega_n v \check{u}) := (-1)^n (p_n(y); \check{\omega}_1, ..., \check{\omega}_{n-1}, \omega_n \check{v}, \check{u}).$$

First, one has $d_i h_n(x) + h_{n-1} d_i(x) = 0$ for $1 \le i \le n-1$. Second, one has

$$(-1)^n d_n h_n(x) = d_n(p_n(y); \check{\omega}_1, ..., \check{\omega}_{n-1}, \omega_n \check{v}, \check{u}) = (y; \check{\omega}_1, ..., \check{\omega}_{n-1}, \omega_n v \check{u}) = x$$

since $d_n p_n(y) = y$ and $\circ_n^{p_n(y)} = \vdash$. So we have proved relation (*) in this case.

• Case (c): $x = (y; \check{\omega}_1, ..., \check{\omega}_{n-1}, \check{u})$ and the last two leaves of y have the shape  . One puts

$$h_n(y; \check{\omega}_1, ..., \check{\omega}_{n-1}, \check{u}) := 0.$$

First, one has $d_i h_n(x) = 0$ for $1 \le i \le n$ and $h_{n-1} d_i(x) = 0$ for $1 \le i \le n-2$. Second, one has

$$(-1)^{n-1} h_{n-1} d_{n-1}(x) = (-1)^{n-1} h_{n-1}(d_{n-1}(y); \check{\omega}_1, ..., \check{\omega}_{n-1} u) = x$$

since (by using case (b)) $s_n d_{n-1}(y) = y$ for such y. So we have proved relation (*) in this case.

• Case (d): $x = (y; \check{\omega}_1, ..., \check{\omega}_{n-1}v, \check{u})$ and the last two leaves of y have the shape  . One puts

$$h_n(y; \check{\omega}_1, ..., \check{\omega}_{n-1}v, \check{u}) := (-1)^n (s_n(y) - p_{n-1}(y); \check{\omega}_1, ..., \check{\omega}_{n-1}, \check{v}, \check{u}).$$

Let us write $h_n = (-1)^n \underline{s}_n + (-1)^{n-1} \underline{p}_{n-1}$. First, one has $d_i h_n(x) + h_{n-1} d_i(x) = 0$ for $1 \leq i \leq n-2$. Second, one has $d_{n-1} h_n(x) + h_{n-1} d_{n-1}(x) = x$ since $d_{n-1} \underline{s}_n(x) = h_{n-1} d_{n-1}(x)$ and $d_{n-1} \underline{p}_{n-1}(x) = x$. Third, one has $d_n h_n(x) = 0$ since $d_n \underline{p}_{n-1}(x) = d_n \underline{s}_n(x)$. So we have proved relation (*) in this case.

• Case (e): $x = (y; \check{\omega}_1, ..., \omega_{n-1}\check{v}, \check{u})$ and the last two leaves of y have the shape  . One puts

$$h_n(y; \check{\omega}_1, ..., \omega_{n-1}\check{v}, \check{u}) := (-1)^n (s_n(y) - s_{n-1}(y); \check{\omega}_1, ..., \check{\omega}_{n-1}\ , \check{v}, \check{u}).$$

Let us write $h_n = (-1)^n \underline{s}_n + (-1)^{n-1} \underline{s}_{n-1}$. First, one has $d_i h_n(x) + h_{n-1} d_i(x) = 0$ for $1 \leq i \leq n-2$. Observe that in many cases $d_i h_n = 0 = h_{n-1} d_i$. Second, one has $d_{n-1} h_n(x) + h_{n-1} d_{n-1}(x) = x$ since $d_{n-1} \underline{s}_n(x) = h_{n-1} d_{n-1}(x)$ and $d_{n-1} \underline{s}_{n-1}(x) = x$. Third, one has $d_n h_n(x) = 0$ since $d_n \underline{s}_{n-1}(x) = d_n \underline{s}_n(x)$. So we have proved relation (*) in this case. □

3.9. Theorem. *For any dialgebra D the graded module $HY_*(D)$ is a graded dual-codialgebra and the graded module $HY^*(D)$ is a graded dendriform algebra (see section 5). As a consequence $HY^*(D)$ is a graded associative algebra.*

Proof. Though one could prove these statements directly, they are consequences of general facts about Koszul operads (see Appendix B). We will show in section 6 that the operad of dendriform algebras is dual to the operad of associative dialgebras. Moreover, by theorem 3.8 these operads are Koszul, hence the statement follows from general properties of Koszul operads (cf. Appendix B5d). The last statement is a consequence of the preceding one and of Lemma 7.3. □

3.10. Simplicial properties of the chain-modules. We have seen in Lemma 3.4 that the face maps $d_i : CY_n(D) \to CY_{n-1}(D)$ satisfy the standard simplicial relations. Suppose that D is equipped with a bar-unit e and let us define $s_j : CY_n(D) \to CY_{n+1}(D)$ by

$$s_j(y; a_1, \cdots, a_n) := (s_j(y); a_1, \cdots, a_j, e, a_{j+1}, \cdots, a_n),\ 0 \leq j \leq n,$$

where $s_j(y)$ is described in 3.2. From the properties of the bar-unit, it is immediate to check that

$$d_i s_j = \begin{cases} s_{j-1} d_i & \text{for } i < j, \\ \text{id} & \text{for } i = j,\ i = j+1, \\ s_j d_{i-1} & \text{for } i > j+1, \end{cases}$$

$$s_i s_j = s_{j+1} s_i \quad \text{for } i < j.$$

So the family $(CY_n(D); d_i, s_j)_{n\geq 0}$ is an *almost simplicial module*, that is the face and degeneracy operators satisfy all the standard relations of a simplicial module, *except* for the relation $s_i s_i = s_{i+1} s_i$ (cf. 3.2).

A variation of the Eilenberg-Zilber theorem is still valid for pseudo-simplicial modules (cf. Inassaridze [I]) and a fortiori for almost simplicial modules. It is used in the proof of the following result which is due to Alessandra Frabetti.

3.11. Theorem [F4]. *If D is a dialgebra equipped with a bar-unit, then*

$$HY_n(D) = 0, \text{ for any } n \geq 0.$$

□

Comment. This result is similar to the vanishing of the bar-homology for a unital associative algebra.

3.12. Generalization of the homology of a dialgebra to the symmetric group. There is a generalization of the complex CY_* consisting in replacing the set of planar binary trees Y_n by the symmetric group S_n, or, equivalently, by the set $\tilde{Y}_n$ of *binary increasing trees* (cf. Appendix A).

The formulas for the maps d_i are the same as in 3.2 and 3.3 once S_n has been identified with $\tilde{Y}_n$ (observe that deleting a leaf in an increasing tree still gives an increasing tree). Hence we get a new complex

$$CS_*(D): \quad \cdots \to K[S_n] \otimes D^{\otimes n} \to \cdots \to K[S_3] \otimes D^{\otimes 3} \to K[S_2] \otimes D^{\otimes 2} \xrightarrow{(\dashv,\vdash)} D$$

for any dialgebra D, and new homology groups $HS_*(D)$.

The boundary map is still the alternate sum of face maps. When the dialgebra is bar-unital, then there also exist degeneracy maps. All these maps satisfy the simplicial relations *except* the relations involving only the degeneracy maps. Such an object is called a *pseudo-simplicial* module (cf. [I]).

Forgetting the levels gives a map $\Psi : S_n = \tilde{Y}_n \to Y_n$ (cf. Appendix A) which induces a chain map $CS_*(D) \to CY_*(D)$ and hence a morphism

$$HS_*(D) \to HY_*(D).$$

This new theory HS_*, or, more accurately, its variant with non trivial coefficients, crops up naturally when one wants to compute the Leibniz homology of the Leibniz algebra of matrices $\mathbf{g}l(D)$ over a dialgebra D (cf. Frabetti [F2]).

3.13. Remark. We will show in section 6 that $CY_*(D)$ is the chain complex of the dialgebra D predicted by the operad theory. One could wonder if there is another notion of algebra for which $CS_*(D)$ would be the predicted chain complex. If this would be the case, then the Poincaré series of the dual operad would be the inverse of the series

$$\sum_{n\geq 1} (-1)^n \# S_n x^n = \sum_{n\geq 1} (-1)^n n! x^n.$$

However this inverse is not of the form $\sum_{n\geq 1} (-1)^n a_n x^n$ with $a_n \in \mathbf{N}$, hence this operad, even if it existed, could not be a Koszul operad.

4. LEIBNIZ ALGEBRAS, ASSOCIATIVE DIALGEBRAS AND HOMOLOGY

A Leibniz algebra is a non-commutative version of a Lie algebra. In this section we show that, when we replace Lie algebras by Leibniz algebras, then the role of associative algebras is played by the associative dialgebras. In particular we show that any Leibniz algebra has a *universal enveloping associative dialgebra*.

4.1. Leibniz algebras [L1], [LP]. Recall that a *Leibniz algebra* over K is a K-module $\mathbf{g}$ equipped with a binary operation (called a bracket):

$$[-,-] : \mathbf{g} \otimes \mathbf{g} \to \mathbf{g},$$

which satisfies the *Leibniz identity*:

$$[x,[y,z]] = [[x,y],z] - [[x,z],y],$$

for all x, y, z in $\mathbf{g}$. This is in fact a *right* Leibniz algebra. For the opposite structure, that is $[x,y]' = [y,x]$, the left Leibniz identity is

$$[[x,y]',z]' = [x,[y,z]']' - [y,[x,z]']'.$$

If the bracket happens to be anticommutative, then $\mathbf{g}$ is a Lie algebra. Quotienting the Leibniz algebra $\mathbf{g}$ by the ideal generated by the elements $[x,x]$ for all $x \in \mathbf{g}$ gives a Lie algebra that we denote by $\mathbf{g}_{Lie}$.

To any Leibniz algebra $\mathbf{g}$ is associated a chain-complex

$$CL_*(\mathbf{g}) : \qquad \ldots \longrightarrow \mathbf{g}^{\otimes n} \xrightarrow{d} \mathbf{g}^{\otimes n-1} \xrightarrow{d} \ldots \xrightarrow{d} \mathbf{g}^{\otimes 2} \xrightarrow{d} \mathbf{g}$$

where

$$d(x_1, \ldots, x_n) = \sum_{1 \le i < j \le n} (-1)^j (x_1, \ldots, x_{i-1}, [x_i, x_j], x_{i+1}, \ldots, \hat{x}_j, \ldots, x_n).$$

The homology groups of this complex are denoted by $HL_n(\mathbf{g})$, for $n \geq 1$.

4.2. Proposition. *Let D be a dialgebra. Then the bracket*

$$[x,y] := x \dashv y - y \vdash x$$

makes D into a Leibniz algebra, denoted by D_{Leib}.

Proof. It is a straightforward checking in which axioms (1), (2), (4), (5) are used once and axiom (3) twice since

$$\begin{aligned} [x,[y,z]] &= x \dashv (y \dashv z) - x \dashv (z \vdash y) - (y \dashv z) \vdash x + (z \vdash y) \vdash x, \\ -[[x,y],z]] &= -(x \dashv y) \dashv z + (y \vdash x) \dashv z + z \vdash (x \dashv y) - z \vdash (y \vdash x), \\ [[x,z],y]] &= (x \dashv z) \dashv y - (z \vdash x) \dashv y - y \vdash (x \dashv z) + y \vdash (z \vdash x). \end{aligned}$$

□

This construction defines a functor

$$\mathbf{Dias} \xrightarrow{-} \mathbf{Leib}$$

from the category **Dias** of dialgebras to the category **Leib** of Leibniz algebras.

4.3. Example. For any dialgebra D over K the $n \times n$-matrices with entries in D form a new dialgebra $\mathcal{M}_n(D)$. Its associated Leibniz algebra is denoted $\mathbf{g}l_n(D)$. It is not a Lie algebra in general. The homology of $\mathbf{g}l(D)$, when D has a bar-unit, has been computed by Frabetti [F2].

4.4. Proposition. *The following diagram of categories and functors is commutative*

$$\begin{array}{ccc} \mathbf{Dias} & \xrightarrow{-} & \mathbf{Leib} \\ \uparrow & & \uparrow \\ \mathbf{As} & \xrightarrow{-} & \mathbf{Lie}. \end{array}$$

□

4.5. Remark. In the definition of a dialgebra, axiom (3) could be relaxed slightly, though proposition 4.2 remains valid. It could be replaced by the weaker axiom:

$$((y \vdash x) \dashv z) + (z \vdash (x \dashv y)) = (y \vdash (x \dashv z)) + ((z \vdash x) \dashv y).$$

Observe that in this formula the variables do not stay in the same order in the monomials. Hence the associated operad would not be a non-Σ-operad anymore.

4.6. Universal enveloping associative dialgebra of a Leibniz algebra. The functor $-$: $\mathbf{As} \to \mathbf{Lie}$ has a left adjoint which is the universal enveloping algebra of a Lie algebra:

$$\overline{U}(\mathbf{g}) = \overline{T}(\mathbf{g})/\{[x,y] - x \otimes y + y \otimes x \mid x, y \in \mathbf{g}\}.$$

($\overline{U}(\mathbf{g})$ is the augmentation ideal of the classical enveloping unital algebra $U(\mathbf{g})$). Similarly, define the *universal enveloping dialgebra* of a Leibniz algebra $\mathbf{g}$ as the following quotient of the free dialgebra on $\mathbf{g}$:

$$Ud(\mathbf{g}) := T(\mathbf{g}) \otimes \mathbf{g} \otimes T(\mathbf{g})/\{[x,y] - x \dashv y + y \vdash x \mid x, y \in \mathbf{g}\}.$$

Under our previous notation, the elements generating the ideal are denoted by $\check{[x,y]} - \check{x}y + y\check{x}$.

4.7. Proposition. *The functor* $Ud : \mathbf{Leib} \to \mathbf{Dias}$ *is left adjoint to the functor* $-$: $\mathbf{Dias} \to \mathbf{Leib}$.

Proof. Let $f : \mathbf{g} \to D_{Leib}$ be a morphism of Leibniz algebras. There is a unique extension of f as a morphism of dialgebras from $T(\mathbf{g}) \otimes \mathbf{g} \otimes T(\mathbf{g})$ to D. Since the image of $[x,y]-x\dashv y+y\vdash x$ under this morphism is 0, it defines a morphism from $Ud(\mathbf{g})$ to D.

On the other hand the restriction of the morphism of dialgebras $g: Ud(\mathbf{g})\to D$ to $\mathbf{g}=K\otimes\mathbf{g}\otimes K$ yields a morphism of Leibniz algebras $\mathbf{g}\to D_{Leib}$.

It is now immediate to check that these two constructions give rise to isomorphisms

$$\mathrm{Hom}_{\mathbf{Dias}}(Ud(\mathbf{g}),D)\cong\mathrm{Hom}_{\mathbf{Leib}}(\mathbf{g},D_{Leib}).$$

□

It is well-known that the universal enveloping algebra of a Lie algebra is not only an associative algebra but a Hopf algebra. Similarly the universal enveloping dialgebra of a Leibniz algebra possesses co-operations. They are studied by Goichot in [Go].

4.8. Lemma. *For any Leibniz algebra* $\mathbf{g}$*, one has* $Ud(\mathbf{g})_{As}=U(\mathbf{g}_{Lie})$*.*

Proof. Since the functor $(-)_{As}:\mathbf{Dias}\to\mathbf{As}$ is left adjoint to $inc:\mathbf{As}\to\mathbf{Dias}$ and since $(-)_{Lie}:\mathbf{Leib}\to\mathbf{Lie}$ is left adjoint to $inc:\mathbf{Lie}\to\mathbf{Leib}$ both composites $U\circ(-)_{Lie}$ and $(-)_{As}\circ Ud$ are left adjoint to the composite $\mathbf{Leib}\to\mathbf{Lie}\to\mathbf{As}$, and so are equal. □

4.9. Proposition. *The universal enveloping dialgebra* $Ud(\mathbf{g})$ *is isomorphic to* $U(\mathbf{g}_{Lie})\otimes\mathbf{g}$*, equipped with the dialgebra structure issued from a* $U(\mathbf{g}_{Lie})$*-bimodule structure and the bimodule map* $U(\mathbf{g}_{Lie})\otimes\mathbf{g}\to U(\mathbf{g}_{Lie})$ *(cf. example 2.2.d).*

Proof. Let us define a $U(\mathbf{g}_{Lie})$-bimodule structure on $U(\mathbf{g}_{Lie})\otimes\mathbf{g}$. The left module structure is given by multiplication in the left factor. The right module structure is induced by

$$(\omega\otimes x)\cdot\bar{y}:=\omega\otimes[x,y]+\omega\bar{y}\otimes x,$$

where $\omega\in U(\mathbf{g}_{Lie}), x\in\mathbf{g},\bar{y}\in\mathbf{g}_{Lie}$ and $y\in\mathbf{g}$ is a lifting of $\bar{y}\in\mathbf{g}_{Lie}$. It is a well-defined element because the bracket $[x,y]$ in the Leibniz algebra $\mathbf{g}$ depends only on the class of y in $\mathbf{g}_{Lie}$. Let us check that this formula provides a representation of $\mathbf{g}_{Lie}$.

Let $\bar{y}$ and $\bar{z}$ be elements in $\mathbf{g}_{Lie}$ and y,z be liftings in $\mathbf{g}$. On one hand one gets

$$((\omega\otimes x\cdot\bar{y})\cdot\bar{z}=\omega\otimes[[x,y],z]+\omega\bar{z}\otimes[x,y]+\omega\bar{y}\otimes[x,z]+\omega\bar{y}\bar{z}\otimes x,$$

and

$$((\omega\otimes x\cdot\bar{z})\cdot\bar{y}=\omega\otimes[[x,z],y]+\omega\bar{y}\otimes[x,z]+\omega\bar{z}\otimes[x,y]+\omega\bar{z}\bar{y}\otimes x.$$

Hence one has

$$\begin{aligned}((\omega\otimes x\cdot\bar{z})\cdot\bar{y}-((\omega\otimes x\cdot\bar{z})\cdot\bar{y}&=\omega\otimes([x,y],z]-[x,z],y])-\omega(\bar{y}\bar{z}-\bar{z}\bar{y})\otimes x\\&=\omega\otimes[x,[y,z]]-\omega\overline{[y,z]}\otimes x\\&=(\omega\otimes x)\cdot[y,z]\,.\end{aligned}$$

The right and left module structures are immediately seen to be compatible, hence $U(\mathbf{g}_{Lie}) \otimes \mathbf{g}$ is a $U(\mathbf{g}_{Lie})$-bimodule.

The linear map $U(\mathbf{g}_{Lie}) \otimes \mathbf{g} \to U(\mathbf{g}_{Lie}), \omega \otimes x \mapsto \omega\bar{x}$ is a $U(\mathbf{g}_{Lie})$-bimodule because $\omega\bar{x}\bar{y} = \omega\overline{[x,y]} + \omega\bar{y}\bar{x}$.

So, it follows that $U(\mathbf{g}_{Lie}) \otimes \mathbf{g}$ is equipped with a dialgebra structure (cf. 22.d). The (nonunital) associative algebra associated to this dialgebra is the augmentation ideal of $U(\mathbf{g}_{Lie})$, cf. 2.6.

There is a well-defined dialgebra map

$$Ud(\mathbf{g}) \to U(\mathbf{g}_{Lie}) \otimes \mathbf{g}$$

which sends $\omega \otimes x \otimes 1$ to $\bar{\omega} \otimes x$ (for $\omega \in T(\mathbf{g})$ and $\bar{\omega}$ its image in $U(\mathbf{g}_{Lie})$). Indeed, any element in $Ud(\mathbf{g})$ can be written as a linear combination of elements of the form $\omega \otimes x \otimes 1$ since

$$\omega \otimes x \otimes y = \omega \otimes [x,y] \otimes 1 + \omega y \otimes x \otimes 1.$$

Since by lemma 4.8 one has $Ud(\mathbf{g})_{As} = U(\mathbf{g}_{Lie})$, it follows that the element $\omega\check{x}\omega'$ in $Ud(\mathbf{g})$ depends only on the class of $\omega \in T(\mathbf{g})$ (resp. $\omega' \in T(\mathbf{g})$) in $U(\mathbf{g}_{Lie})$. So, on can define a dialgebra map

$$U(\mathbf{g}_{Lie}) \otimes \mathbf{g} \to Ud(\mathbf{g})$$

by sending $\bar{\omega} \otimes x$ to $\omega\check{x}$, where ω is a lifting of $\bar{\omega}$.

It is immediate to check that both composites are the identity, whence the isomorphism. $\square$

4.10. Free algebras and free dialgebras. Let V be a K-module and let $\overline{T}(V) = V \oplus V^{\otimes 2} \oplus \cdots \oplus V^{\otimes n} \oplus \cdots$ be the tensor module. We denote by γ the endomorphism of $\overline{T}(V)$ defined inductively by $\gamma(v) = v$ for $v \in V$ and $\gamma(\omega \otimes v) = \omega \otimes v - v \otimes \omega$ for $\omega \in V^{\otimes n}$ and $v \in V$. It is well-known that $\mathrm{Im}\,\gamma$ is isomorphic to the free Lie algebra $Lie(V)$ over V. Recall that the free associative algebra over V is $\overline{T}(V)$ equipped with the concatenation product, and the free Leibniz algebra over V is $\overline{T}(V)$ equipped with the unique Leibniz bracket which satisfies $[\omega, v] = \omega \otimes v$ for $\omega \in V^{\otimes n}$ and $v \in V$ (cf. [LP]). In the sequence

$$\overline{T}(V) \xrightarrow{\gamma} Lie(V) \hookrightarrow \overline{T}(V),$$

the first map is a map of Leibniz algebras, the second one is a map of Lie algebras (for the Lie structure of $\overline{T}(V)$ coming from its associative algebra structure). From Proposition 4.4 it follows that there is a commutative diagram

$$\begin{array}{ccccc}
Leib(V) = \overline{T}(V) & \xrightarrow{\check{\gamma}_1} & \overline{T}(V) \otimes V \otimes \overline{T}(V) & = Dias(V) \\
\gamma \downarrow & & \downarrow fusion & \\
Lie(V) & \hookrightarrow & \overline{T}(V) & = As(V)
\end{array}$$

where the maps *fusion* and $\check{\gamma_1}$ are described as follows.

The fusion map consists in forgetting the symbol $\check{}$, that is $\omega\check{v}\omega' \mapsto \omega v\omega'$.

The image of $v_1\cdots v_n$ by $\check{\gamma_1}$ is $\gamma(\check{v}_1 v_2\cdots v_n)$, which means writing $\gamma(v_1\cdots v_n)$ and putting the symbol $\check{}$ on the variable v_1 of each monomial. For instance $\check{\gamma_1}(v_1) = \check{v}_1, \check{\gamma_1}(v_1\otimes v_2) = \check{v}_1 v_2 - v_2\check{v}_1, \check{\gamma_1}(v_1\otimes v_2\otimes v_3) = \check{v}_1 v_2 v_3 - v_2\check{v}_1 v_3 - v_3\check{v}_1 v_2 + v_3 v_2\check{v}_1$.

One observes that this map is very similar to the map α used by Lodder in [Lo] to describe the loop suspension of a wedge product of topological spaces.

4.11. Proposition. *Let V be a K-module and $Leib(V) = \overline{T}(V)$ be the free Leibniz algebra on V. Then one has an isomorphism*

$$Ud(Leib(V)) \cong Dias(V) = T(V)\otimes V\otimes T(V).$$

Proof. The functor *Leib* is left adjoint to the forgetful functor from Leibniz algebras to modules. Similarly the functor Ud is left adjoint to the functor from dialgebras to Leibniz algebras, therefore the composite is left adjoint to the forgetful functor from dialgebras to modules, so it is the free diassociative algebra functor. □

4.12. Theorem. *For any dialgebra D there is a natural transformation*

$$HL_*(D_{Leib}) \longrightarrow HS_*(D)$$

induced by the chain complex map

$$\epsilon_n : CL_n(D_{Leib}) = D^{\otimes n} \longrightarrow K[S_n]\otimes D^{\otimes n} = CS_n(D),$$
$$\epsilon_n(x_1,\cdots,x_n) = \sum_{\sigma\in S_n} \mathrm{sgn}(\sigma)\sigma\otimes\sigma^{-1}(x_1,\cdots,x_n).$$

Proof. The boundary map d_L of the Leibniz complex CL_* is described in 4.1. For any element $(\underline{x},y) := (x_1,\ldots,x_n,y)\in D^{\otimes n+1}$ the map d_L is defined inductively by the formula:

$$d_L(\underline{x},y) = (d_L(\underline{x}),y) + (-1)^n\,\mathrm{ad}\,(y)(\underline{x}), \tag{4.12.1}$$

where $\mathrm{ad}\,(y)(\underline{x}) = \mathrm{ad}\,(y)(x_1,\ldots,x_n) := \sum_{i=1}^n (x_1,\ldots,x_i\dashv y - y\vdash x_i,\ldots,x_n)$.

The boundary map d_S of the symmetric complex CS_* is described in 3.12 and 3.2.

One extends the operator $\mathrm{ad}\,(y)$ to $K[S_n]\otimes D^{\otimes n}$ by putting $\mathrm{ad}\,(y)(\sigma\otimes\underline{x}) = \sigma\otimes\mathrm{ad}\,(y)(\underline{x})\in K[S_n]\otimes D^{\otimes n}$, so that it obviously commutes with ϵ_n:

$$\epsilon_n\,\mathrm{ad}\,(y) = \mathrm{ad}\,(y)\epsilon_n. \tag{4.12.2}$$

It turns out that this new operator is homotopic to 0. The homotopy $h(y) : K[S_n] \otimes D^{\otimes n} \to K[S_{n+1}] \otimes D^{\otimes n+1}$ is given by

$$h(y)(\sigma \otimes \underline{x}) := \sum_{i=0}^{n} (-1)^i s_i(\sigma) \otimes (x_1, \ldots, x_i, y, x_{i+1}, \ldots, x_n),$$

where $s_i(\sigma)$ is the i-th degeneracy of σ (bifurcate the i-th leaf of the corresponding increasing tree, cf. 3.2). The checking of

$$d_S h(y) + h(y) d_S = \mathrm{ad}\,(y) \tag{4.12.3}$$

is tedious but straightforward.

The comparison of the homotopy operator $h(y)$ with the symmetrization operator ϵ_n gives:

$$\epsilon_{n+1}(\underline{x}, y) = (-1)^n h(y) \epsilon_n(\underline{x}). \tag{4.12.4}$$

The proof of the commutation relation

$$d_S \circ \epsilon_n = \epsilon_{n-1} \circ d_L \tag{$(*)_n$}$$

is done by induction on n as follows.

For $n = 1$, one has $\epsilon_1 = \mathrm{Id}$.

For $n = 2$, the map $\epsilon_2 : D^{\otimes 2} \to K[S_2] \otimes D^{\otimes 2}$ is given by

$$\epsilon_2(x, y) = [12] \otimes (x, y) - [21] \otimes (y, x).$$

Since $d_L(x, y) = [x, y]$ and $d_S([12] \otimes (x, y)) = x \dashv y$, $d_S([21] \otimes (x, y)) = x \vdash y$, it follows from $[x, y] = x \dashv y - y \vdash x$ that $d_S \circ \epsilon_2 = \epsilon_1 \circ d_L$.

By induction we suppose that $(*)_n$ holds and we will prove $(*)_{n+1}$.

One gets:

$$\begin{aligned}
d_S \epsilon_{n+1}(\underline{x}, y) &= (-1)^n d_S h(y) \epsilon_n(\underline{x}) && \textit{by } (4.12.4)\\
&= (-1)^n (\mathrm{ad}\,(y) - h(y) d_S) \epsilon_n(\underline{x}) && \textit{by } (4.12.3)\\
&= (-1)^n \,\mathrm{ad}\,(y) \epsilon_n(\underline{x}) + (-1)^{n-1} h(y) \epsilon_{n-1} d_L(\underline{x}) && \textit{by induction}\\
&= (-1)^n \,\mathrm{ad}\,(y) \epsilon_n(\underline{x}) + \epsilon_n(d_L(\underline{x}), y) && \textit{by } (4.12.4)\\
&= (-1)^n \epsilon_n(\underline{x})\,\mathrm{ad}\,(y) + \epsilon_n(d_L(\underline{x}), y) && \textit{by } (4.12.2)\\
&= \epsilon_n d_L(\underline{x}, y) && \textit{by } (4.12.1).
\end{aligned}$$

□

Remark. This proof is mimicked on the proof for the Lie case as done in [L0], Proposition 1.3.5. This Proposition has been extended to homology of dialgebras with coefficients by Alessandra Frabetti in her thesis (unpublished).

4.13. Proposition. *The composite*

$$HL_*(D_{Leib}) \to HS_*(D) \xrightarrow{\psi_*} HY_*(D),$$

where $\psi_n : S_n \twoheadrightarrow Y_n$ *is the surjective map described in Appendix A, is the map induced, through the operad theory, by the morphism of operads* $Dias^! \to Leib^!$.

Proof. We show in section 8 that the complexes CL_* and CY_* are the complexes predicted by the operad theory. So, by Appendix B5e, it suffices to check that the natural map $K[Y_n] \otimes K[S_n] = Dias^!(n) \to Leib^!(n) = K[S_n]$ on the dual operads (see section 8) is given by $y \otimes \omega \mapsto \sum_{\{\sigma \in S_n | \psi(\sigma)=y\}} \sigma\omega$, and this is precisely Theorem 7.5. □

4.14. Comparison of $HL_*(\mathbf{g})$ and $HY_*(Ud(\mathbf{g}))$ for a Leibniz algebra g. For a Lie algebra **h** the composite map

$$H_*^{Lie}(\mathbf{h}) \to H_*^{Lie}(\bar{U}(\mathbf{h})_{Lie}) \to H_*^{As}(\bar{U}(\mathbf{h})),$$

is known to be an isomorphism (cf. for instance [L0]).

However, for a Leibniz algebra **g**, the composite map

$$HL_*(\mathbf{g}) \to HL_*(Ud(\mathbf{g})_{Leib}) \to HY_*(Ud(\mathbf{g}))$$

is no longer an isomorphism, contrarily to what was mistakenly announced in [L2]. The main point in the proof of the Lie case, is that the graded space associated to the filtration of $U(\mathbf{g})$ depends only on the vector space **g**, and not on the Lie structure. In the Leibniz case, the graded space associated to $Ud(\mathbf{g}) = U(\mathbf{g}_{Lie}) \otimes \mathbf{g}$ (cf. Proposition 4.9) depends on $\mathbf{g}_{Lie}$, that is, on the Leibniz structure of **g**. Hence the map $HL_*(\mathbf{g}) \to HY_*(Ud(\mathbf{g}))$ is part of a spectral sequence involving the derived functors of "Lie-zation".

4.15. Poisson dialgebra. By definition a *Poisson dialgebra* P is a vector space P equipped with a dialgebra structure $\dashv$ and $\vdash$, and a Leibniz structure $[-,-]$ which are compatible in the sense that they satisfy the following 4 relations:

$$[x, y \dashv z] = y \vdash [x, z] + [x, y] \dashv z = [x, y \vdash z],$$

$$[x \dashv y, z] = x \dashv [y, z] + [x, z] \dashv y,$$

$$[x \vdash y, z] = x \vdash [y, z] + [x, z] \vdash y.$$

This definition generalizes the "non-commutative Poisson algebra" as defined in [Kub], [KS] and [Ak].

5. DENDRIFORM ALGEBRAS

In this chapter we construct a new type of algebra with two binary operations, which dichotomizes the notion of associative algebra. It is closely related to associative dialgebras. In fact, we show in section 8 that its operad is Koszul dual to the operad of associative dialgebras. The terminology is due to the structure of the free dendriform algebra, which is best described in terms of planar binary trees.

5.1. Definition. A *dendriform algebra* E over K is a K-vector space E equipped with two binary operations

$$\begin{aligned}\prec &: E \otimes E \to E,\\ \succ &: E \otimes E \to E,\end{aligned}$$

which satisfy the following axioms:

$$\begin{aligned}
&\text{(i)} && (a \prec b) \prec c = a \prec (b \prec c) + a \prec (b \succ c),\\
&\text{(ii)} && (a \succ b) \prec c = a \succ (b \prec c),\\
&\text{(iii)} && (a \prec b) \succ c + (a \succ b) \succ c = a \succ (b \succ c),
\end{aligned}$$

for any elements a, b and c in E.

It is sometimes preferable to call this object *dendriform dialgebra* to insist on the fact that it is defined by two operations, but we do not use this terminology here. It is important to observe that, like for associative algebras and associative dialgebras, the monomials involved in the relations keep the variables in the same order. As a consequence the associated operad is a non-Σ-operad.

Observe that there is no "monoid" version of dendriform algebras, since relations (i) and (iii) involve sums. In other words, the associated operad does not come from a set-operad.

By introducing the operation

$$x * y := x \prec y + x \succ y,$$

these relations take the following more concise form:

$$\begin{aligned}
&\text{(i)} && (a \prec b) \prec c = a \prec (b * c),\\
&\text{(ii)} && (a \succ b) \prec c = a \succ (b \prec c),\\
&\text{(iii)} && (a * b) \succ c = a \succ (b \succ c).
\end{aligned}$$

5.2. Lemma. *For any dendriform algebra E the product defined by*

$$x * y := x \prec y + x \succ y.$$

is associative.

Proof. Adding up the three equalities (i), (ii) and (iii) we get $(x * y) * z$ on the left hand side and $x * (y * z)$ on the right hand side, whence the statement. □

It follows from this lemma that a dendriform algebra is in fact an associative algebra, whose product has some special property. The category of dendriform algebras is denoted by **Dend**.

5.3. Proposition. *Let D be a dialgebra and E a dendriform algebra. Then, on the tensor product $D \otimes E$, the bracket*

$$\begin{aligned}[x \otimes a, y \otimes b] :=& (x \dashv y) \otimes (a \prec b) - (y \vdash x) \otimes (b \succ a) \\ & - (y \dashv x) \otimes (b \prec a) + (x \vdash y) \otimes (a \succ b),\end{aligned}$$

where $x, y \in D, a, b \in E$, defines a structure of Lie algebra.

Proof. The bracket is antisymmetric by definition. Hence, it suffices to show that the Jacobi identity is fulfilled.

The Jacobi identity for $x \otimes a$, $y \otimes b$, $z \otimes c$ gives a total of 48 terms, in fact $8 \times 3!$ terms. There are 8 terms for which x, y, z (and also a, b, c) stay in the same order. The other sets of 8 terms are permutations of this set which reads:

$$\begin{array}{lcl} x \dashv (y \dashv z) \otimes a \prec (b \prec c) & - & (x \dashv y) \dashv z \otimes (a \prec b) \prec c, \\ x \vdash (y \dashv z) \otimes a \succ (b \prec c) & - & (x \vdash y) \dashv z \otimes (a \succ b) \prec c, \\ x \dashv (y \vdash z) \otimes a \prec (b \succ c) & - & (x \dashv y) \vdash z \otimes (a \prec b) \succ c, \\ x \vdash (y \vdash z) \otimes a \succ (b \succ c) & - & (x \vdash y) \vdash z \otimes (a \succ b) \succ c. \end{array}$$

The terms 1 and 3 in column 1 together with the term 1 in column 2 cancel due to axioms (1), (2) and (i). Similarly the terms 41, 32 and 42 cancel due to axioms (4), (5) and (iii). Finally the terms 21 and 22 cancel due to axioms (3) and (ii). □

This result may also be seen as a consequence of Koszul duality (cf. Appendix B).

5.4. Examples of dendriform algebras.
(a) Shuffle algebra. Let V be a vector space and let $T'(V)$ be the reduced tensor module over V equipped with the shuffle product (which is associative and commutative). The shuffle of two generating elements $v_1 \cdots v_p$ and $v_{p+1} \cdots v_{p+q}$ can be split into two parts depending on the fact that the first element is v_1 or v_{p+1}. The first part gives the left product and the second part gives the right product. One can show that the shuffle algebra is then a dendriform algebra (this fact had been previously remarked by Gian-Carlo Rota).
(b) Matrices over dendriform algebras. Since in the axioms of a dendriform algebra the variables a, b, c stay in this order in all the monomials, the tensor product of two dendriform algebras is naturally a dendriform algebra. Similarly, let $\mathcal{M}_n(E)$ be the module of $n \times n$-matrices with entries in the dendriform algebra E. Then the formulas

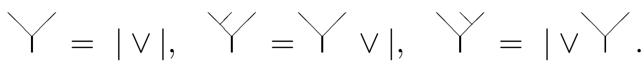
$$(\alpha \prec \beta)_{ij} = \sum_k \alpha_{ik} \prec \beta_{kj} \quad \text{and} \quad (\alpha \succ \beta)_{ij} = \sum_k \alpha_{ik} \succ \beta_{kj}$$

make $\mathcal{M}_n(E)$ into a dendriform algebra.

(c) Free dendriform algebra. Let V be a K-module and denote by $Dend(V)$ the free dendriform algebra over V. It is a dendriform algebra which satisfies the classical universal property. We will prove its existence and give an explicit description in 5.7. As a first step we describe the free dendriform algebra on one generator by using the sets of planar binary trees Y_n, and some operations on them.

5.5. Grafting operation on trees. By definition the *grafting* of the trees $y \in Y_p$ and $z \in Y_q$ is the tree $y \vee z \in Y_{p+q+1}$ obtained from y and z by joining their roots together and adding a new root. Observe that the number of internal vertices of $y \vee z$ is the sum of the numbers of internal vertices of y and of z plus 1.

Given a tree y (different from $|$) there is a unique decomposition $y = y_1 \vee y_2$. For instance one has:



which, with our notation, reads

$$[1] = [0] \vee [0], \quad [12] = [1] \vee [0], \quad [21] = [0] \vee [1].$$

The grafting operation is easy to write down in terms of the permutation-like notation. Indeed, for $y \in Y_p$ and $z \in Y_q$, one has

$$[y] \vee [z] = [y\ p+q+1\ z].$$

For instance one has:

$$[1] \vee [1] = [1\ 3\ 1], \qquad [1\ 3\ 1] \vee [2\ 1] = [1\ 3\ 1\ 6\ 2\ 1].$$

We put $K[Y_\infty] := \oplus_{n \geq 0} K[Y_n]$ and $\overline{K[Y_\infty]} := \oplus_{n \geq 1} K[Y_n]$. We introduce recursively the following operations on $K[Y_\infty]$:

(5.5.1) $\quad y \prec z := y_1 \vee (y_2 * z),$

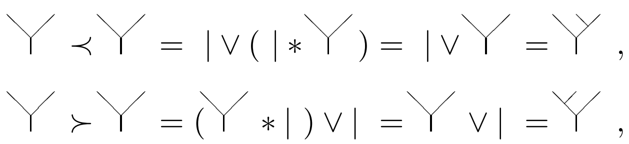
(5.5.2) $\quad y \succ z := (y * z_1) \vee z_2,$

(5.5.3) $\quad y * z := y \prec z + y \succ z,$

(5.5.4) $\quad x \prec | := x =: | \succ x$ and $x \succ | := 0 =: | \prec x,$ for $x \neq |.$

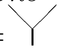
for $y \in Y_p$ and $z \in Y_q$. Observe that $|$ is a unit for $*$.

Since the decomposition $y = y_1 \vee y_2$ is unique, it is clear that these formulas are well-defined by recursion. For instance



or, equivalently, $[1] \prec [1] = [21], \quad [1] \succ [1] = [12]$.

These operations are extended to $K[Y_\infty]$ by linearity. Observe that $[0] * [0] = [0]$, but $[0] \prec [0]$ and $[0] \succ [0]$ are not defined.

5.6. Lemma. *The vector space $\oplus_{n\geq 1} K[Y_n]$ equipped with the two operations $\prec$ and $\succ$ described above is a dendriform algebra, which is generated by* $[1] =$  .

Proof. We prove this assertion by induction on the (total) degree of the trees.

Let $y = y_1 \vee y_2,\ y' = y'_1 \vee y'_2$ and $y'' = y''_1 \vee y''_2$ be planar binary trees. The following equalities follow by induction from the definitions of the operations and the associativity of $*$:

$$\begin{aligned}
(i)\quad & (y \prec y') \prec y'' = (y_1 \vee (y_2 * y')) \prec y'' \\
& = y_1 \vee (y_2 * y' * y'') = y \prec (y' \prec y'') \ + \ y \prec (y' \succ y'').
\end{aligned}$$

$$\begin{aligned}
(ii)\quad & y \succ (y' \prec y'') = y \succ (y'_1 \vee (y'_2 * y'')) = (y * y'_1) \vee (y'_2 * y'') \\
& = ((y * y'_1) \vee y'_2) \prec y'' = (y \succ y') \prec y''.
\end{aligned}$$

$$\begin{aligned}
(iii)\quad & y \succ (y' \succ y'') = y \succ ((y' * y''_1) \vee y''_2) \\
& = (y * y' * y''_1) \vee y''_2 = (y \prec y') \succ y'' + (y \succ y') \succ y''.
\end{aligned}$$

Let us show that $\overline{K[Y_\infty]}$ is generated by $[1]$ under the operations $\prec$ and $\succ$. Let $y = y_1 \vee y_2$ be a tree. From the definitions of the operations we have

$$\begin{aligned}
y_1 \vee y_2 &:= [1] && \text{if } y_1 = [0] = y_2, \\
&:= [1] \prec y_2 && \text{if } y_1 = [0] \neq y_2, \\
&:= y_1 \succ [1] && \text{if } y_1 \neq [0] = y_2, \\
&:= y_1 \succ [1] \prec y_2 && \text{if } y_1 \neq [0] \neq y_2.
\end{aligned}$$

Therefore, by induction, it is clear that $\overline{K[Y_\infty]}$ is generated by $[1]$. □

5.7. Proposition. *The unique dendriform algebra map $Dend(K) \to \oplus_{n\geq 1} K[Y_n]$ which sends the generator x of $Dend(K)$ to $[1]$ is an isomorphism.*

Proof. Let us show that the dendriform algebra $(\overline{K[Y_\infty]}, \prec, \succ)$ defined in 5.5 satisfies the universal condition to be the free dendriform algebra on one generator.

Let D be a dendriform algebra and a an element in D. Define a linear map $\alpha : \overline{K[Y_\infty]} \to D$ by its value on the trees $y = y_1 \vee y_2$ as follows :

$$\begin{aligned}
\alpha(y_1 \vee y_2) &:= a && \text{if } y_1 = [0] = y_2, \\
&:= a \prec \alpha(y_2) && \text{if } y_1 = [0] \neq y_2, \\
&:= \alpha(y_1) \succ a && \text{if } y_1 \neq [0] = y_2, \\
&:= \alpha(y_1) \succ a \prec \alpha(y_2) && \text{if } y_1 \neq [0] \neq y_2.
\end{aligned}$$

We claim that α is a morphism of dendriform algebras. The proof is by induction on the degree of the tree. Indeed, on one hand

$$\begin{aligned}\alpha(y \prec z) &= \alpha(y_1 \vee (y_2 * z))\\ &= \alpha(y_1) \succ a \prec \alpha(y_2 * z)\\ &= \alpha(y_1) \succ a \prec (\alpha(y_2) * \alpha(z)).\end{aligned}$$

On the other hand,

$$\begin{aligned}\alpha(y) \prec \alpha(z) &= (\alpha(y_1) \succ a \prec \alpha(y_2)) \prec \alpha(z)\\ &= ((\alpha(y_1) \succ a) \prec \alpha(y_2)) \prec \alpha(z)\\ &= (\alpha(y_1) \succ a) \prec (\alpha(y_2) * \alpha(z))\\ &= \alpha(y_1) \succ a \prec (\alpha(y_2) * \alpha(z)).\end{aligned}$$

Here we supposed that $y_1 \neq [0] \neq y_2$, but the proof is similar for the other cases.

Since by lemma 5.6 $\overline{K[Y_\infty]}$ is generated by [1], the morphism α such that $\alpha([1]) = a$ is unique.

It follows that $(\overline{K[Y_\infty]}, \prec, \succ)$ is the free dendriform algebra on one generator. □

5.8. Theorem (Free dendriform algebra). *The unique dendriform algebra map*

$$Dend(V) \to \oplus_{n\geq 1} K[Y_n] \otimes V^{\otimes n}$$

which sends the generator $v \in V$ to $[1] \otimes v$ is an isomorphism.

Proof. Define the dendriform algebra structure on $\oplus_{n\geq 1} K[Y_n] \otimes V^{\otimes n}$ by

$$\begin{aligned}y \otimes \omega \prec y' \otimes \omega' &:= (y \prec y') \otimes \omega\omega',\\ y \otimes \omega \succ y' \otimes \omega' &:= (y \succ y') \otimes \omega\omega'.\end{aligned}$$

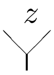
Since in the relations defining a dendriform algebra the variables stay in the same order, the free dendriform algebra over V is completely determined by the free dendriform algebra on one generator:

$$Dend(V) = \bigoplus_{n\geq 1} Dend(K)_n \otimes V^{\otimes n},$$

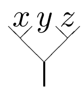
where $Dend(K)_n$ is the subspace of $Dend(K)$ generated by all the possible products of n copies of the generator. Hence, by proposition 5.7, one gets $Dend(V) \cong \oplus_n K[Y_n] \otimes V^{\otimes n}$. □

5.9. Remark. The inverse isomorphism is obtained as follows. From $y \in Y_n$ we construct a monomial in the variables $x_1, \cdots, x_n$ by first putting the variable

x_i in between the leaves $i-1$ and i. Then, for each vertex of depth one, we replace the local patterns with two vertices by local patterns with one vertex:

Continue the process until one reaches 

. The element $z \in Dend(V)$ is the image of $(y; x_1 \otimes \cdots \otimes x_n) \in K[Y_n] \otimes V^{\otimes n}$. Observe that sometimes one needs to choose an order to perform the process. But, thanks to the second axiom of dendriform algebras, the result does not depend on this choice:

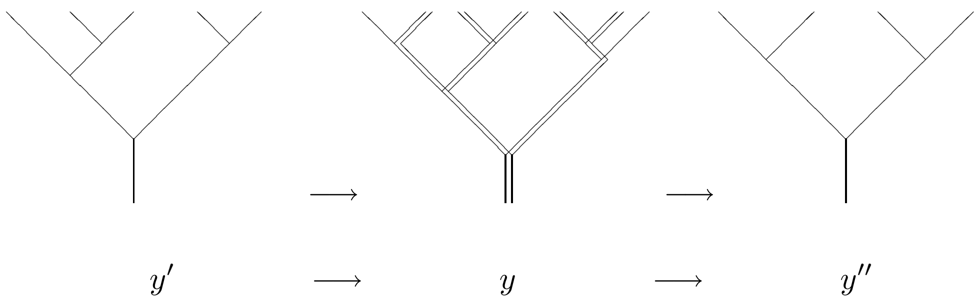


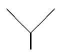
$$\mapsto (x \succ y) \prec z = x \succ (y \prec z) \quad .$$

Observe that the right S_n-module $Dend(n)$ which is such that $Dend(V) = \oplus_n Dend(n) \otimes_{S_n} V^{\otimes n}$, is the regular representation $K[Y_n] \otimes K[S_n]$.

5.10. Nested sub-trees and quotients. Given a planar binary tree with $n+1$ leaves and a consecutive sequence of $k+1$ leaves $\{i, \ldots, i+k\}$, the sub-tree of y which contains these leaves is isomorphic to a unique planar binary tree with $k+1$ leaves, which we denote by y'. We say that the sub-tree y' is *nested* in y at i. By definition the quotient $y'' = y/y'$ is the planar binary tree with $n-k+3$ leaves obtained from y by removing the leaves $\{i+1, \ldots, i+k-1\}$.

Example with $n = 6, i = 1, k = 4$ and $y = [131612]$:



Observe that y is a sub-tree of itself with quotient $[1] =$  , and that there are n nested sub-trees of y of the form $[1]$ (with quotient y).

In terms of the permutation-like notation the names of y' and of y'' are obtained as follows. Let $y = [a_1 \ldots a_n]$. Start with the sequence of integers $[a_{i+1} \ldots a_{i+k}]$ and make it into a name of tree by first replacing the largest integer by k. Then proceed the same way with the two remaining intervals, and so forth (like in A3). Ultimately one gets the name of y'. In our example we get $[2141]$. The quotient tree y'' of y by y' is obtained by making the sequence of numbers $[a_1 \ldots a_i \; a_j \; a_{i+k+1} \ldots a_n]$ into a name of tree as before (a_j is the largest integer in $[a_{i+1} \ldots a_{i+k}]$). In our example we get $[131]$.

5.11. Proposition. *Under the isomorphism of Proposition 5.7 the composition operation in Dend(K) induces the following composition operation in* $\oplus_{n\geq 1} K[Y_n]$*:*

$$y'' \circ_i y' = \sum y,$$

where the sum is extended over all the trees y *which contain* y' *as a nested sub-tree at* i *and for which* $y/y' = y''$*.*

Proof. Since the operad is quadratic it suffices to check this assertion in low dimension. The isomorphism gives $[21] = x \prec x$ and $[12] = x \succ x$. The eight distinct cases of composition are:

$$\begin{aligned}
&[21] \circ_1 [21] = (x \prec x) \prec x = x \prec (x \prec x) + x \prec (x \succ x) = [213] + [312]\\
&[21] \circ_1 [12] = (x \succ x) \prec x = [131]\\
&[21] \circ_2 [21] = x \prec (x \prec x) = [321]\\
&[21] \circ_2 [12] = x \prec (x \succ x) = [312]\\
&[12] \circ_1 [21] = (x \prec x) \succ x = [213]\\
&[12] \circ_1 [12] = (x \succ x) \succ x = [123]\\
&[12] \circ_2 [21] = x \succ (x \prec x) = [131]\\
&[12] \circ_2 [12] = x \succ (x \succ x) = x \prec (x \succ x) + x \succ (x \succ x) = [213] + [312]
\end{aligned}$$

In each case we verify that the trees of the right hand side are precisely such that y' is nested at i with quotient y''. For instance both [213] and [312] have [21] nested at 1. □.

5.12. Associative algebra structure on $\overline{K[Y_\infty]}$ and $K[S_\infty]$. By lemma 5.2 the vector space $\overline{K[Y_\infty]}$ is an associative algebra for the product

$$x * y = x \prec y + x \succ y,$$

hence $K[Y_\infty]$ is a graded associative and unital algebra whose structure is completely determined by the following two conditions: $|$ is a unit, the recursive formula

$$y * z = y_1 \vee (y_2 * z) + (y * z_1) \vee z_2,$$

holds.

Observe that this algebra has an obvious involution: $[i_1, \ldots, i_n] \mapsto [i_n, \ldots, i_1]$ on trees. In theorem 3.8 of [LR] we prove that it is isomorphic to the tensor algebra over $K[Y_{0,\infty}]$, where $Y_{0,n-1} := \{\ y = | \vee y' \in Y_n,\ \text{for } y' \in Y_{n-1}\}$. It is also isomorphic to $T(K[Y_{\infty,0}])$, where $Y_{n-1,0} := \{\ y = y' \vee |\}$.

The map $\psi_n : S_n \to Y_n$ (cf. Appendix A) induces a linear map $\psi : K[S_\infty] \to K[Y_\infty]$. There is an associative and unital algebra structure on $K[S_\infty]$ given by

$$x * y := sh_{n,m} \cdot (x \times y) \in K[S_{n+m}],$$

where $x \in S_n$, $y \in S_m$ and $sh_{n,m} \in K[S_{n+m}]$ is the sum of all the (n,m)-shuffles. It is proved in [LR] that ψ is an associative algebra homomorphism.

5.13. Hopf structure on $Dend(V)$**.** It is well-known that the free associative algebra $T(V)$ is in fact a cocommutative Hopf algebra, where the coproduct is given by the shuffle. Similarly there exists a structure of Hopf algebra on the associative unital algebra $K \oplus Dend(V) = \oplus_{n\geq 0} K[Y_n] \otimes V^{\otimes n}$. The coproduct is completely determined by the shuffles and the coproduct on $\oplus_{n\geq 0} K[Y_n]$ was constructed in [LR]. Observe that this coproduct is not cocommutative. It is related to the "brace algebras" and has been studied in details in [R].

6. (CO)HOMOLOGY OF DENDRIFORM ALGEBRAS

In this section we show that there exists a chain complex (of Hochschild type), for any dendriform algebra. It enables us to construct a homology and a cohomology theory for dendriform algebras. It will be proved in section 8 that these theories are the ones predicted by the operad theory in characteristic zero.

6.1. The chain complex of a dendriform algebra. Let E be a dendriform algebra and let C_n be the set $\{1, \cdots, n\}$. We define the module of n-chains of E as

$$C_n^{Dend}(E) := K[C_n] \otimes E^{\otimes n},$$

and the differential $d = -\sum_{i=1}^{n-1}(-1)^i d_i : C_n^{Dend}(E) \to C_{n-1}^{Dend}(E)$ as follows. First, we define the face operators $d_i, 1 \le i \le n-1$, on $r \in C_n$ by

$$d_i(r) = \begin{cases} r-1 & \text{if } i \le r-1, \\ r & \text{if } i \ge r. \end{cases}$$

These maps are extended linearly to maps

$$d_i : K[C_n] \to K[C_{n-1}],\ 1 \le i \le n-1.$$

Second, we define the symbol $\circ_i^r$ as follows:

$$\circ_i^r = \begin{cases} * & \text{if } i < r-1, \\ \succ & \text{if } i = r-1, \\ \prec & \text{if } i = r, \\ * & \text{if } i > r. \end{cases}$$

Recall that $x * y = x \prec y + x \succ y$.

Finally the map $d_i : C_n^{Dend}(E) \to C_{n-1}^{Dend}(E)$ is given by

$$d_i(r; x_1 \otimes \cdots x_n) := (d_i(r); x_1 \otimes \cdots \otimes x_{i-1} \otimes x_i \circ_i^r x_{i+1} \otimes \cdots \otimes x_n)$$

for $1 \le i \le n-1$.

6.2. Lemma. *The maps $d_i : C_n^{Dend}(E) \to C_{n-1}^{Dend}(E)$ satisfy the simplicial relations $d_i d_j = d_{j-1} d_i$, for $i < j$, and so $(C_*^{Dend}(E), d)$ is a chain complex.*

Proof. Let us first prove the lowest dimensional case, that is $d_1 d_1 = d_1 d_2$ on $C_3^{Dend}(E) = 3\ E^{\otimes 3}$.

On the first component ($r = 1$) one gets:

$$d_1 d_1(1; a \otimes b \otimes c) = d_1(1; (a \prec b) \otimes c) = (1; (a \prec b) \prec c),$$
$$d_1 d_2(1; a \otimes b \otimes c) = d_1(1; a \otimes (b * c)) = (1; a \prec (b * c)),$$

hence $d_1 d_1 = d_1 d_2$ by axiom (i).

On the second component ($r = 2$) one gets:

$$d_1 d_1(2; a \otimes b \otimes c) = d_1(1; (a \succ b) \otimes c) = (1; (a \succ b) \prec c),$$
$$d_1 d_2(2; a \otimes b \otimes c) = d_1(2; a \otimes (b \prec c)) = (1; a \succ (b \prec c)),$$

hence $d_1 d_1 = d_1 d_2$ by axiom (ii).

On the third component ($r = 3$) one gets:

$$d_1 d_1(3; a \otimes b \otimes c) = d_1(2; (a * b) \otimes c) = (1; (a * b) \succ c),$$
$$d_1 d_2(3; a \otimes b \otimes c) = d_1(2; a \otimes (b \succ c)) = (1; a \succ (b \succ c)),$$

hence $d_1 d_1 = d_1 d_2$ by axiom (iii).

Higher up, the verification of $d_i d_j = d_{j-1} d_i$ splits up into two different cases. First, if $j = i+1$, then it is the same kind of computation as above, so it is a consequence of the axioms of a dendriform algebra. Second, if $j > i+1$, then both operations agree on C_n (direct checking) and the image of $(a_1 \otimes \cdots \otimes a_n)$ is, in both cases,

$$(a_1 \otimes \cdots \otimes a_i \circ_i^r a_{i+1} \otimes \cdots \otimes a_j \circ_j^r a_{j+1} \otimes \cdots \otimes a_n).$$

□

6.3. The chain bicomplex of a dendriform algebra. Observe that $C_*^{Dend}(E)$ is in fact the total complex of a bicomplex. Indeed, let

$$C_{p,q}^{Dend}(E) = \{p\} E^{\otimes p+q} \quad \text{for} \quad p \geq 1, q \geq 0.$$

The map

$$d^h : C_{p,q}^{Dend}(E) \to C_{p-1,q}^{Dend}(E) \quad \text{by} \quad d^h := \sum_{i=1}^{p-1} (-1)^i d_i,$$

is well-defined because $d_i(p) = p - 1$, when $i \leq p - 1$, and

$$d^v : C_{p,q}^{Dend}(E) \to C_{p,q-1}^{Dend}(E) \quad \text{by} \quad d^v := \sum_{i=p}^{p+q} (-1)^i d_i.$$

is well-defined because $d_i(p) = p$, when $i \geq p$. In other words, the p-th component of $C_n^{Dend}(E)$ is put in bidegree $(p, n - p)$.

6.4. (Co)homology of dendriform algebras. By definition the homology (with trivial coefficients) of a dendriform algebra E is

$$H_*^{Dend}(E) := H_*(C_*^{Dend}(E), d) ,$$

and the cohomology of a dendriform algebra E (with trivial coefficients) is

$$H^*_{Dend}(E) := H^*(\mathrm{Hom}(C^{Dend}_*(E), K)).$$

Let us use freely the interpretation of the preceding results in terms of operads as devised in the next section. From Koszul duality and Appendix B5d, the graded module $H^*_{Dend}(E)$ is naturally equipped with a structure of graded dialgebra (and hence a structure of graded Leibniz algebra).

6.5. Theorem. *The dendriform algebra homology of a free dendriform algebra is trivial. More precisely*

$$H_n^{Dend}(Dend(V)) = 0 \ for \ n > 0,$$
$$H_1^{Dend}(Dend(V)) = V.$$

Proof. Since we know already that the analogous theorem for associative dialgebras is true (Theorem 3.8), it is a consequence (by the operad theory, cf. Appendix B), of the operad duality between *Dias* and *Dend* proved in Proposition 8.3. □

7. ZINBIEL ALGEBRAS, DENDRIFORM ALGEBRAS AND HOMOLOGY

In this section we introduce Zinbiel (i.e. dual-Leibniz) algebras and we compare them with dendriform algebras. In particular we compute the natural map from a free dendriform algebra to the free Zinbiel algebra, considered as a dendriform algebra. Finally we compare the homology theories.

7.1. Zinbiel algebras [L3]. A *Zinbiel algebra* R (*) (also called *dual-Leibniz algebra*) is a module over K equipped with a binary operation $(x, y) \mapsto x \cdot y$, which satisfies the identity

$$(x \cdot y) \cdot z = x \cdot (y \cdot z) + x \cdot (z \cdot y)\ , \quad for\ all\ x, y, z \in R. \tag{7.1.1}$$

The category of Zinbiel algebras is denoted **Zinb**. The free Zinbiel algebra over the vector space V is $\overline{T}(V) = \oplus_{n\geq 1} V^{\otimes n}$ equipped with the following product

$$(x_0 \dots x_p) \cdot (x_{p+1} \dots x_{p+q}) = x_0 sh_{p,q}(x_1 \dots x_{p+q})$$

where $sh_{p,q}$ is the sum over all (p, q)-shuffles. We denote it by $Zinb(V)$. Observe that $Zinb(V) = \bigoplus_n Zinb(n) \otimes_{S_n} V^{\otimes n}$ for $Zinb(n) = K[S_n]$. It is immediate to check that the symmetrized product

$$xy := x \cdot y + y \cdot x \tag{7.1.2}$$

is associative (cf. [R], [L3]), so, under the symmetrized product, R becomes an associative and commutative algebra. This construction gives a functor

$$\mathbf{Zinb} \xrightarrow{+} \mathbf{Com}.$$

Let us construct functors to and from the category of dendriform algebras **Dend**.

7.2. Lemma. *Let R be a Zinbiel algebra and put*

$$x \prec y := x \cdot y\ ,\ x \succ y := y \cdot x\ ,\ \forall\ x, y \in R.$$

Then $(R, \prec, \succ)$ is a dendriform algebra denoted R_{Dend}. Conversely, a commutative dendriform algebra (i.e. a dendriform algebra for which $x \succ y = y \prec x$) is a Zinbiel algebra.

Proof. Indeed, relation (i) is exactly relation (3.3.1) and so is relation (iii). Relation (ii) also follows from (3.3.1) since

$$x \succ (y \prec z) = (yz)x, \quad (x \succ y) \prec z = (yx)z,$$

(*) Terminology proposed by J.-M. Lemaire and the relation $(y \cdot z) \cdot x = (y \cdot x) \cdot z$ follows from (3.3.1) for y, z, x and y, x, z. $\square$

So we have constructed a functor

$$\mathbf{Zinb} \to \mathbf{Dend}.$$

From Lemma 5.2 we get a functor

$$\mathbf{Dend} \xrightarrow{+} \mathbf{As}.$$

Summarizing we get the following

7.4. Proposition. *The following diagram of functors between categories of algebras is commutative*

$$\begin{array}{ccc} \mathbf{Zinb} & \hookrightarrow & \mathbf{Dend} \\ \downarrow + & & \downarrow + \\ \mathbf{Com} & \hookrightarrow & \mathbf{As} \end{array}$$

Proof. The commutativity of the diagram is immediate since for a Zinbiel algebra the associated associative products are equal :

$$x * y = x \prec y + x \succ y = x \cdot y + y \cdot x = xy \quad .$$

$\square$

7.5. Theorem. *For any vector space V, the natural map of dendriform algebras $Dend(V) \to Zinb(V)_{Dend}$ is induced, in degree n, by the map*

$$K[Y_n] \otimes K[S_n] \to K[S_n], \quad y \otimes \omega \mapsto \sum_{\{\sigma \in S_n \ \mid \ \psi'(\sigma) = y\}} \sigma\omega \ ,$$

where $\psi' : S_n \twoheadrightarrow Y_n$ is the surjective map described in Appendix A6.

Proof. First, observe that the map $V \to Zinb(V)$ gives a map (the same on the underlying vector spaces) $V \to Zinb(V)_{Dend}$. Since this latter object is a dendriform algebra, the map factors through the free dendriform algebra on V, whence a natural dialgebra map $\phi : Dend(V) \to Zinb(V)_{Dend}$.

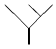
The proof will be done by induction on n.

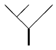
Restricting ϕ to the degree n part gives a commutative square (cf. 5.8 and 5.9):

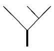
$$\begin{array}{ccc} K[Y_n] \otimes V^{\otimes n} & \xrightarrow{\phi_n} & V^{\otimes n} \\ \bigcap & & \bigcap \\ Dend(V) & \xrightarrow{\phi} & Zinb(V)_{Dend} \ . \end{array}$$

For $n = 1$, ϕ_1 is clearly the identity of V.

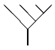
For $n = 2$, $\phi_2 : K[Y_2] \otimes V^{\otimes 2} \to V^{\otimes 2}$ is given by

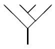
 $$\otimes (x, y) = x \prec y \mapsto x \cdot y = [12].(x, y),$$

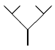
 $$\otimes (x, y) = x \succ y \mapsto y \cdot x = [21].(x, y),$$

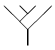
so  $\mapsto [12]$ and  $\mapsto [21]$, which is the map ψ' of Appendix A6.

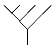
For n=3, one has

 $$\otimes (x, y, z) = x \prec (y \prec z) \mapsto x \cdot (y \cdot z) = [123].(x, y, z),$$

 $$\otimes (x, y, z) = x \prec (y \succ z) \mapsto x \cdot (z \cdot y) = [132].(x, y, z),$$

 $$\otimes (x, y, z) = (x \succ y) \prec z \mapsto (y \cdot x) \cdot z = y \cdot (x \cdot z + z \cdot x) = ([213] + [312]).(x, y, z),$$

 $$\otimes (x, y, z) = (z \prec x) \succ y \mapsto x \cdot (y \cdot z) = [231].(x, y, z),$$

 $$\otimes (x, y, z) = (z \succ y) \succ x \mapsto x \cdot (y \cdot z) = [321].(x, y, z),$$

so the map is precisely ψ'.

Let us now prove it for any n. We suppose that the theorem has been proved for any $p < n$. We are going to prove it for n. We use the *grafting* operation on trees (cf. Appendix A).

The element $(y_1 \vee y_2; x_1 \dots x_{p+q+1})$ can be written as

$$(y_1; x_1 \dots x_p) \succ x_{p+1} \prec (y_2; x_{p+2} \dots x_{p+q+1})$$

in $Dend(V)$ when $p \geq 1, q \geq 1$. Here x_{p+1} stands for $([1]; x_{p+1})$. We do not need to put any parenthesis because of the second relation of dendriform algebras. The image of this element under ϕ is

$$\begin{aligned}(x_{p+1} \cdot \phi_q(y_2; x_{p+2} \dots x_{p+q+1})) \cdot \phi_p(y_1; x_1 \dots x_{p+q+1}) \\ = x_{p+1} \cdot (\phi_q(y_2; x_{p+2} \dots x_{p+q+1}) \cdot \phi_p(y_1; x_1 \dots x_{p+q+1}) \\ + \phi_p(y_1; x_1 \dots x_{p+q+1}) \cdot \phi_q(y_2; x_{p+2} \dots x_{p+q+1})).\end{aligned}$$

By induction we know that $\phi_p(y_1; x_1 \dots x_p) = \sum_{\sigma_1 \in \psi'^{-1}(y_1)} \sigma_1(x_1 \dots x_p)$ and that $\phi_q(y_2; x_{p+2} \dots x_{p+q+1}) = \sum_{\sigma_2 \in \psi'^{-1}(y_2)} \sigma_2(x_{p+2} \dots x_{p+q+1})$. For $z_i \in V$ one has, in the free Zinbiel algebra, the equality

$$z_0 \cdot (z_1 \dots z_p \cdot z_{p+1} \dots z_{p+q} + z_{p+1} \dots z_{p+q} \cdot z_1 \dots z_p) = \sum_\tau \tau(z_0 \dots z_{p+q}),$$

where the sum is extended over all the (p, q)-shuffles τ (acting on the set $\{1, \dots, p + q\}$. Hence we get

$$\phi_n(y_1 \vee y_2; x_1 \dots x_n) = \sum_\pi \pi(x_1 \dots x_n),$$

where the permutation π is of the following form :

$$\pi(i) = \tau\sigma_1(i) \text{ for } 1 \leq i \leq p,$$
$$\pi(p+1) = 1,$$
$$\pi(i) = \tau(p + \sigma_2(i)) \text{ for } p+2 \leq i \leq p+q+1,$$

where $\sigma_1 \in \psi'^{-1}(y_1)$, $\sigma_2 \in \psi'^{-1}(y_2)$ and τ is a (p,q)-shuffle.

On the other hand one easily checks that all the permutations σ which belong to $\psi'^{-1}(y_1 \vee y_2)$ are precisely obtained by choosing such a σ_1 and such a σ_2, and then shuffle the associated levels. So we have proved the formula for n.

In the preceding proof we assumed that $p \geq 1, q \geq 1$. We let to the reader the task of modifying the proof when either $p = 0$ or $q = 0$. □

7.6. Comparison of homology theories. As mentioned in Appendix B, for any algebra over a Koszul operad, there is a small chain complex, modelled on the dual operad, whose homology is the homology of the algebra. For Zinbiel algebras it takes the following form (cf. [Li2]):

$$C_*^{Zinb}(R): \qquad \cdots \longrightarrow R^{\otimes n} \xrightarrow{d} R^{\otimes n-1} \xrightarrow{d} \cdots \xrightarrow{d} R^{\otimes 2} \xrightarrow{\cdot} R$$

where

$$d(x_1, \ldots, x_n) = (x_1 \cdot x_2, x_3, \ldots, x_n) + \sum_{i=2}^{n-1} (-1)^{i-1}(x_1, \ldots, x_i * x_{i+1}, \ldots, x_n).$$

The homology groups of this complex are denoted $H_n^{Zinb}(R)$, for $n \geq 1$. By Appendix B5e, there is a natural map of complexes

$$C_*^{Dend}(R_{Dend}) \longrightarrow C_*^{Zinb}(R)$$

inducing

$$H_*^{Dend}(R_{Dend}) \longrightarrow H_*^{Zinb}(R).$$

Let us describe explicitly the chain complex map.

7.7. Proposition. *Let $\theta_n^r \in K[S_n]$ be the elements defined recursively by the formulas :*

$$\theta_1^1(1) = (1), \qquad \theta_n^r(1, \ldots, n) = (r, \tilde\theta_n^r(1, \ldots, \widehat{r} \ldots, n)),$$
$$\tilde\theta_n^r(1, \ldots, n-1) = (r-1, \tilde\theta_{n-1}^{r-1}(1, \ldots, \widehat{r-1} \ldots, n)) + (r, \tilde\theta_{n-1}^r(1, \ldots, \widehat{r} \ldots, n)).$$

In particular, $\theta_n^1(1, \ldots, n) = (1, \ldots, n)$ and $\theta_n^n(1, \ldots, n) = (n, \ldots, 1)$. The map $\Theta_n : K[C_n] \otimes R^{\otimes n} \to R^{\otimes n}$ defined by

$$\Theta_n([r] \otimes (x_1, \cdots x_n)) := \theta_n^r(x_1, \cdots x_n)$$

is the chain complex map $\Theta_* : C_*^{Dend}(R_{Dend}) \longrightarrow C_*^{Zinb}(R)$ induced by the operad morphism $Dend \to Zinb$.

Proof. This is a consequence of the explicit description of the morphism of Leibniz algebras $Leib(V) \to Dias(V)_{Leib}$, cf. Appendix B5e.

Observe that θ_n^r is the sum of the signed action of $\binom{n}{r}$ permutations. Since $\sum \binom{n}{r} = 2^n$, we recover (up to permutation) the 2^n monomials of $[x_1, [x_2, [\ldots, x_n]]]$ (cf. 4.10). □

8. KOSZUL DUALITY FOR THE DIALGEBRA OPERAD

In this section we show that the operad associated to dendriform algebras is dual, in the operadic sense, to the operad associated to dialgebras. Moreover we use the results of section 4 to show that the operad of associative dialgebras is a Koszul operad (and so is the operad of dendriform algebras). The reader not familiar with the notions of operad and Koszul duality may have a look at Appendix B, from which we take the notation.

8.1. The associative dialgebra operad. A dialgebra is determined by two operations (left and right product) on two variables and by relations which make use of the composition of two such operations. Hence the operad *Dias* associated to the notion of dialgebra is a binary (operations on two variables) quadratic (relations involving two operations) operad. Moreover, there is no symmetry property for these operations, and, in the relations, the variables stay in the same order. Hence the operad is a *non-*Σ*-operad*, that is, as a representation of S_n, the space $Dias(n)$ is a sum of copies of the regular representation. It was proved in 2.5 that the free dialgebra over the vector space V is $Dias(V) = T(V) \otimes V \otimes T(V)$. The degree n part of it is $\bigoplus_{i+1+j=n} V^{\otimes i} \otimes V \otimes V^{\otimes j}$. Hence the operad *Dias* is such that

$$Dias(n) = nK[S_n] \qquad (n \text{ copies of the regular representation}).$$

In particular $Dias(1) = K$ (the only unary operation on a dialgebra is the identity), $E := Dias(2) = E' \otimes K[S_2]$, where E' is 2-dimensional generated by $\dashv$ and $\vdash$. The space $\operatorname{Ind}_{S_2}^{S_3}(E \otimes E)$ is the sum of 8 copies of the regular representation of S_3. Each copy corresponds to a choice of parenthesizing: $(- \circ_1 (- \circ_2 -))$ or $((- \circ_1 -) \circ_2 -)$, and a choice for the two operations $\circ_1$ and $\circ_2$. The space of relations $R \subset \operatorname{Ind}_{S_2}^{S_3}(E \otimes E)$ is of the form $R' \otimes K[S_3]$, where R' is the subspace of $E'^{\otimes 2} \oplus E'^{\otimes 2}$ determined by the relations 1 to 5 of a dialgebra (cf. 2.1).

8.2. The dendriform algebra operad. Analogously the operad *Dend* associated to the notion of dendriform algebra is binary and quadratic, and is a *non-*Σ*-operad* since there is no symmetry for the operations and, in the relations, the variables stay in the same order. It was proved in 5.7 that the free dendriform algebra over the vector space V is $Dend(V) = \bigoplus_{n\geq 1} K[Y_n] \otimes V^{\otimes n}$. Hence the operad *Dend* is such that

$$Dend(n) = K[Y_n] \otimes K[S_n].$$

In particular $Dend(1) = K, Dend(2) = F' \otimes K[S_2]$, where F' is 2-dimensional generated by $\prec$ and $\succ$. The space

$$\operatorname{Ind}_{S_2}^{S_3}(Dend(2) \otimes Dend(2)) \cong (F'^{\otimes 2} \oplus F'^{\otimes 2}) \otimes K[S_3]$$

is the sum of 8 copies of the regular representation of S_3. The space of relations is of the form $S' \otimes K[S_3]$, where $S' \subset (F'^{\otimes 2} \oplus F'^{\otimes 2})$ is the subspace determined by the 3 relations of a dendriform algebra (cf. 5.1).

8.3. Proposition. *The operad Dend of dendriform algebras is dual, in the operad sense, to the operad Dias of dialgebras : $Dias^! = Dend$.*

Proof. Let us identify $F' = E'^{\vee}$ with E' by identifying the basis $(\prec, \succ)$ with the basis $(\dashv, \vdash)$. Since $R = R' \otimes K[S_3]$, the space of relations for the dual operad is of the form $R'^{ann} \otimes K[S_3]$, where, according to Proposition B3, R'^{ann} is the annihilator of R'. Recall from Proposition B3 that the scalar product on $E'^{\otimes 2} \oplus E'^{\otimes 2}$ is given by the matrix $\begin{bmatrix} \mathrm{Id} & 0 \\ 0 & -\mathrm{Id} \end{bmatrix}$.

With obvious notation, the subspace R' is determined by the relations

$$\begin{cases} (\dashv,\dashv)_2 - (\dashv,\dashv)_1 = 0, \\ (\dashv,\dashv)_1 - (\dashv,\vdash)_2 = 0, \\ (\vdash,\dashv)_1 - (\vdash,\dashv)_2 = 0, \\ (\dashv,\vdash)_1 - (\vdash,\vdash)_2 = 0, \\ (\vdash,\vdash)_2 - (\vdash,\vdash)_1 = 0. \end{cases}$$

It is immediate to verify that its annihilator $R'^{\perp}$ with respect to the given scalar product is the subspace determined by the relations

$$\begin{cases} (\dashv,\dashv)_1 - (\dashv,\dashv)_2 - (\dashv,\vdash)_2 = 0, \\ (\vdash,\dashv)_1 - (\vdash,\dashv)_2 = 0, \\ (\vdash,\vdash)_2 - (\dashv,\vdash)_1 - (\vdash,\vdash)_1 = 0. \end{cases}$$

This is precisely the relations of dendriform algebras (once we have changed $\dashv$ into $\prec$ and $\vdash$ into $\succ$). □

8.4. Proposition. *The chain complex $CY_*(D)$ associated to a dialgebra D is the chain complex of D in the operad sense (cf. Appendix B4). Hence HY is the (co)homology theory for dialgebras predicted by the operad theory.*

Proof. From the theory of operads recalled in Appendix B we have

$$C_n^{Dias}(D) = Dend(n)^{\vee} \otimes_{S_n} D^{\otimes n}.$$

By Proposition 5.7 we get

$$C_n^{Dias}(D) = K[Y_n] \otimes K[S_n] \otimes_{S_n} D^{\otimes n} = K[Y_n] \otimes D^{\otimes n} = CY_n(D).$$

So, it suffices to prove that the boundary operator d of $CY_*(D)$ agrees with the dialgebra structure of D on $CY_2(D)$ and that it is a coalgebra derivation.

The boundary map on $CY_2(D) = D^{\otimes 2} \oplus D^{\otimes 2}$ is given by $x \otimes y \mapsto x \dashv y$ on the first component and by $x \otimes y \mapsto x \vdash y$ on the second one. So the first condition is fulfilled. Checking the coderivation property is analogous to the associative case (cf. B4). □

8.5. Theorem. *The operad Dias of dialgebras is a Koszul operad, and so is the operad Dend of dendriform algebras.*

Proof. From the definition of a Koszul operad given in B4, this theorem follows from the vanishing of $H_*^{Dias} = HY_*$ of a free dialgebra, as proved in Theorem 3.8. Since $\mathcal{P}$ being Koszul implies $\mathcal{P}^!$ is Koszul (cf. Appendix B5b), the operad *Dend* is Koszul. □

Observe that this theorem, together with the general property of Koszul operads recalled in B4a, implies proposition 5.2.

8.6. Poincaré series. From the description of the operad *Dias* it follows that its Poincaré series is

$$g_{Dias}(x) = \sum_{n\geq 1} (-1)^n n n! \frac{x^n}{n!} = \sum_{n\geq 1} (-1)^n n x^n = \frac{-x}{(1+x)^2}.$$

On the other hand, the Poincaré series of *Dend* is

$$g_{Dend}(x) = \sum_{n\geq 1} (-1)^n c_n n! \frac{x^n}{n!} = \sum_{n\geq 1} (-1)^n c_n x^n = \frac{-1-2x+\sqrt{1+4x}}{2x}.$$

As expected (cf. Appendix B5c) we verify that $g_{Dend}(g_{Dias}(x)) = x$.

8.7. Operad morphisms. Any morphism of operads induces a functor between the associated categories of algebras. Taking the dual gives also a morphism, but in the other direction. For instance the dual of the inclusion of the category of commutative algebras into the category of associative algebras is the "-" functor which transforms an associative algebra into a Lie algebra. All the functors between categories of algebras that we met in the previous sections come from morphisms of operads. They assemble into the following commutative diagram of functors (cf. Proposition 4.4 and Proposition 7.4):

		Dend				**Dias**		
	inc ↗		+ ↘		*inc* ↗		− ↘	
Zinb				**As**				**Leib**
	+ ↘		*inc* ↗		− ↘		*inc* ↗	
		Com				**Lie**		

The symmetry (around a vertical axis passing through $\mathbf{As}^! = \mathbf{As}$) reflects the Koszul duality of quadratic operads : $Lie^! = Com, Dias^! = Dend, Leib^! = Zinb$.

9. STRONG HOMOTOPY ASSOCIATIVE DIALGEBRAS

Strong homotopy $\mathcal{P}$-algebras are governed by the Koszul dual operad $\mathcal{P}^!$ (cf. Appendix B6). Since for *Dias* we know of an explicit description of its dual $Dias^!$, we are able to describe explicitly the notion of *strong homotopy dialgebra*. In order to do it we use the notion of nested sub-tree of a planar binary tree.

9.1. Nested sub-trees. Recall that in section 5 we defined the notion of nested sub-tree y' of a tree y with quotient $y'' = y/y'$ and we described composition in the free dendriform algebra in terms of nested sub-trees, cf. 5.10 and Proposition 5.11.

9.2. Theorem. *A strong homotopy dialgebra is a graded vector space* $A = \oplus_{i\in\mathbf{Z}} A_i$ *equipped with operations*

$$m_y : A^{\otimes n} \to A, \ for\ any\ y \in Y_n,\ n \geq 1,$$

which are homogeneous of degree $n-2$ *and which satisfy the following relations for any* y*:*

$$(*)_y \qquad \sum_{y'\subset y,\ y''=y/y'} \pm m_{y''}(a_1, \ldots, a_i, m_{y'}(a_{i+1}, \ldots, a_{i+k}), \ldots, a_n) = 0,$$

where the sign $\pm$ *is* $+$ *or* $-$ *according to the parity of* $(k+1)(i+1) + k(n + \sum_{j=1}^{k} |a_j|)$*.*

In this relation the tree y *is fixed and the sum runs over all the nested sub-trees* y' *of* y *with* $y'' = y/y'$ *(as described in 5.10).*

Compare with the definition of a A_∞-algebra [St, p. 294].

Proof. Since the operad *Dias* is Koszul (cf. Theorem 8.5), we may apply Theorem B.7. By Proposition 5.8 the dual operad *Dend* is generated in dimension n (as a free S_n-module) by the set of n-trees Y_n. Hence the operations on a $\mathcal{B}(\mathcal{P}^!)^*$-algebra A are generated by operations m_y, for $y \in Y_n$,

$$m_y : A^{\otimes |y|} \to A \quad \text{of degree } |y| - 2.$$

The relations satisfied by these operations are obtained as follows. First one extends them in order to get a coderivation

$$m : \oplus_n K[Y_n] \otimes A^{\otimes n} \longrightarrow \oplus_n K[Y_n] \otimes A^{\otimes n},$$

and then one writes $m \circ m = 0$. The component in $K[Y_1] \otimes A = A$ of the image under m of an element $K\{y\} \otimes A^{\otimes n} \subset K[Y_n] \otimes A^{\otimes n}$ is given by m_y (up to sign). More precisely we put

$$m_y(a_1, \cdots, a_n) := (-1)^{(k-1)|a_1| + (k-2)|a_2| + \cdots a_{k-1}} m(y; a_1, \cdots, a_n)_1.$$

In order to obtain the component in $K\{y'\} \otimes A^{\otimes k}$ we need to look at the composition in the cofree co-dendriform algebra, or dually, in the free dendriform algebra:

$$Dias^!(k) \otimes Dias^!(i_1) \otimes \cdots \otimes Dias^!(i_k) \longrightarrow Dias^!(i_1 + \cdots + i_k)$$

or equivalently,

$$K[Y_k] \otimes K[Y_{i_1}] \otimes \cdots \otimes K[Y_{i_k}] \longrightarrow K[Y_{i_1+\cdots+i_k}].$$

It is sufficient to write the relation $m \circ m = 0$ for the component in $K[Y_1] \otimes A = A$ of the image, since the vanishing of the other components is a consequence of that one. Hence it is sufficient to compute the composition product for $i_u = 1$ for all i_u except one of them, let us say $i_j = m$. In this case it is precisely the result of Proposition 5.11. □

9.3. Strong homotopy associative dialgebra in low dimensions. Let us write $m_y = m_{ij\cdots k}$ in place of $m_{[ij\cdots k]}$, when $y = [ij\cdots k]$.

For $n = 1$ the operation $\delta := m_1 : A \to A$ is of degree -1. Since the only nested sub-tree of [1] is [1] itself, the relation $(*)_1$ is

$$\delta \circ \delta = 0.$$

Hence (A, δ) is a chain complex.

For $n = 2$, there are two maps m_{12} and $m_{21} : A^{\otimes 2} \to A$, which are of degree 0. The tree [12] (resp. [21]) has three nested sub-trees, itself with quotient [1], and two (different) copies of [1], both with quotient [12] (resp. [21]). Hence the relation $(*)_{12}$ takes the form (where 1 stands for Id):

$$\delta \circ m_{12} - m_{12} \circ (\delta \otimes 1) - m_{12} \circ (1 \otimes \delta) = 0,$$

and similarly for [21]. In other words δ is a derivation for m_{12} and for m_{21}.

For $n = 3$, there are five maps $A^{\otimes 3} \to A$, denoted $m_{123}, m_{213}, m_{131}, m_{312}, m_{321}$, corresponding to the five trees of Y_3. The five relations $(*)_y$ for y a tree of degree 3 are:

$$\delta \circ m_{123} + m_{123} \circ (\delta \otimes 1 \otimes 1 + 1 \otimes \delta \otimes 1 + 1 \otimes 1 \otimes \delta) = m_{12} \circ (m_{12} \otimes 1) - m_{12} \circ (1 \otimes m_{12}),$$

$$\delta \circ m_{213} + m_{213} \circ (\delta \otimes 1 \otimes 1 + 1 \otimes \delta \otimes 1 + 1 \otimes 1 \otimes \delta) = m_{12} \circ (m_{21} \otimes 1) - m_{12} \circ (1 \otimes m_{12}),$$

$$\delta \circ m_{131} + m_{131} \circ (\delta \otimes 1 \otimes 1 + 1 \otimes \delta \otimes 1 + 1 \otimes 1 \otimes \delta) = m_{21} \circ (m_{12} \otimes 1) - m_{12} \circ (1 \otimes m_{21}),$$

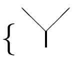
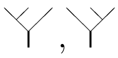
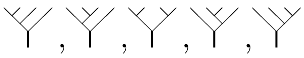
$$\delta \circ m_{312} + m_{312} \circ (\delta \otimes 1 \otimes 1 + 1 \otimes \delta \otimes 1 + 1 \otimes 1 \otimes \delta) = m_{21} \circ (m_{21} \otimes 1) - m_{21} \circ (1 \otimes m_{12}),$$

$$\delta \circ m_{321} + m_{321} \circ (\delta \otimes 1 \otimes 1 + 1 \otimes \delta \otimes 1 + 1 \otimes 1 \otimes \delta) = m_{21} \circ (m_{21} \otimes 1) - m_{21} \circ (1 \otimes m_{21}).$$

One observes that, as expected, if all the maps m_{ijk} are trivial (and also higher up), then $m_{12} = \vdash$, $m_{21} = \dashv$, and the algebra is a graded dialgebra.

Appendix A. PLANAR BINARY TREES AND PERMUTATIONS.

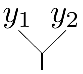
A.1. Planar binary trees. A planar tree is *binary* if any vertex is trivalent. We denote by Y_n the set of planar binary trees with n vertices, that is with $n+1$ leaves (and one root). Since we only use planar binary trees in this section we abbreviate it into tree (or n-tree). The integer n is called the degree of the tree. For any $y \in Y_n$ we label the $n+1$ leaves by $\{0, 1, \ldots, n\}$ from left to right. We label the vertices by $\{1, \ldots, n\}$ so that the i-th vertex is in between the leaves $i-1$ and i. In low dimension these sets are:

$Y_0 = \{|\}$, $Y_1 = \{$  $\}$, $Y_2 = \{$  $\}$, $Y_3 = \{$  $\}$.

The number of elements in Y_n is $c_n = \frac{(2n)!}{n!(n+1)!}$, so $\mathbf{c} = (1, 2, 5, 14, 42, 132, \ldots)$ is the sequence of the Catalan numbers.

We first introduce a notation for these trees as follows. The only element $|$ of Y_0 is denoted by [0]. The only element of Y_1 is denoted by [1].

The *grafting* of a p-tree y_1 and a q-tree y_2 is a $(p+q+1)$-tree denoted by $y_1 \vee y_2$ obtained by joining the roots of y_1 and y_2 and creating a new root from that vertex.

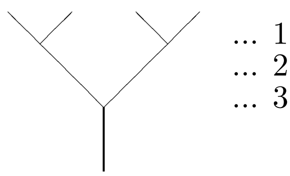
$y_1 \vee y_2 =$ 

Its name is written $[\ y_1\ p+q+1\ y_2\ ]$ with the convention that all 0's are deleted (except for the element in Y_0). For instance one has: $[0] \vee [0] = [1]$, $[1] \vee [0] = [12]$, $[0] \vee [1] = [21]$, $[1] \vee [1] = [131]$, and so on. So the names of the trees pictured above are, from left to right:

$$[0]\ ,[1]\ ,[12]\ ,[21]\ ,[123]\ ,[213]\ ,[131]\ ,[312]\ ,[321]\ .$$

This labelling has several advantages. For instance if we draw the tree metrically, with the leaves regulary spaced, and the lines at 45 degree angle, then the integers in the sequence are precisely the depth of the successive vertices.

Example



[131]

The orientation of the leaves can be read from the name as follows. Let $y = [a_1 \ldots a_n]$. The i-th leaf is oriented SW-NE (resp. SE-NW) when $a_i < a_{i+1}$ (resp. $a_i > a_{i+1}$).

The following is an inductive criterion to check whether a sequence of integers is the name of a tree.

A.2. Proposition. *A sequence of positive integers* $[a_1 \ldots a_n]$ *is the name of a tree if and only if it satisfies the following conditions:*

– *there is a unique integer* p *such that* $a_p = n$,

– *the two sequences* $a_1 \ldots a_{p-1}$ *and* $a_{p+1} \ldots a_n$ *are either empty or name of trees.*

Proof. It suffices to remark that any tree y is of the form $y_1 \vee y_2$ for two uniquely determined trees y_1 and y_2, whose degree is strictly smaller than the degree of y. □

A.3. From permutations to trees. There is defined a surjective map

$$\psi : S_n \twoheadrightarrow Y_n$$

as follows. The image of $\{1, \ldots, n\}$ under the permutation σ is a sequence of positive integers $[\sigma_1 \ldots \sigma_n]$. We convert it into the name of a tree by the following inductive rule. Replace the largest integer in the interval $\sigma_1 \ldots \sigma_n$, say σ_p, by the length of the interval (which is n here). Then repeat the modification for the intervals $\sigma_1 \ldots \sigma_{p-1}$ and $\sigma_{p+1} \ldots \sigma_n$, and so on, until each integer has been modified. This gives the name of a tree (hence a tree), since it obviously satisfies the above criterion. Let us perform this construction on an example, where the successive modifications are underlined:

$$\sigma = [341652] \mapsto [341\underline{6}52] \mapsto [3\underline{3}1\underline{6}\underline{2}2] \mapsto [\underline{131621}] = \psi(\sigma).$$

A.4. Planar binary increasing trees. Let us introduce a variation of trees: the *planar binary trees with levels* also called *increasing trees* in the literature. A tree with levels is an n-tree together with a given level for each vertex. This level takes value in $\{1, \ldots, n\}$, and we suppose that each vertex has a different level, and that the levels are increasing, that is they respect the partial order structure of the tree (the level is the depth of the vertex).

Example: the following are two distinct increasing trees

We denote by $\widetilde{Y}_n$ the set of increasing n-trees.

The following is a well-known result which was brought to my attention by Phil Hanlon [Ha].

A.5. Proposition. *The map which assigns a level to each vertex determines a permutation. This gives a bijection* $\widetilde{Y}_n \cong S_n$ *between increasing trees and permutations.*

Proof. Label the vertices by their right leaf (i.e. vertex i is in between the leaf $i-1$ and the leaf i). Since each vertex has a different level it is clear that we get a bijection. So any increasing tree gives rise to a permutation.

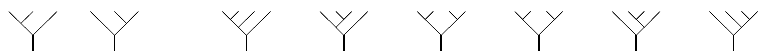
On the other hand a permutation gives rise to a tree under ψ, cf. A3. Labelling the level of the i-th vertex by $\sigma(i)$ gives an increasing tree.

It is immediately seen that the two constructions are inverse to each other. □

Hence the map $\psi : S_n \to Y_n$ is the composite of the bijection from the set of permutations to the set of increasing trees with the forgetful map (forgetting about the levels).

In low dimension ψ is given by:

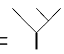
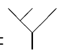
σ	[12]	[21]	[123]	[213]	[132]	[231]	[312]	[321]
$\psi(\sigma)$	[12]	[21]	[123]	[213]	[131]	[131]	[312]	[321]

tree 

A.6. The other choice ψ' to code trees. It is sometimes useful to code the vertices of a planar binary trees by *height* rather than by depth. The resulting map $\psi' : S_n \to Y_n$ is related by the formula

$$\psi'(\sigma) = \psi(\omega\sigma\omega^{-1}), \quad for \quad \omega = [n \cdots 2\ 1].$$

For instance

$\psi'([12]) =$  *and* $\psi'([21]) =$  .

A.7. Geometric interpretation of ψ. The map ψ can be considered as the restriction to the vertices of a cellular map between polytopes. Indeed, starting from the set $< n >= \{1, ..., n\}$ let us define a poset as follows. An element of the poset is an ordered partition of $< n >$. The element X is less than the element Y if Y can be obtained from X by reuniting successive subsets. For instance $\{(35)(2)(14)(7)(6)\} < \{(235)(14)(67)\}$. Note that the minimal elements are precisely the permutations. It can be shown that if we exclude the trivial partition $\{(12...n)\}$ from the poset, then the geometric realization is homeomorphic to the sphere S^{n-2} (it is called the *permutohedron*).

Similarly, starting with planar trees with n interior vertices one defines a poset as follows: a tree x is less than a tree y if y can be obtained from x by scratching some interior edges. Note that the minimal elements are precisely the planar binary trees. It can be shown that if we exclude the trivial tree with only one vertex from the poset, then the geometric realization is a polytope homeomorphic to the sphere S^{n-2} (it is called the Stasheff polytope, or the *associahedron*). The map ψ described above can be obviously extended to a map of posets, and its geometric realization is a homotopy equivalence of spaces, both homeomorphic to the sphere S^{n-2}. One can even compare the associahedron with the hypercubes by looking at the orientation of the leaves, cf. [LR].

Appendix B. ALGEBRAIC OPERADS

In this short appendix we briefly survey Koszul duality for algebraic operads as studied by Ginzburg and Kapranov [GK] and Getzler-Jones [GJ].

The ground field K is supposed to be of characteristic zero. The category of finite dimensional vector spaces over K is denoted by **Vect**. When V is graded its suspension sV is such that $(sV)_n = V_{n-1}$.

B.1. Algebraic operads. For a given "type of algebra" $\mathcal{P}$ (for instance associative algebras, or Lie algebras, etc ...), let $\mathcal{P}(V)$ be the free algebra over the vector space V. One can view $\mathcal{P}$ as a functor from the category **Vect** to itself, which preserves filtered limits. The map $V \to \mathcal{P}(V)$ gives a natural transformation $\mathrm{Id} \to \mathcal{P}$. The functor $\mathcal{P}$ is *analytical*. In characteristic zero it is equivalent (by Schur's lemma) to: $\mathcal{P}$ is of the form

$$(B.1.1) \qquad \mathcal{P}(V) = \bigoplus_{n \geq 0} \mathcal{P}(n) \otimes_{S_n} V^{\otimes n},$$

for some right S_n-module $\mathcal{P}(n)$.

From the universal property of the free algebra applied to $\mathrm{Id} : \mathcal{P}(V) \to \mathcal{P}(V)$ one gets a natural map $\mathcal{P}(\mathcal{P}(V)) \to \mathcal{P}(V)$, that is a transformation of functors $\gamma : \mathcal{P} \circ \mathcal{P} \to \mathcal{P}$. By universality property of the free algebra functor, one sees that the operation γ is associative and has a unit.

In order to make precise the notion of "type of algebras", one axiomatizes the above properties and puts up the following definition.

By definition an *algebraic operad* over a characteristic zero field is an analytical functor $\mathcal{P} : \mathbf{Vect} \to \mathbf{Vect}$, such that $\mathcal{P}(0) = 0$, equipped with a natural transformation of functor $\gamma : \mathcal{P} \circ \mathcal{P} \to \mathcal{P}$ which is associative and has a unit $1 : \mathrm{Id} \to \mathcal{P}$. In other words $(\mathcal{P}, \gamma, 1)$ is a *monad* in the tensor category $(\mathbf{Funct}, \circ)$, where **Funct** is the category of analytical functors from **Vect** to itself, and $\circ$ stands for composition of functors.

By definition a $\mathcal{P}$-*algebra* (that is an algebra over the operad $\mathcal{P}$) is a vector space A equipped with a map $\gamma_A : \mathcal{P}(A) \to A$ compatible with the composition γ in the following sense: the diagram

$$\begin{array}{ccc} \mathcal{P}(\mathcal{P}(A)) & \xrightarrow{\mathcal{P}(\gamma_A)} & \mathcal{P}(A) \\ \downarrow \gamma(A) & & \downarrow \gamma_A \\ \mathcal{P}(A) & \xrightarrow{\gamma_A} & A \end{array}$$

is commutative.

By writing $\mathcal{P}(V)$ and $\mathcal{P} \circ \mathcal{P}(V)$ in terms of the vector spaves $\mathcal{P}(n)$'s we see that the operation γ is determined by linear maps

$$\mathcal{P}(n) \otimes \mathcal{P}(i_1) \otimes \cdots \otimes \mathcal{P}(i_n) \longrightarrow \mathcal{P}(i_1 + \cdots + i_n)$$

which satisfy some axioms deduced from the S_n-module structure of $\mathcal{P}(n)$ and from the associativity of γ (cf. for instance [M]).

From this point of view, a $\mathcal{P}$-algebra is determined by linear maps

$$\mathcal{P}(n) \otimes_{S_n} A^{\otimes n} \to A$$

satisfying compatibility properties (cf. [M]). Observe that the space $\mathcal{P}(n)$ is the space of all operations that can be performed on n variables.

The family of S_n-modules $\{\mathcal{P}(n)\}_{n\geq 1}$ is called an **S**-module. There is an obvious forgetful functor from operads to **S**-modules. The left adjoint functor exists and gives rise to the *free operad* over an **S**-module (cf. for instance [BJT]).

We are only dealing here with *binary* operads, that is operads generated by operations on two variables. More explicitly, let E be an S_2-module (module of generating operations) and let $\mathcal{T}(E)$ be the free operad on the **S**-module

$$(0, E, 0, \ldots \quad).$$

The first components of $\mathcal{T}(E)$ are

$$\begin{aligned}
&\mathcal{T}(E)(1) = K,\\
&\mathcal{T}(E)(2) = E,\\
&\mathcal{T}(E)(3) = \mathrm{Ind}_{S_2}^{S_3}(E \otimes E) = 3E \otimes E,
\end{aligned}$$

where the action of S_2 on $E \otimes E$ is on the second factor only. Explicitly, $\mathcal{T}(E)(3)$ is the space of all the operations on 3 variables that can be performed out of the operations on 2 variables in E.

A binary operad is *quadratic* if it is the quotient of $\mathcal{T}(E)$ (for some S_2-module E) by the ideal generated by some S_3-submodule R of $\mathcal{T}(E)(3)$. The associated operad is denoted $\mathcal{P}(E, R)$.

For instance, for $\mathcal{P} = As$, one has $E = K[S_2]$, the regular representation, and so $\mathrm{Ind}_{S_2}^{S_3}(E \otimes E) = K[S_3] \oplus K[S_3]$. Labelling the generators of the first summand by $x_i(x_j x_k)$ and the generators of the second summand by $(x_i x_j)x_k$, the space R is the subspace generated by all the elements $x_i(x_j x_k) - (x_i x_j)x_k$.

All the operads $Com, Lie, As, Pois, Leib, Zinb, Dias, Dend$ are binary and quadratic. For all these cases the operadic notion of algebra coincides with the one we started with.

B.2. Dual operad. For any right S_n-module V we denote by $V^\vee$ the right S_n-module $V^* \otimes (sgn)$, where (sgn) is the one-dimensional signature representation. Explicitly, if ${}^\sigma f$ is given by ${}^\sigma f(x) = f(x^\sigma)$, then $f^\sigma = sgn(\sigma)({}^{\sigma^{-1}}f)$. The pairing between $V^\vee$ and V given by:

$$< -, - >: V^\vee \otimes V \longrightarrow K, \quad < f, x >= f(x),$$

is sign-invariant, that is $< f^\sigma, x^\sigma >= sgn(\sigma) < f, x >$.

To any quadratic binary operad $\mathcal{P} = \mathcal{P}(E, R)$ is associated its *dual operad* $\mathcal{P}^!$ defined as follows. Since E is an S_2-module, so is $E^\vee$. There is a canonical isomorphism

$$\mathcal{T}(E^\vee)(3) = \mathrm{Ind}\,_{S_2}^{S_3}(E^\vee \otimes E^\vee) \cong (\mathrm{Ind}\,_{S_2}^{S_3}(E \otimes E))^\vee = \mathcal{T}(E)(3)^\vee,$$

and one defines the orthogonal space of R as

$$R^\perp := \mathrm{Ker}\ (\mathcal{T}(E^\vee)(3) \to R^\vee).$$

By definition one puts

$$\mathcal{P}^! := \mathcal{P}(E^\vee, R^\perp).$$

It is shown in [GK] that $As^! = As$, $Com^! = Lie$, $Lie^! = Com$. The notation *Zinb* indicates that the operad of Zinbiel algebras is precisely the dual, in the above sense, of the operad of Leibniz algebras (cf. [L3]). It is proved in section 6 that the operad associated to dendriform algebras, denoted *Dend* is the dual of the operad associated to dialgebras, as expected. It is a consequence of the following proposition, which makes explicit the dual operad when the operations bear no symmetry. This hypothesis says that the space of operations is of the form $E = E' \otimes K[S_2]$ for some vector space E'. Then, the space of operations on 3 variables is $\mathcal{T}(E)(3) = (E'^{\otimes 2} \oplus E'^{\otimes 2}) \otimes K[S_3]$. For any $\xi \in E'$ we denote by $(\xi)_1$ (resp. $(\xi)_2$ the element corresponding to an operation of the type $(-(--))$ (resp. $((--)-)$). The first (resp. the second) component of $E'^{\otimes 2} \oplus E'^{\otimes 2}$ is made of the elements $(\xi)_1$ (resp. $(\xi)_2$).

B.3. Proposition. *Let $\mathcal{P}$ be an operad whose generating operations have no symmetry, in other words $\mathcal{P} = \mathcal{P}(E' \otimes K[S_2], R)$ for some vector space E', and some S_3-sub-module R of $(E'^{\otimes 2} \oplus E'^{\otimes 2}) \otimes K[S_3]$.*

The dual operad of $\mathcal{P}$ is

$$\mathcal{P}^! = \mathcal{P}(E' \otimes K[S_2], R^{ann}),$$

where R^{ann} is the annihilator of R for the scalar product on $(E'^{\otimes 2} \oplus E'^{\otimes 2}) \otimes K[S_3]$ given by

$$< \xi_1 \otimes \sigma, \xi_1 \otimes \sigma >= \mathrm{sgn}(\sigma),$$
$$< \xi_2 \otimes \sigma, \xi_2 \otimes \sigma >= -\mathrm{sgn}(\sigma),$$

all other scalar products are 0, where ξ_1 (resp ξ_2) is a basis vector of the first (resp. second) summand of $E'^{\otimes 2} \oplus E'^{\otimes 2}$ and $\sigma \in S_3$.

Proof. Let $\phi : E \to E^\vee$ be the isomorphism of S_2-modules deduced from the preferred basis of E. It induces an isomorphism of S_3-modules

$$\mathcal{T}(\phi) : \mathcal{T}(E^\vee)(3) \cong \mathcal{T}(E)(3).$$

Hence, the natural evaluation map $\mathcal{T}(E)(3)^\vee \otimes \mathcal{T}(E)(3) \to K$ gives a scalar product $\mathcal{T}(E)(3) \otimes \mathcal{T}(E)(3) \to K$.

Suppose that $E = E' \otimes K[S_2]$. Then $\mathcal{T}(E)(3) = (E'^{\otimes 2} \oplus E'^{\otimes 2}) \otimes K[S_3]$ as mentioned above. In order to prove the Proposition, it suffices to prove it when E' is one-dimensional, generated by the unique product (x_1x_2). The basis of E is $\{(x_1x_2), (x_2x_1)\}$. The isomorphism ϕ is given by $\phi((x_1x_2)) = (x_1x_2)^*, \phi((x_2x_1)) = -(x_2x_1)^*$. The map $\mathcal{T}(\phi)$ sends a basis element to itself (up to sign), so the scalar product is diagonal. Since it is sign-invariant, it suffices to compute $< (x_1(x_2x_3)), (x_1(x_2x_3)) >$ and $< ((x_1x_2)x_3), ((x_1x_2)x_3) >$. With the choice at hand we find $+1$ in the first case and -1 in the second. □

It is clear from this Proposition that $As^! = As$. Indeed, E' is one dimensional generated by $(\cdot\ \cdot)$, and $E'^{\otimes 2} \oplus E'^{\otimes 2}$ is 2-dimensional generated by $(\cdot(\cdot\ \cdot))$ and $((\cdot\ \cdot)\cdot)$. The space R is of the form $R' \otimes K[S_3]$ because, in the associative relation, the variables stay in order. Since R' is determined by the equation $(\cdot(\cdot\ \cdot)) - ((\cdot\ \cdot)\cdot) = 0$, it is immediate to check that it is its own annihilator.

B.4. Homology and Koszul duality. Let $\mathcal{P}$ be a quadratic binary operad and $\mathcal{P}^!$ its dual operad. Let A be a $\mathcal{P}$-algebra. There is defined a chain-complex $C_*^{\mathcal{P}}(A)$:

$$\cdots \to \mathcal{P}^!(n)^\vee \otimes_{S_n} A^{\otimes n} \xrightarrow{d} \mathcal{P}^!(n-1)^\vee \otimes_{S_{n-1}} A^{\otimes n-1} \to \cdots \to \mathcal{P}^!(1)^\vee \otimes A$$

where the differential d agrees, in low dimension, with the $\mathcal{P}$-algebra structure of A

$$\gamma_A(2) : \mathcal{P}(2) \otimes A^{\otimes 2} \to A.$$

In fact d is characterized by this condition plus the fact that on the cofree coalgebra $\mathcal{P}^{!*}(sA)$ it is a graded coderivation. The associated homology groups are denoted by $H_n^{\mathcal{P}}(A)$, for $n \geq 1$. Taking the linear dual of $C_*^{\mathcal{P}}(A)$ over K gives a cochain complex, which permits us to define the cohomology groups $H^n_{\mathcal{P}}(A)$.

If, for any vector space V, the groups $H_n^{\mathcal{P}}(\mathcal{P}(V))$ are trivial for $n > 1$, then the operad $\mathcal{P}$ is called a *Koszul* operad.

One can check that the chain-complex $C_*^{\mathcal{P}}$ is
– the Hochschild complex (of nonunital algebras) for $\mathcal{P} = As$,
– the Harrison complex (of nonunital commutative algebras) for $\mathcal{P} = Com$,
– the Chevalley-Eilenberg complex for $\mathcal{P} = Lie$,
– the chain-complex constructed in [L1] for $\mathcal{P} = Leib$,
– the chain-complex constructed in [Li2] for $\mathcal{P} = Zinb$.

Let us give some details about the check in the case of nonunital associative algebras. Since $As^! = As$, one has

$$C_n^{As}(A) = As^!(n)^\vee \otimes_{S_n} A^{\otimes n} = K[S_n] \otimes_{S_n} A^{\otimes n} = A^{\otimes n}.$$

Since the lowest differential coincides with the product on A, the map $d_2 : A^{\otimes 2} \to A$ is given by $d_2(x,y) = xy$. The coalgebra structure of $C_*^{As}(A)$ is given by "deconcatenation", that is

$$\Delta(a_1, \ldots, a_n) = \sum_{i=0}^{n} (a_1, \ldots, a_i) \otimes (a_{i+1}, \ldots, a_n).$$

Since d is a graded coderivation, we have on $A^{\otimes 3}$

$$\begin{aligned} \Delta d_3(a_1, a_2, a_3) &= (d_2 \otimes 1 + 1 \otimes d_2)\Delta(a_1, a_2, a_3) \\ &= (d_2 \otimes 1 + 1 \otimes d_2)((a_1, a_2) \otimes a_3 + a_1 \otimes (a_2, a_3)) \\ &= (a_1 a_2) \otimes a_3 - a_1 \otimes (a_2 a_3). \end{aligned}$$

Hence we obtain $d_3(a_1, a_2, a_3) = (a_1 a_2, a_3) - (a_1, a_2 a_3)$.

More generally, the same kind of computation shows that

$$d_n(a_1, \ldots, a_n) = \sum_{i=1}^{n-1} (-1)^{i-1} (a_1, \ldots, a_i a_{i+1}, \ldots, a_n).$$

So $(C_*^{\mathcal{P}}(A), d)$ is precisely the Hochschild complex for non-unital algebras (also called the b'-complex in the literature, cf. [L0] for instance).

B.5. Properties of the operad dualization. Here are some properties of the dualization of operads.

(a) *Lie algebra property.* Let A be a $\mathcal{P}$-algebra and B be a $\mathcal{P}^!$-algebra. The following bracket makes $A \otimes B$ into a Lie algebra :

$$[a \otimes b, a' \otimes b'] := \sum \big(\mu(a, a') \otimes \mu^\vee(b, b') - \mu(a', a) \otimes \mu^\vee(b', b)\big),$$

where the sum is over a basis $\{\mu\}$ of the binary operations of $\mathcal{P}$, $\{\mu^\vee\}$ being the dual basis in $\mathcal{P}^!$.

(b) *Koszulness.* $\mathcal{P}^{!!} = \mathcal{P}$. If $\mathcal{P}$ is Koszul, then so is $\mathcal{P}^!$.

(c) *Poincaré series.* Define the Poincaré series of $\mathcal{P}$ as

$$g_{\mathcal{P}}(x) := \sum_{n \geq 1} (-1)^n \dim\ \mathcal{P}(n) \frac{x^n}{n!}.$$

If $\mathcal{P}$ is Koszul, then the following formula holds:

$$g_{\mathcal{P}}(g_{\mathcal{P}^!}(x)) = x.$$

For instance $g_{As}(x) = \frac{-x}{1+x}$, $g_{Comm}(x) = \exp(-x) - 1$, $g_{Lie}(x) = -\log(1+x)$.

(d) *Multiplicative structure.* For any $\mathcal{P}$-algebra A the homology groups $H_*^{\mathcal{P}}(A)$ form a graded $\mathcal{P}^!$-coalgebra. Equivalently, $H^*_{\mathcal{P}}(A)$ is a graded $\mathcal{P}^!$-algebra.

(e) *Functoriality.* Let $\phi : \mathcal{P} \to \mathcal{Q}$ be a morphism of operads, inducing a functor ($\mathcal{Q}$-algebras) $\to$ ($\mathcal{P}$-algebras), $A \mapsto A_{\mathcal{P}}$ between the categories of algebras. For any $\mathcal{Q}$-algebra A there is a well-defined graded morphism

$$\phi_* : H_*^{\mathcal{P}}(A_{\mathcal{P}}) \to H_*^{\mathcal{Q}}(A),$$

which is induced, at the complex level, by the unique morphism of $\mathcal{Q}^!$-algebra

$$\mathcal{Q}^!(V) \to \mathcal{P}^!(V)_{\mathcal{Q}^!},$$

which commutes with the embeddings of the vector space V.

(f) *Quillen homology.* If $\mathcal{P}$ is Koszul, then the homology theory $H_*^{\mathcal{P}}$ for $\mathcal{P}$-algebras, as defined above, coincides with the Quillen homology with trivial coefficients (cf. [Q], [Li3], [FM]). This is a consequence of the existence of a quasi-isomorphism $\mathcal{B}(\mathcal{P}^!)^* \to \mathcal{P}$ (see below).

B.6. Homotopy algebras over an operad. A *quasi-free resolution* $\mathcal{Q} \to \mathcal{P}$ of the operad $\mathcal{P}$ is a differential graded operad $\mathcal{Q}$ which is a free operad over some graded $\mathbf{S}$-module V, and such that, for all n, $\mathcal{Q}(n)$ is a chain-complex whose homology is trivial, except H_0 which is equal to $\mathcal{P}(n)$. The isomorphism $H_0(\mathcal{Q}) \cong \mathcal{P}$ is supposed to be an isomorphism of operads. Observe that one does not require $\mathcal{Q}$ to be free in the category of differential graded operads. By definition a *homotopy $\mathcal{P}$-algebra* is a $\mathcal{Q}$-algebra for some quasi-free resolution $\mathcal{Q}$ of $\mathcal{P}$.

A quasi-free resolution $\mathcal{Q}$ over V, with augmentation ideal $\bar{\mathcal{Q}}$ is called *minimal* when the differential d is *quadratic*, that is, satisfies $d(V) \subset \bar{\mathcal{Q}} \cdot \bar{\mathcal{Q}}$. For a given $\mathcal{P}$ such a minimal model always exists in characteristic zero and is unique up to homotopy.

By definition a *strong homotopy $\mathcal{P}$-algebra* is a $\mathcal{Q}$-algebra for the minimal model $\mathcal{Q}$ of $\mathcal{P}$.

If $\mathcal{P}$ is Koszul, then there is a way of constructing the minimal model as follows. Let $\mathcal{P}^!$ be the Koszul dual of $\mathcal{P}$ and let $\mathcal{B}(\mathcal{P}^!)$ be the bar-construction over $\mathcal{P}^!$. It is the cofree cooperad $\mathcal{T}'(s\bar{\mathcal{P}}^!)$ equipped with the unique coderivation d which is 0 on $\mathcal{P}^!$ and coincides with the composition on $\mathcal{P}^! \circ \mathcal{P}^!$. It is immediate that $d^2 = 0$, and hence $\mathcal{B}(\mathcal{P}^!) := (\mathcal{T}'(s\bar{\mathcal{P}}^!), d)$ is a complex called the *bar-complex* of $\mathcal{P}^!$. It is shown in [GK] (cf. also [GJ]), that, if the operad $\mathcal{P}$ is Koszul, then the differential graded operad $\mathcal{B}(\mathcal{P}^!)^*$ is a resolution of $\mathcal{P}$. Since it is quasi-free and since the differential m of $\mathcal{B}(\mathcal{P}^!)^*$ is quadratic, $\mathcal{B}(\mathcal{P}^!)^*$ is the minimal model of $\mathcal{P}$.

B.7. Theorem. *For a Koszul operad $\mathcal{P}$, strong homotopy $\mathcal{P}$-algebras are equivalent to $\mathcal{B}(\mathcal{P}^!)^*$-algebras.*

B.8. Strong homotopy associative algebras. The associative operad $\mathcal{P} = \mathbf{As}$ is Koszul and self-dual. Since $\mathbf{As}(n)$ is one-dimensional as a free S_n-module, it gives rise to a generating operation $m_n : A^{\otimes n} \to A$ of the strong homotopy associative algebra A. The condition $m \circ m = 0$ gives, for each $n \geq 1$

$$\sum_{k+l=n-1} m_l \circ m_k = 0.$$

Hence a strong homotopy associative algebra is exactly what is called an A_∞-algebra in the literature (cf. [St]).

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Institut de Recherche Mathématique Avancée
CNRS et Université Louis Pasteur
7 rue René-Descartes
67084 Strasbourg Cedex, France
E-mail : loday@math.u-strasbg.fr

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