K-Theory **30:** 105–127, 2003.

Algebraic K-Theory and the Conjectural Leibniz K-Theory

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(Received: March 2003)

Abstract. The analogy between algebraic K-theory and cyclic homology is used to build a program aiming at understanding the algebraic K-theory of fields and the periodicity phenomena in algebraic K-theory. In particular, we conjecture the existence of a Leibniz K-theory which would play the role of Hochschild homology. We propose a motivated presentation for the Leibniz K_2-group of a field.

Mathematics Subject Classifications (2000): 18F25, 17A32, 19C99, 19Dxx.

Key words: algebraic K-theory, cyclic homology, motivic complex, Leibniz K-theory.

1. Introduction

The algebraic K-groups of a field F, denoted $K_n(F)$, are well known for $n = 0, 1$, and for $n = 2$, Matsumoto's theorem gives a small presentation. It would be desirable to have such a small presentation for higher n, or, at least, a small presentation of a chain complex whose homology would be these K-groups. By small we essentially mean no matrices in the presentation. Many efforts by different people have been made in this direction, and it was shown that the k-logarithm functions (more precisely, their functional equation) play a prominent role. Among them are the motivic complexes of Goncharov which are a family of finite chain complexes whose homology would conjecturally give the graded quotient K-groups associated to their γ-filtration.

Topological K-theory is periodic (of period 2 over the complex numbers and of period 8 over the reals). However, algebraic K-theory is not periodic as one can see from the computation of the K-theory of finite fields. Nevertheless, some periodicity phenomena appear in the computation. For instance, the rational algebraic K-groups of the integers are periodic of period 4 for $n \geqslant 2$.

The algebraic K-groups are defined out of the general linear group. If one replaces it by the Lie algebra of matrices, then one gets 'additive algebraic K-theory'. It turns out that for this additive theory both problems discussed above, that is computation in terms of a small complex and periodicity phenomena, can be solved. The solution is given by the so-called *cyclic homology*.

The purpose of this paper is, first, to recall briefly what is known about these problems and their solution in the additive framework, and then to take advantage of this solution to propose a program in the multiplicative framework. This program, far from being completed, is made of several steps. First we conjecture the existence of a bicomplex, in the multiplicative framework, analogous to the so-called (b, B)-bicomplex whose homology gives cyclic homology. The homology of this bicomplex should give the algebraic K-groups with the γ-filtration. Second we investigate in more detail the first column of this conjectural bicomplex. Its homology should be some 'Leibniz K-theory', that is an analogue, in the multiplicative framework, of the Hochschild homology. It should be related to some 'Leibniz homology' of the general linear group, like Hochschild homology is related to Leibniz homology of Lie algebras of matrices. In particular we propose a presentation for the Milnor–Leibniz K-group KL_2^M of a field (cf. Section 5.5).

Leibniz homology of a Lie algebra is in fact the homology theory of another type of algebras called *Leibniz algebras*. Similarly, we expect the Leibniz homology of discrete groups to be the homology of some algebraic objects that we call 'coquecigrues'. In the search of this mythical object we were led to build new types of algebras called 'associative dialgebras' and its dual type 'dendriform algebras', cf. [1]. Surprisingly, these new types of algebraic structure proved helpful in completely different topics:

— associative dialgebras appeared in quantum information theory, see the work of Leroux [2], with potential application to the genome combinatorics,
— dendriform algebras appeared in renormalization theory, see the work of Brouder and Frabetti [3].

This paper is mainly expository and conjectural, since most of the material is already in print, but disseminated in several papers. A very similar circle of ideas is showing up in Cathelineau's paper [4].

2. Algebraic *K*-Theory of Fields

First we briefly recall the definition of the algebraic K-groups, some of their general properties and a few facts about the computation of the algebraic K-theory of fields. Second we recall some conjectures about the motivic complex whose homology would give the algebraic K-groups of fields.

2.1. ALGEBRAIC *K*-THEORY

For any ring A, let $GL_n(A)$ be the group of invertible matrices with entries in A. Bordering such a matrix by 0's and 1 on the diagonal gives rise to an inclusion $GL_n(A) \to GL_{n+1}(A)$. The union over all n is denoted by $GL(A)$. Quillen [5] has proved that by adding a few cells to the classifying space $BGL(A)$ one can construct a new space $BGL(A)^+$ with the following properties:

— the homology of $BGL(A)^+$ is the homology of the discrete group $GL(A)$,
— the space $BGL(A)^+$ is an H-space.

In fact the space $BGL(A)^+$ is uniquely defined (up to homotopy) by the fact that its fundamental group is Abelian, and that for any $GL(A)_{ab}$-module M the map $BGL(A) \to BGL(A)^+$ induces an isomorphism on $H_*(-, M)$, cf. [5, 6].

By definition the algebraic K-groups $K_n(A)$ of the ring A are defined for $n \geqslant 1$ by

$$K_n(A) := \pi_n(BGL(A)^+).$$

As a consequence of these properties the graded group $H_*(GL(A), \mathbf{Q})$ is a graded connected Hopf algebra whose primitive part is precisely $\oplus_{n \geqslant 1} K_n(A)_{\mathbf{Q}}$. In other words there is a graded isomorphism,

$$H_*(GL(A), \mathbf{Q}) \cong \Lambda(K_*(A)_{\mathbf{Q}}),$$

where Λ stands for the graded symmetric functor: $\Lambda M = S(M^{\mathrm{ev}}) \otimes E(M^{\mathrm{odd}})$, where S is the symmetric power functor and E is the exterior power functor. Observe that, rationally, the K-groups can be defined without using the $+$-construction.

When A is commutative there is a *product structure* defined on K-theory

$$\cup\colon K_p(A) \times K_q(A) \to K_{p+q}(A), \tag{2.1.1}$$

which permits us to construct some elements in the K-groups (cf. [6]). One can also define the so-called *Adams operations* (cf. [7]), which enable us to put a γ-filtration on the K-groups. The associated graded group is denoted $\oplus_i\, gr_i^{\gamma} K_n(A)$.

Unlike topological K-theory, algebraic K-theory is not periodic, since, for instance, the K-theory of a finite field $\mathbb{F}_q$ is given by (cf. [5])

$$K_i(\mathbb{F}_q) = \begin{cases} 0 & i > 0,\ i \text{ even}, \\ \mathbf{Z}/(q^k - 1) & i = 2k-1,\ k \geqslant 1. \end{cases}$$

However there are some periodicity phenomenons. Let us recall two of them.

For any ring of integers $\mathcal{O}$ in a number field, Borel [8] computed the rational K-groups out of the homology of the general linear groups. The results show that $K_n(\mathcal{O})_{\mathbf{Q}}$ is periodic of period 4 provided that $n \geqslant 2$:

$$K_n(\mathcal{O})_{\mathbf{Q}} = \begin{cases} 0 & \text{for } n = 4i \text{ and for } n = 4i+2, \\ \mathbf{Q}^{r_1+r_2} & \text{for } n = 4i+1, \\ \mathbf{Q}^{r_1} & \text{for } n = 4i+3, \end{cases}$$

where r_1 (respectively r_2) is the number of real (respectively, complex) embeddings.

Let I be a nilpotent ideal in the rational algebra A. Then one can define the relative K-groups $K_n(A, I)$ which fit into a long exact sequence involving $K_n(A)$

and $K_n(A, I)$. Goodwillie [9] showed that, rationally, relative K-theory and relative cyclic homology are isomorphic for nilpotent ideals. Though cyclic homology is not periodic, there exists a periodicity map and the obstruction to periodicity is computable (see next section and [10]).

2.2. MILNOR K-THEORY [11]

Let F be a field and let $F^\times$ be the multiplicative group of its invertible elements (i.e. $GL_1(F)$). Consider $F^\times$ as an Abelian group, that is a module over $\mathbf{Z}$ and let $T(F^\times)$ be its tensor algebra. By definition the *Milnor K-group* $K_n^M(F)$ of F is the degree n part of the quotient of $T(F^\times)$ by the 2-sided ideal generated by the elements of degree 2 of the form $u \otimes (1-u)$ for $u \in F^\times \backslash 1$. We adopt the following multiplicative notation: for $u_1, \dots, u_n \in F^\times$ we denote by $\{u_1, \dots, u_n\}$ the class of $u_1 \otimes \cdots \otimes u_n$ in $K_n^M(F)$.

The inclusion $F^\times = GL_1(F) \to GL(F)$ induces a map $K_1^M(F) \to K_1(F)$. Since $K_1(F) = GL(F)/[GL(F), GL(F)]$, this map is an isomorphism and its inverse is induced by the determinant. By using the product in K-theory constructed in [6] one deduces a map

$$K_n^M(F) \to K_n(F), \qquad \{u_1, \dots, u_n\} \mapsto u_1 \cup \cdots \cup u_n, \quad n \geqslant 1.$$

The Abelian group $K_2^M(F)$ is defined by the following relations

$$\{u_1u_2, v\} = \{u_1, v\}\{u_2, v\}, \qquad \{u, v_1v_2\} = \{u, v_1\}\{u, v_2\},$$
$$\{u, 1-u\} = 1.$$

The celebrated Matsumoto theorem (cf. [11, 12]) asserts that the map

$$K_2^M(F) \to K_2(F), \qquad \{u, v\} \mapsto u \cup v$$

is an isomorphism.

For higher n the Milnor K-groups are related to the homology of $GL(F)$ as follows. First there is a stability theorem. For any infinite field F there are isomorphisms

$$H_n(GL_n(F)) \cong H_n(GL_{n+1}(F)) \cong \cdots \cong H_n(GL(F)),$$

and the first obstruction to stability is given by the exact sequence

$$H_n(GL_{n-1}(F)) \to H_n(GL_n(F)) \to K_n^M(F) \to 0.$$

From these results (due to Suslin [13]) and the Hurewicz map one gets a well-defined map $K_n(M) \to K_n^M(F)$. The composite

$$K_n^M(F) \to K_n(M) \to K_n^M(F)$$

is shown to be $(n-1)!$ Id.

2.3. MORE ABOUT K_2

There is a different way of presenting $K_2(F)$ by generators and relations, avoiding matrices, through the so-called Dennis–Stein symbols [14]. Let A be a commutative ring and $a, b \in A$ be two elements such that $1 - ab$ is invertible in A. Define the Abelian group $K_2^{DS}(A)$ by generators $\langle a, b\rangle$ and relations

$$\langle a, 1\rangle = 0 \quad (DS0),$$
$$\langle ab, c\rangle - \langle a, bc\rangle + \langle ca, b\rangle = 0 \quad (DS1),$$
$$\langle a, b\rangle + \langle a', b\rangle - \langle a + a' - aba', b\rangle = 0 \quad (DS2).$$

There exists a well-defined map $K_2^{DS}(A) \to K_2(A)$ (see below) which happens to become an isomorphism when A is a local ring. For instance when $A = F$ is a field this isomorphism is given by $\langle a, b\rangle \mapsto \{a, 1 - ab\}$ if a is different from 0.

In order to describe the map $K_2^{DS}(A) \to K_2(A)$ in full generality we need to recall the definition of the *Steinberg group* $St(A)$ (cf. [11]). It is defined by generators and relations as follows. The generators are $x_{ij}(a), i \geqslant 1, j \geqslant 1, i \neq j, a \in A$ and the relations

$$\begin{aligned} x_{ij}(a)x_{ij}(b) &= x_{ij}(a + b), \\ x_{ij}(a)x_{kl}(b) &= x_{kl}(b)x_{ij}(a), \quad \text{if } i \neq l \text{ and } j \neq k, \\ x_{ij}(a)x_{jk}(b) &= x_{jk}(b)x_{ik}(ab)x_{ij}(a). \end{aligned}$$

There is a well-defined map $St(A) \to GL(A)$ sending the generator $x_{ij}(a)$ to the elementary matrix $e_{ij}(a)$. It is known that the cokernel of this map is $K_1(A)$ and the kernel is $K_2(A)$. In order to get $K_2^{DS}(A) \to K_2(A)$ one sends $K_2^{DS}(A)$ to $St(A)$ by sending the generator $\langle a, b\rangle$ to the element $H_{12}(a, b)H_{12}(ab, 1)^{-1}$, where

$$H_{12}(a, b) := x_{21}(-b(1 - ab)^{-1})x_{12}(-a)x_{21}(b)x_{12}((1 - ab)^{-1}a).$$

Let us observe that a similar presentation holds for the relative K-group $K_2(A, I)$ when I is nilpotent (cf. [15]).

2.4. ABOUT K_3

It would be highly desirable to have a similar description of the higher K-groups of a field for $n \geqslant 3$, that is a presentation by generators and relations with no matrices around. Results in this direction have been first obtained by Bloch [16] for $K_3(F)$ (see also [17]). Let us describe it rationally. By the result of Suslin mentioned above, the Milnor K-group K_3^M splits off from the group K_3. Hence

$$K_3(F)_{\mathbf{Q}} = K_3^M(F)_{\mathbf{Q}} \oplus K_3^{\text{indec}}(F)_{\mathbf{Q}},$$

where the latter group is called the *indecomposable* part of K_3. Suslin showed that $K_3^{\text{indec}}(F)$ is the kernel of the Bloch map

$$\mathcal{B}_2(F) := \frac{\mathbf{Z}[F^\times \backslash 1]}{\mathcal{R}_2} \xrightarrow{\delta_2} F^\times \wedge F^\times, \qquad \delta_2[x] = (1 - x) \wedge x,$$

where $\mathcal{R}_2$ is the so-called 'functional equation of the dilogarithm', that is

$$[x] - [y] + \left[\frac{y}{x}\right] - \left[\frac{1-y}{1-x}\right] + \left[\frac{x(1-y)}{(1-x)y}\right] = 0 \quad (\mathcal{R}_2).$$

Let us recall that $K_3(A)$ is isomorphic to $H_3(St(A))$, a result due to Gersten.

2.5. MOTIVIC COMPLEXES OF GONCHAROV

For higher n conjectures have been made in the same spirit by Zagier [18] and Goncharov and others for higher K-groups. In [19, 20] Goncharov conjectures the existence of $\mathbf{Q}$-vector spaces $G_n = G_n(F)$ defined by generators and relations (see below) for $n \geqslant 1$, such that on $G_*(F) := \oplus_{n=1}^{\infty} G_n(F)$ there is a Lie coalgebra structure, that is homomorphisms

$$\delta_n \colon G_n \longrightarrow \bigoplus_{i \leqslant n/2} G_i \wedge G_{n-i}$$

such that, if $\delta := \oplus \delta_n$, then

$$G_* \xrightarrow{\delta} \Lambda^2 G_* \xrightarrow{\delta \otimes \mathrm{Id} - \mathrm{Id} \otimes \delta} \Lambda^3 G_* \longrightarrow \cdots \tag{2.5.1}$$

is a chain complex, called the *motivic complex*.

The relationship with the algebraic K-groups would be given by

$$K_{2n-i}^{(n)}(F) := gr_n^{\gamma} K_{2n-i}(F)_{\mathbf{Q}} = H_{(n)}^{i}(G_*(F)), \tag{2.5.2}$$

where (n) is the weight of the chain complex. In order to compare these complexes with their additive analogues, we picture them as follows

$$\begin{array}{ccccccc}
\cdots & & \cdots & & \cdots & & \cdots \\
\Lambda^4 G_1 & \xleftarrow{\delta} & G_2 \otimes \Lambda^2 G_1 & \xleftarrow{\delta} & G_3 \otimes G_1 \oplus \Lambda^2 G_2 & \xleftarrow{\delta} & G_4 \\
\Lambda^3 G_1 & \xleftarrow{\delta} & G_2 \otimes G_1 & \xleftarrow{\delta} & G_3 & & \\
\Lambda^2 G_1 & \xleftarrow{\delta} & G_2 & & & & \\
G_1 & & & & & &
\end{array}$$

The homology would give the pieces $K_n^{(p)}(F)$ of the K-groups (in particular, we should have $K_n^{(n)}(F) = K_n^M(F)$):

$$\begin{array}{cccc}
\cdots & \cdots & \cdots & \cdots \\
K_4^{(4)}(F) & K_5^{(4)}(F) & K_6^{(4)}(F) & K_7^{(4)}(F) \\
K_3^{(3)}(F) & K_4^{(3)}(F) & K_5^{(3)}(F) & \\
K_2^{(2)}(F) & K_3^{(2)}(F) & & \\
K_1^{(1)}(F) & & &
\end{array}$$

which is coherent with the vanishing conjecture of Beilinson and Soulé [7].

From the information we know on the lower dimensional K-groups, the following are reasonable candidates:

$$G_1(F) = F^\times \cong \frac{\mathbf{Z}[F^\times]}{\mathcal{R}_1}, \qquad G_2(F) = \mathcal{B}_2(F) \otimes \mathbf{Q} \quad \text{(the Bloch group)}$$

Goncharov found a group $\mathcal{B}_3(F) \otimes \mathbf{Q}$ as a good candidate for $G_3(F)$:

$$\mathcal{B}_3(F) := \frac{\mathbf{Z}[F^\times \backslash 1]}{\mathcal{R}_3},$$

where $\mathcal{R}_3$ is defined in terms of functional equations of the 3-logarithm (cf. [21, p. 134] and [22]).

The maps δ_n are given by $\delta_2(\{x\}_2) = (1-x) \wedge x$ and $\delta_3(\{x\}_3) = \{x\}_2 \otimes x$. Here $\{x\}_i$ denotes the class of $x \in F^\times$ in $\mathcal{B}_i(F)$.

2.6. THE MODIFIED k-LOGARITHM FUNCTION

The classical k-logarithm $\mathrm{Li}_k(z)$ is defined on the unit disk $|z| < 1$ by

$$\mathrm{Li}_k(z) := \sum_{n=1}^{\infty} \frac{z^n}{n^k}.$$

Observe that $\mathrm{Li}_1(z) = -\log(1-z)$ and $z(\mathrm{d}/\mathrm{d}z)\mathrm{Li}_k(z) = \mathrm{Li}_{k-1}(z)$. By using the formula

$$\mathrm{Li}_k(z) = \int_0^z \mathrm{Li}_{k-1}(z) \frac{\mathrm{d}z}{z},$$

one can continue analytically $\mathrm{Li}_k(z)$ to a multivalued analytic function on the space $\mathbf{C}P^1 \backslash \{0, 1, \infty\}$. However, Zagier discovered that the modified k-logarithm $\mathcal{L}_k(z)$ defined as

$$\begin{aligned} \mathcal{L}_1(z) &:= \log|z|, \\ \mathcal{L}_2(z) &:= \mathrm{Im}\,(\mathrm{Li}_2(z)) + \arg(1-z)\log|z|, \\ \mathcal{L}_3(z) &:= \mathrm{Re}\big(\mathrm{Li}_3(z) - \log|z|\,\mathrm{Li}_2(z) - \tfrac{1}{3}\log^2(|z|)\log(1-z)\big), \end{aligned}$$

and more generally as

$$\mathcal{L}_k(z) := \left\{ \begin{matrix} \mathrm{Re} \\ \mathrm{Im} \end{matrix} \right\} \left(\sum_{n=0}^{n=k-1} \frac{2^n B_n}{n!} \log^n |z| \, \mathrm{Li}_{k-n}(z) \right) \quad \left\{ \begin{matrix} k \ \text{odd} \\ k \ \text{even} \end{matrix} \right\},$$

is single-valued and continuous on $\mathbf{C}P^1$ for $k \geqslant 2$. Here $B_0 = 1$, $B_1 = -(1/2)$, $B_2 = 1/6, \ldots$ are the Bernoulli numbers.

In [19], Goncharov defined implicitly an Abelian subgroup $\mathcal{R}_k(F)$ of $\mathbf{Z}[F^\times \backslash 1]$ such that for $F = \mathbf{C}$ the map

$$\mathbf{Z}[\mathbf{C}^\times \backslash 1]/\mathcal{R}_k(\mathbf{C}) \longrightarrow \mathbf{R}, \qquad [x] \mapsto \mathcal{L}_k(x)$$

is well defined (see also [18]).

The Abelian group $\mathbf{Z}[F^{\times}\backslash 1]/\mathcal{R}_k(F)$ is a candidate for $G_k(F)$.
Information about $\mathcal{R}_4$ may be found in [23].

3. The Additive Case, Cyclic Homology

In this section, we treat an analogue of K-theory obtained by replacing the general linear group by the Lie algebra of matrices. The new K-theory is called *additive K-theory* and denoted $K_n^+(A)$. In this additive case, not only we know how to compute $K_1^+(A)$ or even $K_2^+(A)$, but in fact all groups $K_n^+(A)$ in the following sense: we know how to construct a *small* (meaning no matrices) chain complex out of A, whose homology gives additive K-theory. This is *cyclic homology*. As a consequence it gives a solution to the problems posed in the preceding section. The additive motivic complex turns out to be the truncated de Rham complex. The periodicity phenomena are well explained by the so-called periodicity exact sequence which shows that, in this additive framework, obstruction to periodicity is Hochschild homology.

The comprehensive knowledge of the solution in the additive case gives a hint about how to handle the multiplicative case, in particular it gives the idea of the motivic bicomplex that is to be discussed in the next section. We introduce, first, Hochschild and cyclic homology and, second, additive K-theory.

3.1. HOCHSCHILD AND CYCLIC HOMOLOGY [10, 24]

In this section K is a field, and A is an associative unital algebra over K. By M we denote a unitary A-bimodule.

By definition the *Hochschild complex* is the chain complex

$$C_*(A,M)\colon \cdots \to M\otimes A^{\otimes n} \xrightarrow{b} M\otimes A^{\otimes n-1} \to \cdots \to M\otimes A \to M,$$

where the boundary map b is given by

$$b(a_0,a_1,\ldots,a_n) = \sum_{i=0}^{n-1}(-1)^i(a_0,\ldots,a_ia_{i+1},\ldots,a_n) + {} + (-1)^n(a_na_0,a_1,\ldots,a_{n-1}),$$

where $a_0\in M$ and $a_i\in A$ for $i\geqslant 1$. Its homology is called *Hochschild homology* and denoted $H_*(A,M)$. For $M=A$, considered as a bimodule over itself, we adopt the notation $HH_*(A)=H_*(A,A)$, so that HH is a functor on the category of associative K-algebras.

The Hochschild homology groups are *Morita invariant* in the following sense. Let $\mathcal{M}_r(A)$ be the associative algebra of $r\times r$-matrices over A. There is a well-defined map tr: $\mathcal{M}_r(A)^{\otimes n}\to A^{\otimes n}$, given by

$$\operatorname{tr}(\alpha,\beta,\ldots,\zeta) = \sum_{(i_1,\ldots,i_n)} \alpha_{i_1i_2}\otimes\beta_{i_2i_3}\otimes\cdots\otimes\zeta_{i_ni_1},$$

which commutes with the Hochschild boundary. One can prove that the induced map on homology is an isomorphism

$$\mathrm{tr}_*\colon HH_*(\mathcal{M}_r(A)) \xrightarrow{\sim} HH_*(A).$$

Let $t\colon A^{\otimes n+1} \to A^{\otimes n+1}$ be the cyclic operator given by

$$t(a_0, \dots, a_n) = (-1)^n (a_n, a_0, \dots, a_{n-1}).$$

Define $C_n^\lambda(A) := A^{\otimes n+1}/(1-t)$. In [25], Connes remarked that the operator b is still well-defined on the quotient C_n^λ, and defines a chain complex C_*^λ that we call *Connes' complex*. Its homology is denoted $H_*^\lambda(A) := H_*(C_*^\lambda(A), b)$. The quotient map induces a canonical map $HH_*(A) \to H_*^\lambda(A)$ (which is not surjective in general).

There is another way of using the cyclic operator t. Let $b'\colon A^{\otimes n+1} \to A^{\otimes n}$ be given by

$$b'(a_0, \dots, a_n) := \sum_{i=0}^{n-1} (-1)^i (a_0, \dots, a_i a_{i+1}, \dots, a_n),$$

and put $N = 1 + t + \cdots + t^n$ (remark that $t^{n+1} = \mathrm{Id}$). Then one can show that the following relations hold:

$$(1-t)b' = b(1-t), \qquad b'N = Nb.$$

As a consequence the following is a well-defined chain bicomplex denoted $CC(A)$, and called the *cyclic bicomplex*:

$$\begin{array}{ccccccc}
b\downarrow & & -b'\downarrow & & b\downarrow & & \\
A^{\otimes 3} & \xleftarrow{1-t} & A^{\otimes 3} & \xleftarrow{N} & A^{\otimes 3} & \xleftarrow{1-t} & \cdots \\
b\downarrow & & -b'\downarrow & & b\downarrow & & \\
A^{\otimes 2} & \xleftarrow{1-t} & A^{\otimes 2} & \xleftarrow{N} & A^{\otimes 2} & \xleftarrow{1-t} & \cdots \\
b\downarrow & & -b'\downarrow & & b\downarrow & & \\
A & \xleftarrow{1-t} & A & \xleftarrow{N} & A & \xleftarrow{1-t} & \cdots
\end{array}$$

The homology of the associated total complex (sum over anti-diagonals) is called *cyclic homology* of A and denoted

$$HC_*(A) := H_*(\mathrm{Tot}\,(CC(A))).$$

Again, passing to the quotient of the cokernel of the first column gives a well-defined map $HC_*(A) \to H_*^\lambda(A)$ which turns out to be an isomorphism if the ground field K contains $\mathbf{Q}$.

By using the fact that the b'-complex (every second column in $CC(A)$) is acyclic since A is unital, and the fact that $CC(A)$ modulo the first two columns is nothing but $CC(A)$ shifted, one proves that there is an exact sequence

$$\cdots \to HH_n(A) \xrightarrow{I} HC_n(A) \xrightarrow{S} HC_{n-2}(A) \xrightarrow{B} HH_{n-1}(A) \to \cdots.$$

called *Connes' periodicity exact sequence*.

One can simplify greatly the cyclic bicomplex $CC(A)$ by 'killing' the b'-complexes and by using $\overline{A} := A/K$. Indeed, let us define

$$B\colon A \otimes \overline{A}^{\otimes n} \to A \otimes \overline{A}^{\otimes n+1},$$

by

$$B(a_0, a_1, \ldots, a_n) := \sum_{i=0}^{n} (-1)^{ni}(1, a_i, a_{i+1}, \ldots, a_n a_0, \ldots, a_{i-1}),$$

and call it the *Connes boundary map*.

The following relations hold:

$$b^2 = 0 = B^2 \quad \text{and} \quad bB + Bb = 0.$$

As a consequence the following is a well-defined chain complex denoted $\mathcal{B}(A)$ and called the *(b, B)-bicomplex*:

$$\begin{array}{ccccc}
\ldots & & \ldots & & \ldots \\
b\downarrow & & b\downarrow & & b\downarrow \\
A \otimes \bar{A}^{\otimes 2} & \xleftarrow{B} & A \otimes \bar{A} & \xleftarrow{B} & A \\
b\downarrow & & b\downarrow & & \\
A \otimes \bar{A} & \xleftarrow{B} & A & & \\
b\downarrow & & & & \\
A & & & &
\end{array}$$

There is a natural map of complexes $\operatorname{Tot} CC(A) \to \operatorname{Tot}\mathcal{B}(A)$ which induces an isomorphism on homology. Hence, $H_*(\operatorname{Tot}\mathcal{B}(A)) = HC_*(A)$. The map B in the periodicity exact sequence (which is easily constructed from the $\mathcal{B}(A)$-complex) is obviously induced by Connes' boundary map B.

As in the Hochschild case, cyclic homology is Morita invariant, that is there is a natural isomorphism

$$HC_*(\mathcal{M}_r(A)) \xrightarrow{\sim} HC_*(A).$$

It is immediate from the definitions that $HH_0(A) = HC_0(A) = A/[A, A]$, where $[A, A]$ is generated as a K-module by the elements $xy - yx$.

If A is *commutative*, then $HH_1(A)$ is isomorphic to the A-module of Kähler differentials $\Omega^1_{A/K}$, and $HC_1(A) \cong \Omega^1_{A/K}/\mathrm{d}A$.

Let $A = K$. Then it is immediate to check that $HH_0(K) = K$, and $HH_n(K) = 0$ for $n > 0$. Hence, from the periodicity exact sequence, we deduce that cyclic homology of K is periodic of period 2:

$$HC_{2n}(K) \cong K, \qquad HC_{2n+1}(K) = 0.$$

There exist several operations on cyclic homology (cf. [10, 26]). Let us just mention the product. If A is commutative, then there exists a product

$$\cup\colon HC_{n-1}(A) \times HC_{p-1}(A) \to HC_{n+p-1}(A)$$

(note the degree) which is induced by a variation of the shuffle product. For $n = p = 0$ it is given by $a \cup b = a\,\mathrm{d}b\ (= -b\,\mathrm{d}a) \in \Omega^1_{A/K}/\mathrm{d}A$.

3.2. ADDITIVE ALGEBRAIC K-THEORY

Suppose now that K is a field of characteristic zero and A a unital K-algebra. The Lie algebra of $n \times n$-matrices with entries in A is denoted $gl_n(A)$. Bordering such a matrix by 0's gives rise to an inclusion $gl_n(A) \to gl_{n+1}(A)$. The union over all n is denoted by $gl(A)$. The Lie algebra homology $H_*(gl(A)) = H_*(gl(A), K)$ is a graded connected Hopf algebra whose primitive part is denoted

$$K^+_*(A) = \bigoplus_{n \geqslant 1} K^+_n(A)$$

and called *additive K-theory*. There is a graded isomorphism

$$H_*(gl(A)) \cong \Lambda(K^+_*(A)),$$

where Λ is the graded symmetric power over K. Let $(C^{\mathrm{Lie}}_*(gl(A)), d)$ be the Chevalley–Eilenberg complex whose homology is $H_*(gl(A))$. In degree n one has $C^{\mathrm{Lie}}_n(gl(A)) = \Lambda^n gl(A)$. One can check that the symmetrization map

$$\Lambda^{n+1} gl(A) \to C^\lambda_n(\mathcal{M}(A)),$$
$$\alpha_0 \wedge \cdots \wedge \alpha_n \mapsto \sum_{\sigma \in S_{n+1}} \mathrm{sgn}(\sigma)\sigma(\alpha_0, \ldots, \alpha_n)$$

is a map of complexes and so induces a map

$$H_{n+1}(gl(A)) \to H^\lambda_n(\mathcal{M}(A)) \cong HC_n(\mathcal{M}(A)) \cong HC_n(A).$$

The main theorem asserts that, restricted to the primitive part, this map is an isomorphism:

THEOREM (Loday–Quillen–Tsygan) [24, 26]. *If K contains* $\mathbf{Q}$ *and A is a unital associative K-algebra, then there is a natural isomorphism*:

$$K^+_n(A) \cong HC_{n-1}(A) \quad \textit{for } n \geqslant 1.$$

In particular when A is commutative one has $K^+_1(A) = A$ (as an additive group) and $K^+_2(A) \cong \Omega^1_{A/K}/\mathrm{d}A$ (due first to Bloch [27], then generalized to non-commutative algebras in [28]), which is the additive analogue of Matsumoto's theorem. Indeed $\Omega^1_{A/K}/\mathrm{d}A$ admits the following presentation which is very close to the Dennis–Stein presentation of the group $K^M_2(F)$:

— generators: (a, b) over K for $a, b \in A$,
— relations:

$$(a, 1) = 0 \quad (0),$$
$$(ab, c) - (a, bc) + (ca, b) = 0 \quad (1),$$
$$(a, b) + (a', b) - (a + a', b) = 0 \quad (2).$$

Moreover there is a product on additive K-theory which coincides with the product on cyclic homology under the previous isomorphism:

$$\cup\colon K_n^+(A) \times K_p^+(A) \to K_{n+p}^+(A).$$

As in the multiplicative case there is a Milnor version of additive K-theory (respectively cyclic homology) when A is commutative: it is $HC_n^M(A) = \Omega^n_{A/K}/\mathrm{d}\Omega^{n-1}_{A/K}$. There is a stability theorem: *the following map is an isomorphism provided that* $i > n$:

$$H_i(gl_{n-1}(A)) \to H_i(gl_n(A)),$$

and an analogue of Suslin's theorem about obstruction to stability: *the following sequence is exact*:

$$H_n(gl_{n-1}(A)) \to H_n(gl_n(A)) \to \Omega^{n-1}_{A/k}/\mathrm{d}\Omega^{n-2}_{A/k} \to 0.$$

There also exist Adams operations (respectively λ-operations) on $H_*(gl(A))$ constructed out of exterior powers of matrices as shown in [29]. These operations restrict to the primitive part and so there is a γ-filtration on $K_*^+(A)$. In [10] and [29] these λ-operations on $HC_*(A)$ have been completely determined and made explicit in terms of the action of the symmetric group.

3.3. ADDITIVE MOTIVIC COMPLEX AND BICOMPLEX

Recall that a commutative algebra A is called *formally smooth* (we just say smooth) if it satisfies the following condition: for any commutative algebra R equipped with a square zero ideal I (i.e. $I^2 = 0$), any algebra map $A \to R/I$ has a lifting to R.

Hochschild, Kostant and Rosenberg have computed Hochschild homology of smooth algebras in terms of differential forms: if A is smooth over the characteristic zero field K, then there is a canonical isomorphism

$$HH_n(A) \cong \Omega^n_{A/K} \quad \text{for } n \geqslant 0.$$

From this computation one gets the computation of cyclic homology of a smooth algebra. The main point is that Connes' boundary map B induces the boundary map of the de Rham complex $d\colon \Omega^n \to \Omega^{n+1}$ on differential forms.

We denote by $H^n_{DR}(A)$ the homology groups of the de Rham complex.

THEOREM [24–26]. *If A is smooth over the characteristic zero field K, then there is a canonical isomorphism*

$$HC_n(A) \cong \Omega^n_{A/K}/\mathrm{d}\Omega^{n-1}_{A/K} \oplus H^{n-2}_{DR}(A) \oplus H^{n-4}_{DR}(A) \oplus \cdots .$$

This computation is, in fact, a consequence of the following statement.

When A is smooth over a characteristic 0 field, the (b, B)-bicomplex is quasi-isomorphic to the bicomplex obtained by taking the vertical homology, i.e. the following bicomplex in which the vertical boundary map is 0:

$$\begin{array}{lll} \cdots & \cdots & \cdots \\ \Omega^2_{A/K} \xleftarrow{d} & \Omega^1_{A/K} \xleftarrow{d} & \Omega^0_{A/K} \\ \Omega^1_{A/K} \xleftarrow{d} & \Omega^0_{A/K} & \\ \Omega^0_{A/K}. & & \end{array}$$

The homology of these complexes is

$$\begin{array}{llll} \cdots & \cdots & \cdots & \cdots \\ HC_2^{(3)}(A) & HC_3^{(3)}(A) & HC_4^{(3)}(A) & \\ HC_1^{(2)}(A) & HC_2^{(2)}(A) & & \\ HC_0^{(1)}(A), & & & \end{array}$$

where

$$HC^{(n)}_{2n-i-1} := gr^{\gamma}_n K^{+}_{2n-i}(A)_{\mathbf{Q}} \cong \begin{cases} H^{i-1}_{DR}(A), & 1 \leqslant i < n, \\ \Omega^{n-1}_{A/K} / \mathrm{d}\Omega^{n-2}_{A/K}, & i = n, \\ 0, & \text{otherwise.} \end{cases}$$

since the γ-filtration on HC_* (and hence on K^+_*) coincides with the filtration coming from λ-operations on the Lie algebra (and hence on additive K-theory) (cf. [29]).

There is a product on the homology of these complexes: the *Deligne product* which is the additive analogue of the product in algebraic K-theory.

As a conclusion we see that the results known about additive K-theory and cyclic homology provide a solution to the motivic complex and the periodicity phenomenons in the additive framework.

4. The Motivic Bicomplex

The motivic complex as envisioned in the first section will, at best, give the graded associated group of the K-groups. By analogy with the (b, B)-bicomplex of the additive case, we propose to look for a finer object, that is a *chain bicomplex* whose total homology would give the algebraic K-groups and the filtration by columns would give the γ-filtration. The relationship with the motivic complex would be through the vertical homology.

4.1. ANALOGY: ADDITIVE VERSUS MULTIPLICATIVE

Additive algebraic K-theory is based on the Lie algebra of matrices, that is matrices and Lie bracket. Algebraic K-theory is based on the general linear group, that is matrices and multiplication. In the additive case the first invariant is the trace, in the multiplicative case the first invariant is the determinant. This analogy goes further and, since we know more in the additive case, we will use this knowledge to make conjectures in the multiplicative case.

Here is a brief dictionary:

Additive	$x+y$	gl	tr	K^+	$\Omega^{n-1}/\mathrm{d}\Omega^{n-2}$	$\mathcal{B}_{**}$	HC
Multiplicative	$x+y-xy$	GL	det	K	K_n^M	CM_{**}	HM

In the additive case (and A commutative) one has $K_1^+(A)\cong A$ and the map $gl(A)\to A$ is the trace map. In the multiplicative case (and $A=F$ a field) one has $K_1(F)\cong F^\times$ and the map $GL(F)\to F^\times$ is the determinant map.

In the additive case one has a presentation of $K_2^+(A)$ involving the additive group law $x+y$ (cf. Section 3.2). In the multiplicative case one has a presentation of $K_2(F)$, by the Dennis–Stein symbols, which is the same provided one changes the additive group law into the multiplicative group law $x+y-xy$ (cf. Section 2.3).

In the additive case the obstruction to stability for the homology of matrices is $\Omega^{n-1}/\mathrm{d}\Omega^{n-2}$, in the multiplicative case it is K_n^M (Milnor K-theory). They happen to have the same presentation with either the additive or the multiplicative group law. In both cases the total group is generated by the degree 1 elements.

In the additive case the best computation is given by the (b, B)-bicomplex. Taking the vertical homology (i.e. E^1 of the spectral sequence) gives a sequence of finite length complexes. In the multiplicative case the motivic complex is expected to be of this form. However since the (b, B)-bicomplex is a finer object, we think that we should look for its analogue in the multiplicative framework.

4.2. THE CONJECTURAL MOTIVIC BICOMPLEX

In the spirit of [30] we wish to construct, for any ring A (not necessarily commutative), a bicomplex $CM(A)=CM_{**}(A)$, defined by generators and relations, of the form

$$\begin{array}{ccccc}
\cdots & & \cdots & & \cdots \\
\downarrow d & & \downarrow d & & \downarrow d \\
CM_{13}(A) & \stackrel{\delta}{\leftarrow} & CM_{23}(A) & \stackrel{\delta}{\leftarrow} & CM_{33}(A) \\
\downarrow d & & \downarrow d & & \\
CM_{12}(A) & \stackrel{\delta}{\leftarrow} & CM_{22}(A) & & \\
\downarrow d & & & & \\
CM_{11}(A) & & & &
\end{array}$$

with the following properties. Let us denote by $HM_n(A)$ the nth homology group of the total complex $\mathrm{Tot}\,CM_{**}(A)$. The functor HM is expected to be Morita invariant, that is the trace map induces an isomorphism $HM_*(\mathcal{M}(A)) \to HM_*(A)$. There should exist, for any ring A, a natural map

$$HM_n(A) \to K_n(A)$$

(compatible with Morita invariance), which is expected to be an isomorphism in certain instances, namely over $\mathbf{Q}$ when $A = F$ is a field. This isomorphism should be compatible with the filtration coming from the bicomplex structure on the left side and the γ-filtration on K-theory on the right side. There should also exist a relative theory $HM_*(A, I)$ for I a 2-sided ideal of A, together with a natural map $HM_*(A, I) \to K_*(A, I)$.

4.3. RELATIONSHIP WITH THE POLYLOGARITHMS

The column number k of the motivic bicomplex, that is $CM_{k*}(A)$, is associated, in a certain sense, to the polylogarithm function $\mathcal{L}_k$. For instance, in the field case, a candidate for the last group is $CM_{kk} = \mathbf{Z}[F^\times]/\mathcal{R}_k$, where $\mathcal{R}_k$ is generated by the functional equation(s) of the polylogarithm function $\mathcal{L}_k$ (cf. Section 2.6). In particular when $K = \mathbf{C}$ the map $\mathcal{L}_k$: $CM_{kk} \to \mathbf{C}$ is well defined. At this point it is helpful to remark the following facts. The functions $\mathcal{L}_k$ and $\mathcal{L}_{k-1}$ are linked by the formula $\mathcal{L}_{k-1} = x(\mathrm{d}/\mathrm{d}x)\mathcal{L}_k$ ($x(\mathrm{d}/\mathrm{d}x)$ is the so-called Euler operator). In the additive case the function for each column is $f(x) = x$ and we also have $f(x) = x(\mathrm{d}/\mathrm{d}x)f(x)$.

4.4. RELATIONSHIP WITH GONCHAROV MOTIVIC COMPLEX

Taking the vertical homology of $CM_{**}(F)$ gives a sequence of finite length (horizontal) complexes which should give a candidate for the motivic complex $G_*(F)$ (see Section 2.5).

First, there is a coherence concerning the lengths and also concerning the last groups, which, in Goncharov's setting, is the generating part of the complexes.

In [31] we proposed a first attempt for a presentation of the first two columns.

4.5. THE NILPOTENT CASE

When I is a nilpotent ideal in the ring A, then the relative K-theory group $K_2(A, I)$ admits also a presentation by means of the Dennis–Stein symbols, with the same type of relations (cf. [15]). So we can apply the same philosophy as before to find a bicomplex computing the relative K-groups $K_n(A, I)$ when I is nilpotent. Over $\mathbf{Q}$ this can be done since, by a theorem of Goodwillie (cf. [9]) one has an isomorphism $K_n(A, I)_{\mathbf{Q}} \cong HC_{n-1}(A, I)$ when A is a $\mathbf{Q}$-algebra. Moreover, the γ-filtrations that one can define on K-theory and on cyclic homology are compatible via this isomorphism, cf. [32].

One way of interpreting Goodwillie's results is to say that the tangent functor to algebraic K-theory and to cyclic homology is the same, namely Hochschild homology. In [4] Cathelineau takes the problem along these lines and proposes a multiplicative presentation for Kähler differentials.

5. Leibniz Homology, Leibniz K-Theory, Coquecigrue

We now focus on the first column CM_{1*} of the motivic bicomplex, whose additive analogue is the Hochschild complex. First we recall the matrix Lie algebra interpretation of Hochschild homology. It is similar to the Loday–Quillen–Tsygan theorem, but one has to replace the Lie homology by a different one, namely the Leibniz homology. This suggests the existence of 'Leibniz homology for discrete groups'. We expect the Leibniz homology of the general linear group to be strongly related with the homology of CM_{1*} in the case of a field: this would be 'Leibniz K-theory'. An interpretation of the Leibniz homology of discrete groups as the homology theory of some algebraic structure would give an answer to the search of 'coquecigrues'.

5.1. LEIBNIZ HOMOLOGY OF MATRICES

By definition the Leibniz complex of the Lie algebra g is

$$CL_*(g)\colon \cdots \to g^{\otimes n} \xrightarrow{d} g^{\otimes n-1} \to \cdots \to g \to 0,$$

where the boundary map d is given by

$$d(x_1, \ldots, x_n) = \sum_{1 \leqslant i < j \leqslant n} (-1)^j (x_1, \ldots, x_{i-1}, [x_i, x_j], x_{i+1}, \ldots, \hat{x}_j, \ldots, x_n).$$

We constructed this complex in [33] and we observed that it is well-defined for a larger class of algebras, namely the Leibniz algebras whose definition is as follows. A *Leibniz algebra* g over K is a vector space equipped with a binary map $[-,-]\colon g \otimes g \to g$ satisfying the *Leibniz identity*:

$$[[x, y], z] = [[x, z], y] + [x, [y, z]]$$

(i.e. $[-, z]$ is a derivation for the bracket). A Lie algebra is an example of a Leibniz algebra since, when the bracket is skew-symmetric, the Leibniz relation is equivalent to the Jacobi identity. The Leibniz complex $CL_*(g)$ is well-defined for any Leibniz algebra and its homology is called *Leibniz homology* and denoted $HL_n(g)$ for $n \geqslant 1$.

THEOREM [10, 34]. *For any associative unital algebra A over a characteristic zero field K there is an isomorphism of graded modules*:

$$HL_*(gl(A)) \cong T(HH_{*-1}(A)),$$

where HH is the Hochschild homology.

Observe that, when one changes Lie homology into Leibniz homology, there are two modifications: not only HC is changed into HH but Λ is changed into T (tensor module functor). The core of the proof is still invariant theory.

5.2. HOCHSCHILD HOMOLOGY AND ADDITIVE MILNOR K-THEORY

For any algebra A there is a natural quotient of $HH_n(A)$, which is the obstruction to the stability of the Leibniz homology of the Lie algebra of matrices, cf. [10, Section 10.6.19]. When A is commutative, this quotient turns out to be simply the module of differential forms $\Omega^n_{A/K}$: *for any* A *there is a stability isomorphism* $HL_n(gl_n(A)) \to HL_n(gl_{n+i}(A))$ *and*, *when* A *is commutative*, *there is an exact sequence*

$$HL_n(gl_{n-1}(A) \to HL_n(gl_n(A)) \to \Omega^{n-1}_{A/K} \to 0.$$

In low dimension the Leibniz homology of the Lie algebra of matrices has the same kind of properties as the Lie homology and the group homology (cf. [28]):

$$HL_2(stl(A)) = HH_1(A), \qquad HL_3(stl(A)) = HH_2(A),$$

where $stl(A)$ is the universal central extension of the perfect Leibniz algebra $sl(A)$.

For the definition and study of universal central extensions of Leibniz algebras, see [35] and [36]. The analogous statements for Lie algebras can be found in [27] when A is commutative and in [28] for the general case.

5.3. LEIBNIZ HOMOLOGY OF DISCRETE GROUPS

By analogy with the Lie algebra case one can expect the existence of a 'Leibniz homology of discrete groups', denoted $HL_*(G)$. It should come with a natural transformation

$$\alpha_*\colon HL_*(G) \to H_*(G),$$

which would be an isomorphism for $*=1$ and an epimorphism for $*=2$. Both $H_*(G)$ and $H_*(g)$ are graded commutative coalgebras (here g is a Lie algebra). Since, by [37], $HL_*(g)$ is a graded Zinbiel coalgebra we can expect $HL_*(G)$ to be also a graded Zinbiel coalgebra (and *a fortiori* a graded commutative coalgebra), the natural transformation α being a homomorphism of graded commutative coalgebras.

Recall that a *Zinbiel algebra* is a vector space equipped with a binary operation $x \cdot y$ satisfying $(x \cdot y) \cdot z = x \cdot (y \cdot z) + x \cdot (z \cdot y)$. Dually a *Zinbiel coalgebra* is a vector space C equipped with a linear map $\delta\colon C \to C \otimes C$ satisfying:

$$(\delta \otimes \mathrm{Id}) \circ \delta = (\mathrm{Id} \otimes \delta) \circ \delta + (\mathrm{Id} \otimes \tau\delta) \circ \delta,$$

where $\tau(x \otimes y) = y \otimes x$.

If G is a group with direct sum (for instance $GL(A)$ or $G =$ Abelian group), then it is expected that $HL_*(G)$ is a free Zinbiel coalgebra, that is to be of the form $T(V)$ for some graded subspace V of $HL_*(G)$. When G is Abelian we should find $HL_n(G) = G^{\otimes n}$.

5.4. LEIBNIZ K-THEORY

By analogy with the Lie case we expect the Leibniz homology of the general linear group to be of the form

$$HL_*(GL(A), \mathbf{Q}) \cong T(KL_*(A)_{\mathbf{Q}}),$$

where $KL_*(A)_{\mathbf{Q}}$ is, in a certain sense, the primitive part of $HL_*(GL(A), \mathbf{Q})$. We call these conjectural groups $KL_*(A)$ the *Leibniz K-groups* of the ring A. The natural map from HL to H should induce a map from KL to K.

The following table summarizes our knowledge and our conjectures:

	Commutative	Noncommutative
$+$	$H_*^{\mathrm{Lie}}(gl(A)) \cong \Lambda(HC_{*-1}(A))$	$HL_*(gl(A)) \cong T(HH_{*-1}(A))$
	Lie algebra, cyclic homology	Leibniz algebra, Hochschild homology
$\times$	$H_*(GL(A)) \cong \Lambda(K_*(A))$	$HL_*(GL(A)) \cong T(KL_*(A))$
	group, rational K-theory	coquecigrue ?, Leibniz K-theory ??

The notion of coquecigrue is going to be dealt with in Section 5.7.

With these conjectural objects at hand, several results holding in the three known cases can be expected in the noncommutative multiplicative setting. For instance, the map $HL_n(GL_n(F)) \to HL_n(GL_{n+i}(F))$ is expected to be an isomorphism and the cokernel of $HL_n(GL_{n-1}(F)) \to HL_n(GL_n(F))$ is expected to be some analogue of the Milnor K-group, denoted $KL_n^M(F)$ (see Section 5.6), which should coincide with the subgroup of $KL_n(F)$ generated by the elements of $KL_1(F) = F^\times$.

5.5. THE FIRST COLUMN OF THE CONJECTURAL MOTIVIC BICOMPLEX

In the additive motivic complex, i.e. the (b, B)-complex, all the columns are the Hochschild complex (up to a shift of degree). In the multiplicative motivic bicomplex it is expected that the kth column is related to the k-logarithm. We focus now on the case $k = 1$.

By analogy with the additive case, we expect the homology of the first column $CM_{1*}(F)$ of the conjectural bicomplex to be the Leibniz K-group $KL_*(F)$. Though we expect the groups $KL_*(A)$ to be defined for any ring A and over $\mathbf{Z}$, we will mainly investigate its properties when $A = F$ is a field.

From our former discussion in Section 4 it is clear that we expect $KL_1(F) = F^\times$.

5.6. LEIBNIZ K_2 GROUP

We have recalled in 2.3 the presentation of $K_2(F)$ in terms of the Dennis–Stein symbols. The only difference with the presentation of $HC_1(A)$ is the presence of the multiplicative formal group law in place of the additive formal group law, that is:

$$(a + a' - aba')b = ab + a'b - ab\,a'b.$$

in place of (cf. Section 3.2):

$$(a + a')b = ab + a'b.$$

Since $HH_1(A)$ admits the following presentation (over $\mathbf{Z}$ for instance):

$$\begin{aligned}&(ab, c) - (a, bc) + (ca, b) = 0 \quad (1),\\&(a, b) + (a', b) - (a + a', b) = 0 \quad (2),\\&(a, b) + (a, b') - (a, b + b') = 0 \quad (2'),\end{aligned}$$

by analogy we expect $KL_2(A)$ to be presented by

$$\begin{aligned}&\langle ab, c\rangle - \langle a, bc\rangle + \langle ca, b\rangle = 0 \quad (\mathrm{DS1}),\\&\langle a, b\rangle + \langle a', b\rangle - \langle a + a' - aba', b\rangle = 0 \quad (\mathrm{DS2}),\\&\langle a, b\rangle + \langle a, b'\rangle - \langle a, b + b' - bab'\rangle = 0 \quad (\mathrm{DS2}'),\end{aligned}$$

at least when A is a local ring. Translated in terms of the Steinberg symbols $\{-, -\}$ by $\langle a, b\rangle \mapsto \{1 - ab, b\}$ it leads to the following definition.

By definition the *Milnor–Leibniz K-group* $KL_2^M(F)$ of the field F is an Abelian group presented by the generators $\{u, v\}$ for $u, v \in F^\times$ and the relations:

$$\begin{aligned}&\{u_1u_2, v\} = \{u_1, v\}\{u_2, v\} \quad (\text{left linearity}),\\&\{u, v_1v_2\} = \{u, v_1\}\{u, v_2\} \quad (\text{right linearity}),\\&\{u, 1 - u\}\{v, 1 - v\} = \{uv, 1 - uv\} \quad (\text{Steinberg linearity}).\end{aligned}$$

Indeed (DS1) is right multiplicativity, while (DS2) is left multiplicativity in terms of the Steinberg symbols. Then (DS2$'$) is

$$\{1 - ab, b\}\{1 - ab', b'\} = \left\{(1 - ab)(1 - ab'), \frac{1 - (1 - ab)(1 - ab')}{a}\right\}.$$

Putting $u = 1 - ab$ and $v = 1 - a'b$, we get:

$$\left\{u, \frac{1 - u}{a}\right\}\left\{v, \frac{1 - v}{a}\right\} = \left\{uv, \frac{1 - uv}{a}\right\}.$$

Assuming multiplicativity, this relation is equivalent to

$$\{u, 1 - u\}\{u, a\}^{-1}\{v, 1 - v\}\{v, a\}^{-1} = \{uv, 1 - uv\}\{uv, a\}^{-1},$$

which is equivalent to (Steinberg linearity) under (left linearity).

Observe that in the presentation of $K_2^M(F)$ the Steinberg linearity relation is implied by the relation $\{u, 1 - u\} = 1$. It is immediate to give a definition of $KL_n^M(F)$ for any n. A good definition of Leibniz homology of groups would lead to $KL_2(F) := HL_2(SL(F))$ such that $KL_2(F) = KL_2^M(F)$. Also, rationally, we expect $KL_n^M(F)$ to split in $KL_n(F)$.

For cyclic homology, Connes' periodicity exact sequence takes the following form in low dimension:

$$HC_0(A) \to HH_1(A) \to HC_1(A) \to 0.$$

With this definition of $KL_2(F)$ we get a similar exact sequence

$$K_1(F) \to KL_2(F) \to K_2(F) \to 1,$$

where $[x] \in F^\times$ is sent to $\{x, 1 - x\}$ in $KL_2(F)$.

5.7. COQUECIGRUE

In the additive case we mentioned that the Leibniz homology theory of Lie algebras is in fact the homology of a specific type of algebras, namely the Leibniz algebras. So it is reasonable to expect that, similarly, the Leibniz homology of discrete groups is the homology theory of some (still unknown) algebraic objects that we call 'coquecigrues'.

In order to fulfill its role the notion of coquecigrue should have groups as examples. From the above analogy, it is reasonable to expect that a coquecigrue is equipped with a lower central series whose graded associated object is a Leibniz algebra. Moreover, a free coquecigrue should give rise to a free Leibniz algebra (analogue of the Witt theorem which says that the Lie algebra associated to a free group is free).

Hopefully there should be a notion of exact sequence of coquecigrues and even of universal central extension of a perfect coquecigrue. Its kernel should be exactly $HL_2(G)$. The group $E(A)$ should be such a perfect coquecigrue and $HL_2(E(A))$ should be $KL_2(A)$. The perfect coquecigrue $E(A)$ should have a universal central extension, denoted $StL(A)$, with kernel $KL_2(A)$ and such that $KL_3(A) = HL_3(StL(A))$.

So one can think of a coquecigrue as an integration of a Leibniz algebra. One may consult [38] for a proposal in this direction. Let us also mention that precrossed modules of groups give rise to a notion which almost fulfill the requirements about Leibniz algebras, cf. [39].

5.8. DIALGEBRAS

Both homology of a group G and homology of a Lie algebra g can be expressed as the homology of an associative algebra ($\mathbf{Z}[G]$ and $U(g)$, respectively):

$$H_*(G, \mathbf{Z}) \cong H_*(\mathbf{Z}[G], \mathbf{Z}), \qquad H_*(g, \mathbf{Z}) \cong H_*(U(g), \mathbf{Z}).$$

So the category of associative algebras provides a link between groups and Lie algebras:

$$\mathrm{Lie} \;\; \begin{array}{c} \xrightarrow{U(-)} \\ \longleftarrow \end{array} \;\; \mathrm{As} \;\; \begin{array}{c} \xleftarrow{\mathbf{Z}[-]} \\ \longrightarrow \end{array} \;\; \mathrm{Gp}$$

Obviously, for the noncommutative setting the category of Lie algebras Lie is to be replaced by the category of Leibniz algebras Leib and the first step of the program is the following:

— determine what plays the role of As,
— construct the pair of adjoint functors relating this new category to Leib,
— construct the homology theory for these new algebras.

So far we have completed this first step (cf. [1]). It consists of the notion of an 'associative dialgebra' and the associated homology theory. By definition an *associative dialgebra* is a vector space equipped with two binary operations denoted $\dashv$ and $\vdash$ satisfying the five axioms

$$\begin{aligned} &x \dashv (y \dashv z) = (x \dashv y) \dashv z = x \dashv (y \vdash z), \\ &(x \vdash y) \dashv z = x \vdash (y \dashv z), \\ &(x \dashv y) \vdash z = x \vdash (y \vdash z) = (x \vdash y) \vdash z. \end{aligned}$$

It is easy to see that defining $[x, y] := x \dashv y - x \vdash y$ gives rise to a Leibniz structure. Let us denote by Ud: Leib $\to$ Dias the left adjoint functor to the functor $-$: Dias $\to$ Leib. We have the following diagram where the coquecigrues should take the place of (??):

$$\begin{array}{ccccc} \mathrm{Lie} & \begin{array}{c} \xrightarrow{U(-)} \\ \longleftarrow \end{array} & \mathrm{As} & \begin{array}{c} \xleftarrow{\mathbf{Z}[-]} \\ \longrightarrow \end{array} & \mathrm{Gp} \\ \cap & & \cap & & \cap \\ \mathrm{Leib} & \begin{array}{c} \xrightarrow{Ud(-)} \\ \longleftarrow \end{array} & \mathrm{Dias} & \begin{array}{c} \xleftarrow{?} \\ \longrightarrow \end{array} & (??) \end{array}$$

Though this notion of a dialgebra (and its dual counterpart: a dendriform algebra) proved to be a very interesting structure, it does not completely fulfill our requirements because the homology of the universal enveloping dialgebra of a Leibniz algebra is not the homology of the Leibniz algebra. So it does not seem that the dialgebra homology of the group algebra of G is the expected Leibniz homology theory of a group. But there are other candidates for playing the role of dialgebras. They were not experimented yet.

Acknowledgement

Thanks to Herbert Gangl for his fruitful comments and remarks.

References

1. Loday, J.-L.: Dialgebras. In: *Dialgebras and Related Operads*, Lecture Notes in Math. 1763, Springer, New York, 2001, pp. 7–66.
2. Leroux, P.: Co-associativity breaking and oriented graphs, e-print (2002), ArXiv math. QA/0204342.
3. Brouder, C. and Frabetti, A.: Renormalization of QED with trees, *Europ. Phys. J. C* **19** (2001), 715–741.
4. Cathelineau, J.-L.: Infinitesimal polylogarithms, multiplicative presentations of Kähler differentials and Goncharov complexes. Talk at the Workshop on Polylogarithms, Essen 1997. http://www.exp-math.uni-essen.de/˜herbert/polyloquy.html.
5. Quillen, D.: *Cohomology of Groups*, Actes du Congrès Intern. des Mathématiciens (Nice, 1970), Tome 2, Gauthier-Villars, Paris, 1971, pp. 47–51.
6. Loday, J.-L.: K-théorie algébrique et représentations de groupes, *Ann. Sci. École Norm. Sup. (4)* **9**(3) (1976), 309–377.
7. Soulé, C.: Opérations en K-théorie algébrique, *Canad. J. Math.* **37**(3) (1985), 488–550.
8. Borel, A.: Stable real cohomology of arithmetic groups, *Ann. Sci. École Norm. Sup. (4)* **7** (1974), 235–272.
9. Goodwillie, T. G.: On the general linear group and Hochschild homology, *Ann. Math. (2)* **121**(2) (1985), 383–407.
10. Loday, J.-L.: *Cyclic Homology*, 2nd edn, Grundl. Math. Wiss. 301, Springer-Verlag, Berlin, 1998.
11. Milnor, J.: *Introduction to Algebraic K-Theory*, Ann. of Math. Stud. 72, Princeton University Press, Princeton, NJ; University of Tokyo Press, Tokyo, 1971.
12. Matsumoto, H.: Sur les sous-groupes arithmétiques des groupes semi-simples déployés, *Ann. Sci. École Norm. Sup. (4)* **2** (1969), 1–62.
13. Suslin, A. A.: Homology of GL_n, characteristic classes and Milnor K-theory. In: *Algebraic K-Theory, Number Theory, Geometry and Analysis (Bielefeld, 1982)*, Lecture Notes in Math. 1046, Springer, Berlin, 1984, pp. 357–375.
14. Stein, M. R. and Dennis, R. K.: K_2 of Radical ideals and semi-local rings revisited, In: *Algebraic K-theory, II: 'Classical' Algebraic K-theory and Connections with Arithmetic*, Proc. Conf., Battelle Memorial Inst., Seattle, WA, 1972, Lecture Notes in Math. 342, Springer, Berlin, 1973, pp. 281–303.
15. Maazen, H. and Stienstra, J.: A presentation for K_2 of split radical pairs, *J. Pure Appl. Algebra* **10**(3) (1977/1978), 271–294.
16. Bloch, S.: *Higher Regulators, Algebraic K-Theory, and Zeta Functions of Elliptic Curves*, CRM Monogr. Ser. 11, Amer. Math. Soc., Providence, RI, 2000.
17. Suslin, A. A.: K_3 of a field, and the Bloch group. Translated in Proc. Steklov Inst. Math. 1991, No. 4, pp. 217–239. Galois theory, rings, algebraic groups and their applications. *Trudy Mat. Inst. Steklov.* **183** (1990), 180–199, 229.
18. Zagier, D.: Polylogarithms, Dedekind zeta functions and the algebraic K-theory of fields, In: *Arithmetic Algebraic Geometry (Texel, 1989)*, Progr. Math. 89, Birkhäuser, Boston, MA, 1991, pp. 391–430.
19. Goncharov, A. B.: Polylogarithms and motivic Galois groups, In: *Motives* (*Seattle*, *WA*, *1991*), Proc. Sympos. Pure Math. 55, Amer. Math. Soc., Providence, 1994, pp. 43–96.
20. Goncharov, A. B.: Geometry of configurations, polylogarithms and motivic cohomology, *Adv. Math.* **114** (1995), 197–318.
21. Goncharov, A. B.: Geometry of the trilogarithm and the motivic Lie algebra of a field, In: *Regulators in Analysis, Geometry and Number Theory*, Progr. Math. 171, Birkhäuser, Boston, MA, 2000, pp. 127–165.

22. Zhao, J.: Motivic complexes of weight three and pairs of simplices in projective 3-space, *Adv. Math.* **161** (2001), 141–208.
23. Gangl, H.: Some computations in weight 4 motivic complexes. In: *Regulators in Analysis, Geometry and Number Theory*, Progr. Math. 171, Birhäuser, Boston, MA, 2000, pp. 117–125.
24. Feĭgin, B. L. and Tsygan, B. L.: Additive K-theory, In: *K-theory, Arithmetic and Geometry (Moscow, 1984–1986)*, Lecture Notes in Math. 1289, Springer, Berlin, 1987, pp. 67–209.
25. Connes, A.: Noncommutative differential geometry, *Inst. Hautes Études Sci. Publ. Math.* **62** (1985), 257–360.
26. Loday, J.-L. and Quillen, D.: Cyclic homology and the Lie algebra homology of matrices, *Comment. Math. Helv.* **59**(4) (1984), 569–591.
27. Bloch, S.: The dilogarithm and extensions of Lie algebras, In: *Algebraic K-Theory* (*Evanston, 1980*), Proc. Conf., Northwestern Univ., Evanston, IL, 1980, Lecture Notes in Math. 854, Springer, New York, 1981, pp. 1–23.
28. Kassel, C. and Loday, J.-L.: Extensions centrales d'algèbres de Lie, *Ann. Inst. Fourier (Grenoble)* **32**(4) (1982), 119–142.
29. Loday, J.-L. and Procesi, C.: Cyclic homology and lambda operations, In: *Algebraic K-Theory: Connections with Geometry and Topology (Lake Louise, AB, 1987)*, NATO Adv. Sci. Inst. Ser. C Math. Phys. Sci., Vol. 279, Kluwer Acad. Publ., Dordrecht, 1989, pp. 209–224.
30. Beĭlinson, A., MacPherson, R. and Schechtman, V.: Notes on motivic cohomology, *Duke Math. J.* **54** (1987), 679–710.
31. Loday, J.-L.: Comparaison des homologies du groupe linéaire et de son algèbre de Lie, *Ann. Inst. Fourier (Grenoble)* **37**(4) (1987), 167–190.
32. Cathelineau, J.-L.: λ-structures in algebraic K-theory and cyclic homology, *K-Theory* **4**(6) (1990/1991), 591–606.
33. Loday, J.-L.: Une version non commutative des algèbres de Lie: les algèbres de Leibniz, *Enseign. Math. (2)* **39**(3–4) (1993), 269–293.
34. Cuvier, C.: Algèbres de Leibniz: définitions, propriétés, *Ann. Sci. École Norm. Sup. (4)* **27**(1) (1994), 1–45.
35. Gnedbaye, A. V.: Third homology groups of universal central extensions of a Lie algebra, *Afrika Mat. (3)* **10** (1999), 46–63.
36. Gnedbaye, A. V.: A non-abelian tensor product of Leibniz algebras, *Ann. Inst. Fourier (Grenoble)* **49** (1999), 1149–1177.
37. Loday, J.-L.: Cup-product for Leibniz cohomology and dual Leibniz algebras, *Math. Scand.* **77** (1995), 189–196.
38. Kinyon, M. K. and Weinstein, A.: Leibniz algebras, Courant algebroids, and multiplications on reductive homogeneous spaces, *Amer. J. Math.* **123**(3) (2001), 525–550.
39. Baues, H. J. and Conduché, D.: The central series for Peiffer commutators in groups with operators, *J. Algebra* **133**(1) (1990), 1–34.