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Arithmetree

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Abstract

We construct an addition and a multiplication on the set of planar binary trees, closely related to addition and multiplication on the integers. This gives rise to a new kind of (noncommutative) arithmetic theory. The price to pay for this generalization is that, first, the addition is not commutative, second, the multiplication is distributive with the addition only on the left. This algebraic structure is the "exponent part" of the free dendriform algebra on one generator, a notion related to several other types of algebras. In the second part we extend this theory to all the planar trees. Then it is related to the free dendriform trialgebra as constructed in [J.-L. Loday, M.O. Ronco, C.R. Acad. Sci. Paris Ser. I 333 (2001) 81–86].

Introduction

Elementary arithmetic deals with the natural numbers:

$$0, 1, 2, \ldots, n, \ldots,$$

on which one knows how to define an addition $+$ and a mutiplication $\times$. In this paper we propose the following generalization: we replace the integers by the planar binary trees. Recall that there are

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$$c_n = \frac{(2n!)}{n!(n+1)!}$$

planar binary trees with $n+1$ leaves. The integer c_n is classically called the Catalan number. We first construct the sum of two planar binary trees. In general, this sum is not just a tree but a union of planar binary trees. However, it comes with the following feature: all the trees appearing in this sum are different. In other words, it is a subset of the set of planar binary trees. We call such a subset a *grove*. We show that this sum indeed extends to groves and is associative. It is not commutative, but there is an involution compatible with the sum. The construction of the sum takes advantage of a poset structure on the set of planar binary trees.

Next, we show that there also exists a multiplication on planar binary trees. Again, the product of two trees is not a tree in general, but a grove. We show that the product of two groves is still a grove. This product is associative, distributive on the left with the sum, but not distributive on the right.

The existence of this multiplication is due to a very peculiar property of the addition. The sum of two planar binary trees turns out to be in fact the union of the results of two other operations. Roughly speaking, it is like making a difference between adding x on the left to y and adding y on the right to x. These two operations happen to satisfy some relations. When we take the polynomial algebra with planar binary trees as exponents in place of integers, then we obtain what we call a *dendriform algebra*. The fact that this dendriform algebra is nothing but the free dendriform algebra on one generator enables us to define the multiplication on planar binary trees.

The set of integers is very often used as an indexing set. However, sometimes it is not sufficient and one has to move to planar binary trees. This happens, for instance, in solving differential equations by means of series (cf. [Br,BF]), and in algebraic topology (generalization of the simplicial category (cf. [Fr,J]), of operads, of PROPs). As soon as one wants to manipulate these objects, one needs to add and multiply the planar binary trees. This is one of the motivations of the present work.

It is tempting to find out whether the addition (respectively the multiplication) can easily be described on some of the other interpretations of the Catalan sets. We give one of them by introducing the "permutation-like notation" of the elements in the Catalan sets.

In the second part of this paper we extend this arithmetic to *all* the planar trees, the case of planar binary trees becoming a quotient of it. Since there are 3 different planar trees with three leaves, this theory is related to a type of algebra defined by 3 operations. They are called *dendriform trialgebras* and were introduced in [LR3].

Thanks to Patrick Ion for suggesting the terminology "grove". Though it is not apparent in the text, this paper owes much to the book "On Numbers and Games" by J.H.C. Conway [Co].

I. Arithmetic of planar binary trees

1. The poset of planar binary trees

1.1. Catalan sets

Let Y_0 be a set with one element. The sets Y_n for $n \geqslant 1$ are defined inductively by the formula

$$Y_n := Y_{n-1} \times Y_0 \cup \cdots \cup Y_{n-i} \times Y_{i-1} \cup \cdots \cup Y_0 \times Y_{n-1}.$$

If we denote by a the unique element of Y_0, then an element of Y_n can be described as a (complete) parenthesizing of the word $aa \cdots a$ of length $n+1$. Let $x \in Y_p$ and $y \in Y_q$. The element $(x, y) \in Y_p \times Y_q$ viewed as an element in Y_{p+q+1} is denoted $x \vee y \in Y_{p+q+1}$. In terms of parenthesizing it simply consists in concatenating the two words and putting a parenthesis at both ends.

There are many other combinatorial descriptions of the sets Y_n. We will use two of them as described below, one classical: the planar binary trees, and one less classical: the permutation-like notation. Others include: the triangulations of an $(n+2)$-gon, the vertices of the Stasheff polytope of dimension $n-1$, see [St] for many more.

Let c_n be the number of elements of Y_n. One has immediately:

$$c_0 = 1 \quad \text{and} \quad c_n := c_{n-1}c_0 + \cdots + c_{n-i}c_{i-1} + \cdots + c_0c_{n-1}. \tag{1.1.1}$$

Hence the generating series $f(x) := \sum_{n \geqslant 0} c_n x^n$ satisfies the functional equation $xf(x)^2 = f(x) - 1$, and we obtain

$$f(x) = \frac{1 - \sqrt{1-4x}}{2x}.$$

As a consequence we obtain

$$c_n = \frac{(2n)!}{n!(n+1)!}.$$

It is classically called the Catalan number, so Y_n is called the *Catalan set*.

1.2. Planar binary trees

A *planar binary tree* (p.b.tree for short) is an oriented planar graph drawn in the plane with $n+1$ leaves and one root, such that each internal vertex has two leaves and one root. We consider these trees up to planar isotopy. An example is in Fig. 1.

We define the *degree* of a tree as being the number of leaves minus 1. When the tree is binary the degree is precisely the number n of internal vertices ($n = 5$ in our example).

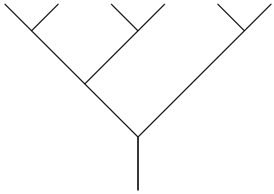
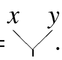
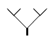


Fig. 1.

There is only one p.b.tree with one leaf: | , and only one p.b.tree with two leaves: . The main operation on p.b.trees is called *grafting*. The grafting of x and y, denoted by $x \vee y$, is the tree obtained by joining the two roots to a new vertex:

$x \vee y :=$ .

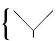
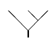
For instance: $| \vee | =$ ,  $\vee$  $=$ . Observe that the degree of $x \vee y$ is $\deg x + \deg y + 1$.

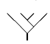
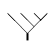
The main point about p.b.trees is the following: the decomposition

$$x = x^l \vee x^r$$

exists (provided that $x \neq |$) and is *unique*. Moreover, one has $\deg x^l < \deg x$ and $\deg x^r < \deg x$. From this property of the p.b.trees it is clear that there is a one-to-one correspondence between the Catalan set Y_n, as defined in Section 1.1, and the set of p.b.trees of degree n.

In low degree one has:

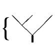
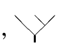
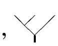
$Y_0 = \{|\}$, $Y_1 = \{$  $\}$, $Y_2 = \{$ ,  $\}$,

$Y_3 = \{$ , , , ,  $\}$.

The union of all the sets Y_n, $n \geqslant 0$, is denoted Y_∞.

1.3. *Grove (bosquet)*

In the sequel we will deal with the subsets of Y_n. By definition a *binary grove* of degree n (or simply a grove when the context is clear) is a non-empty subset of Y_n. We will refer to a grove as a disjoint union of trees. Hence a grove is a non-empty union of p.b.trees of the same degree, such that each tree appears at most once. We denote the set of groves of degree n by $\mathbb{Y}_n$. The number of elements of $\mathbb{Y}_n$ is $2^{c_n} - 1$.

Example. $\mathbb{Y}_0 = \{|\}$, $\mathbb{Y}_1 = \{$  $\}$, $\mathbb{Y}_2 = \{$ , ,  $\cup$  $\}$.

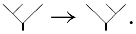
An important role is going to be played by the following special grove:

$$\underline{n} := \bigcup_{y \in Y_n} y.$$

We call it the *total grove* of degree n.

1.4. Partial order structure on Y_n

We put a *partial order* structure on the set Y_n of planar binary trees of degree n as follows. We say that $x < y$ (also denoted $x \to y$) if the tree y is obtained from x by moving an edge from left to right over a vertex, as in the basic example:



The partial order relation is induced by this relation. More formally the partial order on Y_n is induced by the following relations:

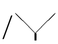
$$(x \vee y) \vee z \leqslant x \vee (y \vee z),$$
$$x < y \Longrightarrow x \vee z < y \vee z,$$
$$x < y \Longrightarrow z \vee x < z \vee y.$$

For $n = 3$ we obtain the classical poset (pentagon) (see Fig. 2).

One can show that, equipped with this poset structure, Y_n is a lattice, sometimes called the Tamari lattice in the literature (cf., for instance, [BW]).

1.5. The Over and Under operations

Let us introduce two new operations on planar binary trees. For $x \in Y_p$ and $y \in Y_q$ the tree x/y (read x *over* y) in Y_{p+q} is obtained by identifying the root of x with the leftmost leaf of y. Similarly, the tree $x\backslash y$ (read x *under* y) in Y_{p+q} is obtained by identifying the rightmost leaf of x with the root of y.

Example.  /  =  and  \  = .

Observe that both operations are associative. The tree | is the neutral element for both operations and on both sides since

$$x/| = |/x = x = |\backslash x = x\backslash |.$$

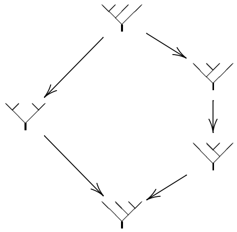


Fig. 2.

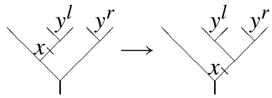


Fig. 3.

We will also use the following immediate property:

$$x/(y \vee z) = (x/y) \vee z \quad \text{and} \quad x \vee (y\backslash z) = (x \vee y)\backslash z.$$

1.6. Proposition. *For any p.b.trees x and y one has*

$$x/y \leqslant x\backslash y.$$

Proof. The proof is by induction on the degree of y. If $y = |$, then both elements are equal to x. If the degree of y is strictly positive, then one can write $y = y^l \vee y^r$. Since, by induction hypothesis, $x/y^l \leqslant x\backslash y^l$ and since $x/y = (x/y^l) \vee y^r$, one obtains

$$x/y = \left(x/y^l\right) \vee y^r \leqslant \left(x\backslash y^l\right) \vee y^r.$$

In $(x\backslash y^l) \vee y^r$ we move all the vertices standing on the right leg of x from left to right over the lowest vertex to obtain $x\backslash(y^l \vee y^r)$, as in Fig. 3. Since the latter tree is $x\backslash y$, we have finished the proof. □

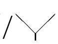
2. Addition

In this section we define a binary operation on planar binary trees and we extend it to groves. We call it *addition* or *sum* though it is not commutative.

2.1. Definition. By definition, the *sum* of two p.b.trees x and y is the following disjoint union of p.b.trees:

$$x + y := \bigcup_{x/y \leqslant z \leqslant x\backslash y} z.$$

All the elements in the sum have the same degree which happens to be $\deg x + \deg y$. The associativity of the sum follows immediately from the associativity of the Over and Under operations.

Example. Since  /  =  and  \  = , one obtains

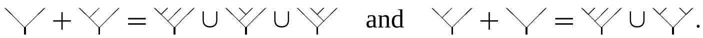
 +  =  ∪ .

Addition is extended to groves by distributivity on both sides:

$$\bigcup_i x_i + \bigcup_j y_j := \bigcup_{ij} (x_i + y_j).$$

It is clear that the tree | is the neutral element for $+$ since it is the neutral element for both the Over operation and the Under operation.

Despite the notation $+$ that we use, the addition on groves is not commutative, for instance:



However, there is an involution, as we will see below.

Notation. From now on we often denote the tree | by 0 and the tree  by 1.

2.2. Theorem. *The sum of two groves* (*a fortiori the sum of two p.b.trees*) *is still a grove*:

$$+ : \mathbb{Y}_n \times \mathbb{Y}_m \to \mathbb{Y}_{n+m}.$$

Proof. Observe that it is not immediate a priori that the trees appearing in the union defining the sum are all different.

This theorem is a consequence of Proposition 2.3 below. Indeed, any grove is a subset of the total grove. Since, in the sum of two total groves, a given tree appears at most once, the same property is true for the sum of any two groves. Hence this sum is a grove. □

2.3. Proposition. *Let* $\underline{n} := \bigcup_{y \in Y_n} y$ *be the total grove of degree* n. *Then one has*

$$\underline{n} + \underline{m} = \underline{n+m}.$$

Proof. It is sufficient to prove the proposition for $m = 1$. Indeed, by induction and associativity of the addition, it comes

$$\underline{n} + \underline{m} = \underline{n} + \underline{1} + \cdots + \underline{1} = \underline{n+1} + \underline{1} + \cdots + \underline{1} = \cdots = \underline{n+m}.$$

Let us show that $\underline{n} + \underline{1} = \underline{n+1}$.

We want to prove that

$$\bigcup_{y \in Y_n} (y+1) = \bigcup_{z \in Y_{n+1}} z.$$

Therefore it is sufficient to show that for any element $z \in Y_{n+1}$ there exists a unique element $y \in Y_n$ such that $y/1 \leqslant z \leqslant y\backslash 1$. We first prove the existence of y. We work by induction on the degree of z. Let $z = z^l \vee z^r$. By induction we know that, if $z^r \neq 0$, then there exists t such that $t/1 \leqslant z^r \leqslant t\backslash 1$. One has the following relations:

$$\begin{aligned}\big(z^l \vee t\big)/1 &= \big(z^l \vee t\big) \vee 0 \leqslant z^l \vee (t \vee 0) = z^l \vee (t/1) \leqslant z^l \vee z^r \\ &\leqslant z^l \vee (t\backslash 1) = \big(z^l \vee t\big)\backslash 1.\end{aligned}$$

Therefore one can take $y = z^l \vee t$.

If $z^r = 0$, then $z = z^l \vee 0 = z^l/1$. Since $z^l/1 \leqslant z^l\backslash 1$, $y = z^l$ is a solution.

Let us show now the uniqueness of the solution. For any $y \in Y_n$ let $E(y) = \{z \in Y_{n+1} \mid y/1 \leqslant z \leqslant y\backslash 1\}$. Suppose that we have proved the equality $\sum_{y\in Y_n} \#E(y) = c_{n+1}$. Then, since any element $z \in Y_{n+1}$ belongs to some $E(y)$, it cannot belong to two of them (since $\#Y_{n+1} = c_{n+1}$). Hence the union $\bigcup_{y\in Y_n} E(y)$ is a disjoint union which covers Y_{n+1}. It implies $\underline{n} + \underline{1} = \underline{n+1}$.

Let us now prove that $\sum_{y\in Y_n} \#E(y) = c_{n+1}$. Let $y = y^l \vee y^r$. One has

$$y/1 = \bigl(y^l \vee y^r\bigr)/1 < y^l \vee (y^r/1)$$

and there is no element between these two. Hence $\#E(y) = \#E(y^r) + 1$. Working by induction, we suppose that $\sum_{y\in Y_i} \#E(y) = c_{i+1}$ for $i < n$. By using the decomposition $Y_n = Y_{n-1} \times Y_0 \cup \cdots \cup Y_{n-i} \times Y_{i-1} \cup \cdots \cup Y_0 \times Y_{n-1}$ we obtain

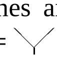
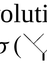
$$\begin{aligned}
\sum_{y\in Y_n} \#E(y) &= \sum_{i=1}^{n} c_{n-i} \times \Biggl(\sum_{z\in Y_{i-1}} \bigl(\#E(z) + 1\bigr)\Biggr) \\
&= \sum_{i=1}^{n} c_{n-i} \times (c_i + c_{i-1}) \quad \text{(by induction hypothesis)} \\
&= c_n + c_{n-1} \times c_1 + \cdots + c_0 \times c_n \quad \text{(by formula (1.1.1))} \\
&= c_{n+1} \quad \text{(again by formula (1.1.1) since } c_0 = 1\text{)}. \qquad \square
\end{aligned}$$

2.4. Corollary. *Let n and m be two integers and let $z \in Y_{n+m}$. Then there exist unique elements $x \in Y_n$ and $y \in Y_m$ such that $x/y \leqslant z \leqslant x\backslash y$.*

2.5. *Cylindrical structure*

As mentioned in the introduction, the elements of the Catalan set Y_{n+1} are in one-to-one correspondence with the vertices of the Stasheff polytope $\mathcal{K}^n$ (associahedron) of dimension n. One can view $\mathcal{K}^n$ as a cylinder $\mathcal{K}^{n-1} \times I$ such that the vertices in $\mathcal{K}^{n-1} \times \{0\}$ (respectively $\mathcal{K}^{n-1} \times \{1\}$) correspond to the elements $y/1$ (respectively $y\backslash 1$). Then the elements between $y/1$ and $y\backslash 1$ are lying on the edge joining them. In particular, this subset is totally ordered. This cylindrical point of view appears also in [SU].

2.6. *Involution*

Observe that for a p.b.tree *symmetry* around the axis passing through the root defines an involution σ on Y_n and therefore also on $\mathbb{Y}_n$. For instance, $\sigma($$) =$  and $\sigma($$) =$ . It is clear that

$$\begin{aligned}
\sigma(x \vee y) &= \sigma(y) \vee \sigma(x), \\
\sigma(x/y) &= \sigma(y)\backslash\sigma(x), \\
\sigma(x\backslash y) &= \sigma(y)/\sigma(x);
\end{aligned}$$

therefore

$$\sigma(x+y) = \sigma(y) + \sigma(x).$$

Summarizing what we have proved until now, we obtain the following corollary.

2.7. Corollary. *The set* $\mathbb{Y}_\infty := \bigcup_{n \geqslant 0} \mathbb{Y}_n$ *of groves is an involutive graded monoid for* $+$. *The maps*

$$\begin{aligned} &\mathbb{N} \longrightarrow \mathbb{Y}_\infty \longrightarrow \mathbb{N} \\ &n \ \mapsto \underline{n}, \quad y \mapsto \deg y \end{aligned}$$

are morphisms of monoids.

3. Permutation-like notation of trees

Though the trees help in figuring out the operations on the elements of Y_n, when one wants to work explicitly it is like working with Roman numerals. So there is a need for a more useful notation. The following permutation-like notation permits us to code linearly the elements of Y_n and so to use computer computation. It is similar to the decimal notation for integers.

3.1. Definition. By definition the *name* of the unique element of Y_0 is 0, and the name $w(x)$ of the element $x \in Y_n$, $n \geqslant 1$, is a finite sequence of strictly positive integers obtained inductively as follows:

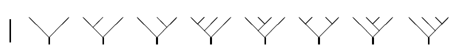
$$\text{if} \quad x = (y,z) \in Y_{n-i} \times Y_{i-1} \subset Y_n \quad \text{then} \quad w(x) := \big(w(y), n, w(z)\big),$$

with the convention that we do not write the zeros.

If there is no possibility of confusion we simply write $w(x) := w(y)\,n\,w(z)$ (concatenation). Observe that, except for 0, such a sequence is made of n integers and the integer n appears once and only once. The name of the unique element of Y_1 is therefore 1, and this is in accordance with our previous notation.

3.2. Bijection with the planar binary trees

When an element in Y_n corresponds to a p.b.tree x and to a name $w(x)$, we will say that $w(x)$ is the name of the tree x. In low dimension we obtain:

$x =$ 
$w(x) =$ 0 1 12 21 123 213 131 312 321.

The relationship with grafting is obviously given by

$$w(x) = w\big(x^l\big)\,n\,w(x^r) \quad \text{for } x = x^l \vee x^r \in Y_n.$$

Recall that symmetry around the root axis induces an involution on Y_n. If $a_1 \cdots a_n$ is the name of the tree x, then the name of $\sigma(x)$ is $a_n \cdots a_1$.

3.3. Weight

From the picture of a tree one can obtain its name quickly as follows. Define its *ith weight* as being the degree minus 1 of the minimal subtree which contains the leaves number $i-1$ and i (we number them from left to right starting with 0). Then the name of the tree is precisely the sequence of weights.

3.4. Test for sequences

Given a sequence of integers, is it the name of a p.b.tree? First, check that the largest integer in the sequence is the length of the sequence. Second, check that the left and right subsequences (not containing the largest integer) are names of trees. Example: 15321812 is the name of a tree, but 15121812 is not.

3.5. Relationship with the symmetric group

Any permutation σ of the set $\{1, \ldots, n\}$ with values $\sigma(1) \cdots \sigma(n)$ can be made uniquely into the name of a tree by the following algorithm. Keep the largest integer in place in this sequence. On each side of it replace the largest integer by the length of the side, and so on. For instance 123 stays the same 123, however 132 becomes 131, and 23154 becomes 13151. The properties of the map $\Sigma_n \to Y_n$ so obtained and its relationship with the polytope decompositions of the sphere as permutohedron and Stasheff polytope respectively has been investigated in [LR1].

4. The universal property

First, we give a recursive formula for computing the sum of groves. Second, we state the universal property of the addition which will enable us to construct the multiplication.

4.1. Theorem [LR2]. *For any p.b.trees x and y different from* $|$ (*i.e.* 0) *the following formula holds*:

$$x + y = x^l \vee (x^r + y) \cup \left(x + y^l\right) \vee y^r,$$

where $x = x^l \vee x^r$ and $y = y^l \vee y^r$.

The proof follows from [LR2, Theorem 5.1], where it is written in terms of associative algebras as explained in the next section.

4.2. The Left and Right operations

From the preceding theorem it is immediately seen that the sum of two trees is given as the union of two groves. In order to identify the two parts we define two operations $\dashv$ and $\vdash$ as follows:

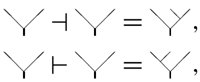
$$x \dashv y := x^l \vee (x^r + y) \quad \text{when } x \neq 0,$$
$$x \vdash y := \bigl(x + y^l\bigr) \vee y^r \quad \text{when } y \neq 0, \quad \text{and}$$
$$0 \dashv x = 0 = x \vdash 0.$$

So we have

$$x + y = x \dashv y \cup x \vdash y.$$

(Pictorially the sign $+$ splits into the two signs $\dashv$ and $\vdash$). Observe that $0 \dashv 0$ and $0 \vdash 0$ are not defined though $0 + 0 = 0$, see Remark 9.3.

The operations $\dashv$ and $\vdash$ are called respectively the *Left* and the *Right* sum (according to the direction in which they point). For instance, one obtains



i.e. $1 \dashv 1 = 21$,

i.e. $1 \vdash 1 = 12$.

They are extended to groves by distributivity with respect to disjoint union:

$$\bigcup_i x_i \dashv \bigcup_j y_j := \bigcup_{ij} (x_i \dashv y_j)$$

and similarly for $\vdash$.

Since the Left (respectively Right) sum of two groves is a subset of the sum, the Left (respectively Right) sum of groves is a grove.

It is clear that the relationship with the involution is as follows:

$$\sigma(x \dashv y) = \sigma(y) \vdash \sigma(x) \quad \text{and} \quad \sigma(x \vdash y) = \sigma(y) \dashv \sigma(x). \tag{4.2.1}$$

Recall that $x/y = (x/y^l) \vee y^r$ and $x^l \vee (x^r \backslash y) = x \backslash y$. One can show that in terms of the operations $/$ and $\backslash$ we obtain

$$x \dashv y = \bigcup_{x^l \vee (x^r/y) \leqslant z \leqslant x \backslash y} z \quad \text{and} \quad x \vdash y = \bigcup_{x/y \leqslant z \leqslant (x \backslash y^l) \vee y^r} z. \tag{4.2.2}$$

In Appendix A the addition table is written such that the first line gives $x \vdash y$ and the second (union third if any) line gives $x \dashv y$.

4.3. Tricks for computation

For some special trees the computation of the Left or Right sum is easy. First recall that $- \vee - = -\mathtt{max}-$, where $\mathtt{max}$ stands for the largest integer (i.e. the length) of the word to which it pertains. For instance $1 \vee 1 = 131$ and $1 \vee 0 = 12$. Then it is easy to check that

$$(-\mathtt{max}) \dashv - = -\mathtt{max}-, \qquad - \vdash (\mathtt{max}-) = -\mathtt{max}-.$$

For instance:

$$213 \dashv 12 = 21512, \qquad 1412 \vdash 54131 = 141294131.$$

For higher degree trees it is safer to write $(a_1, \dots, a_n)$ instead of $a_1 \cdots a_n$ to avoid confusion.

4.4. Proposition. *The Left and Right sums on groves satisfy the following relations*:

$$(x \dashv y) \dashv z = x \dashv (y + z),$$
$$(x \vdash y) \dashv z = x \vdash (y \dashv z),$$
$$(x + y) \vdash z = x \vdash (y \vdash z),$$

and $0 \vdash x = x = x \dashv 0$ *for* $x \neq 0$.

The proof follows from [L2] as we will show in Section 5.

4.5. Theorem. *For any p.b.tree x of degree n there is a unique way of writing it as a composition of n copies of* 1 *with the Left and the Right sum. It is called the "universal expression" of x and denoted $w_x(1)$.*

The universal expression is unique modulo the relations of Proposition 4.4.

The proof follows from [L2] as we will show in Section 5. Observe that this theorem gives still another combinatorial description of the Catalan sets (not in the list of [St]).

The inductive algorithm to construct $w_x(1)$ is given by

$$w_0(1) = 0 \quad \text{and} \quad w_x(1) := w_{x^l}(1) \vdash 1 \dashv w_{x^r}(1). \tag{4.5.1}$$

Examples.

$$12 = 1 \vdash 1$$
$$21 = 1 \dashv 1$$
$$123 = 12 \vdash 1 = (1 \vdash 1) \vdash 1$$
$$213 = 21 \dashv 1 = (1 \dashv 1) \vdash 1$$
$$131 = 1 \vdash 1 \dashv 1$$
$$312 = 1 \dashv 12 = 1 \dashv (1 \vdash 1)$$
$$321 = 1 \dashv 21 = 1 \dashv (1 \dashv 1)$$
$$131492141 = \big((1 \vdash 1 \dashv 1) \vdash 1\big) \vdash 1 \dashv \big((1 \dashv 1) \vdash 1 \dashv 1\big).$$

4.6. Remark. Let $a_1 a_2 \cdots a_n$ be the name of the tree x. We say that there is an *ascent* (respectively a *descent*) at i if $a_i < a_{i+1}$ (respectively $a_i > a_{i+1}$). One can show that, in the universal expression of x, the signs are $\vdash$ when there is an ascent and $\dashv$ when there is a descent.

5. Polynomial algebra with tree exponents

In this section we show how the results of Section 4 are just translation of results contained in [LR2,LR1,L2].

Let K be a commutative ring. It is well known that the polynomial ring $K[X]$ has a linear basis indexed by $\mathbb{N}$: $\{X^n\}_{n\in\mathbb{N}}$. Multiplication of monomials in $K[X]$ corresponds to addition in $\mathbb{N}$: $X^n X^m = X^{n+m}$. Observe that if we were using the group algebra notation, then the polynomial algebra would be denoted $K[\mathbb{N}]$.

Similarly let us introduce $K[Y_\infty]$ the vector space with basis X^y, $y \in Y_n$, for all $n \geqslant 0$. We put a product on it by the formula:

$$X^y * X^z := X^{y+z}.$$

Since $y + z$ need not be a tree but a grove, we define $X^{y\cup y'}$ as

$$X^{y\cup y'} := X^y + X^{y'},$$

so that the previous formula has meaning. Under this formula, any grove determines an element in $K[Y_\infty]$. This associative algebra has already been encountered in the framework of dendriform algebras as we now explain.

5.1. *Dendriform algebras* [L1,L2]

By definition, a *dendriform algebra* (also called *dendriform dialgebra*) is a K-vector space A equipped with two binary operations

$$\prec : A \otimes A \to A, \qquad \succ : A \otimes A \to A,$$

which satisfy the following axioms:

$$\text{(i)} \quad (a \prec b) \prec c = a \prec (b * c),$$
$$\text{(ii)} \quad (a \succ b) \prec c = a \succ (b \prec c),$$
$$\text{(iii)} \quad (a * b) \succ c = a \succ (b \succ c)$$

for any elements a, b and c in A, with the notation

$$a * b := a \prec b + a \succ b.$$

Adding the three relations show that the operation $*$ is associative.

5.2. A dendriform algebra associated to p.b.trees

Let $K[Y'_\infty]$ be the vector space generated by the elements X^x, for $x \in Y_n$ ($n \geqslant 1$) (we exclude $0 \in Y_0$ for a while). We define two operations on $K[Y'_\infty]$ by the formulas

$$X^x \prec X^y := X^{x^l \vee (x^r + y)}, \qquad X^x \succ X^y := X^{(x+y^l)\vee y^r},$$

with the convention that, for any two trees y and y', one has

$$X^{y\cup y'} := X^y + X^{y'}.$$

These operations are extended to $K[Y'_\infty]$ by distributivity:

$$(a+b) \prec c = a \prec c + b \prec c \quad \text{and} \quad a \prec (b+c) = a \prec b + a \prec c,$$

and similarly for $\succ$.

5.3. Theorem (Universal property) [L2, Proposition 5.7]. *The vector space $K[Y'_\infty]$ equipped with the two operations $\prec$ and $\succ$ as above is a dendriform algebra. Moreover it is the free dendriform algebra on one generator* (*namely X*).

5.4. Proof of the results of Section 4

The associative product on $K[Y'_\infty]$ is given by

$$X^x * X^y = X^x \prec X^y + X^x \succ X^y = X^{x\dashv y} + X^{x\vdash y} = X^{x+y}.$$

It is a nonunital associative algebra. We add a unit $1 = X^0$ to it, so that $K[Y_\infty] = K[Y'_\infty] \oplus K \cdot 1$ becomes an augmented unital associative algebra. The operations $\prec$ and $\succ$ can be partially extended to $K[Y_\infty]$ by $X^0 \succ X^x = X^x = X^x \prec X^0$ for $x \neq 0$.

In [LR2, Theorem 5.1] it is proved that this algebra structure on $K[Y_\infty]$ is the same as the one given by the rule

$$X^x * X^y = \sum_{x/y \leqslant z \leqslant x\backslash y} X^z.$$

Therefore the results of Section 4 are just translations of the results of [LR2] and the results of [LR1] cited above.

5.5. Remarks about notation

In [LR1,LR2,L1,L2] the linear generators are denoted x instead of X^x. We adopt this new notation here to avoid confusion with the operations in $\mathbb{Y}_\infty$.

In [L1,L2] the symbols $\dashv$ and $\vdash$ are used to denote operations of an associative dialgebra. We have given them a completely different meaning here. Let us recall here that the operad of associative dialgebras is Koszul dual to the operad of dendriform dialgebras (see [L1,L2]).

6. Multiplication

Since the polynomial algebra is the free associative algebra on one generator, one can define the composition of polynomials. It turns out that composite of monomials is still a monomial. It is related to multiplication of integers by $(X^n)^m = X^{nm}$.

Similarly, since the associative algebra $K[Y'_\infty]$ is free on one generator when considered as a dendriform algebra, one can perform composition of polynomials with tree exponents. Though the composite of monomials, that is $(X^x)^y$ where x and y are p.b.trees, is not a monomial, it turns out that it is X to the power of some grove. Hence one can define the multiplication of p.b.trees as being this grove and then extend this multiplication to any groves.

6.1. Definition. Let x and y be planar binary trees. By definition the *product* $x \times y$ is

$$x \times y := w_x(y),$$

where $w_x(1)$ is the universal expression of x (cf. Theorem 4.5). In other words we replace all the copies of 1 by copies of y in this universal expression.

Observe that the above definition of the product has a meaning even when y is a grove since Right sum and Left sum of groves are well-defined. We extend the multiplication to x being a grove by distributivity on the left with respect to disjoint union:

$$(x \cup x') \times y = w_x(y) \cup w_{x'}(y).$$

So we have defined the product of two groves. This product is a grove since it is obtained by the operations $\dashv$ and $\vdash$. It is clear that the degree of the product is the product of the degrees, so we have defined a map

$$\times : \mathbb{Y}_n \times \mathbb{Y}_m \to \mathbb{Y}_{nm}.$$

Observe that the product is not commutative.

Examples. Since $21 = 1 \dashv 1$, one has $21 \times x = x \dashv x$, so, for instance, $21 \times 12 = 12 \dashv 12 = 1412$. On the other hand, $12 \times 21 = 21 \vdash 21 = 2141$.

Since $131 = 1 \vdash 1 \dashv 1$, one has $131 \times x = x \vdash x \dashv x$, so, for instance, $131 \times 21 = 12 \vdash 12 \dashv 12 = 1412 \dashv 12 = (1412 + 1) \vee 0 = (1412 \dashv 1)6 \cup 141256 = (1 \vee (12 + 1))6 \cup 141256 = 151316 \cup 151236 \cup 141256$.

6.2. Proposition. *The multiplication $\times$ on groves is distributive on the left with respect to the Left sum, to the Right sum and to the sum, but is not distributive on the right.*

Proof. The formula $w_{x+x'}(1) = w_x(1) + w_{x'}(1)$ follows from

$$w_{x \dashv x'}(1) = w_x(1) \dashv w_{x'}(1) \quad \text{and} \quad w_{x \vdash x'}(1) = w_x(1) \vdash w_{x'}(1).$$

These last two formulas follow inductively from the properties of the function w, namely (4.5.1) and $w_1(1) = 1$. $\square$

6.3. Proposition. *The multiplication of groves is associative with neutral element on both sides the tree* $\vee = 1$.

Proof. Interpreted in terms of dendriform algebra, the multiplication of p.b.trees is composition of monomials. Indeed, since $K[Y'_\infty]$ is the free dendriform algebra on the generator X, there exists a unique morphism of dendriform algebras $W_x : K[Y'_\infty] \to K[Y'_\infty]$ sending X to X^x. The image of X^y by W_x is precisely $X^{w_x(y)}$. Since composition of dendriform algebra morphisms is associative, the multiplication of p.b.trees is associative.

Since $w_1(1) = 1$, we obtain $w_1(y) = y$ and so $1 \times y = y$. On the other hand, $x \times 1 = x$ is a tautology. $\square$

6.4. Theorem. *With the notation* $\underline{n} = \bigcup_{y \in Y_n} y$, *one has* $\underline{n} \times \underline{m} = \underline{nm}$.

Proof. Since the multiplication is distributive on the left with respect to the addition, we obtain

$$\begin{aligned} \underline{n} \times \underline{m} &= (\underline{1} + \cdots + \underline{1}) \times \underline{m} = \underline{1} \times \underline{m} + \cdots + \underline{1} \times \underline{m} \\ &= \underline{m} + \cdots + \underline{m} = \underline{nm}, \end{aligned}$$

since $\underline{1} \times \underline{m} = \underline{m}$ by Proposition 6.3. $\square$

6.5. Proposition (recursive property). *Let* $x = x^l \vee x^r$ *be a p.b.tree and let* y *be a grove. The multiplication is given recursively by the formulas*

$$x \times y = \bigl(x^l \times y\bigr) \vdash y \dashv (x^r \times y) \quad \textit{and} \quad 0 \times y = 0.$$

Proof. The universal expression satisfies

$$w_{x^l \vee x^r}(1) = w_{x^l}(1) \vdash 1 \dashv w_{x^r}(1)$$

because the name of $x^l \vee x^r$ is $w(x^l)\, n\, w(x^r)$ (cf. Section 4.3). Hence we obtain

$$w_{x^l \vee x^r}(y) = w_{x^l}(y) \vdash y \dashv w_{x^r}(y) = \bigl(x^l \times y\bigr) \vdash y \dashv (x^r \times y). \qquad \square$$

Exercise. Starting with the formula of Proposition 6.5 as a definition for $\times$, prove associativity. Hint: use the formula

$$a \vdash b \dashv c = \bigl(a + b^l\bigr) \vee (b^r + c).$$

Show that, as a consequence, one has

$$(x \times y)^l = x^l \times y + y^l \quad \text{and} \quad (x \times y)^r = x^r \times y + y^r.$$

6.6. Proposition (involution). *For any groves x and y one has*

$$\sigma(x \times y) = \sigma(x) \times \sigma(y).$$

Proof. It obviously suffices to prove the formula when x is a p.b.tree. We work by induction on the degree of x. The formula is a tautology for $x = 1$. By Proposition 6.5 we obtain

$$\begin{aligned}
\sigma(x \times y) &= \sigma\big(x^l \times y \vdash y \dashv x^r \times y\big) \\
&= \sigma(x^r \times y) \vdash \sigma(y) \dashv \sigma\big(x^l \times y\big) \\
&= \big(\sigma(x^r) \times \sigma(y)\big) \vdash \sigma(y) \dashv \big(\sigma\big(x^l\big) \times \sigma(y)\big) \\
&= \big(\sigma(x^r) \vee \sigma\big(x^l\big)\big) \times \sigma(y) \\
&= \sigma(x) \times \sigma(y). \qquad \square
\end{aligned}$$

6.7. Summary

On the set of groves $\mathbb{Y}_\infty = \bigcup_{n \geqslant 0} \mathbb{Y}_n$ there are defined operations $+$ and $\times$ such that

- the addition $+$ is associative, distributive both sides with respect to $\cup$, with neutral element $0 = |$, but is not commutative,
- the multiplication $\times$ is associative, distributive on the left with respect to the sum $+$ and to the disjoint union $\cup$ (but not right distributive), with neutral element (both sides) $1 = \vee$, but is not commutative,
- the involution σ on $\mathbb{Y}_\infty$ satisfies $\sigma(x + y) = \sigma(y) + \sigma(x)$ and $\sigma(x \times y) = \sigma(x) \times \sigma(y)$,
- the maps $\mathbb{N} \to \mathbb{Y}$, $n \mapsto \underline{n} = \bigcup_{y \in Y_n} y$, and $\deg : \mathbb{Y} \to \mathbb{N}$ (degree) are compatible with $+$ and $\times$. The composite is the identity of $\mathbb{N}$.

6.8. Questions and problems

By definition a grove is *prime* if it is not the product of two groves different from 1. Obviously any tree (respectively grove) whose degree is prime is a prime tree (respectively grove). However there are also prime trees and groves of nonprime degree, for instance 1234. The tree 1241 is not prime since $1241 = 12 \times 21$.

It would be interesting to study the factorization of a grove into the product of two. In particular it seems that, when a grove is a product of prime groves, then the ordered sequence of factors is unique.

7. Elementary combinatorial applications

Here are simple combinatorial applications of the formulas $\underline{n}+\underline{m}=\underline{n+m}$ and $\underline{n}\times\underline{m}=\underline{nm}$.

7.1. Proposition. *For any pair of p.b.trees $x\in Y_n$ and $y\in Y_m$, let $c_{x,y}$ be the number of trees in the grove $x+y$. Then the following combinatorial formula holds*:

$$\sum_{x\in Y_n,\ y\in Y_m} c_{x,y}=c_{n+m}.$$

We see, for instance, that $c_{12,12}=3$, $c_{12,21}=2$, $c_{21,12}=6$, $c_{21,21}=4$, and so $3+2+6+3=14=c_4$, as expected.

7.2. Proposition. *For any p.b.trees $x\in Y_n$ and any integer m let $d_{x,\underline{m}}$ be the number of trees in the grove $x\times\underline{m}$. Then the following combinatorial formula holds*:

$$\sum_{x\in Y_n} d_{x,\underline{m}}=c_{nm}.$$

We see, for instance, that $d_{12,\underline{2}}=7$, $d_{12,\underline{2}}=7$, and so $7+7=14=c_4$, as expected; $d_{123,\underline{2}}=22$, $d_{213,\underline{2}}=33$, $d_{131,\underline{2}}=20$, $d_{312,\underline{2}}=33$, $d_{321,\underline{2}}=22$, and so $22+33+20+33+22=132=c_6$, as expected.

II. Arithmetic of planar trees

In this second part we extend the addition and the multiplication on planar binary trees to all planar trees. The binary case becomes a quotient of this case. The formulas are slightly more complicated since there is now one more tree in degree 2 which accounts for one more operation that we call the Middle sum. As before, this theory is governed by some free object on one generator in a category of algebras, they are the *dendriform trialgebras* introduced in [LR3].

Instead of using a poset structure, we begin right away with the recursive definition of the sum. We skip the arguments which are similar to the arguments in the first part. Except for the poset definition of the operations, this part is parallel to the first part and therefore we indicate only the main changes.

8. The set of planar trees

8.1. Super Catalan sets

Let T_0 be a set with one element. The sets T_n for $n \geqslant 1$ are defined inductively by the formula

$$T_n := \bigcup_{i_0+\cdots+i_k=n-k} T_{i_0} \times \cdots \times T_{i_k}$$

where the disjoint union is extended to all possibilities with $k \geqslant 1$ and $i_j \geqslant 0$ for all j.

If we denote by a the unique element of T_0, then an element of T_n can be described as a partial parenthesizing of the word $aa\cdots a$ of length $n+1$. Let $x^{(0)} \in T_{i_0}, \ldots, x^{(k)} \in T_{i_k}$. The element $(x^{(0)}, \ldots, x^{(k)}) \in T_{i_0} \times \cdots \times T_{i_k}$ viewed as an element in T_n is denoted $x^{(0)} \vee \cdots \vee x^{(k)} \in T_n$. In terms of parenthesizing it simply consists in concatenating the words and putting parenthesizes at both ends.

There are many other combinatorial descriptions of the sets T_n. We will use two of them as described below, one classical: the planar trees, and one less classical: the permutation-like notation. Others include the cells of the $(n-1)$-dimensional Stasheff polytope.

Let C_n be the number of elements of T_n. One has immediately:

$$C_0 = 1 \quad \text{and} \quad C_n = \sum_{i_0+\cdots+i_k=n-k} C_{i_0} \cdots C_{i_k}.$$

It is classically called the Super Catalan number, so T_n is called the *Super Catalan set* (see below for the generating series).

8.2. Planar trees

A *planar tree* is an oriented planar graph drawn in the plane with $n+1$ leaves and one root, such that each internal vertex has at least two leaves and one (and only one) root. We consider these trees up to isotopy. An example is in Fig. 4.

We define the *degree* of a tree as being the number of leaves minus 1 (in our example in Fig. 4, $n = 6$).

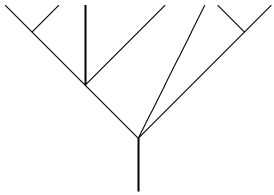
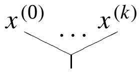


Fig. 4.

There is only one planar tree with one leaf: |, and only one planar tree with two leaves: . The main operation on planar trees is called *grafting*. The grafting of $x^{(0)}, \ldots, x^{(k)}$ denoted by $x^{(0)} \vee \cdots \vee x^{(k)}$ is the tree obtained by joining the roots to a new vertex and creating a new root:

$x^{(0)} \vee \cdots \vee x^{(k)} :=$  .

For instance: $| \vee | \vee | =$. Observe that the degree of $x^{(0)} \vee \cdots \vee x^{(k)}$ is $\sum_i (\deg x^{(i)} + 1) - 1$.

The main point about planar trees is the following: given a planar tree x, the decomposition

$$x = x^{(0)} \vee \cdots \vee x^{(k)}$$

exists and is *unique*. Moreover, when the degree of x is strictly positive, one has $k \geqslant 1$ and $\deg x^{(i)} < \deg x$ for any i. From this property of the planar trees it is clear that there is a one-to-one correspondence between the Super Catalan set T_n, as defined in Section 8.1, and the set of planar trees of degree n. So we identify them.

In low degree one has:

$T_0 = \{|\}$, $T_1 = \{\ \}$, $T_2 = \{\ ,\ ,\ \}$,
$T_3 = \{\ , \ldots,\ , \ldots,\ \}$.

The union of all the sets T_n for $n \geqslant 0$ is denoted T_∞.

Each set T_n is itself graded by the number of internal vertices. Denoting by $T_{n,i}$ the set of planar trees with $n+1$ leaves and $n+1-i$ internal vertices, we obtain a decomposition

$$T_n = T_{n,1} \cup \cdots \cup T_{n,n}.$$

It is clear that $T_{n,1} = Y_n$ (planar binary trees), and that $T_{n,n}$ is made of only one element. In the Stasheff polytope interpretation of the Super Catalan sets the elements of $T_{n,i}$ correspond to the $(i-1)$-cells of the Stasheff polytope of dimension $(n-1)$. Let $a_{n,i}$ be the number of elements of $T_{n,i}$. We define a generating series by

$$f(x,t) := 1 + \sum_{n \geqslant 1} \left(\sum_{i=1}^{i=n} a_{n,i} t^{i-1} \right) x^n.$$

One can show, either by direct inspection as in Section 1.1, or by using Koszul duality of operads (cf. [LR3,LR4]), that

$$f(x,t) = \frac{1 + tx - \sqrt{1 - 2(2+t)x + t^2x^2}}{2(1+t)x}.$$

8.3. Groves (bosquets)

In the sequel we will deal with the subsets of T_n. By definition a *grove* of degree n is a non-empty subset of T_n. We will refer to a grove as a disjoint union of trees. Hence a grove is a non-empty union of planar trees of the same degree, such that each tree appears at most once. We denote the set of groves of degree n by $\mathbb{T}_n$. The number of elements of $\mathbb{T}_n$ is $2^{C_n} - 1$. The set of groves made of the trees in $T_{n,i}$ is denoted $\mathbb{T}_{n,i}$.

An important role is going to be played by the following peculiar grove:

$$\underline{n} := \bigcup_{x \in T_n} x.$$

We call it the *total grove* of degree n.

9. Addition

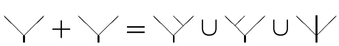
In this section we define a binary operation on planar trees and we extend it to groves. We call it *addition* or *sum* though it is not commutative.

9.1. Definition. By definition the *sum* of two planar trees x and y is given recursively by the formula:

$$\begin{aligned} x + y \;:=\; & x^{(0)} \vee \cdots \vee \bigl(x^{(k)} + y\bigr) \\ & \cup x^{(0)} \vee \cdots \vee \bigl(x^{(k)} + y^{(0)}\bigr) \vee \cdots \vee y^{(\ell)} \\ & \cup \bigl(x + y^{(0)}\bigr) \vee \cdots \vee y^{(\ell)}, \end{aligned}$$

where $x = x^{(0)} \vee \cdots \vee x^{(k)} \in T_p$ and $y = y^{(0)} \vee \cdots \vee y^{(\ell)} \in T_q$ (for $p \neq 0 \neq q$). Moreover, $0 = | \in T_0$ is a neutral element for $+$.

One observes that the sum of two trees is a (disjoint) union of trees. Their degree is the sum of the degrees of the starting trees.

Example. 

The addition is extended to the union of trees by distributivity on both sides:

$$\bigcup_i x_i + \bigcup_j y_j := \bigcup_{ij} (x_i + y_j).$$

Notation. From now on we often denote the tree | by 0 and the tree  by 1.

Despite the notation $+$ that we use, the addition on groves is not commutative, however there is an involution, as we will see below.

9.2. The three operations $\dashv$, $\vdash$, $\perp$

From the definition of the sum it is clear that one has a disjoint union of three different pieces. Let us define the following three operations (for $x \neq 0$ and $y \neq 0$):

$$x \dashv y := x^{(0)} \vee \cdots \vee \left(x^{(k)} + y\right),$$
$$x \vdash y := \left(x + y^{(0)}\right) \vee \cdots \vee y^{(\ell)},$$
$$x \perp y := x^{(0)} \vee \cdots \vee \left(x^{(k)} + y^{(0)}\right) \vee \cdots \vee y^{(\ell)}.$$

We call $\dashv$ the *Left sum*, $\vdash$ *the Right sum*, and $\perp$ *the Middle sum*. They are well-defined for planar trees and extended to groves by distributivity. By construction one has

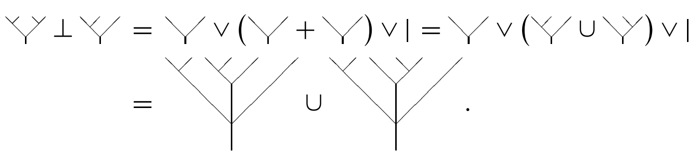
$$x + y := x \dashv y \ \cup\ x \vdash y \ \cup\ x \perp y.$$

We extend these definitions to x or y being 0 in the following cases: for $x \neq 0$:

$$x \dashv 0 = x, \qquad x \vdash 0 = 0, \qquad x \perp 0 = 0,$$
$$0 \dashv x = 0, \qquad 0 \vdash x = x, \qquad 0 \perp x = 0.$$

For $x = 0$ these elements are not defined, however $0 + 0 = 0$.

Here is an example of a Middle sum:



9.3. Remark. In order to give a meaning to $0 \dashv 0$, $0 \vdash 0$, and $0 \perp 0$, we could use the following trick. Think of a planar tree as being defined by its internal vertices and the relationship between them. So the tree | is in fact the empty set $\emptyset$ since it has no vertex. Then one can put, without contradiction, $\emptyset \dashv \emptyset = \emptyset \vdash \emptyset = \emptyset \perp \emptyset = \emptyset$. We still have $\emptyset + \emptyset = \emptyset$ since the union of the empty set with itself is still the empty set.

9.4. Proposition. *The sum of union of trees is associative. The neutral element is* 0.

Proof. In the next section we will show that this associativity property follows from [LR3,LR4] (cf. Section 11.4). $\square$

9.5. Theorem. *The Left sum, the Middle sum, the Right sum and the sum of two groves* (*a fortiori of two planar trees*) *is still a grove.*

Proof. This theorem is a consequence of Proposition 9.6 below. Indeed, any grove is a subset of the total grove. Since, in the sum of two total groves, a given tree appears at most once, the same property is true for the sum of any two groves. Hence this sum is a grove. □

9.6. Proposition. *Let* $\underline{n} := \bigcup_{x\in T_n} x$ *be the total grove of degree* n*. Then one has*

$$\underline{n} + \underline{m} = \underline{n+m}.$$

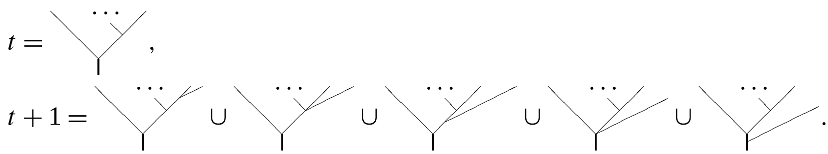
Proof. It is sufficient to prove the proposition for $m = 1$. Let us show that $\underline{n} + \underline{1} = \underline{n+1}$.

We want to prove that

$$\bigcup_{t\in T_n} (t+1) = \bigcup_{s\in T_{n+1}} s.$$

Therefore it is sufficient to show that for any element $s \in T_{n+1}$ there exists a unique element $t \in T_n$ such that $s \in t + 1$.

First we show that, for a given tree t, the trees appearing in $t + 1$ are obtained from t by adding a new leaf starting from the right side either from a vertex (lying on this right side) or from the middle of an edge (including the root and the leaf). By definition $t + 1 = t \dashv 1 \cup t \perp 1 \cup t \vdash 1$. The third component $t \vdash 1$ is $(t + 0) \vee 0 = t \vee 0$, which is precisely the tree obtained by the adjunction of a leaf starting from the (middle of) the root. Then, the second component $t \perp 1$ is $t^{(0)} \vee \cdots \vee (t^{(k)} + 0) \vee 0 = t^{(0)} \vee \cdots \vee t^{(k)} \vee 0$, which is precisely the tree obtained by the adjunction of a leaf starting from the lowest vertex. Finally, the first component $t \dashv 1$ is $t^{(0)} \vee \cdots \vee (t^{(k)} + 1)$ and here we use induction (on the degree of the last piece, that is $t^{(k)}$) to prove that we obtain the union of the trees obtained from t by adjoining a leaf to the other vertices and edges:



Let us start with a tree $s \in T_{n+1}$. By deleting the rightmost leaf we obtain a tree $t \in T_n$ such that s belongs to $t + 1$ by the preceding argument. This proves existence. On the other hand, if $t \in T_n$ is such that s belongs to $t + 1$, then obviously by deleting the rightmost leaf of s we recover t. This proves unicity. □

9.7. Remark. The proof of $\underline{n} + \underline{1} = \underline{n+1}$ in the binary case as given in Proposition 2.3 was using the definition of the addition in terms of the poset structure of Y_n. Here the proof of the analogous result is done by using the recursive definition of the sum. Since, in the binary case, one can also define the addition through a recursive formula, this proof works also under the following modification: the trees in $s+1$ are obtained from s by adding a new leaf starting from the middle of the edges on the right side. So all these trees are still binary.

9.8. *Involution*

Observe that for a planar tree *symmetry* around the axis passing through the root defines an involution σ on T_n and therefore also on $\mathbb{T}_n$. For instance $\sigma($$) =$ , $\sigma($$) =$ , and $\sigma($$) =$ .

By induction one has immediately

$$\sigma(x \dashv y) = \sigma(y) \vdash \sigma(x),$$
$$\sigma(x \vdash y) = \sigma(y) \dashv \sigma(x),$$
$$\sigma(x \perp y) = \sigma(y) \perp \sigma(x),$$

and therefore one obtains

$$\sigma(x + y) = \sigma(y) + \sigma(x).$$

9.9. *Filtration*

Define a filtration on $\mathbb{T}_n$ by

$$F^i\mathbb{T}_n := \mathbb{T}_{n,1} \cup \cdots \cup \mathbb{T}_{n,i}.$$

So we have

$$\emptyset = F^0\mathbb{T}_n \subset \mathbb{Y}_n = F^1\mathbb{T}_n \subset \cdots \subset F^i\mathbb{T}_n \subset \cdots \subset F^n\mathbb{T}_n = \mathbb{T}_n.$$

Claim. *If $x \in F^i\mathbb{T}_p$ and $y \in F^j\mathbb{T}_q$, then $x + y \in F^{\sup(i,j)}\mathbb{T}_{p+q}$.*

The proof is straightforward by induction. In fact if $x \in \mathbb{T}_{p,i}$ and $y \in \mathbb{T}_{q,j}$, then $x + y \in \mathbb{T}_{p+q,\sup(i,j)} \cup \cdots \cup \mathbb{T}_{p+q,i+j}$.

In particular, fix i and consider the sets $\mathbb{T}_{n,i}$ for $n \geqslant i$. Then there is a well-defined sum

$$\mathbb{T}_{p,i} \times \mathbb{T}_{q,i} \longrightarrow \mathbb{T}_{p+q,i}, \quad (x, y) \mapsto x +_i y,$$

which consists in taking in $x + y$ only the trees belonging to $\mathbb{T}_{p+q,i}$. From the preceding claim, the operation $+_i$ is associative. Indeed, in $x + y + z$ the elements in $\mathbb{T}_{p+q+r,i}$ cannot come from elements in $\mathbb{T}_{p,j} \times \mathbb{T}_{q,i} \times \mathbb{T}_{r,i}$ with $j > i$, for instance, since $\sup(j, i, i) = j > i$.

For $i = 1$ one has $\mathbb{T}_{n,1} = \mathbb{Y}_n$ and one recovers the sum of planar binary trees devised in Definition 2.1.

Summarizing what we have proved until now, we obtain the following corollary.

9.10. Corollary. *The set* $\mathbb{T}_\infty := \bigcup_{n\geqslant 0} \mathbb{T}_n$ *of groves is an involutive graded monoid for* $+$. *The maps*

$$\begin{aligned} \mathbb{N} &\longrightarrow \mathbb{T} \longrightarrow \mathbb{N} \\ n &\mapsto \underline{n},\ t \mapsto \deg t \end{aligned}$$

are morphisms of monoids. The set $\mathbb{T}_{\infty,i} := \mathbb{T}_0 \cup \bigcup_{n\geqslant i} \mathbb{T}_{n,i}$ *is an involutive graded monoid for* $+_i$.

10. Permutation-like notation of trees

We extend the permutation-like notation introduced in Section 3 to all planar trees.

10.1. Definition. By definition the *name* of the unique element of T_0 is 0, and the name $w(x)$ of the element $x \in T_n$, $n \geqslant 1$, is a finite sequence of strictly positive integers obtained inductively as follows:

$$\begin{aligned} &\text{if} \quad x = x^{(0)} \vee x^{(1)} \vee \cdots \vee x^{(k)} \in T_n, \\ &\text{then} \quad w(x) := \big(w\big(x^{(0)}\big), n, w\big(x^{(1)}\big), n, \ldots, n, w\big(x^{(k)}\big)\big), \end{aligned}$$

with the convention that we do not write the zeros.

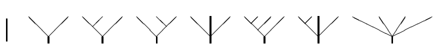
If there is no possibility of confusion we simply write

$$w(x) := w\big(x^{(0)}\big)\, n \cdots n\, w\big(x^{(k)}\big) \quad \text{(concatenation)}.$$

Observe that, except for 0, such a sequence is made of n integers and the integer n appears k times. The name of the unique element of T_1 is therefore 1.

10.2. Bijection with the planar trees

When an element in T_n corresponds to a planar tree x and to a name $w(x)$, we will say that $w(x)$ is the name of the tree x. In low dimension we obtain, for instance:

$x =$ 

$w(x) =$ 0 1 12 21 22 123 133 333.

Recall that symmetry around the root axis induces an involution on T_n. If $a_1 \cdots a_n$ is the name of the tree x, then the name of $\sigma(x)$ is $a_n \cdots a_1$.

10.3. Test for sequences

Given a sequence of integers, is it the name of a planar tree? First, check that the largest integers in the sequence are the length of the sequence. Second, check that the maximal subsequences not containing the largest integers are names of trees. Example: 14218812 is the name of a tree, 323 is not.

10.4. Relationship with the ordered partitions

In Section 3.5 we mentioned the relationship between the permutations and the planar binary trees. There is a similar relationship between ordered partitions of $\{1, 2, \ldots, n\}$ and T_n. It is exploited in [LR4].

10.5. Tricks for computation

For some special trees the computation of the Left or Right or Middle sum is easy. First recall that

$$-\vee-\cdots-\vee- = -\mathtt{max}-\cdots-\mathtt{max}-,$$

where max stands for the largest integer (i.e the length) of the word to which it belongs. Then it is easy to check that

$$(-\mathtt{max}-\cdots-\mathtt{max}) \dashv - = -\mathtt{max}-\cdots-\mathtt{max}-,$$
$$- \vdash (\mathtt{max}-\cdots-\mathtt{max}-) = -\mathtt{max}-\cdots-\mathtt{max}-,$$
$$(-\mathtt{max}-\cdots-\mathtt{max}) \perp (-\mathtt{max}-\cdots-\mathtt{max}-)$$
$$= -\mathtt{max}-\cdots-\mathtt{max}-\mathtt{max}-\cdots-\mathtt{max}-),$$
$$(-\mathtt{max}-\cdots-\mathtt{max}-) \perp (\mathtt{max}-\cdots-\mathtt{max}-)$$
$$= -\mathtt{max}-\cdots-\mathtt{max}-\mathtt{max}-\cdots-\mathtt{max}-).$$

For instance, $313 \dashv 12 = 51512$, $313 \perp 12 = 51515$.

11. Polynomial algebra with tree exponents

In this section we recall some results on dendriform trialgebras from [LR3, LR4] and deduce algebraic relations between the operations on $\mathbb{T}_\infty$.

11.1. Dendriform trialgebras

By definition a *dendriform trialgebra* is a K-vector space A equipped with three binary operations

$$\prec : A \otimes A \to A, \qquad \succ : A \otimes A \to A, \qquad \cdot : A \otimes A \to A,$$

which satisfy the following axioms:

$$(a \prec b) \prec c = a \prec (b * c),$$
$$(a \succ b) \prec c = a \succ (b \prec c),$$
$$(a * b) \succ c = a \succ (b \succ c),$$
$$(a \succ b) \cdot c = a \succ (b \cdot c),$$
$$(a \prec b) \cdot c = a \cdot (b \succ c),$$
$$(a \cdot b) \prec c = a \cdot (b \prec c),$$
$$(a \cdot b) \cdot c = a \cdot (b \cdot c),$$

for any elements a, b and c in A, with the notation

$$a * b := a \prec b + a \succ b + a \cdot b.$$

Adding up all the relations shows that the operation $*$ is associative.

11.2. A dendriform trialgebra associated to planar trees

Let $K[T'_\infty]$ be the vector space generated by the elements X^x, for $x \in T_n$ ($n \geqslant 1$) (we exclude $1 \in T_0$ for a while). For a union of trees $\omega = \bigcup_i x_i$, we introduce the notation

$$X^\omega := \sum_i X^{x_i}.$$

We define three operations on $K[T'_\infty]$ by the formulas:

$$X^x \prec X^y := X^{x \dashv y}, \qquad X^x \succ X^y := X^{x \vdash y}, \qquad X^x \cdot X^y := X^{x \perp y},$$

and distributivity.

We now recall the following main result.

11.3. Theorem (Universal property) [LR3,LR4]. *The vector space $K[T'_\infty]$ equipped with the three operations $\prec$, $\succ$, and $\cdot$ as above is a dendriform trialgebra. Moreover, it is the free dendriform trialgebra on one generator* (*namely X*).

Therefore $K[T'_\infty]$ equipped with $*$ is an associative algebra. We make it into an associative and unital algebra $K[T_\infty] = K \oplus K[T'_\infty]$ by adding the vector space K, generated by $1 = X^0$.

11.4. Relations in $\mathbb{T}_\infty$

The dictionnary comparing the operations on trees and on the free dendriform trialgebra is the following:

$\mathbb{T}_\infty$	$\cup$	$\dashv$	$\vdash$	$\perp$	$+$
$K[T_\infty]$	$+$	$\prec$	$\succ$	$\cdot$	$*$

By Theorem 11.3 associativity of the operation $*$ implies associativity of the operation $+$ on planar trees (this proves Proposition 9.4). Moreover, the relations in the dendriform algebra give for $x, y, z \in \mathbb{T}_\infty$:

$$\begin{aligned}
&(x \dashv y) \dashv z = x \dashv (y + z),\\
&(x \vdash y) \dashv z = x \vdash (y \dashv z),\\
&(x + y) \vdash z = x \vdash (y \vdash z),\\
&(x \vdash y) \perp z = x \vdash (y \perp z),\\
&(x \dashv y) \perp z = x \perp (y \vdash z),\\
&(x \perp y) \dashv z = x \perp (y \dashv z),\\
&(x \perp y) \perp z = x \perp (y \perp z).
\end{aligned} \tag{11.4.1}$$

11.5. Corollary. *Any planar tree $x \in T_n$ can be written as a composite* (*using* $\dashv$, $\vdash$, $\perp$) *of n copies of $1 \in T_1$ with a suitable parenthesizing. This universal expression of x, denoted $w_x(1)$, is unique modulo the relations* (11.4.1).

Examples. $w_{21}(1) = 1 \dashv 1$, $w_{12}(1) = 1 \vdash 1$, $w_{22}(1) = 1 \perp 1$, $w_{122}(1) = 1 \vdash 1 \perp 1$, $w_{322}(1) = 1 \dashv (1 \perp 1)$.

11.6. Remarks about notation

In [LR3,LR4] the linear generators are denoted x instead of X^x. We adopt this different notation here to avoid confusion with the operations in $\mathbb{T}_\infty$.

In [LR3,LR4] the symbols $\dashv$, $\vdash$, and $\perp$ are used to denote operations of an associative trialgebra which is the Koszul dual structure of dendriform trialgebra. We have given them a completely different meaning here.

12. Multiplication

Since the associative algebra $K[T'_\infty]$ is free on one generator when considered as a dendriform trialgebra, one can perform composition of polynomials with tree exponents. Though the composite of monomials, that is $(X^x)^y$ where x and y are planar trees, is not a monomial, it turns out that it is X to the power of some grove. Hence one can define the multiplication of planar trees as being this grove and then extend this multiplication to any groves.

12.1. Definition. Let x and y be planar trees. By definition the *product* $x \times y$ is

$$x \times y := w_x(y),$$

where $w_x(1)$ is the universal expression of x (cf. Corollary 11.5). In other words, we replace all the copies of 1 by copies of y in this universal expression.

Observe that the above definition of the product has a meaning even when y is a grove since the Right sum, the Left sum and the Middle sum of groves are well-defined. We extend the multiplication to x being a grove by distributivity on the left with respect to disjoint union:

$$(x \cup x') \times y = w_x(y) \cup w_{x'}(y).$$

So we have defined the product of two groves. This product is a grove since it is obtained by the operations $\dashv$, $\vdash$, and $\perp$. Observe that the product is not commutative.

12.2. Proposition. *The multiplication $\times$ on groves is distributive on the left with respect to the Left sum, the Right sum, the Middle sum and the sum* (*but not on the right*)*.*

Proof. The formula $w_{x+x'}(1) = w_x(1) + w_{x'}(1)$ follows from

$$w_{x\dashv x'}(1) = w_x(1) \dashv w_{x'}(1),$$
$$w_{x\vdash x'}(1) = w_x(1) \vdash w_{x'}(1),$$
$$w_{x\perp x'}(1) = w_x(1) \perp w_{x'}(1).$$

These three formulas follow inductively from the properties of the function w. $\square$

12.3. Proposition. *The multiplication of groves is associative with neutral element on both sides the tree* $\curlyvee = 1$.

Proof. Interpreted in terms of dendriform trialgebra, the multiplication of planar trees is composition of monomials. Since composition is associative, the multiplication of planar trees is associative.

Since $w_1(1) = 1$, we obtain $w_1(y) = y$ and so $1 \times y = y$. On the other side, $x \times 1 = x$ is a tautology. $\square$

12.4. Theorem. *With the notation $\underline{n} = \bigcup_{x \in T_n} x$, one has $\underline{n} \times \underline{m} = \underline{nm}$.*

Proof. Since, by Proposition 12.2 the multiplication is distributive on the left with respect to addition, we obtain

$$\begin{aligned} \underline{n} \times \underline{m} &= (\underline{1} + \cdots + \underline{1}) \times \underline{m} = \underline{1} \times \underline{m} + \cdots + \underline{1} \times \underline{m} \\ &= \underline{m} + \cdots + \underline{m} = \underline{nm}. \qquad \square \end{aligned}$$

12.5. Proposition (recursive property). *Let $x = x^{(0)} \vee \cdots \vee x^{(k)}$ be a planar tree and let y be a grove. The multiplication is given recursively by the formulas*:

$$\begin{aligned} &x \times y = \left(x^{(0)} \times y\right) \vdash y \perp \left(x^{(1)} \times y\right) \perp y \perp \cdots \perp y \dashv \left(x^{(k)} \times y\right), \\ &0 \times y = 0. \end{aligned}$$

Proof. First, observe that because of the relations $(*)$ there is no need for parenthesis in this formula. It suffices to show the equality

$$w_x(y) = w_{x^{(0)}}(y) \vdash y \perp w_{x^{(1)}}(y) \perp y \perp \cdots \perp y \dashv w_{x^{(k)}}(y),$$

which is a consequence of

$$x = x^{(0)} \vee \cdots \vee x^{(k)} = x^{(0)} \vdash 1 \perp x^{(1)} \perp 1 \perp \cdots \perp 1 \dashv x^{(k)}.$$

This last formula is easily proved by using the definition of the three operations. $\square$

12.6. Proposition (involution). *For any groves x and y one has*

$$\sigma(x \times y) = \sigma(x) \times \sigma(y).$$

Proof. It is a consequence of the previous proposition and of the relations between the involution and the three operations (cf. Section 9.8). $\square$

12.7. Summary

On the set of groves $\mathbb{T}_\infty = \bigcup_{n \geqslant 0} \mathbb{T}_n$ there are defined operations $+$ and $\times$ such that

- the addition $+$ is associative, distributive both sides with respect to $\cup$, with neutral element $0 = |$, but is not commutative;
- the multiplication $\times$ is associative, distributive on the left with respect to the sum $+$ and to the disjoint union $\cup$ (but not right distributive), with neutral element (both sides) $1 = \vee$, but is not commutative;
- the involution σ on $\mathbb{T}_\infty$ satisfies $\sigma(x + y) = \sigma(y) + \sigma(x)$ and $\sigma(x \times y) = \sigma(x) \times \sigma(y)$;
- the maps $\mathbb{N} \to \mathbb{T}_\infty$, $n \mapsto \underline{n} = \bigcup_{x \in Y_n} x$, and $\deg : \mathbb{T}_\infty \to \mathbb{N}$ (degree) are compatible with $+$ and $\times$. The composite is the identity of $\mathbb{N}$;
- the quotient map $\mathbb{T}_\infty \to \mathbb{Y}_\infty$, which consists in forgetting about the planar trees which are not binary, is both additive and multiplicative.

12.8. Question. It would be interesting to know if one can also put an internal multiplication on $\mathbb{T}_{\infty,i}$ for $i > 1$, cf. Section 9.9.

III. Final comments

• It is sometimes helpful to index chain complexes, not by the integers or even pair of integers (as in a bicomplex), but by trees. Examples and the simplicial properties of the planar binary trees have been investigated by A. Frabetti in [Fr]. Similarly, many small categories in algebraic topology have the natural numbers as objects (for instance Δ, Γ). In the work of André Joyal [J] appears a category Θ whose objects are the planar trees as considered here.

• The relationship of dendriform algebras and trialgebras with other types of algebras like associative dialgebras, associative trialgebras, Leibniz algebras, Zinbiel algebras, associative algebras has been treated in [L2,LR4] in terms of operads. See also [Ch].

• The operad of associative algebras comes from a set-operad by the functor which associates to a set the vector space based on it. The operad of dendriform dialgebras (respectively dendriform trialgebras) does not come from a set operad since the sum of two trees is not a tree. However, since the sum of two groves is a grove, it is very close to being a set operad.

• The associative algebra $K[Y_\infty]$ has a dendriform structure, but has also a Hopf algebra structure (cf. [LR1]). Moreover, these two structures are compatible, as was discovered by M. Ronco [R1]. It turns out that $K[Y_\infty]$ is the universal enveloping dendriform dialgebra of the free *brace algebra* on one generator as proved by M. Ronco in [R2,R3]. A similar result holds for the Hopf algebra $K[T_\infty]$.

• Planar binary trees have been used by Christian Brouder [Br] in place of natural numbers in order to index series which are solutions of some differential equations in quantum field theory (the Schwinger–Dyson equations). The *renormalization* of quantum electrodynamics is governed by a certain non-commutative and non-cocommutative Hopf algebra, cf. [BF]. It turns out that this Hopf algebra is isomorphic to $K[Y_\infty]$, cf. [Fo,H].

Appendix A. Tables for planar binary trees

A.1. Addition table

Recall that 0 is the neutral element for $+$, so

$$0 + x = x = x + 0.$$

Table 1
Addition table

$x+y$	1	12	21
1	12 21	123 213 312	131 321
12	123 131	1234 1314 1412	1241 1421
21	213 312 321	2134 3124 3214 4123 4213 4312	2141 4131 4321
123	1234 1241	12345 12415 12512	12351 12521
213	2134 2141	21345 21415 21512	21351 21521
131	1314 1412 1421	13145 14125 14215 15123 15213 15312	13151 15131 15321
312	3124 4123 4131	31245 41235 41315 51234 51314 51412	31251 51241 51421
321	3214 4213 4312 4321	32145 42135 43125 43215 52134 53214 53124 54123 54213 54312	32151 54131 52141 54321

In Table 1 we omit the $\cup$ sign, x/y is the first element of the first line, $x\backslash y$ is the last element of the last line. The first line is $x \vdash y$ and the second line (union third if any) is $x \dashv y$. Recall that $\sigma(x+y) = \sigma(y) + \sigma(x)$, so a sum like $21 + 123$ is easily obtainable from this table.
Example of computation by using the recursive formulas:

$$\begin{aligned} 131 \vdash 12 &= 131 \vdash (1 \vee 0) = (131 + 1) \vee 0 = (131 \dashv 1 \cup 131 \vdash 1) \vee 0 \\ &= \big((1 \vee 1) \dashv 1 \cup 1314\big) \vee 0 = \big(1 \vee (1+1) \cup 1314\big) \vee 0 \\ &= (1412 \cup 1421 \cup 1314) \vee 0 = 14125 \cup 14215 \cup 13145. \end{aligned}$$

A.2. Mutiplication table

Recall that 1 is the neutral element for $\times$, so

$$1 \times x = x = x \times 1.$$

Recall that the recursive formula is $x \times y = (x^l \times y) \vdash y \dashv (x^r \times y)$ and $0 \times x = 0$, $0 \vdash x = x = x \dashv 0$.

The products $312 \times y$ and $321 \times y$ for $y = 12$ or 21 or $12 \cup 21$ can be obtained from Tables 2 and 3 by the formula $\sigma(x \times y) = \sigma(x) \times \sigma(y)$.

Table 2
Multiplication table

$x \times y$	12	21	$12 \cup 21$			123	213	131	312	321
12	1234	2141	1234	1314	1241	123456	213516	131461	312612	321621
	1314		2134	3124	3214	124156	215216	141261		
				2141		125126		142161		
21	1412	4131	1412	4123	4213	126123	216213	161241	612512	621521
		4321	4312	1421	4131			162141	615312	651421
				4321				164131		654321

Table 3
Multiplication table

$x \times y$	12	21	$12 \cup 21$					
123	123456	214161	123516	123456	131516	131456	125126	125216
	131456		124156	213516	213456	312516	312456	321516
	123516		321456	215126	215216	214156	123461	131461
	131516		124161	213461	312461	321461	214161	
213	141256	413161	151236	151316	141256	152136	153126	153216
	151236	432161	142156	512346	512416	412356	521346	521416
	151316		421356	531246	541236	541316	431256	513146
			514126	514216	413156	532146	542136	543126
			543216	432156	141261	142161	412361	421361
				431261	413161	432161		
131	123612	216131	123612	123621	131612	131621	126123	126213
	131612	216321	126312	126131	126321	213612	213621	312612
			312621	321612	321621	216123	216213	216312
					216131	216321		

A.3. Exercise

The following grove ($\cup$ is omitted):

$$\begin{matrix} 141294131 & 141291241 & 141292141 \\ 142194131 & 142191241 & 142192141 \\ 131494131 & 131491241 & 131492141 \end{matrix}$$

is a product of two groves, which ones?

Appendix B. Tables for planar trees

B.1. Addition table

Recall that 0 is the neutral element for $+$, so

$$0 + x = x = x + 0.$$

Table 4
Addition table

$x+y$	1	12	21	22
1	12	123 213 223	131	133
	21	312	321	322
	22	313	331	333
12	123	1234 1314 1334	1241	1244
	131	1412	1421	1422
	133	1414	1441	1444
21	213	2134 3124 3214 3224 3134	2141	2144
	312 321 322	4123 4213 4312 4223 4313	4131 4321 4331	4133 4322 4333
	313	4124 4214 4224	4141	4144
22	223	2234 3314 3334	2241	2244
	331	4412	4421	4422
	333	4414	4441	4444

In Tables 4 and 5 we omit the $\cup$ sign. The first line is $x \vdash y$, the second line is $x \dashv y$ and the third line is $x \perp y$.

B.2. Mutiplication table

Recall that 1 is the neutral element for $\times$, so

$$1 \times x = x = x \times 1.$$

Table 5
Multiplication table

$x \times y$	12	21	22	$12 \cup 21$				123		133
12	1234	2141	2244	1234	1314	1241	1334	125126	124166	134166
	1314			2134	3124	3214	3224	123456	124456	133466
	1334				2141	3134		125156		134466
21	1412	4131	4422	1412	4123	4213	4223	126123		166133
		4321		4312	1421	4131	4313			
		4331			4321	4331				
22	1414	4141	4444	1414	1441	4124	4214	126126		166166
					4224	4141				

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