

Leibniz n -algebras

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Abstract. A *Leibniz n -algebra* is a vector space equipped with an n -ary operation which has the property of being a derivation for itself. This property is crucial in Nambu mechanics. For $n = 2$ this is the notion of Leibniz algebra. In this paper we prove that the free Leibniz $(n + 1)$ -algebra can be described in terms of the n -magma, that is the set of n -ary planar trees. Then it is shown that the n -tensor power functor, which makes a Leibniz $(n + 1)$ -algebra into a Leibniz algebra, sends a free object to a free object. This result is used in the last section, together with former results of Loday and Pirashvili, to construct a small complex which computes Quillen cohomology with coefficients for any Leibniz n -algebra.

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1 Introduction

Leibniz algebras were introduced by the second author in [4]. They play an important role in Hochschild homology theory [4], [5] as well as in Nambu mechanics ([6][10], see also [1]). Let us recall that a *Leibniz algebra* is a vector space \mathfrak{g} equipped with a bilinear map $[-, -] : \mathfrak{g} \otimes \mathfrak{g} \rightarrow \mathfrak{g}$ satisfying the identity:

$$(1.1) \quad [x, [y, z]] = [[x, y], z] - [[x, z], y].$$

One easily sees that Lie algebras are exactly Leibniz algebras satisfying the relation $[x, x] = 0$. Hence Leibniz algebras are a non-commutative version of Lie algebras.

Recently there have been several works dealing with various generalization of Lie structures by extending the binary bracket to an n -bracket (see [1], [2], [9], [11]).

In this paper we introduce the notion of a Leibniz n -algebra—a natural generalization of both concepts. For $n = 2$ one recovers Leibniz algebras. Any Leibniz algebra \mathfrak{g} is also a Leibniz n -algebra under the following n -bracket:

$$[x_1, x_2, \dots, x_n] := [x_1, [x_2, \dots, [x_{n-1}, x_n] \cdots]].$$

Conversely, if \mathcal{L} is a Leibniz $(n + 1)$ -algebra, then on $\mathcal{D}_n(\mathcal{L}) = \mathcal{L}^{\otimes n}$ the following

bracket

$$[a_1 \otimes \cdots \otimes a_n, b_1 \otimes \cdots \otimes b_n] := \sum_{i=1}^n a_1 \otimes \cdots \otimes [a_i, b_1, \dots, b_n] \otimes \cdots \otimes a_n$$

makes $\mathcal{D}_n(\mathcal{L})$ into a Leibniz algebra. This construction goes back to Gautheron [2] and plays an important role in [1].

The main result of this paper is to show that if \mathcal{L} is a free Leibniz n -algebra, then $\mathcal{D}_{n-1}(\mathcal{L})$ is a free Leibniz algebra too (on a different vector space). This result plays an essential role in the cohomological investigation of Leibniz n -algebras, which we consider in the last section.

We first introduce a notion of *representation* of a Leibniz n -algebra \mathcal{L} . This notion for $n = 2$ was already considered in [5]. One observes that if M is a representation of a Leibniz n -algebra \mathcal{L} , then $\text{Hom}(\mathcal{L}, M)$ can be considered as a representation of the Leibniz algebra $\mathcal{D}_{n-1}(\mathcal{L})$. The work of [2] and [1] suggests defining the cohomology of \mathcal{L} with coefficients in M to be $HL^*(\mathcal{D}_{n-1}(\mathcal{L}), \text{Hom}(\mathcal{L}, M))$. We deduce from our main theorem that this theory is exactly the Quillen cohomology for Leibniz n -algebras.

The referee drew our attention to the paper [11] which surveys different kinds of n -ary algebras.

2 Derivations

In the whole paper K is a field. All tensor products are taken over K . Let \mathcal{A} be a vector space equipped with an n -linear operation $\omega : \mathcal{A}^{\otimes n} \rightarrow \mathcal{A}$. A map $f : \mathcal{A} \rightarrow \mathcal{A}$ is called a *derivation* with respect to ω if

$$f(\omega(a_1, \dots, a_n)) = \sum_{i=1}^n \omega(a_1, \dots, f(a_i), \dots, a_n).$$

In this case we also say that f is an ω -*derivation*. We let $\mathcal{D}er_\omega(\mathcal{A})$ be the set of all ω -derivations. The following is well-known.

Proposition 2.1. i) *The subset $\mathcal{D}er_\omega(\mathcal{A})$ of the Lie algebra of endomorphisms $\text{End}(\mathcal{A})$ is a Lie subalgebra.*

ii) *If $f \in \mathcal{D}er_\omega(\mathcal{A})$ and $f \in \mathcal{D}er_\sigma(\mathcal{A})$, then $f \in \mathcal{D}er_{\omega+\sigma}(\mathcal{A})$. Here $\sigma : \mathcal{A}^{\otimes n} \rightarrow \mathcal{A}$ is also an n -linear operation.*

Proposition 2.2. *Let $[-, -] : \mathcal{A}^{\otimes 2} \rightarrow \mathcal{A}$ be a bilinear operation and let $\omega : \mathcal{A}^{\otimes n} \rightarrow \mathcal{A}$ be given by*

$$\omega(x_1, \dots, x_n) := [x_1, [x_2, \dots, [x_{n-1}, x_n] \cdots]].$$

If f is a derivation with respect to $[-, -]$, then $f \in \mathcal{D}er_\omega(\mathcal{A})$.

Proof. One has

$$\begin{aligned}
 f(\omega(x_1, \dots, x_n)) &= f([x_1, [x_2, \dots, [x_{n-1}, x_n] \cdots]]) \\
 &= [f(x_1), [x_2, \dots, [x_{n-1}, x_n] \cdots]] \\
 &\quad + [x_1, f([x_2, \dots, [x_{n-1}, x_n] \cdots])] \\
 &= \sum_i [x_1, \dots, [f(x_i), \dots, x_n]] = \sum_i \omega(x_1, \dots, f(x_i), \dots, x_n).
 \end{aligned}$$

The following is an immediate generalization of Proposition 2.2. Since it has the same proof, we omit it here.

Proposition 2.3. *Let $\omega_i : \mathcal{A}^{\otimes n_i} \rightarrow \mathcal{A}$ be n_i -ary operations for $i = 1, \dots, k$ and let $\omega : \mathcal{A}^{\otimes k} \rightarrow \mathcal{A}$ be a k -ary operation. If f is a derivation with respect to $\omega_1, \dots, \omega_n, \omega$, then it is also a derivation with respect to the composite $\sigma : \mathcal{A}^{\otimes n} \rightarrow \mathcal{A}$. Here $n = n_1 + \dots + n_k$,*

$$\sigma(a_1, \dots, a_n) := \omega(\omega_1(a_1, \dots, a_{n_1}), \dots, \omega_k(a_s, \dots, a_n))$$

and $s = n - n_k + 1 = n_1 + \dots + n_{k-1} + 1$.

Proposition 2.4. *Let $\omega : \mathcal{A}^{\otimes(n+1)} \rightarrow \mathcal{A}$ be an $(n+1)$ -linear map and let $\mu_i : \mathfrak{g} \otimes \mathfrak{g} \rightarrow \mathfrak{g}$ be the bilinear map given by*

$$\mu_i(a_1 \otimes \cdots \otimes a_n, b_1 \otimes \cdots \otimes b_n) = a_1 \otimes \cdots \otimes \omega(a_i, b_1 \otimes \cdots \otimes b_n) \otimes \cdots \otimes a_n.$$

Here $\mathfrak{g} = \mathcal{A}^{\otimes n}$ and $1 \leq i \leq n$. Suppose that $f : \mathcal{A} \rightarrow \mathcal{A}$ is an ω -derivation and $\varphi : \mathfrak{g} \rightarrow \mathfrak{g}$ is given by

$$\varphi(a_1 \otimes \cdots \otimes a_n) = \sum_{j=1}^n a_1 \otimes \cdots \otimes f(a_j) \otimes \cdots \otimes a_n.$$

Then φ is a derivation with respect to μ_i , for any $1 \leq i \leq n$.

Proof. One has

$$\varphi(\mu_i(a_1 \otimes \cdots \otimes a_n, b_1 \otimes \cdots \otimes b_n)) = \varphi(a_1 \otimes \cdots \otimes \omega(a_i, b_1 \otimes \cdots \otimes b_n) \otimes \cdots \otimes a_n).$$

Since f is an ω -derivation, we see that this expression is equal to

$$\begin{aligned}
 &f(a_1) \otimes \cdots \otimes \omega(a_i, b_1 \otimes \cdots \otimes b_n) \otimes \cdots \otimes a_n + \cdots + \\
 &\quad + a_1 \otimes \cdots \otimes \omega(f(a_i), b_1 \otimes \cdots \otimes b_n) \otimes \cdots \otimes a_n +
 \end{aligned}$$

$$\begin{aligned}
 &+ a_1 \otimes \cdots \otimes \omega(a_i, f(b_1) \otimes \cdots \otimes b_n) \otimes \cdots \otimes a_n + \cdots + \\
 &+ a_1 \otimes \cdots \otimes \omega(a_i, b_1 \otimes \cdots \otimes f(b_n)) \otimes \cdots \otimes a_n + \cdots + \\
 &+ a_1 \otimes \cdots \otimes \omega(a_i, b_1 \otimes \cdots \otimes b_n) \otimes \cdots \otimes f(a_n).
 \end{aligned}$$

On the other hand the expression

$$\mu_i(\varphi(a_1 \otimes \cdots \otimes a_n), b_1 \otimes \cdots \otimes b_n) + \mu_i(a_1 \otimes \cdots \otimes a_n, \varphi(b_1 \otimes \cdots \otimes b_n)),$$

is clearly equal to the previous expression thanks to the definition of μ_i and φ . This proves that φ is a derivation with respect to μ_i .

3 Leibniz n -algebras

A *Leibniz algebra of order n* , or simply a *Leibniz n -algebra*, is a vector space \mathcal{L} equipped with an n -linear operation $[-, \dots, -]: \mathcal{L}^{\otimes n} \rightarrow \mathcal{L}$ such that for all x_1, \dots, x_{n-1} the map $ad(x_1, \dots, x_{n-1}): \mathcal{L} \rightarrow \mathcal{L}$ given by $ad(x_1, \dots, x_{n-1})(x) = [x, x_1, \dots, x_{n-1}]$ is a derivation with respect to $[-, \dots, -]$. This means that the following *Leibniz n -identity* holds:

$$\begin{aligned}
 (3.1) \quad & [[x_1, x_2, \dots, x_n], y_1, y_2, \dots, y_{n-1}] \\
 &= \sum_{i=1}^n [x_1, \dots, x_{i-1}, [x_i, y_1, y_2, \dots, y_{n-1}], x_{i+1}, \dots, x_n].
 \end{aligned}$$

We let ${}_n\mathbf{Lb}$ be the category of Leibniz n -algebras. Let us observe that for $n = 2$ the identity (3.1) is equivalent to (1.1). So a Leibniz 2-algebra is simply a Leibniz algebra in the sense of [4], and so Leibniz 2-algebras are called just Leibniz algebras, and we use \mathbf{Lb} instead of ${}_2\mathbf{Lb}$.

Clearly a Lie algebra is a Leibniz algebra such that $[x, x] = 0$ holds. Similarly for $n \geq 3$ an *n -Lie* or an *n -Nambu-Lie algebra* is a Leibniz n -algebra such that $[x_1, \dots, x_i, x_{i+1}, \dots, x_n] = 0$ as soon as $x_i = x_{i+1}$ for $1 \leq i \leq n - 1$. Such algebras appear in what is known as *Nambu mechanics* and there exists several interesting papers about them (see [1], [2] and references given there).

Another big class of Leibniz 3-algebras which were considered in the literature are the so called *Lie triple systems*. Let us recall that a Lie triple system [3] is a vector space equipped with a bracket $[-, -, -]$ that satisfies the same identity (3.1) and, instead of skew-symmetry, satisfies the conditions

$$[x, y, z] + [y, z, x] + [z, x, y] = 0$$

and

$$[x, y, y] = 0.$$

Proposition 3.2. *Let \mathfrak{g} be a Leibniz algebra. Then \mathfrak{g} is also a Leibniz $(n + 1)$ -algebra with respect to the operation $\omega : \mathfrak{g}^{\otimes(n+1)} \rightarrow \mathfrak{g}$ given by*

$$\omega(x_0, x_1, \dots, x_n) := [x_0, [x_1, \dots [x_{n-1}, x_n]]].$$

Proof. From the definition of Leibniz algebra we know that

$$ad(x) = [-, x] : \mathfrak{g} \rightarrow \mathfrak{g}$$

is a derivation with respect to the 2-bracket. By Proposition 2.2 we know that it is also a derivation with respect to ω . Since for all $x_1, x_2, \dots, x_n \in \mathfrak{g}$ one has $ad(x_1, x_2, \dots, x_n) = [x_1, \dots [x_{n-1}, x_n] \dots]$ the Proposition follows. Here $ad(x_1, x_2, \dots, x_n) = \omega(-, x_1, x_2, \dots, x_n) : \mathfrak{g} \rightarrow \mathfrak{g}$.

Proposition 3.2 shows that there exists a “forgetful” functor

$$U_n : \mathbf{Lb} \rightarrow {}_n\mathbf{Lb}.$$

Here are more examples of Leibniz 3-algebras.

Examples 3.3. i) Let \mathfrak{g} be a Leibniz algebra with involution σ . This means that σ is an automorphism of \mathfrak{g} and $\sigma^2 = id$. Then

$$\mathcal{L} := \{x \in \mathfrak{g} \mid x + \sigma(x) = 0\}$$

is a Leibniz 3-algebra with respect to the bracket

$$[x, y, z] := [x, [y, z]].$$

ii) Let V be a 4-dimensional vector space with basis i, j, k, l . Then we define $[x, y, z] := \det(A)$, where A is the following matrix

$$\begin{pmatrix} i & j & k & l \\ x_1 & x_2 & x_3 & x_4 \\ y_1 & y_2 & y_3 & y_4 \\ z_1 & z_2 & z_3 & z_4 \end{pmatrix}$$

One sees that this gives rise to a Leibniz 3-algebra. Moreover it is a Nambu-Lie algebra. Here $x = x_1i + x_2j + x_3k + x_4l$ and so on. One easily generalizes this example to obtain an n -Nambu-Lie algebra starting with an $(n + 1)$ -dimensional vector space. This example was a starting point for investigating n -Lie (or Nambu-Lie) algebras.

Let \mathcal{L} be a Leibniz n -algebra. Thanks to Proposition 2.1 i) we know that

$$Der(\mathcal{L}) = \{f : \mathcal{L} \rightarrow \mathcal{L} \mid f \text{ is a derivation}\}$$

is a Lie algebra.

Proposition 3.4. *Let \mathcal{L} be a Leibniz $(n + 1)$ -algebra. Then $\mathcal{D}_n(\mathcal{L}) = \mathcal{L}^{\otimes n}$ is a Leibniz algebra with respect to the bracket*

$$[a_1 \otimes \cdots \otimes a_n, b_1 \otimes \cdots \otimes b_n] := \sum_{i=1}^n a_1 \otimes \cdots \otimes [a_i, b_1, \dots, b_n] \otimes \cdots \otimes a_n$$

Moreover

$$ad : \mathcal{L}^{\otimes n} \rightarrow \mathcal{D}er(\mathcal{L}), \quad x_1 \otimes \cdots \otimes x_n \mapsto ad(x_1, x_2, \dots, x_n)$$

is a homomorphism of Leibniz algebras.

Proof. Fix $x_1, \dots, x_n \in \mathcal{L}$. We have to prove that

$$\varphi : \mathcal{D}_n(\mathcal{L}) \rightarrow \mathcal{D}_n(\mathcal{L})$$

given by

$$\varphi(a_1 \otimes \cdots \otimes a_n) = \sum_{j=1}^n a_1 \otimes \cdots \otimes f(a_j) \otimes \cdots \otimes a_n,$$

is a derivation with respect to $[-, -]$. Here $f = [-, x_1, \dots, x_n] : \mathcal{L} \rightarrow \mathcal{L}$. Thanks to Proposition 2.4 we know that φ is a derivation with respect of all μ_i , $1 \leq i \leq n$, where

$$\mu_i(a_1 \otimes \cdots \otimes a_n, b_1 \otimes \cdots \otimes b_n) = a_1 \otimes \cdots \otimes \omega(a_i, b_1 \otimes \cdots \otimes b_n) \otimes \cdots \otimes a_n.$$

Then φ is also a derivation with respect to $[-, -] = \sum_{i=1}^n \mu_i$ thanks to Proposition 2.1 ii) and the first part of the Proposition follows. Let us show that ad is a homomorphism of Leibniz algebras. Indeed, one sees that

$$ad([a_1 \otimes \cdots \otimes a_n, b_1 \otimes \cdots \otimes b_n])(a) = \sum_{i=1}^n [a, [a_1, \dots, [a_i, b_1, \dots, b_n], \dots, a_n]].$$

On the other hand

$$\begin{aligned} & [ad(a_1 \otimes \cdots \otimes a_n), ad(b_1 \otimes \cdots \otimes b_n)](a) \\ &= ad(a_1 \otimes \cdots \otimes a_n)ad(b_1 \otimes \cdots \otimes b_n)(a) \\ & \quad - ad(b_1 \otimes \cdots \otimes b_n)ad(a_1 \otimes \cdots \otimes a_n)(a) \\ &= [[a, [b_1, \dots, b_n]], a_1 \otimes \cdots \otimes a_n] - [[a, a_1 \otimes \cdots \otimes a_n], b_1, \dots, b_n]. \end{aligned}$$

Therefore (3.1) shows that

$$ad([a_1 \otimes \cdots \otimes a_n, b_1 \otimes \cdots \otimes b_n]) = [ad(a_1 \otimes \cdots \otimes a_n), ad(b_1 \otimes \cdots \otimes b_n)].$$

Hence $ad : \mathcal{D}_n \mathcal{L} \rightarrow \mathcal{D}er(\mathcal{L})$ is a homomorphism of Leibniz algebras.

Remark 3.5. One can prove that if \mathcal{A} is a Leibniz $(kn + 1)$ -algebra, then $\mathcal{A}^{\otimes k}$ is a Leibniz $(n + 1)$ -algebra with respect to the following bracket

$$\begin{aligned} & [x_{01} \otimes x_{02} \cdots \otimes x_{0k}, \dots, x_{n1} \otimes x_{n2} \cdots \otimes x_{nk}] \\ & := [x_{01}, x_{11}, \dots, x_{1k}, \dots, x_{n1}, \dots, x_{nk}] \otimes x_{02} \otimes \cdots \otimes x_{0k} \\ & \quad + \cdots + x_{01} \otimes \cdots \otimes x_{0k-1} \otimes [x_{0k}, x_{11}, \dots, x_{nk}]. \end{aligned}$$

By Proposition 3.4 the map $\mathcal{L} \mapsto \mathcal{D}_n(\mathcal{L})$ from Leibniz $(n + 1)$ -algebras to Leibniz algebras is a functor that we denote by \mathcal{D}_n . More generally, by Remark 3.5, there exist functors $\mathcal{D}_{kn}^n : {}_{kn+1}\mathbf{Lb} \rightarrow {}_{n+1}\mathbf{Lb}$ (where $\mathcal{D}_n = \mathcal{D}_n^1$) and we have $\mathcal{D}_n^1 \circ \mathcal{D}_{kn}^n = \mathcal{D}_{kn}^1$.

4 The main theorem

The goal of this section is to prove that the functor $\mathcal{D}_n : {}_{n+1}\mathbf{Lb} \rightarrow \mathbf{Lb}$ sends free objects to free objects. For more specific statements see Theorem 4.4 and Theorem 4.8 below. Since \mathcal{D}_1 is nothing but the identity functor, we have to consider the case $n \geq 2$. To avoid long formulas we will first restrict ourself to the case $n = 2$ and, second, we indicate how to modify the argument for $n \geq 3$.

Let us recall (see [8]) that a *magma* \mathcal{M} is a set together with a map (binary operation)

$$\mathcal{M} \times \mathcal{M} \rightarrow \mathcal{M}, \quad (x, y) \mapsto x \star y.$$

Let Y be the free magma with one generator e . We recall from [8] the construction of Y . First one defines the sequence of sets $(Y_m)_{m \geq 1}$ as follows:

$$Y_1 = \{e\}, \quad Y_m = \coprod_{p+q=m} Y_p \times Y_q, \quad (m \geq 2; p, q \geq 1).$$

We let Y be the disjoint union

$$Y = \coprod_{m \geq 1} Y_m.$$

One defines $\star : Y \times Y \rightarrow Y$ by means of

$$Y_p \times Y_q \rightarrow Y_{p+q} \subseteq Y.$$

Then Y is a magma, which is freely generated by e . Let C_m be the number of elements

of Y_{m+1} . Clearly $C_0 = 1, C_1 = 1$ and

$$C_{m+1} = \sum_{i+j=m} C_i C_j.$$

Hence the function $f(t) = \sum_{m=0}^{\infty} C_m t^m$ satisfies the functional equation

$$(4.1) \quad f(t) - 1 = t f^2(t).$$

Of course this equation is well known, as well as the fact that C_m is equal to the Catalan number, that is

$$C_m = \frac{(2m)!}{m!(m+1)!}.$$

So one has

$$C_0 = 1, C_1 = 1, C_2 = 2, C_3 = 5, C_4 = 14, C_5 = 42, C_6 = 132, \dots$$

If $\omega \in Y_m$, then we say that ω is of length m and we write $l(\omega) = m$. Clearly if $\omega \in Y$, then $\omega = e$ or $\omega = \omega_1 * \omega_2$, with unique $(\omega_1, \omega_2) \in Y \times Y$. Moreover $l(\omega) = l(\omega_1) + l(\omega_2)$. Recall that the elements of Y_n can be interpreted as planar binary trees with n leaves. Under this interpretation the operation \star is simply the grafting operation (join the roots to a new vertex and add a new root).

The following proposition is the analogue for Leibniz 3-algebras of Lemma 1.3 in [5] concerning Leibniz algebras.

Proposition 4.2. *Let $K[Y]$ be the vector space spanned by Y . Then there exists a unique structure of Leibniz 3-algebra on $K[Y]$ such that*

$$[\omega_1, \omega_2, e] = \omega_1 \star \omega_2.$$

Moreover $K[Y]$ with this structure is a free Leibniz 3-algebra generated by e .

Proof. We use the method devised in [5] for the case of Leibniz algebras. Let us observe that (3.1) for Leibniz 3-algebras is equivalent to

$$(4.2.1) \quad [a, b, [c, x, y]] = [[a, b, c], x, y] - [[a, x, y], b, c] - [a, [b, x, y], c].$$

The 3-bracket $[\omega_1, \omega_2, \omega_3]$ has been already defined for $\omega_3 = e$. If $\omega_3 \neq e$, then it is of the form $\omega \star \omega'$ for some elements ω and ω' . Hence

$$[\omega_1, \omega_2, \omega_3] = [\omega_1, \omega_2, \omega \star \omega'] = [\omega_1, \omega_2, [\omega, \omega', e]],$$

and one can use (4.2.1) to rewrite it with 3-brackets whose last variable is either ω or

ω' . Since $l(\omega)$ and $l(\omega')$ are less than $l(\omega_3)$, we get, by recursivity, the element $[\omega_1, \omega_2, \omega_3]$ as a unique algebraic sum of elements in Y .

We now have to prove that, with this definition, the 3-bracket satisfies the Leibniz 3-identity (4.2.1). Clearly it holds when $y = e$, since it is precisely this formula which was used to compute the left part. So we can work by induction with respect to $l(y)$. If $l(y) \geq 2$ then $y = y_1 \star y_2$ and therefore

$$\begin{aligned}
[a, b, [c, x, y]] &= [a, b, [c, x, [y_1, y_2, e]]] \\
&= [a, b, [[c, x, y_1], y_2, e]] - [a, b, [[c, y_2, e], x, y_1]] - [a, b, [c, [x, y_2, e], y_1]] \\
&= [[a, b, [c, x, y_1]], y_2, e] - [[a, y_2, e], b, [c, x, y_1]] - [a, [b, y_2, e], [c, x, y_1]] \\
&\quad - [[a, b, [c, y_2, e]], x, y_1] + [[a, x, y_1], b, [c, y_2, e]] + [a, [b, x, y_1], [c, y_2, e]] \\
&\quad - [[[a, b, c], [x, y_2, e], y_1] + [[a, [x, y_2, e], y_1], b, c] - [a, [b, [x, y_2, c], y_1], c] \\
&= [[a, b, c], x, y_1], y_2, e] - [[[a, x, y_1], b, c], y_2, e] - [[a, [b, x, y_1], c], y_2, e] \\
&\quad - [[[a, y_2, e], b, c], x, y_1] + [[[a, y_2, e], x, y_1], b, c] + [[a, y_2, e], [b, x, y_1], c] \\
&\quad - [[a, [b, y_2, e], c], x, y_1] + [[a, x, y_1], [b, y_2, e], c] + [a, [[b, y_2, e], x, y_1], c] \\
&\quad - [[[a, b, c], y_2, e], x, y_1] + [[[a, y_2, e], b, c], x, y_1] + [[a, [b, y_2, e], c], x, y_1] \\
&\quad + [[[a, x, y_1], b, c], y_2, e] - [[[a, x, y_1], y_2, e], b, c] - [[a, x, y_1], [b, y_2, e], c] \\
&\quad + [[a, [b, x, y_1], c], y_2, e] - [[a, y_2, e], [b, x, y_1], c] - [a, [[b, x, y_1], y_2, e], c] \\
&\quad - [[a, b, c], [x, y_2, e], y_1] + [[a, [x, y_2, e], y_1], b, c] + [a, [b, [x, y_2, e], y_1], c].
\end{aligned}$$

One sees that 2^{nd} and 13^{th} , as well as 3^{rd} and 16^{th} , 4^{th} and 11^{th} , 6^{th} and 17^{th} , 7^{th} and 12^{th} , 8^{th} and 15^{th} terms cancel. Hence we have

$$\begin{aligned}
[a, b, [c, x, y]] &= [[[a, b, c], x, y_1], y_2, e] + [[[a, y_2, e], x, y_1], b, c] + [a, [[b, y_2, e], x, y_1], c] \\
&\quad - [[[a, b, c], y_2, e], x, y_1] - [[[a, x, y_1], y_2, e], b, c] - [a, [[b, x, y_1], y_2, e], c] \\
&\quad - [[a, b, c], [x, y_2, e], y_1] + [[a, [x, y_2, e], y_1], b, c] + [a, [b, [x, y_2, e], y_1], c].
\end{aligned}$$

On the other hand we have

$$\begin{aligned}
[[a, b, c], x, y] &= [[[a, b, c], x, y_1], y_2, e] - [[[a, b, c], y_2, e], x, y_1] - [[a, b, c], [x, y_2, e], y_1].
\end{aligned}$$

Similarly

$$\begin{aligned} & -[[a, x, y], b, c] \\ & = -[[[a, x, y_1], y_2, e], b, c] + [[[a, y_2, e], x, y_1], b, c] + [[a, [x, y_2, e], y_1], b, c] \end{aligned}$$

and

$$\begin{aligned} & -[a, [b, x, y], c] \\ & = -[a, [[b, x, y_1], y_2, e], c] + [a, [[b, y_2, e], x, y_1], c] + [a, [b, [x, y_2, e], y_1], c]. \end{aligned}$$

One checks that after substitution in (3.1) all terms cancel and therefore $K[Y]$ has a well defined structure of Leibniz 3-algebra. If \mathcal{L} is any Leibniz 3-algebra and $x \in \mathcal{L}$, then by induction one can check that there exists a unique homomorphism

$$f : K[Y] \rightarrow \mathcal{L}$$

such that $f(e) = x$ and Proposition 4.2 is proved.

Now we can formulate the following

Theorem 4.3. *The vector space spanned by the set $\bar{Y} = Y - \{e\}$ has a unique Leibniz algebra structure such that*

$$[x \star y, z \star e] = (x \star z) \star y + x \star (y \star z).$$

It is a free Leibniz algebra over the set $Y' = \{x \star e \mid x \in Y\} \subset \bar{Y}$. Moreover one has isomorphisms of Leibniz algebras

$$\mathcal{D}_2(K[Y]) \cong K[\bar{Y}] \cong T(K[Y']).$$

Proof. 1st Step: Uniqueness. Assume that such a Leibniz algebra structure exists. Then $ad(u) = [-, u]$ is uniquely determined when $u = x \star e$. We will prove by induction on $l(q)$ that $ad(u)$ is uniquely determined when $u = x \star q$. If $l(q) > 1$, then $q = y \star z$ and by assumption

$$u = [x \star y, z \star e] - (x \star z) \star y.$$

Therefore

$$\begin{aligned} ad(u) & = -ad((x \star z) \star y) + ad[x \star y, z \star e] \\ & = -ad((x \star z) \star y) + ad(z \star e) \cdot ad(x \star y) - ad(x \star y) \cdot ad(z \star e) \end{aligned}$$

and by induction assumption $ad(u)$ is uniquely determined.

2nd Step: Bijection. There is a linear isomorphism $K[Y] \otimes K[Y] \cong K[\bar{Y}]$, which is given by

$$(x, y) \mapsto x \star y, \quad \text{for } x, y \in Y.$$

It yields indeed a bijection because

$$\begin{aligned} K[Y] \otimes K[Y] &\cong \bigoplus_{p, q \geq 1} K[Y_p] \otimes K[Y_q] \cong \bigoplus_{p, q \geq 1} K[Y_p \times Y_q] \\ &\cong \bigoplus_{m \geq 2} \bigoplus_{p+q=m} K[Y_p \times Y_q] \cong \bigoplus_{m \geq 2} K[Y_m] = K[\bar{Y}]. \end{aligned}$$

3rd Step: Algebra isomorphism $\mathcal{D}_2(K[Y]) \cong K[\bar{Y}]$. Let us consider $K[\bar{Y}]$ as a Leibniz algebra induced by the linear isomorphism from Step 2. Since $x \star y$ and $z \star e$ are the images of $x \otimes y$ and $z \otimes e \in K[Y] \otimes K[Y]$ under the isomorphism of Step 2, we have to show that, in this algebra, the following identity

$$[x \star y, z \star e] = (x \star z) \star y + x \star (y \star z)$$

holds. By definition of the functor \mathcal{D}_2 one has

$$[x \otimes y, z \otimes e] = [x, z, e] \otimes y + x \otimes [y, z, e] = (x \star z) \otimes y + x \otimes (y \star z)$$

and this element goes to $(x \star z) \star y + x \star (y \star z)$ under the isomorphism of Step 2. This proves also the existence part of the Theorem.

4th Step: Y' generates $K[\bar{Y}]$. Indeed let X be the subalgebra of $K[\bar{Y}]$ generated by Y' . We have to prove that $\bar{Y} \subset X$. Let $x \star y$ be an element in \bar{Y} . We will show by induction on $l(y)$ that $x \star y \in X$. If $l(y) = 1$, then $x \star y = x \star e \in Y' \subset X$. If $l(y) > 1$, then $y = y_1 \star z$ and by the assumption

$$x \star y = -(x \star z) \star y_1 + [x \star y_1, z \star e].$$

But, by the induction assumption, one has $(x \star z) \star y_1, x \star y_1, z \star e \in X$. Therefore $x \star y \in X$ as well.

5th Step: $K[\bar{Y}]$ as a graded Leibniz algebra. For $x \in \bar{Y}$ we let $d(x)$ to be $l(x) - 1$. Then $K[\bar{Y}]$ can be considered as a graded vector space by declaring that the degree of an element $x \in \bar{Y}$ is $d(x)$. We claim that under this grading $K[\bar{Y}]$ is a graded Leibniz algebra, that is, if $d(x) = k$ and $d(y) = m$, then $[x, y]$ is a linear combination of elements of degree $k + m$. The claim is clear when $y = x' \star y'$ and $l(y') = 1$ and it can be proved by the same induction arguments as in Step 1.

6th Step: Y' freely generates $K[\bar{Y}]$. Let us recall from [5] that for a vector space U the free Leibniz algebra generated by U is the unique Leibniz algebra structure on

$$\bar{T}(U) = \bigoplus_{m \geq 1} U^{\otimes m}$$

such that for any $u \in U$ one has

$$[x, u] = x \otimes u, \quad x \in \bar{T}(U).$$

Take $U = K[Y']$. Then we obtain the natural epimorphism

$$\varphi : \bar{T}(K[Y']) \rightarrow K[\bar{Y}].$$

We have to show that φ is injective. The vector space $K[Y']$ is a graded subspace of $K[\bar{Y}]$. Therefore $\bar{T}(K[Y'])$ is also graded and φ is a morphism of graded Leibniz algebras. Since each component is of finite dimension, it is enough to show that the degree p part of $\bar{T}(K[Y'])$ and $K[\bar{Y}]$ have the same dimension.

The dimension of the degree p part of $K[\bar{Y}]$ is equal to C_p (the Catalan number), while the dimension of the degree p part of $\bar{T}(K[Y'])$ is equal to the coefficient of t^p in the expansion of

$$g(t) = \sum_{m=1}^{\infty} \left(\sum_{k=1}^{\infty} C_{k-1} t^k \right)^m,$$

because the degree m part of $K[Y']$ is of dimension C_{m-1} . Hence we have to prove that $g(t) = f(t) - 1$. But

$$g(t) = \sum_{m=1}^{\infty} (tf(t))^m = \frac{tf(t)}{1 - tf(t)}$$

and it is equal to $f(t) - 1$, thanks to (4.1).

Let us now show the following parametrized version of Theorem 4.3.

Theorem 4.4. *Let V be a vector space, and put*

$$F(V) := \bigoplus_{m \geq 1} K[Y_m] \otimes V^{\otimes 2m-1}.$$

(i) *There exists a unique Leibniz 3-algebra structure on $F(V)$ such that*

$$[\omega_1 \otimes x_1, \omega_2 \otimes x_2, e \otimes x] = (\omega_1 \star \omega_2) \otimes x_1 \otimes x_2 \otimes x,$$

where $\omega_1 \in K[Y_p]$, $x_1 \in V^{\otimes (2p-1)}$, $\omega_2 \in K[Y_q]$, $x_2 \in V^{\otimes (2q-1)}$ and $x \in V$.

(ii) *Equipped with this structure $F(V)$ is a free Leibniz 3-algebra generated by V .*

(iii) *The Leibniz algebra $\mathcal{D}_2(F(V))$ is isomorphic to the free Leibniz algebra generated by the vector space*

$$E = \bigoplus_{m \geq 1} K[Y_m] \otimes V^{\otimes 2m}.$$

Proof. The proof is similar to the proof of Theorem 4.2. At the end, in order to show that the vector spaces $F(V)$ and $T(E)$ have the same dimension we use the following identity of formal power series:

$$(x + x^3 + 2x^5 + 5x^7 + 14x^9 + \dots)^2 = \sum_{m=1}^{\infty} (x^2 + x^4 + 2x^6 + 5x^8 + 14x^{10} + \dots)^m,$$

which is an immediate consequence of the functional equation (4.1).

Let us now state the results for $(n + 1)$ -Leibniz algebras. Since the proofs follow the same pattern as in the case $n + 1 = 2 + 1$, we mention only the main modifications. By definition an n -magma is a set \mathcal{M} together with a map (n -ary operation)

$$(-, \dots, -) : \underbrace{\mathcal{M} \times \dots \times \mathcal{M}}_{n \text{ copies}} \rightarrow \mathcal{M}.$$

Let Z be the free n -magma with one generator e . It can be described as follows. The sequence of sets $(Z_m)_{m \geq 1}$ is given by:

$$Z_1 = \{e\}, \quad Z_m = \coprod_{p_1 + \dots + p_n = m} Z_{p_1} \times \dots \times Z_{p_n}, \quad (m \geq 2; p_i \geq 1).$$

Observe that $Z_m = \emptyset$ unless $m = (n - 1)k + 1$ for some $k \geq 0$. We let Z be the disjoint union

$$Z = \coprod_{m \geq 1} Z_m.$$

One defines $(-, \dots, -) : Z \times \dots \times Z \rightarrow Z$ by means of

$$Z_{(n-1)k_1+1} \times \dots \times Z_{(n-1)k_n+1} \rightarrow Z_{(n-1)(k_1+\dots+k_n+1)+1} \subseteq Z.$$

Then Z is a n -magma, which is freely generated by e . Let D_k be the number of elements of $Z_{(n-1)k+1}$. Clearly $D_0 = 1$, $D_1 = 1$ and

$$D_{k+1} = \sum_{k_1 + \dots + k_n = k} D_{k_1} \dots D_{k_n}.$$

Hence the function $f(t) = \sum_{k=0}^{\infty} D_k t^k$ satisfies the functional equation

$$(4.5) \quad f(t) - 1 = t f(t)^n.$$

If $\omega \in Z_{(n-1)k+1}$, then we say that ω is of length k and we write $l(\omega) = k$. Clearly if $\omega \in Z$, then $\omega = e$ or $\omega = (\omega_1, \dots, \omega_n)$, for some unique elements $\omega_1, \dots, \omega_n$ in Z . Moreover $l(\omega) = l(\omega_1) + \dots + l(\omega_n) + 1$. Recall that the elements of Z_m can be interpreted as n -ary planar trees, that is each vertex has one root and n leaves. Under this interpretation the operation $(-, \dots, -)$ is simply the grafting operation. Observe that the number of vertices (resp. edges) of a tree in $Z_{(n-1)k+1}$ is k (resp. $kn + 1$).

Proposition 4.6. *Let $K[Z]$ be the vector space spanned by Z . Then there exists a unique structure of $(n + 1)$ -Leibniz algebra on $K[Z]$ such that*

$$[\omega_1, \dots, \omega_n, e] = (\omega_1, \dots, \omega_n).$$

Moreover $K[Z]$ with this structure is a free $(n + 1)$ -Leibniz algebra generated by e .

Theorem 4.7. *The vector space spanned by the set $\bar{Z} = Z - \{e\}$ has a unique Leibniz algebra structure such that*

$$[(\omega_1, \dots, \omega_n), (\omega'_1, \dots, \omega'_{n-1}, e)] = \sum_{i=1}^n (\omega_1, \dots, (\omega_i, \omega'_1, \dots, \omega'_{n-1}), \dots, \omega_n).$$

As a Leibniz algebra it is free over the set $Z' = \{(\omega_1, \dots, \omega_{n-1}, e) \mid \omega_i \in Z\}$. Moreover one has isomorphisms of Leibniz algebras:

$$\mathcal{D}_n(K[Z]) \cong K[\bar{Z}] \cong T(K[Z']).$$

Proof. Let us just mention the computation of the dimensions of the vector spaces. Let E_k be the number of n -ary trees with k vertices which are of the form $(\omega_1, \dots, \omega_{n-1}, e)$. One has

$$E_k = \sum_{k_1 + \dots + k_{n-1} + 1 = k} D_{k_1} \times \dots \times D_{k_{n-1}}.$$

Hence we get $\sum E_k t^k = t \sum D_{k_1} t^{k_1} \dots D_{k_{n-1}} t^{k_{n-1}} = tf(t)^{n-1}$. Therefore the generating series for $T(K[Z'])$ is $\frac{tf(t)^{n-1}}{1 - tf(t)^{n-1}}$. By the functional equation (4.6) this is equal to $f(t) - 1$, which is the generating series for $K[\bar{Z}]$.

Theorem 4.8. *Let V be a vector space, and put*

$$F(V) := \bigoplus_{k \geq 0} K[Z_{(n-1)k+1}] \otimes V^{\otimes nk+1}.$$

(i) *There exists a unique $(n + 1)$ -Leibniz algebra structure on $F(V)$ such that*

$$[\omega_1 \otimes x_1, \dots, \omega_n \otimes x_n, e \otimes x] = (\omega_1, \dots, \omega_n) \otimes x_1 \otimes \dots \otimes x_n \otimes x,$$

where $\omega_i \in K[Z_{(n-1)k_i+1}]$, $x_i \in V^{\otimes nk_i+1}$ and $x \in V$.

- (ii) *Equipped with this structure $F(V)$ is a free $(n + 1)$ -Leibniz algebra generated by V .*
- (iii) *The Leibniz algebra $\mathcal{D}_n(F(V))$ is isomorphic to the free Leibniz algebra generated by the vector space*

$$E = \bigoplus_{k \geq 0} K[Z_{(n-1)k+1}] \otimes V^{\otimes n(k+1)}.$$

Remark 4.9. By the same kind of argument one can show that the functor $\mathcal{D}_{kn}^k :_{kn+1} \mathbf{Lb} \rightarrow_{k+1} \mathbf{Lb}$ sends free objects to free objects.

5 Cohomology of Leibniz n -algebras

An abelian extension of Leibniz n -algebras

$$(5.1) \quad 0 \rightarrow M \rightarrow \mathcal{K} \rightarrow \mathcal{L} \rightarrow 0$$

is an exact sequence of Leibniz n -algebras such that $[a_1, \dots, a_n] = 0$ as soon as $a_i \in M$ and $a_j \in \mathcal{K}$ for some $1 \leq i \neq j \leq n$. Here $a_1, \dots, a_n \in \mathcal{K}$. Clearly then M is an *abelian* Leibniz n -algebra, that is the bracket vanishes on M . Let us observe that the converse is true only for $n = 2$.

If (5.1) is an abelian extension of Leibniz n -algebras, then M is equipped with n actions

$$[-, \dots, -] : \mathcal{L}^{\otimes i} \otimes M \otimes \mathcal{L}^{\otimes n-1-i} \rightarrow M, \quad 0 \leq i \leq n - 1$$

satisfying $(2n - 1)$ equations, which are obtained from (3.1) by letting exactly one of the variables $x_1, \dots, x_n, y_1, \dots, y_{n-1}$ be in M and all the others in \mathcal{L} .

By definition a *representation* of the Leibniz n -algebra \mathcal{L} is a vector space M equipped with n actions of $[-, \dots, -] : \mathcal{L}^{\otimes i} \otimes M \otimes \mathcal{L}^{\otimes n-1-i} \rightarrow M$ satisfying these $(2n - 1)$ axioms. For example \mathcal{L} is a representation of \mathcal{L} . The notion of representation of a Leibniz n -algebra for $n = 2$ coincides with the corresponding notion given in [5]. Let \mathcal{L} be a Leibniz n -algebra and let M be a representation of \mathcal{L} . Let

$$(\mathcal{K}) \quad 0 \rightarrow M \rightarrow \mathcal{K} \rightarrow \mathcal{L} \rightarrow 0$$

be an abelian extension, such that the induced structure of representation of \mathcal{L} on M induced by the extension is the prescribed one. If this condition holds, then we say that we have an abelian extension of \mathcal{L} by M . Two such extensions (\mathcal{K}) and (\mathcal{K}') are isomorphic when there exists a Leibniz n -algebra map from \mathcal{K} to \mathcal{K}' which is compatible with the identity on M and on \mathcal{L} . One denotes by $\text{Ext}(\mathcal{L}, M)$ the set of isomorphism classes of extensions of \mathcal{L} by M .

Let $f : \mathcal{L}^{\otimes n} \rightarrow M$ be a linear map. We define an n -bracket on $\mathcal{K} = M \oplus \mathcal{L}$ by

$$\begin{aligned} & [(m_1, x_1), (m_2, x_2), \dots, (m_n, x_n)] \\ & := \left(\sum_{i=1}^n [x_1, \dots, m_i, \dots, x_n] + f(x_1, \dots, x_n), [x_1, \dots, x_n] \right). \end{aligned}$$

Then \mathcal{K} is a Leibniz n -algebra if and only if

$$\begin{aligned} (5.2) \quad & f([x_1, \dots, x_n], y_1, \dots, y_{n-1}) + [f(x_1, \dots, x_n), y_1, \dots, y_{n-1}] \\ & = \sum_{i=1}^n (f(x_1, \dots, [x_i, y_1, \dots, y_{n-1}], \dots, x_n) \\ & \quad + [x_1, \dots, f(x_i, y_1, \dots, y_{n-1}), \dots, x_n]) \end{aligned}$$

for all $x_1, \dots, x_n, y_1, \dots, y_{n-1} \in \mathcal{L}$. If this condition holds, then we obtain an extension

$$0 \rightarrow M \rightarrow \mathcal{K} \rightarrow \mathcal{L} \rightarrow 0$$

of Leibniz n -algebras. Moreover this extension is split in the category of Leibniz n -algebras if and only if there exists a linear map $g : \mathcal{L} \rightarrow M$ such that

$$(5.3) \quad f(x_1, \dots, x_n) = \sum_{i=1}^n [x_1, \dots, g(x_i), \dots, x_n] - g([x_1, \dots, x_n]).$$

An easy consequence of these facts is the following natural bijection:

$$(5.4) \quad \text{Ext}(\mathcal{L}, M) \cong Z(\mathcal{L}, M)/B(\mathcal{L}, M).$$

Here $Z(\mathcal{L}, M)$ is the set of all linear maps $f : \mathcal{L}^{\otimes n} \rightarrow M$ satisfying (5.2) and $B(\mathcal{L}, M)$ is the set of such f which satisfy (5.3) for some k -linear map $g : \mathcal{L} \rightarrow M$.

Let \mathcal{L} be a Leibniz n -algebra and let M be a representation of \mathcal{L} . A map $f : \mathcal{L} \rightarrow M$ is called a *derivation* if

$$f([a_1, \dots, a_n]) = \sum_{i=1}^n [a_1, \dots, f(a_i), \dots, a_n].$$

We let $\text{Der}(\mathcal{L}, M)$ be the vector space of all derivations from \mathcal{L} to M .

The next goal is to construct a cochain complex for Leibniz n -algebras so that the derivations and the elements of Z are cocycles in this complex. It turns out that this problem reduces to the case $n = 2$, that is for Leibniz algebras, which was the subject of the paper [5]. Let us recall the main construction of [5]. Let \mathfrak{g} be a Leibniz algebra

and let M be a representation of \mathfrak{g} . We let $CL^*(\mathfrak{g}, M)$ be a cochain complex given by

$$CL^m(\mathfrak{g}, M) := \text{Hom}(\mathfrak{g}^{\otimes m}, M), \quad m \geq 0,$$

where the coboundary operator $d^m : CL^m(\mathfrak{g}, M) \rightarrow CL^{m+1}(\mathfrak{g}, M)$ is defined by

$$\begin{aligned} (d^m f)(x_1, \dots, x_{m+1}) \\ &:= [x_1, f(x_2, \dots, x_{m+1})] + \sum_{i=2}^{m+1} (-1)^i [f(x_1, \dots, \hat{x}_i, \dots, x_{m+1}), x_i] \\ &\quad + \sum_{1 \leq i < j \leq m} (-1)^{j+1} f(x_1, \dots, x_{i-1}, [x_i, x_j], x_{i+1}, \dots, \hat{x}_j, \dots, x_m). \end{aligned}$$

According to [5] *cohomology of the Leibniz algebra* \mathfrak{g} with coefficients in the representation M is defined by

$$HL^*(\mathfrak{g}, M) := H^*(CL^*(\mathfrak{g}, M), d).$$

In order to generalize this notion to Leibniz n -algebras for $n \geq 3$ we need the following Proposition. Let us recall that if \mathcal{L} is an $(n+1)$ -Leibniz algebra, then the Leibniz algebra $\mathcal{D}_n(\mathcal{L})$ was defined in Section 3. Let \mathcal{L} be an $(n+1)$ -Leibniz algebra and let M be a representation of \mathcal{L} . One defines the maps

$$[-, -] : \text{Hom}(\mathcal{L}, M) \otimes \mathcal{D}_n(\mathcal{L}) \rightarrow \text{Hom}(\mathcal{L}, M)$$

$$[-, -] : \mathcal{D}_n(\mathcal{L}) \otimes \text{Hom}(\mathcal{L}, M) \rightarrow \text{Hom}(\mathcal{L}, M)$$

by

$$[f, x_1 \otimes \cdots \otimes x_n](x) := [f(x), x_1, \dots, x_n] - f([x, x_1, \dots, x_n]),$$

$$[x_1 \otimes \cdots \otimes x_n, f](x)$$

$$:= f([x, x_1, \dots, x_n]) - [f(x), x_1, \dots, x_n] - \cdots - [x, x_1, \dots, f(x_n)].$$

The proof of the next result is a straightforward (but somehow tedious) calculation.

Proposition 5.5. *Let \mathcal{L} be an $(n+1)$ -Leibniz algebra and let M be a representation of \mathcal{L} . Then the above homomorphisms define a structure of representation of $\mathcal{D}_n(\mathcal{L})$ on $\text{Hom}(\mathcal{L}, M)$.*

Let \mathcal{L} be a Leibniz n -algebra and let M be a representation of \mathcal{L} . One defines the cochain complex ${}_n CL^*(\mathcal{L}, M)$ to be $CL^*(\mathcal{D}_{n-1}(\mathcal{L}), \text{Hom}(\mathcal{L}, M))$. We also put

$${}_nHL^*(\mathcal{L}, M) := H^*({}_nCL^*(\mathcal{L}, M))$$

Thus, by definition one has ${}_nHL^*(\mathcal{L}, M) \cong HL^*(\mathcal{D}_{n-1}(\mathcal{L}), \text{Hom}(\mathcal{L}, M))$. Let us observe that for $n = 2$, one has ${}_2CL^m(\mathcal{L}, M) \cong CL^{m+1}(\mathcal{L}, M)$ for all $m \geq 0$. Thus

$${}_2HL^m(\mathcal{L}, M) \cong HL^{m+1}(\mathcal{L}, M), \quad m \geq 1$$

and ${}_2HL^0(\mathcal{L}, M) \cong \text{Der}(\mathcal{L}, M)$. Comparison of the definitions shows that

$${}_nHL^0(\mathcal{L}, M) \cong \text{Der}(\mathcal{L}, M)$$

holds for any Leibniz n -algebras \mathcal{L} . Similarly one has

$$\text{Ker}(d : {}_nCL^1(\mathcal{L}, M) \rightarrow {}_nCL^2(\mathcal{L}, M)) \cong Z(\mathcal{L}, M)$$

and therefore

$$(5.6) \quad \text{Ext}(\mathcal{L}, M) \cong {}_nHL^1(\mathcal{L}, M).$$

Proposition 5.7. *Let \mathcal{L} be a free n -Leibniz algebra and let M be a representation of \mathcal{L} . Then*

$${}_nHL^m(\mathcal{L}, M) = 0, \quad m \geq 1.$$

Proof. The main result of Section 4 shows that $\mathcal{D}_{n-1}(\mathcal{L})$ is a free Leibniz algebra. Thanks to Corollary 3.5 of [5] we have $HL^i(\mathcal{D}_{n-1}(\mathcal{L}), -) = 0$ for $i \geq 2$ and thus ${}_nHL^m(\mathcal{L}, -) = 0, m \geq 1$.

Let us recall that in [7] Quillen developed the cohomology theory in a very general framework. This theory can be applied to Leibniz n -algebras. It has the following description. Let \mathcal{L} be a Leibniz n -algebra and let M be a representation of \mathcal{L} . Then Quillen cohomology of \mathcal{L} with coefficients in M is defined by

$$H^*_{\text{Quillen}}(\mathcal{L}, M) := H^*(\text{Der}(P_*, M)).$$

Here $P_* \rightarrow \mathcal{L}$ is an augmented simplicial n -Leibniz algebra, such that $P_* \rightarrow \mathcal{L}$ is a weak equivalence and each component of P_* is a free Leibniz n -algebra.

Corollary 5.8. *Let \mathcal{L} be a Leibniz n -algebra and let M be a representation of \mathcal{L} . Then*

$$H^*_{\text{Quillen}}(\mathcal{L}, M) \cong {}_nHL^*(\mathcal{L}, M).$$

Proof. Since $P_* \rightarrow \mathcal{L}$ is a weak equivalence, we obtain from the Künneth theorem that ${}_nCL^m(P_*, M) \rightarrow {}_nCL^m(\mathcal{L}, M)$ is also a weak equivalence. This fact, together with Proposition 5.7, shows that both spectral sequences for the bicomplex ${}_nCL^*(P_*, M)$ degenerate and give the expected isomorphism.

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References

- [1] Daletskii, Y. I., Takhtajan, L. A.: Leibniz and Lie algebra structures for Nambu algebra. *Letters in Math. Physics* **39** (1997), 127–141
- [2] Gautheron, P.: Some remarks concerning Nambu mechanics. *Letters in Math. Physics* **37** (1996), 103–116
- [3] Lister, W.: A structure theory of Lie triple systems. *Trans. Amer. Math. Soc.* **72** (1952), 217–242
- [4] Loday, J.-L.: *Cyclic homology*. Grundlehren Math. Wiss. Bd. 301, Springer, 1992
- [5] Loday, J.-L., Pirashvili, T.: Universal enveloping algebras of Leibniz algebras and (co)-homology. *Math. Ann.* **296** (1993), 139–158
- [6] Nambu, Y.: Generalized Hamiltonian dynamics. *Phys. Rev. D* (3) **7** (1973), 2405–2412
- [7] Quillen, D.: On the (co)homology of commutative rings. *Proc. Symp. Pure Math. XVIII*, N.Y. (1970), 65–87
- [8] Serre, J.-P.: *Lie algebras and Lie groups*. Lect. Notes in Math. 1500, Springer, 1992
- [9] Stasheff, J.: The (secret?) homological algebra of the Batalin-Vilkovisky approach. In: *Secondary calculus and cohomological physics (Moscow, 1997)*, 195–210. *Contemp. Math.* **219**, Amer. Math. Soc. Providence, RI 1998
- [10] Takhtajan, L.: On foundation of the generalized Nambu mechanics. *Comm. Math. Phys.* **160** (1994), 295–315
- [11] Vinogradov, A., Vinogradov, M.: On multiple generalizations of Lie algebras and Poisson manifolds. In: *Secondary calculus and cohomological physics (Moscow, 1997)*, 273–287. *Contemp. Math.* **219**, Amer. Math. Soc. Providence, RI 1998

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