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Quadri-algebras

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Abstract

We introduce the notion of quadri-algebras. These are associative algebras for which the multiplication can be decomposed as the sum of four operations in a certain coherent manner. We present several examples of quadri-algebras: the algebra of permutations, the shuffle algebra, tensor products of dendriform algebras. We show that a pair of commuting Baxter operators on an associative algebra gives rise to a canonical quadri-algebra structure on the underlying space of the algebra. The main example is provided by the algebra $\mathsf{End}(A)$ of linear endomorphisms of an infinitesimal bialgebra A. This algebra carries a canonical pair of commuting Baxter operators: $\beta(T)=T*\mathrm{id}$ and $\gamma(T)=\mathrm{id}*T$, where $*$ denotes the convolution of endomorphisms. It follows that $\mathsf{End}(A)$ is a quadri-algebra, whenever A is an infinitesimal bialgebra. We also discuss commutative quadri-algebras and state some conjectures on the free quadri-algebra.

MSC: 17A30; 18D50

0. Introduction

The study of the space of endomorphisms of an infinitesimal bialgebra revealed the existence of peculiar algebraic structures. More specifically, the convolution of endomorphisms gives rise to a pair of commuting Baxter operators

$$\beta(T)=T*\mathrm{id} \quad \text{and} \quad \gamma(T)=\mathrm{id}*T$$

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and each of these determines a dendriform structure on the space of endomorphisms. Obviously, these two structures are somehow intertwined, but under which rule?

In this work, we answer this question. We introduce the notion of "quadri-algebras". A quadri-algebra is an associative algebra whose multiplication is the sum of four operations. These operations satisfy nine relations. We show that the space of endomorphisms of an infinitesimal bialgebra has a natural structure of quadri-algebra which encompasses the two dendriform structures.

In Section 1, we recall the definition of dendriform algebras and we introduce quadri-algebras. We also provide a number of examples of quadri-algebras and constructions relating quadri-algebras to dendriform algebras. We also discuss "commutative" quadri-algebras.

In Section 2, we recall the definition of Baxter operators and we show that a pair of commuting Baxter operators gives rise to a quadri-algebra.

In Section 3, we apply the preceding result to show that the space of endomorphisms of an infinitesimal bialgebra may be naturally endowed with a quadri-algebra structure.

In Section 4, we conclude with some conjectures related to the free quadri-algebra on one generator. In particular, we conjecture that the dimension of its homogeneous component of degree n is equal to the number of non-crossing connected graphs on $n+1$ vertices.

1. Quadri-algebras

1.1. Dendriform algebras. A *dendriform algebra* is a vector space D together with two operations $\prec\ : D\otimes D\to D$ and $\succ\ : D\otimes D\to D$, called *left* and *right*, respectively, such that

$$\begin{aligned}(x\prec y)\prec z &= x\prec (y\prec z)+x\prec (y\succ z),\\ (x\succ y)\prec z &= x\succ (y\prec z),\\ (x\prec y)\succ z+(x\succ y)\succ z &= x\succ (y\succ z).\end{aligned} \tag{1}$$

Dendriform algebras were introduced by the second author [13, Chapter 5]. See [18,14,15,11,4] for additional work on this subject.

Defining a new operation by

$$x\star y := x\prec y + x\succ y \tag{2}$$

permits us to rewrite axioms (1) as

$$\begin{aligned}(x\prec y)\prec z &= x\prec (y\star z),\\ (x\succ y)\prec z &= x\succ (y\prec z),\\ (x\star y)\succ z &= x\succ (y\succ z).\end{aligned}$$

By adding the three relations we see that the operation $\star$ is associative. For this reason, a dendriform algebra may be regarded as an associative algebra $(D,\star)$ for which the multiplication $\star$ can be decomposed as the sum of two coherent operations.

We now introduce a class of algebras with an associative multiplication which can be coherently decomposed as the sum of four operations. As a consequence, these algebras carry two distinct dendriform structures.

1.2. Quadri-algebras. Definition. A *quadri-algebra* is a vector space Q together with four operations $\searrow$, $\nearrow$, $\nwarrow$ and $\swarrow$ $: Q \otimes Q \to Q$ satisfying the axioms below (8). In order to state them, consider the following operations:

$$x \succ y := x \nearrow y + x \searrow y, \tag{3}$$

$$x \prec y := x \nwarrow y + x \swarrow y, \tag{4}$$

$$x \vee y := x \searrow y + x \swarrow y, \tag{5}$$

$$x \wedge y := x \nearrow y + x \nwarrow y \tag{6}$$

and

$$\begin{aligned} x \star y &:= x \searrow y + x \nearrow y + x \nwarrow y + x \swarrow y \\ &= x \succ y + x \prec y = x \vee y + x \wedge y. \end{aligned} \tag{7}$$

The axioms are

$$\begin{array}{lll} (x \nwarrow y) \nwarrow z = x \nwarrow (y \star z) & (x \nearrow y) \nwarrow z = x \nearrow (y \prec z) & (x \wedge y) \nearrow z = x \nearrow (y \succ z) \\ (x \swarrow y) \nwarrow z = x \swarrow (y \wedge z) & (x \searrow y) \nwarrow z = x \searrow (y \nwarrow z) & (x \vee y) \nearrow z = x \searrow (y \nearrow z) \\ (x \prec y) \swarrow z = x \swarrow (y \vee z) & (x \succ y) \swarrow z = x \searrow (y \swarrow z) & (x \star y) \searrow z = x \searrow (y \searrow z) \end{array} \tag{8}$$

We refer to the operations $\searrow$, $\nearrow$, $\nwarrow$, $\swarrow$, as *southeast*, *northeast*, *northwest*, and *southwest*, respectively. Accordingly, we use *north*, *south*, *west*, and *east* for $\wedge$, $\vee$, $\prec$ and $\succ$.

The axioms are displayed in the form of a 3×3 matrix. We will make use of standard matrix terminology (entries, rows and columns) to refer to them.

1.3. From quadri-algebras to dendriform algebras. The three column sums of the matrix of axioms (8) yield

$$(x \prec y) \prec z = x \prec (y \star z), \quad (x \succ y) \prec z = x \succ (y \prec z)$$

and

$$(x \star y) \succ z = x \succ (y \succ z).$$

Thus, endowed with the operations west for left and east for right, Q is a dendriform algebra. We denote it by Q_{h} and call it the *horizontal* dendriform algebra associated to Q.

Consider instead the three row sums in (8). We obtain

$$(x \wedge y) \wedge z = x \wedge (y \star z), \quad (x \vee y) \wedge z = x \vee (y \wedge z)$$

and

$$(x \star y) \vee z = x \vee (y \vee z).$$

Thus, endowed with the operations north for left and south for right, Q is a dendriform algebra. We denote it by Q_{v} and call it the *vertical* dendriform algebra associated to Q.

The associative operations corresponding to the dendriform algebras Q_{h} and Q_{v} by means of (2) coincide, according to (7). Thus, a quadri-algebra may be seen as an associative algebra $(Q, \star)$ for which the multiplication $\star$ can be decomposed into four coherent operations.

1.4. Example. The algebra of permutations. Let S_n denote the symmetric group on n letters and denote by $\Bbbk S_n$ the vector space spanned by its elements (over a field $\Bbbk$). Consider the spaces

$$\Bbbk S_\infty := \bigoplus_{n \geqslant 0} \Bbbk S_n \quad \text{and} \quad \Bbbk S_{\geqslant 2} := \bigoplus_{n \geqslant 2} \Bbbk S_n.$$

Let $\mathsf{Sh}(p,q)$ denote the set of (p,q)-shuffles, that is, those permutations $\zeta \in S_{p+q}$ such that

$$\zeta(1) < \cdots < \zeta(p) \quad \text{and} \quad \zeta(p+1) < \cdots < \zeta(p+q).$$

Note that any such shuffle satisfies

$$\zeta^{-1}(1) = 1 \text{ or } p+1 \quad \text{and} \quad \zeta^{-1}(p+q) = p \text{ or } p+q.$$

Therefore, if both p and q are at least 2, the set of (p,q)-shuffles decomposes into the following four disjoint subsets:

$$\begin{aligned}
\mathsf{Sh}^1(p,q) &= \{\zeta \in \mathsf{Sh}(p,q) \mid \zeta^{-1}(1) = 1 \text{ and } \zeta^{-1}(p+q) = p+q\},\\
\mathsf{Sh}^2(p,q) &= \{\zeta \in \mathsf{Sh}(p,q) \mid \zeta^{-1}(1) = p+1 \text{ and } \zeta^{-1}(p+q) = p+q\},\\
\mathsf{Sh}^3(p,q) &= \{\zeta \in \mathsf{Sh}(p,q) \mid \zeta^{-1}(1) = 1 \text{ and } \zeta^{-1}(p+q) = p\},\\
\mathsf{Sh}^4(p,q) &= \{\zeta \in \mathsf{Sh}(p,q) \mid \zeta^{-1}(1) = p+1 \text{ and } \zeta^{-1}(p+q) = p\}.
\end{aligned}$$

For $\sigma \in S_p$ and $\tau \in S_q$, let $\sigma \times \tau$ denote the permutation in S_{p+q} defined by

$$i \mapsto \begin{cases} \sigma(i) & \text{if } i \leqslant p, \\ p + \tau(i-p) & \text{if } i > p. \end{cases}$$

The composition of permutations $\zeta, \xi \in S_n$ is $(\zeta \cdot \xi)(i) = \zeta(\xi(i))$. Let σ and τ be as above, with $p, q \geqslant 2$. Define operations on $\Bbbk S_{\geqslant 2}$ by

$$\sigma \swarrow \tau = \sum_{\zeta \in \mathsf{Sh}^1(p,q)} \zeta \cdot (\sigma \times \tau),$$

$$\sigma \searrow \tau = \sum_{\zeta \in \mathsf{Sh}^2(p,q)} \zeta \cdot (\sigma \times \tau),$$

$$\sigma \nwarrow \tau = \sum_{\zeta \in \mathsf{Sh}^3(p,q)} \zeta \cdot (\sigma \times \tau),$$

$$\sigma \nearrow \tau = \sum_{\zeta \in \mathsf{Sh}^4(p,q)} \zeta \cdot (\sigma \times \tau).$$

This endows $\Bbbk S_{\geqslant 2}$ with a quadri-algebra structure. Axioms (8) are easily verified with the aid of the standard bijection

$$\mathsf{Sh}(p+q,r) \times \mathsf{Sh}(p,q) \cong \mathsf{Sh}(p,q+r) \times \mathsf{Sh}(q,r).$$

The vertical dendriform structure corresponding to this quadri-algebra is the dendriform algebra of permutations introduced in [13, Section 5.4] and further studied in [15] (this structure is in fact defined on $\Bbbk S_{\geqslant 1}$).

The associative structure is

$$\sigma \star \tau = \sum_{\zeta \in \mathsf{Sh}(p,q)} \zeta \cdot (\sigma \times \tau).$$

This is the multiplication of Malvenuto and Reutenauer [17, pp. 977–978] (which is in fact defined on the whole $\Bbbk S_\infty$).

1.5. Tensor product of dendriform algebras. The tensor product of two dendriform algebras carries a natural quadri-algebra structure.

Let A and B be two dendriform algebras. On the space $A \otimes B$, define operations

$$(a_1 \otimes b_1) \nwarrow (a_2 \otimes b_2) = (a_1 \prec a_2) \otimes (b_1 \prec b_2),$$

$$(a_1 \otimes b_1) \swarrow (a_2 \otimes b_2) = (a_1 \prec a_2) \otimes (b_1 \succ b_2),$$

$$(a_1 \otimes b_1) \nearrow (a_2 \otimes b_2) = (a_1 \succ a_2) \otimes (b_1 \prec b_2),$$

$$(a_1 \otimes b_1) \searrow (a_2 \otimes b_2) = (a_1 \succ a_2) \otimes (b_1 \succ b_2).$$

Axioms (8) follow from axioms (1) for A and B.

The corresponding horizontal dendriform structure is

$$(a_1 \otimes b_1) \prec (a_2 \otimes b_2) = (a_1 \prec a_2) \otimes (b_1 \star b_2),$$

$$(a_1 \otimes b_1) \succ (a_2 \otimes b_2) = (a_1 \succ a_2) \otimes (b_1 \star b_2).$$

This is the structure used in the definition of dendriform bialgebras [18].

The corresponding associative structure is simply the tensor product of the associative structures on A and B:

$$(a_1 \otimes b_1) \star (a_2 \otimes b_2) = (a_1 \star a_2) \otimes (b_1 \star b_2).$$

1.6. Opposite and transpose of a quadri-algebra. The matrix of axioms (8) is symmetric with respect to the main diagonal, provided we interchange $\swarrow$ with $\nearrow$, and leave the other two operations unchanged (note this has the effect of interchanging $\wedge$ with $\prec$ and $\vee$ with $\succ$). Therefore, starting from a quadri-algebra $(Q, \searrow, \nearrow, \nwarrow, \swarrow)$ and defining

$$x \searrow^{t} y = x \searrow y, \quad x \nearrow^{t} y = x \swarrow y, \quad x \nwarrow^{t} y = x \nwarrow y$$

and

$$x \swarrow^{t} y = x \nearrow y,$$

one obtains a new quadri-algebra structure on the underlying space of Q. We refer to this new quadri-algebra as the *transpose* of Q and denote it by Q^{t}. The dendriform structures corresponding to Q (as in 1.3) are the same as those for Q^{t}, in the opposite order: $Q_{h} = Q^{t}_{v}$ and $Q_{v} = Q^{t}_{h}$. In fact,

$$x \vee^{t} y = x \searrow^{t} y + x \swarrow^{t} y = x \searrow y + x \nearrow y = x \succ y,$$

$$x \wedge^{t} y = x \nearrow^{t} y + x \nwarrow^{t} y = x \swarrow y + x \nwarrow y = x \prec y,$$

$$x \succ^{t} y = x \nearrow^{t} y + x \searrow^{t} y = x \swarrow y + x \searrow y = x \vee y,$$

$$x \prec^{t} y = x \nwarrow^{t} y + x \swarrow^{t} y = x \nwarrow y + x \nearrow y = x \wedge y.$$

It follows that the associative structures corresponding to Q and Q^{t} coincide.

The matrix of axioms exhibits another symmetry, with respect to the center (the entry $(2,2)$), provided we replace each arrow by its opposite ($\nwarrow$ by $\searrow$ and $\swarrow$ by $\nearrow$) and reverse the order of the variables. Therefore, starting from a quadri-algebra $(Q, \searrow, \nearrow, \nwarrow, \swarrow)$ and defining

$$x \searrow^{op} y = y \nwarrow x, \quad x \nearrow^{op} y = y \swarrow x, \quad x \nwarrow^{op} y = y \searrow x$$

and

$$x \swarrow^{op} y = y \nearrow x,$$

one obtains a new quadri-algebra structure on the underlying space of Q. We refer to this new quadri-algebra as the *opposite* of Q and denote it by Q^{op}.

1.7. Commutative quadri-algebras. A quadri-algebra Q is said to be *commutative* if it coincides with its opposite: $Q = Q^{op}$. Explicitly, this means that

$$x \searrow y = y \nwarrow x \quad \text{and} \quad x \nearrow y = y \swarrow x.$$

One may then restate axioms (8) in terms of the operations $\searrow$ and $\swarrow$ only. One obtains that a commutative quadri-algebra is a space Q equipped with two operations $\searrow$ and $\swarrow$ $: Q \otimes Q \to Q$ such that

$$x \searrow (y \searrow z) = (x \star y) \searrow z,$$

$$x \searrow (y \swarrow z) = (x \succ y) \swarrow z,$$

$$x \searrow (y \swarrow z) = y \swarrow (x \vee z), \tag{9}$$

where we have set, in agreement with the notation for general quadri-algebras,

$$x \succ y := x \searrow y + y \swarrow x,$$

$$x \vee y := x \searrow y + x \swarrow y$$

and

$$x \star y := x \searrow y + x \swarrow y + y \searrow x + y \swarrow x$$
$$= x \succ y + y \succ x = x \vee y + y \vee x.$$

It follows from (9) that the operation $\star$ is associative and commutative, and that each of the operations $\succ$ and $\vee$ define a left Zinbiel structure on A (see [13, Section 7] for the definition of Zinbiel algebras).

1.8. Example. The shuffle algebra. The shuffle algebra of a vector space V provides an example of a commutative quadri-algebra.

Consider the vector spaces

$$T(V) := \bigoplus_{n\geqslant 0} V^{\otimes n} \quad \text{and} \quad T^{\geqslant 2}(V) := \bigoplus_{n\geqslant 2} V^{\otimes n}.$$

We adopt the concatenation notation for elements in $V^{\otimes n}$. In the sequel $a,b,c,d \in V$ and $\omega, \theta \in V^{\otimes n}$ for some $n \geqslant 0$. On the space $T^{\geqslant 2}(V)$ define operations as follows:

$$a\omega b \swarrow c\theta d = a(\omega b \star c\theta)d,$$

$$a\omega b \searrow c\theta d = c(a\omega b \star \theta)d,$$

$$a\omega b \nwarrow c\theta d = a(\omega \star c\theta d)b,$$

$$a\omega b \nearrow c\theta d = c(a\omega \star \theta d)b,$$

where, as before, $x \star y := x \searrow y + x \nearrow y + x \nwarrow y + x \swarrow y$. In low dimensions, we start with $a \star b = ab + ba$ and 1 is a unit for $\star$. It follows that $\star$ is the shuffle product (which is defined on the whole space $T(V)$) and

$$a\omega b \prec c\theta d = a(\omega b \star c\theta d),$$

$$a\omega b \succ c\theta d = c(a\omega b \star \theta d),$$

$$a\omega b \vee c\theta d = (a\omega \star c\theta d)b,$$

$$a\omega b \wedge c\theta d = (a\omega b \star c\theta)d.$$

Axioms (8) can now be easily verified. For instance,

$$(a\omega b \nwarrow c\theta d) \nwarrow e\xi f = a(\omega \star c\theta d)b \nwarrow e\xi f = a(\omega \star c\theta d \star e\xi f)b,$$

while

$$a\omega b \nwarrow (c\theta d \star e\xi f) = a\omega b \nwarrow (c\theta d \star e\xi f) = a(\omega \star c\theta d \star e\xi f)b.$$

Thus, the axiom in entry $(1,1)$ holds by associativity of the shuffle product. The other axioms can be verified similarly. In fact the nine monomials obtained by formulas (8) begin with a (resp. c, resp. e) in the first (resp. second, resp. third) column and end with b (resp. d, resp. f) in the first (resp. second, resp. third) row.

This is actually an example of a commutative quadri-algebra. In fact,

$$a\omega b \searrow c\theta d = c(a\omega b \star \theta)d = c(\theta \star a\omega b)d = c\theta d \nwarrow a\omega b,$$

by the commutativity of the shuffle product. Similarly, $a\omega b \nearrow c\theta d = c\theta d \swarrow a\omega b$.

Observe that the dendriform structures are defined on $T^{\geqslant 1}(V)$ and the associative structure is defined on $T(V)$ (the shuffle algebra).

1.9. A generalization of quadri-algebras. There is a generalization of dendriform algebras called *dendriform trialgebras*; they carry three operations satisfying seven relations [16]. The free dendriform trialgebra can be explicitly described by means of planar rooted trees (not necessarily binary). It is natural to expect that the notion of quadri-algebra has a similar generalization involving $3^2=9$ operations satisfying $7^2=49$ relations. This has been found recently by Leroux [12].

2. Baxter operators

2.1. Baxter operators on associative algebras. Let $(A,\cdot)$ be an associative algebra. A *Baxter operator* is a map $\beta : A \to A$ such that

$$\beta(x)\cdot\beta(y) = \beta(x\cdot\beta(y) + \beta(x)\cdot y). \tag{10}$$

This identity appeared originally in the work of Baxter [5]. The importance of Baxter operators was emphasized by Rota [19,20].

Given such A and β, one may define new operations on A by

$$x \prec_\beta y = x\cdot\beta(y) \quad \text{and} \quad x \succ_\beta y = \beta(x)\cdot y.$$

It is easy to see that then $(A, \prec_\beta, \succ_\beta)$ is a dendriform algebra [4, Proposition 4.5]. The resulting associative structure $\star_\beta$ on A is related to the original one as follows: $\beta : (A, \star_\beta) \to (A,\cdot)$ is a morphism of associative algebras.

Similarly, starting from a Baxter operator on a dendriform algebra, one can construct a quadri-algebra structure on the same space, as we explain next.

2.2. Baxter operators on dendriform algebras. Let $(D,\prec,\succ)$ be a dendriform algebra. A Baxter operator is a map $\gamma : D \to D$ such that

$$\gamma(x) \succ \gamma(y) = \gamma(x \succ \gamma(y) + \gamma(x) \succ y), \tag{11}$$

$$\gamma(x) \prec \gamma(y) = \gamma(x \prec \gamma(y) + \gamma(x) \prec y). \tag{12}$$

Adding these equations we see that γ is also a Baxter operator on the associative algebra $(D, \star)$:

$$\gamma(x) \star \gamma(y) = \gamma(x \star \gamma(y) + \gamma(x) \star y). \tag{13}$$

2.3. Proposition. *Let* $(D, \prec, \succ)$ *be a dendriform algebra and* $\gamma : D \to D$ *a Baxter operator. Define new operations on A by*

$$x \searrow_\gamma y = \gamma(x) \succ y, \quad x \nearrow_\gamma y = x \succ \gamma(y), \quad x \swarrow_\gamma y = \gamma(x) \prec y$$

and

$$x \nwarrow_\gamma y = x \prec \gamma(y).$$

Then $(D, \searrow_\gamma, \nearrow_\gamma, \nwarrow_\gamma, \swarrow_\gamma)$ *is a quadri-algebra.*

Proof. We verify the axioms corresponding to the entries $(1,1)$ and $(2,2)$ in (8); the others are similar. We have

$$(x \nwarrow_\gamma y) \nwarrow_\gamma z = (x \prec \gamma(y)) \prec \gamma(z) \overset{(1)}{=} x \prec (\gamma(y) \star \gamma(z))$$
$$\overset{(13)}{=} x \prec \gamma(x \star \gamma(y) + \gamma(x) \star y) = x \prec \gamma(y \star_\gamma z) = x \nwarrow_\gamma (y \star_\gamma z).$$

Also,

$$(x \searrow_\gamma y) \nwarrow_\gamma z = (\gamma(x) \succ y) \prec \gamma(z)$$
$$\overset{(1)}{=} \gamma(x) \succ (y \prec \gamma(z)) = x \searrow_\gamma (y \nwarrow_\gamma z),$$

as needed. □

Let Q denote the resulting quadri-algebra. The horizontal dendriform structure associated to Q is

$$x \prec_\gamma y = x \nwarrow_\gamma y + x \swarrow_\gamma y = x \prec \gamma(y) + \gamma(x) \prec y,$$

$$x \succ_\gamma y = x \searrow_\gamma y + x \nearrow_\gamma y = \gamma(x) \succ y + x \succ \gamma(y).$$

Therefore, axioms (11) and (12) can be rewritten as follows:

$$\gamma(x) \succ \gamma(y) = \gamma(x \succ_\gamma y) \quad \text{and} \quad \gamma(x) \prec \gamma(y) = \gamma(x \prec_\gamma y).$$

Thus, γ is a morphism of dendriform algebras $Q_\mathrm{h} \to D$.

On the other hand, the vertical dendriform structure associated to Q is

$$x \wedge_\gamma y = x \nearrow_\gamma y + x \nwarrow_\gamma y = x \succ \gamma(y) + x \prec \gamma(y) = x \star \gamma(y),$$

$$x \vee_\gamma y = x \searrow_\gamma y + x \swarrow_\gamma y = \gamma(x) \succ y + \gamma(x) \prec y = \gamma(x) \star y.$$

Thus, Q_v is the dendriform structure corresponding to γ viewed as a Baxter operator on the associative algebra $(D, \star)$.

From either of the two previous remarks it follows that γ is a morphism of associative algebras $(Q, \star_\gamma) \to (D, \star)$ (a fact that was implicitly used in the proof of 2.3).

2.4. Pairs of commuting Baxter operators. Let A be an associative algebra and β and γ two Baxter operators on A that commute, that is,

$$\beta\gamma = \gamma\beta.$$

In this situation, it is possible to construct a quadri-algebra structure on the underlying space of A, by successively applying the constructions of 2.1 and 2.3, as we explain next.

2.5. Proposition. *Let β and γ be a pair of commuting Baxter operators on an associative algebra A. Then γ is a Baxter operator on the dendriform algebra $(A, \prec_\beta, \succ_\beta)$ corresponding to β as in* 2.1.

Proof. We verify axiom (11); axiom (12) is similar.

$$\begin{aligned}\gamma(x) \succ_\beta \gamma(y) &= \beta(\gamma(x)) \cdot \gamma(y) = \gamma(\beta(x)) \cdot \gamma(y)\\ &\overset{(10)}{=} \gamma(\beta(x) \cdot \gamma(y) + \gamma(\beta(x)) \cdot y) = \gamma(\beta(x) \cdot \gamma(y) + \beta(\gamma(x)) \cdot y)\\ &= \gamma(x \succ_\beta \gamma(y) + \gamma(x) \succ_\beta y). \qquad \square\end{aligned}$$

2.6. Corollary. *Let β and γ be a pair of commuting Baxter operators on an associative algebra A. Then there is a quadri-algebra structure on the underlying space of A, with operations defined by*

$$x \searrow y = \beta(\gamma(x))y = \gamma(\beta(x))y,$$

$$x \nearrow y = \beta(x)\gamma(y),$$

$$x \swarrow y = \gamma(x)\beta(y),$$

$$x \nwarrow y = x\beta(\gamma(y)) = x\gamma(\beta(y)).$$

Proof. Apply Proposition 2.3 to the Baxter operator γ on the dendriform algebra $(A, \prec_\beta, \succ_\beta)$. $\square$

One may start by first constructing the dendriform algebra $(A, \prec_\gamma, \succ_\gamma)$ instead. By Proposition 2.5, β is a Baxter operator on this dendriform algebra. Hence, Proposition 2.3 yields a new quadri-algebra structure on the underlying space of A. It is easy to see that this is the transpose of the structure of Corollary 2.6, in the sense of 1.6.

Let $(A, \star)$ denote the associative structure corresponding to the quadri-algebra structure on A (according to 1.6, this is the same for one structure and its transpose). It follows from the remarks in 2.2 that both γ and β are morphisms of associative algebras $(A, \star) \to (A, \cdot)$. In this sense, the pair of commuting Baxter operators β and γ breaks the associativity of A into four pieces.

2.7. Example. Let A be an associative algebra and let $r = \sum_i u_i \otimes v_i \in A \otimes A$ be a solution of the associative Yang–Baxter equation [1,3]. The map $\beta_r : A \to A$ given by

$$\beta_r(x) := \sum_i u_i x v_i$$

is a Baxter operator on A [2, Section 5]. Suppose that $s \in A \otimes A$ is another solution of the associative Yang–Baxter equation. If r and s commute as elements of $A \otimes A^{\mathrm{op}}$ then the operators β_r and β_s commute and Corollary 2.6 applies.

3. Quadri-algebras from infinitesimal bialgebras

3.1. Infinitesimal bialgebras. An *infinitesimal bialgebra* (abbreviated ε-bialgebra) is a triple (A, μ, Δ) where (A, μ) is an associative algebra, (A, Δ) is a coassociative coalgebra, and for each $a, b \in A$,

$$\Delta(ab) = ab_1 \otimes b_2 + a_1 \otimes a_2 b. \tag{14}$$

We write $\Delta(a) = a_1 \otimes a_2$ (simplified Sweedler's notation) and $(\Delta \otimes \mathrm{id})\Delta(a) = a_1 \otimes a_2 \otimes a_3$. Infinitesimal bialgebras originated in the work of Joni and Rota [10]. See [1,3,4] for the basic theory of ε-bialgebras.

We view the space $\mathsf{End}(A)$ of linear endomorphisms of A as an associative algebra under composition, denoted simply by concatenation:

$$TS : A \xrightarrow{S} A \xrightarrow{T} A.$$

We make use of a second associative product on $\mathsf{End}(A)$, the convolution of endomorphisms, defined by

$$T * S : A \xrightarrow{\Delta} A \otimes A \xrightarrow{T \otimes S} A \otimes A \xrightarrow{\mu} A.$$

Note that the associativity of this product does not depend on any compatibility condition between Δ and μ.

3.2. Proposition. *Let A be an ε-bialgebra and consider $\mathsf{End}(A)$ as an associative algebra under composition. There is a pair of commuting Baxter operators β and γ on $\mathsf{End}(A)$, defined by*

$$\beta(T) = \mathrm{id} * T \quad \text{and} \quad \gamma(T) = T * \mathrm{id}.$$

Proof. The operators commute because the convolution product is associative. Let us check axiom (10) for the operator β; the verification for γ is similar. We have $\beta(S)(a) = a_1 S(a_2)$. Hence, by (14),

$$\Delta(\beta(S)(a)) = a_1 S(a_2)_1 \otimes S(a_2)_2 + a_1 \otimes a_2 S(a_3).$$

Therefore,

$$\begin{aligned}\beta(T)\beta(S)(a) &= a_1 S(a_2)_1 T(S(a_2)_2) + a_1 T(a_2 S(a_3)) \\ &= a_1(\beta(T)S)(a_2) + a_1(T\beta(S))(a_2) \\ &= \beta(\beta(T)S)(a) + \beta(T\beta(S))(a),\end{aligned}$$

as needed. □

3.3. Corollary. *Let A be an ε-bialgebra. There is a quadri-algebra structure on the space* $\mathrm{End}(A)$ *defined by*

$$T \searrow S = (\mathrm{id} * T * \mathrm{id})S,$$

$$T \nearrow S = (\mathrm{id} * T)(S * \mathrm{id}),$$

$$T \swarrow S = (T * \mathrm{id})(\mathrm{id} * S),$$

$$T \nwarrow S = T(\mathrm{id} * S * \mathrm{id}).$$

Proof. Apply Corollary 2.6 to the pair of commuting Baxter operators of Proposition 3.2. □

The horizontal dendriform structure associated to this quadri-algebra structure is

$$T \prec S = (T * \mathrm{id})(\mathrm{id} * S) + T(\mathrm{id} * S * \mathrm{id}),$$

$$T \succ S = (\mathrm{id} * T * \mathrm{id})S + (\mathrm{id} * T)(S * \mathrm{id}).$$

This structure was found in [4, Corollary 4.14] by other means. The present work reveals that it is in fact the structure associated to a (more fundamental) quadri-algebra structure.

The Baxter operators β and γ also give rise to dendriform structures on $\mathrm{End}(A)$; these are obtained by the construction in 2.1. The dendriform structures are, respectively,

$$T \prec S = T(\mathrm{id} * S), \quad T \succ S = (\mathrm{id} * T)S$$

and

$$T \prec S = T(S * \mathrm{id}), \quad T \succ S = (T * \mathrm{id})S.$$

The existence of these structures was announced in [4, Remark 4.15]. □

4. The free quadri-algebra on one generator

4.1. Free quadri-algebras. Let V be a vector space. The *free quadri-algebra* $\mathcal{Q}(V)$ on V is a quadri-algebra equipped with a map $i : V \to \mathcal{Q}(V)$ which satisfies the following universal property: for any linear map $f : V \to A$ where A is a quadri-algebra, there is a unique quadri-algebra morphism $\bar{f} : \mathcal{Q}(V) \to A$ such that $\bar{f} \circ i = f$. In other words, the functor $\mathcal{Q}$ from vector spaces to quadri-algebras is left adjoint to the forgetful functor.

The operad of quadri-algebras is a *non-Σ-operad*; in other words, the four operations of a quadri-algebra have no symmetry, and the relations (8) involve only monomials where x, y and z stay in the same order. For this reason, the free quadri-algebra $\mathcal{Q}(V)$ is of the form

$$\mathcal{Q}(V) = \bigoplus_{n \geqslant 1} \mathcal{Q}_n \otimes V^{\otimes n}.$$

Hence, $\mathcal{Q}(V)$ is completely determined by the free quadri-algebra on one generator $\mathcal{Q}(\Bbbk) = \bigoplus_{n \geqslant 1} \mathcal{Q}_n$. Let x be this generator. Then x is a linear generator of $\mathcal{Q}_1$ and the elements $x \searrow x$, $x \nearrow x$, $x \nwarrow x$, $x \swarrow x$ form a basis of $\mathcal{Q}_2$. The space of four operations on three variables (with no relations) is of dimension $2 \times 4^2 = 32$. Since for quadri-algebras we have nine linearly independent relations, the space $\mathcal{Q}_3$ is of dimension $32 - 9 = 23$.

4.2. Conjecture. *The dimension of the vector space $\mathcal{Q}_n$ is*

$$d_n := \frac{1}{n} \sum_{j=n}^{2n-1} \binom{3n}{n+1+j} \binom{j-1}{j-n}. \tag{15}$$

The first elements of this sequence are

$1, 4, 23, 156, 1162, 9192, \ldots$.

According to [7, Theorem 2], d_n is the number of *non-crossing connected graphs* on $n+1$ vertices.

Let us give some evidence in favor of this conjecture.

4.3. Dual quadri-algebras. The operad $\mathcal{Q}$ of quadri-algebras is binary (since it is generated by binary operations) and is quadratic (since the relations involve only monomials with two operations). Hence, it has a dual operad $\mathcal{Q}^!$ (see [9] for the seminal paper on the subject, or [13, Appendix B] for a short résumé, or [8] for a recent comprehensive treatment).

Let us still denote the four dual operations by $\searrow$, $\nearrow$, $\nwarrow$, $\swarrow$; these are linear generators of $\mathcal{Q}_2^!$. The number of relations defining the binary quadratic operad $\mathcal{Q}^!$ is 23 and so the dimension of $\mathcal{Q}_3^!$ is $32 - 23 = 9$. Let us pick the following elements as linear generators of $\mathcal{Q}_3^!$:

$$\begin{aligned}
&(x \nwarrow y) \nwarrow z, \quad (x \nearrow y) \nwarrow z, \quad x \nearrow (y \nearrow z),\\
&(x \swarrow y) \nwarrow z, \quad (x \searrow y) \nwarrow z, \quad x \searrow (y \nearrow z),\\
&(x \swarrow y) \swarrow z, \quad x \searrow (y \swarrow z), \quad x \searrow (y \searrow z).
\end{aligned} \tag{16}$$

From the 23 relations we deduce that any other monomial of degree 3 can be written as an algebraic sum of these nine elements.

4.4. Proposition. *The dimension of $\mathcal{Q}_n^!$ is less than or equal to n^2.*

Proof. Let y be a monomial of degree n. It is determined by a planar binary tree (or parenthesizing) where each node is labelled by one of the four generating operations. If, locally, a pattern corresponding to one of the 23 discarded monomials of degree 3 appears, then we know that this monomial can be rewritten in terms of other elements. Therefore, to generate $\mathcal{Q}_n^!$ linearly, it suffices to take the s_n monomials where only the nine local patterns mentioned above appear.

Let u_n, v_n, w_n, t_n be the number of these monomials whose lowest node is labelled, respectively, by $\nwarrow$, $\swarrow$, $\nearrow$ and $\searrow$. We have $s_n = u_n + v_n + w_n + t_n$. From the choice of the nine patterns it follows:

$$u_{n+1} = u_n + v_n + w_n + t_n,$$

$$v_{n+1} = v_n,$$

$$w_{n+1} = w_n,$$

$$t_{n+1} = v_n + w_n + t_n.$$

From these inductive relations (and $u_2 = v_2 = w_2 = t_2 = 1$) it follows immediately that $s_n = n^2$. □

4.5. Conjecture. $\dim \mathcal{Q}_n^! = n^2$.

4.6. Koszul duality. We further conjecture that the operad $\mathcal{Q}$ is Koszul (cf. loc. cit.) Together with Conjecture 4.5, this implies Conjecture 4.2. Indeed, they imply that the generating series of $\mathcal{Q}$ and $\mathcal{Q}^!$, that is,

$$f(t) := \sum_{n\geqslant 1} (-1)^n \dim \mathcal{Q}_n t^n$$

and

$$g(t) := \sum_{n\geqslant 1} (-1)^n \dim \mathcal{Q}_n^! t^n = \sum_{n\geqslant 1} (-1)^n n^2 t^n = \frac{t(-1+t)}{(1+t)^3}$$

are inverse to each other with respect to composition:

$$f(g(t)) = t.$$

From here it follows that $\dim \mathcal{Q}_n = d_n$ as in (15); see the Encyclopedia of Integer Sequences [21, A007297].

4.7. Remark. According to [6, Eq. (49)], the numbers d_n satisfy the recursion

$$d_n = \sum_{i=1}^{n-1} d_i d_{n-i} + \sum_{i,j=0}^{n-1} d_i d_j d_{n-1-i-j}, \quad d_0 = 1.$$

This is to be compared with the familiar recursion

$$c_n = \sum_{i=0}^{n-1} c_i c_{n-1-i}, \quad c_0 = 1$$

for the Catalan numbers c_n. It is known that c_n is the dimension of the homogeneous component of degree n of the dendriform operad [13] and that the inverse series is $\sum_{n\geqslant 1} (-1)^n n t^n$ (compare with 4.6).

4.8. A quadri-algebra of shuffles. Recall the quadri-algebra $\Bbbk S_{\geqslant 2} = \bigoplus_{n\geqslant 2} \Bbbk S_n$ of Section 1.4. Consider the subspace Q_n of $\Bbbk S_{2n}$ spanned by the set $\mathrm{Sh}(2, 2, \ldots, 2)$ of $(2, 2, \ldots, 2)$-shuffles. These are permutations $\zeta \in S_{2n}$ of the form

$$\zeta = a_1 b_1 a_2 b_2 \ldots a_n b_n$$

with $a_i < b_i$ for every i. Let $Q := \bigoplus_{n\geqslant 1} Q_n$. Clearly, Q is a quadri-subalgebra of $\Bbbk S_{\geqslant 2}$. Moreover, Q is graded if we declare that the elements of Q_n have degree n. Note that

$$\dim Q_n = \frac{(2n)!}{2^n}.$$

Let F be the free quadri-algebra on one generator. Let $\imath : F \to Q$ be the unique morphism of quadri-algebras that sends the generator to 12 (the identity permutation in S_2).

4.9. Conjecture. *The map $\imath : F \to Q$ is injective. In other words, the quadri-subalgebras of Q generated by* 12 *is free.*

We discuss some evidence for this conjecture, which relates F to the free dendriform algebra and Q to the dendriform algebra of permutations.

Let Y be the free dendriform algebra on one generator [13]. Consider the unique morphisms of dendriform algebras

$$\psi_{\mathrm{h}} : Y \to F_{\mathrm{h}} \quad \text{and} \quad \psi_{\mathrm{v}} : Y \to F_{\mathrm{v}}$$

which send the generator of Y to the generator of the quadri-algebra F.

Let $A := \bigoplus_{n\geqslant 1} \Bbbk S_n$. There are two dendriform structures on this space. With the notations of 1.4, these structures are

$$\sigma \prec \tau = \sum_{\zeta\in \mathsf{Sh}(p,q),\zeta^{-1}(1)=1} \zeta\cdot(\sigma\times\tau), \quad \sigma \succ \tau = \sum_{\zeta\in \mathsf{Sh}(p,q),\zeta^{-1}(1)=p+1} \zeta\cdot(\sigma\times\tau)$$

and

$$\sigma \wedge \tau = \sum_{\zeta\in \mathsf{Sh}(p,q),\zeta^{-1}(p+q)=p} \zeta\cdot(\sigma\times\tau), \quad \sigma \vee \tau = \sum_{\zeta\in \mathsf{Sh}(p,q),\zeta^{-1}(p+q)=p+q} \zeta\cdot(\sigma\times\tau).$$

We denote the first structure by A_{h} and the second by A_{v}. They are the horizontal and vertical dendriform algebra structures corresponding to the quadri-algebra $\Bbbk S_{\geqslant 2}$ of 1.4, enlarged by the component $\Bbbk S_1$ of degree 1.

Let

$$\alpha_{\mathrm{h}} : Y \to A_{\mathrm{h}} \quad \text{and} \quad \alpha_{\mathrm{v}} : Y \to A_{\mathrm{v}}$$

be the unique morphisms of dendriform algebras which send the generator of Y to the permutation $1 \in S_1$.

Consider now the quadri-algebra Q of 4.8 and the corresponding dendriform algebras Q_{h} and Q_{v}.

4.10. Proposition. *There are morphisms of dendriform algebras*

$$\varphi_{\mathrm{h}} : A_{\mathrm{h}} \to Q_{\mathrm{h}} \quad \textit{and} \quad \varphi_{\mathrm{v}} : A_{\mathrm{v}} \to Q_{\mathrm{v}}$$

such that the following diagrams commute:

$$\begin{array}{ccc} F_{\rm h} & \xrightarrow{\imath} & Q_{\rm h} \\ {\scriptstyle\psi_{\rm h}}\uparrow & & \uparrow{\scriptstyle\varphi_{\rm h}} \\ Y & \xrightarrow[\alpha_{\rm h}]{} & A_{\rm h} \end{array} \qquad \begin{array}{ccc} F_{\rm v} & \xrightarrow{\imath} & Q_{\rm v} \\ {\scriptstyle\psi_{\rm v}}\uparrow & & \uparrow{\scriptstyle\varphi_{\rm v}} \\ Y & \xrightarrow[\alpha_{\rm v}]{} & A_{\rm v} \end{array}$$

Proof. Consider the maps $\hat{\varphi}_{\rm h}$ and $\hat{\varphi}_{\rm v} : \mathsf{Sh}(2,2,\ldots,2) \to S_n$ defined by

$$\hat{\varphi}_{\rm h}(a_1b_1a_2b_2\ldots a_nb_n) = \mathrm{st}(a_1a_2\ldots a_n) \text{ and } \hat{\varphi}_{\rm v}(a_1b_1a_2b_2\ldots a_nb_n) = \mathrm{st}(b_1b_2\ldots b_n),$$

where for any sequence of n distinct integers a_i, $\mathrm{st}(a_1a_2\ldots a_n)$ denotes the unique permutation $\sigma \in S_n$ such that

$$\sigma(i) < \sigma(j) \iff a_i < a_j.$$

Define $\varphi_{\rm h} : A_{\rm h} \to Q_{\rm h}$ and $\varphi_{\rm v} : A_{\rm v} \to Q_{\rm v}$ by

$$\varphi_{\rm h}(\sigma) = \sum_{\hat{\varphi}_{\rm h}(\zeta)=\sigma} \zeta \quad \text{and} \quad \varphi_{\rm v}(\sigma) = \sum_{\hat{\varphi}_{\rm v}(\zeta)=\sigma} \zeta.$$

It is easy to check that these are morphisms of dendriform algebras.

Now, the composite $\varphi_{\rm h}\alpha_{\rm h}$ is a morphism of dendriform algebras which sends the generator of Y to $\varphi_{\rm h}(1) = 12$. Since the same is true of the composite $\imath\psi_{\rm h}$, the first square above must commute. The other commutativity is similar. □

It is known that the maps $\alpha_{\rm h}$ and $\alpha_{\rm v}$ are injective. We view this as evidence in favor of Conjecture 4.9.

The maps $\alpha_{\rm h}$ and $\alpha_{\rm v}$ admit explicit combinatorial descriptions in terms of planar binary trees. We expect similar descriptions for the maps $\psi_{\rm h}$, $\psi_{\rm v}$ and $\imath$.

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