

Trialgebras and families of polytopes

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Abstract. We show that the family of standard simplices and the family of Stasheff polytopes are dual to each other in the following sense. The chain modules of the standard simplices, resp. the Stasheff polytopes, assemble to give an operad. We show that these operads are dual of each other in the operadic sense. The main result of this paper is to show that they are both Koszul operads. This theorem appears as a refinement of the functional equation which relates the generating series of the standard simplices to the generating series of the Stasheff polytopes. The two operads give rise to new types of algebras with 3 generating operations, 11 relations, respectively 7 relations, that we call *associative trialgebras* and *dendriform trialgebras* respectively. Similarly, the family of cubes gives rise to an operad which happens to be self-dual for Koszul duality.

Introduction. We introduce a new type of associative algebras characterized by the fact that the associative product $*$ is the sum of three binary operations :

$$x * y := x \prec y + x \succ y + x \cdot y ,$$

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and that the associativity property of $*$ is a consequence of 7 relations satisfied by \prec, \succ and \cdot , cf. 2.1. Such an algebra is called a *dendriform trialgebra*. An example of a dendriform trialgebra is given by the algebra of quasi-shuffles (cf. 2.3).

Our first result is to show that the free dendriform trialgebra on one generator can be described as an algebra over the set of *planar rooted trees*. Equivalently, one can think of these linear generators as being the cells of the Stasheff polytopes (associahedra), since there is a bijection between the k -cells of the Stasheff polytope of dimension n and the planar trees with $n + 2$ leaves and $n - k$ internal vertices.

The knowledge of the free dendriform trialgebra permits us to construct the algebras over the dual operad (in the sense of Ginzburg and Kapranov [G-K]) and therefore to construct the chain complex of a dendriform trialgebra. This dual type is called the *associative trialgebra* since there is again three generating operations, and since all the relations are of the associativity type (cf. 1.1). We show that the free associative trialgebra on one generator is linearly generated by the cells of the standard simplices.

The main result of this paper is to show that the operad of dendriform trialgebras (resp. associative trialgebras) is a Koszul operad, or, equivalently, that the homology of the free dendriform trialgebra is trivial.

As a consequence of the description of the free trialgebras in the dendriform and associative framework, the generating series of the associated operads are the generating series of the family of the Stasheff polytopes and of the standard simplices respectively:

$$f_t^K(x) = \sum_{n \geq 1} (-1)^n p(\mathcal{K}^{n-1}, t) x^n, \quad f_t^\Delta(x) = \sum_{n \geq 1} (-1)^n p(\Delta^{n-1}, t) x^n.$$

Here $p(X, t)$ denotes the Poincaré polynomial of the polytope X .

The acyclicity of the Koszul complex for the dendriform trialgebra operad implies that

$$f_t^\Delta(f_t^K(x)) = x.$$

Since $p(\Delta^n, t) = ((1 + t)^{n+1} - 1)/t$ one gets

$$f_t^\Delta(x) = \frac{-x}{(1 + x)(1 + (1 + t)x)}$$

and therefore

$$f_t^{\mathcal{K}}(x) = \frac{-(1 + (2 + t)x) + \sqrt{1 + 2(2 + t)x + t^2x^2}}{2(1 + t)x}.$$

The series $f^{\mathcal{K}}$ can also be obtained combinatorially by counting trees, and the functional equation relating f^{Δ} and $f^{\mathcal{K}}$ can be checked directly. The acyclicity of the Koszul complex is a refinement of this elementary fact.

In [L1, L2] we dealt with dialgebras, that is with algebras defined by two generating operations. In the associative framework the dialgebra case is a quotient of the trialgebra case and in the dendriform framework the dialgebra case is a subcase of the trialgebra case.

If we split the associative relation for the operation $*$ into 9 relations instead of 7, then we can devise a similar theory in which the family of Stasheff polytopes is replaced by the family of cubes. So we get a new type of algebras that we call the *cubical trialgebras*. It turns out that the associated operad is self-dual (so the family of standard simplices is to be replaced by the family of cubes). The generating series of this operad is the generating series of the family of cubes: $f_t^I(x) = \frac{-x}{1+(t+2)x}$. It is immediate to check that $f_t^I(f_t^I(x)) = x$, hence one can presume that this is a Koszul operad. Indeed we can prove that the Koszul complex of the cubical trialgebra operad is acyclic.

As in the dialgebra case the associative algebra on planar trees can be endowed with a comultiplication which makes it into a Hopf algebra. This comultiplication satisfies some compatibility properties with respect to the three operations \prec, \succ and \cdot , (cf. [L4]).

Here is the content of the paper.

1. Associative trialgebras and standard simplices
2. Dendriform trialgebras and Stasheff polytopes
3. Homology and Koszul duality
4. Acyclicity of the Koszul complex
5. Cubical trialgebras and hypercubes

In the first section we introduce the notion of associative trialgebra and we compute the free algebra. This result gives the relationship with the family of standard simplices.

In the second section we introduce the notion of dendriform trialgebra and we compute the free algebra, which is based on planar

rooted trees. This result gives the relationship with the family of Stasheff polytopes.

In the third section we show that the associated operads are dual to each other for Koszul duality. Then we construct the chain complexes which compute the homology of these algebras. The acyclicity of the Koszul complex of the operad is equivalent to the acyclicity of the chain complex of the free associative trialgebra.

This acyclicity property is the main result of this paper, it is proved in the fourth section. After a few manipulations involving the join of simplicial sets we reduce this theorem to proving the contractibility of some explicit simplicial complexes. This is done by producing a sequence of retractions by deformation.

In the fifth section we treat the case of the family of hypercubes, along the same lines.

These results have been announced in [LR2].

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Convention. The category of vector spaces over the field K is denoted by Vect , and the tensor product of vector spaces over K is denoted by \otimes . The symmetric group acting on n elements is denoted by S_n .

1. Associative trialgebras and standard simplices.

In this section we introduce the definition of associative trialgebras, and we relate it to associative dialgebras introduced in [L1, L2]. We make explicit the free associative trialgebra and we explain the relationship with the standard simplices.

1.1 Definitions. An *associative trialgebra* (resp. an *associative trioid*) is a vector space A (resp. a set X) equipped with 3 binary operations : \dashv called *left*, \vdash called *right* and \perp called *middle*, satisfying the following 11 relations :

$$\left\{ \begin{array}{l} (x \dashv y) \dashv z = x \dashv (y \dashv z), \\ (x \dashv y) \dashv z = x \dashv (y \vdash z), \\ (x \vdash y) \dashv z = x \vdash (y \dashv z), \\ (x \dashv y) \vdash z = x \vdash (y \vdash z), \\ (x \vdash y) \vdash z = x \vdash (y \vdash z), \end{array} \right.$$

$$\left\{ \begin{array}{l} (x \dashv y) \dashv z = x \dashv (y \perp z), \\ (x \perp y) \dashv z = x \perp (y \dashv z), \\ (x \dashv y) \perp z = x \perp (y \vdash z), \\ (x \vdash y) \perp z = x \vdash (y \perp z), \\ (x \perp y) \vdash z = x \vdash (y \vdash z), \end{array} \right.$$

$$\{ (x \perp y) \perp z = x \perp (y \perp z). \}$$

First, observe that the three operations \dashv, \vdash and \perp are associative. Second, observe that, in the relations, the following rule holds: “on the bar side of the symbols, does not matter which product”. Third, each relation has its symmetric counterpart which consists in reversing the order of the parenthesizing, exchanging \vdash and \dashv , leaving \perp unchanged.

A morphism between two associative trialgebras is a linear map which is compatible with the three operations. We denote by **Trias** the category of associative trialgebras.

1.2 Relationship with associative dialgebras. In [L1, L2] the first author introduced the notion of associative dialgebra. They are defined by two operations \dashv, \vdash satisfying the first five relations listed in 1.1. So any associative trialgebra gives rise to an associative dialgebra by forgetting the operation \perp . In other words, this gives a forgetful functor:

$$\mathbf{Trias} \rightarrow \mathbf{Dias}$$

from the category of trialgebras to the category of dialgebras.

1.3 Examples of associative trialgebras.

(a) If $\dashv = \perp = \vdash$, then we get simply an associative algebra (nonunital). So we get a functor between the categories of algebras :

$$\mathbf{As} \rightarrow \mathbf{Trias} .$$

(b) The tensor product of two associative trialgebras has a natural structure of associative trialgebra. For instance the tensor product of an associative algebra with an associative trialgebra is again an associative trialgebra. As a consequence the $n \times n$ -matrices over an associative trialgebra still form an associative trialgebra.

(c) The vector space spanned by an associative trioid is obviously an associative trialgebra.

(d) Let $V = \bigoplus_{n \geq 0} K \cdot \omega_n$ be the graded K -vector space such that the subspace of homogeneous elements of degree n is the vector space of dimension one, spanned by the generator ω_n , for all $n \geq 0$. Consider the tensor algebra $T(V)$, with the operations \perp , \dashv and \vdash given by:

$$(\omega_{n_1} \otimes \dots \otimes \omega_{n_r}) \perp (\omega_{m_1} \otimes \dots \otimes \omega_{m_k}) := \omega_{n_1} \otimes \dots \otimes \omega_{n_r} \otimes \omega_{m_1} \otimes \dots \otimes \omega_{m_k},$$

$$(\omega_{n_1} \otimes \dots \otimes \omega_{n_r}) \dashv (\omega_{m_1} \otimes \dots \otimes \omega_{m_k}) := \omega_{n_1} \otimes \dots \otimes \omega_{n_r} \otimes \omega_{m_1 + \dots + m_k},$$

$$(\omega_{n_1} \otimes \dots \otimes \omega_{n_r}) \vdash (\omega_{m_1} \otimes \dots \otimes \omega_{m_k}) := \omega_{n_1 + \dots + n_r} \otimes \omega_{m_1} \otimes \dots \otimes \omega_{m_k},$$

for $n_1, \dots, n_r, m_1, \dots, m_k \geq 0$. It is easy to check that $(T(V), \perp, \dashv, \vdash)$ is an associative trialgebra.

1.4 Notation. Let $[n-1] := \{0, \dots, n-1\}$ be a set with n elements. The set of non-empty subsets of $[n-1]$ is denoted by P_n . Observe that P_n is graded by the cardinality of its members. We denote by $P_{n,k}$ the subset of P_n whose members have cardinality k . So $P_n = P_{n,1} \cup \dots \cup P_{n,n}$.

1.5 Free associative trialgebra. By definition the free associative trialgebra over the vector space V is an associative trialgebra $Trias(V)$ equipped with a map $V \rightarrow Trias(V)$, which satisfies the following universal property. For any map $V \rightarrow A$, where A is an associative trialgebra, there is a unique extension $Trias(V) \rightarrow A$ which is a morphism of associative trialgebras.

Since the operations have no symmetry and since the relations leave the variables in the same order, $Trias(V)$ is completely determined by the free associative trialgebra on one generator (i.e. $V = K$). The latter is a graded vector space of the form

$$Trias(K) = \bigoplus_{n \geq 1} Trias_n.$$

For $n = 1, 2, 3$, a basis of $Trias_n$ is given by the elements of P_1, P_2 and P_3 respectively (i.e. the cells of the simplices $\Delta^0, \Delta^1, \Delta^2$ respectively).

Let us denote by

$$\text{bij} : [i_1 - 1] \cup \dots \cup [i_n - 1] \rightarrow [i_1 + \dots + i_n - 1]$$

the bijection which sends $k \in [i_j - 1]$ to $i_1 + \dots + i_{j-1} + k \in [i_1 + \dots + i_n - 1]$.

1.6 Theorem. *The free associative trialgebra $\text{Trias}(K)$ on one generator is $\bigoplus_{n \geq 1} K[P_n]$ as a vector space. The binary operations \dashv, \perp and \vdash from $K[P_p] \otimes K[P_q]$ to $K[P_{p+q}]$ are given by*

$$X \dashv Y = \text{bij}(X), \quad X \perp Y = \text{bij}(X \cup Y) \quad X \vdash Y = \text{bij}(Y),$$

where $X \in P_p$ and $Y \in P_q$ and $\text{bij} : [p-1] \times [q-1] \rightarrow [p+q-1]$.

1.7 Corollary. *The free associative trialgebra $\text{Trias}(V)$ on the vector space V is*

$$\text{Trias}(V) = \bigoplus_{n \geq 1} K[P_n] \otimes V^{\otimes n},$$

and the operations are induced by the operations on $\text{Trias}(K)$ and concatenation.

Proof. It suffices to make explicit the free trioid in one generator, see Proposition 1.8 below. Indeed, this proves Theorem 1.6 by applying the functor which sends a set Z to the vector space $K[Z]$ spanned by the elements of Z . Then the Corollary is a consequence of the Theorem because all the relations in the definition of an associative trialgebra leave the variables in the same order. \square

1.8 Proposition. *The free trioid \mathcal{T} on one generator x is isomorphic to the trioid $P = \bigcup_{n \geq 1} P_n$ equipped with the operations described in Theorem 1.6 above.*

Proof. First we prove that $(P; \dashv, \vdash, \perp)$ is a trioid generated by the element $\{0\} \in P_1$. For convenience let us denote this generator by x and by $x \cdots \check{x} \cdots \check{x} \cdots x$ the element corresponding to $X \in P_n$, where there are n copies of x and, if $i \in X$, then the i th factor is checked. For instance $\{0, 2\} \in P_5$ corresponds to $\check{x}x\check{x}xx$. Under this notation the operations are easy to describe: one concatenates the two elements, keeping only the marking on the left side for \dashv , on the right side for \vdash , on both sides for \perp . For instance

$$\begin{cases} \check{x}x \dashv x\check{x}x = \check{x}xxxx, \\ \check{x}x \vdash x\check{x}x = xxx\check{x}x, \\ \check{x}x \perp x\check{x}x = \check{x}xx\check{x}x. \end{cases}$$

It is immediate to verify that the eleven relations are fulfilled.

Since \mathcal{T} is the free trioid generated by x , there exists a unique trioid morphism $\phi : \mathcal{T} \rightarrow P$. Each map $\phi_n : \mathcal{T}_n \rightarrow P_n$ is surjective since, in P , $x = \{0\} \in P_1$ is also the generator. In order to prove that ϕ is an isomorphism, it suffices to show that $\#\mathcal{T}_n \leq \#P_n$.

1.9 Lemma. Any complete parenthesizing of

$$\underbrace{(x \vdash \cdots \vdash x)}_{a_0} \vdash \underbrace{(x \dashv \cdots \dashv x)}_{a_1} \perp \underbrace{(x \dashv \cdots \dashv x)}_{a_2} \perp \cdots \perp \underbrace{(x \dashv \cdots \dashv x)}_{a_k}$$

where $a_0 \geq 0, a_i \geq 1$ for $i = 1, \dots, k$, gives the same element, denoted ω , in \mathcal{T} . We call it the normal form of ω . Its image under ϕ in P is

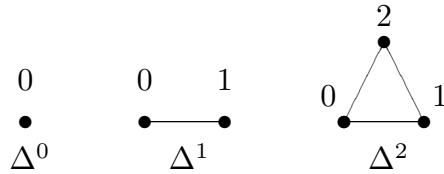
$$\underbrace{x \cdots x}_{a_0} \underbrace{\check{x} \cdots \check{x}}_{a_1} \underbrace{\check{x} \cdots \check{x}}_{a_2} \cdots x \underbrace{\check{x} \cdots \check{x}}_{a_k}.$$

Proof. Putting parentheses outside (resp. inside) the existing parentheses does not change the value of the element by virtue of relations 9 and 11 (resp. 1 and 5). The second statement is immediate by direct inspection. \square

End of the proof of Proposition 1.8. Since any element in P_n is the image of an element of the type indicated in Lemma 1.9, it suffices to show that any element in \mathcal{T}_n can be written under this form. We work by induction on n . It is clear for $n = 1$. We suppose that it is true for all $p < n$. Any $\sigma \in \mathcal{T}_n$ is of the form $\sigma' \dashv \sigma''$ or $\sigma' \vdash \sigma''$ or $\sigma' \perp \sigma''$ for some $\sigma' \in \mathcal{T}_p, \sigma'' \in \mathcal{T}_q$. We write σ' and σ'' in a normal form as in Lemma 1.9 and we compute the three elements $\sigma' \dashv \sigma'', \sigma' \vdash \sigma''$ and $\sigma' \perp \sigma''$. By using the relations 1 to 11 it is easy to show that they can be written under a normal form. So the proof of Proposition 1.8 is complete. \square

1.10 Filtration. The set P_n can be filtered by $F_k P_n := \cup_{i \leq k} P_{n,i}$, cf. 1.4. Since, in any product of two elements, the number of marked variables is less than or equal to the sum of the numbers of the components, the image of $F_k P_n \times F_l P_m$ is in $F_{k+l} P_{n+m}$.

1.11 The family of standard simplices. Let $\Delta^n = \{(x_0, \dots, x_n) \in \mathbf{R}^{n+1} \mid x_0 + \cdots + x_n = 1, 0 \leq x_i \leq 1\}$ be the standard n -simplex. As usual we label its vertices by the integers 0 to n . So the vertex i has coordinates 0 except $x_i = 1$. An i -cell in Δ^n is completely determined by its vertices, hence by a non-empty subset of $[n] = \{0, \dots, n\}$, that is, following our notation, an element of P_{n+1} . So there is a bijection between the k -cells of Δ^n and the set $P_{n+1, k+1}$.



Observe that the Poincaré polynomial of Δ^n is

$$p(\Delta^n, t) := \sum_{k \geq 0} \#(k\text{-cells})t^k = \frac{((1+t)^{n+1} - 1)}{t}.$$

If we define the generating series of a family of polytopes $X(n), n \geq 0$ by $f_t^X(x) = \sum_{n \geq 1} (-1)^n p(X(n-1), t)x^n$, then we get the following for the family of standard simplices:

$$f_t^\Delta(x) = \frac{-x}{(1+x)(1+(1+t)x)}.$$

1.12 Generating series of a filtered operad. Recall that various types of algebras can be described by “algebraic operads”. An algebraic operad \mathcal{P} is a functor $\mathcal{P} : \text{Vect} \rightarrow \text{Vect}$ of the form $\mathcal{P}(V) = \bigoplus_{n \geq 1} \mathcal{P}(n) \otimes_{S_n} V^{\otimes n}$ (here $\mathcal{P}(n)$ is a right S_n -module) together with an associative and unital transformation of functors $\gamma : \mathcal{P} \circ \mathcal{P} \rightarrow \mathcal{P}$, cf. [G-K, O]. The free \mathcal{P} -algebra over V is precisely $\mathcal{P}(V)$.

By definition the generating series of an operad \mathcal{P} is

$$f^{\mathcal{P}}(x) := \sum_{n \geq 1} (-1)^n (1/n!) \dim \mathcal{P}(n) x^n.$$

When the operad is filtered, we can define a finer invariant by replacing the dimension of $\mathcal{P}(n)$ by its Poincaré polynomial

$$p(\mathcal{P}(n), t) := \sum_{k \geq 0} \dim(F_k \mathcal{P}_n / F_{k-1} \mathcal{P}_n) t^k$$

to get a series with polynomials in t as coefficients:

$$f_t^{\mathcal{P}}(x) := \sum_{n \geq 1} (-1)^n (1/n!) p(\mathcal{P}(n), t) x^n.$$

In [G-K] it is shown that if the quadratic operad \mathcal{P} is a Koszul operad, then the generating series of \mathcal{P} and of its dual $\mathcal{P}^!$ are related by $f^{\mathcal{P}}(f^{\mathcal{P}^!}(x)) = x$. This formula is obtained by computing the Euler-Poincaré characteristic of the Koszul complex of \mathcal{P} , which gives the left hand-side. Since this complex is acyclic, its homology is trivial, and this gives the right-hand side.

If the quadratic operad \mathcal{P} is filtered, then a refinement of this argument gives the functional equation:

$$f_t^{\mathcal{P}}(f_t^{\mathcal{P}^!}(x)) = x .$$

1.13 The operad of associative trialgebras. Let $Trias$ be the operad associated to the associative trialgebras. By Corollary 1.7 we have $Trias(n) = K[P_n] \otimes K[S_n]$, where S_n is the symmetric group. So, as an S_n -module, $Trias(n)$ is the direct sum of several copies of the regular representation, one for each element in P_n . In other words $Trias$ comes from a non- Σ operad in the sense of [O, p.4]. The filtration of the free trioid described in 1.10 induces a filtration on the operad $Trias$. Since this filtration corresponds precisely to the filtration of the standard simplex by the dimension of the cells, the generating series are equal:

$$f_t^{Trias}(x) = f_t^{\Delta}(x) = \frac{-x}{(1+x)(1+(1+t)x)} .$$

1.14 Relationship with the Leibniz and Poisson algebra structures. Recall that a Leibniz bracket is a binary operation $[-, -]$ which satisfies the Leibniz identity:

$$[[x, y], z] = [[x, z], y] + [x, [y, z]].$$

By definition a *Leibniz-Poisson algebra* is an associative algebra A equipped with a Leibniz bracket (compare with [A]). These two operations are supposed to satisfy the relations:

$$[xy, z] = [x, z]y + x[y, z] , \quad (1.14.1)$$

$$[x, yz - zy] = [x, [y, z]] . \quad (1.14.2)$$

1.15 Proposition. *Let $(A, \dashv, \vdash, \perp)$ be an associative trialgebra. By defining*

$$[x, y] := x \dashv y - y \vdash x \quad \text{and} \quad xy := x \perp y$$

we get a Leibniz-Poisson algebra structure on A .

Proof. The relation 1.14.1 is a consequence of the relations number 7, 8 and 9 in 1.1 and the relation 1.14.2 is a consequence of the relations number 6 and 10. \square

1.16 Relationship with the boundary map of the standard simplex. The space $K[P_n]$ is in fact the chain module of the standard simplex Δ^{n-1} , and so it is equipped with a differential map $\delta : K[P_{n,k}] \rightarrow K[P_{n,k-1}]$. Explicitly δ is given by

$$\delta(X) := \sum_{i=1}^k (-1)^{i+1} X \setminus \{n_i\},$$

for $X = \{n_1 < n_2 < \dots < n_k\}$ a subset of $[n-1]$. The relationship of δ with the three operations \dashv , \vdash and \perp is given (for $X \in P_{n,k}$) by:

$$\begin{aligned} \delta(X \dashv Y) &= \delta(X) \dashv Y, \\ \delta(X \vdash Y) &= (-1)^k X \vdash \delta(Y), \\ \delta(X \perp Y) &= \begin{cases} \delta(X) \perp Y + (-1)^k X \perp \delta(Y) & \text{for } \delta(X) \neq 0 \text{ and } \delta(Y) \neq 0, \\ X \vdash Y + (-1)^k X \perp \delta(Y) & \text{for } \delta(X) = 0 \text{ and } \delta(Y) \neq 0, \\ \delta(X) \perp Y + (-1)^k X \dashv Y & \text{for } \delta(X) \neq 0 \text{ and } \delta(Y) = 0, \\ X \vdash Y - X \dashv Y & \text{for } \delta(X) = 0 \text{ and } \delta(Y) = 0. \end{cases} \end{aligned}$$

2. Dendriform trialgebras and Stasheff polytopes. In this section we introduce the definition of dendriform trialgebras, and we relate it to dendriform dialgebras introduced in [L1, L2]. We make explicit the free dendriform trialgebra in terms of the planar rooted trees and we explain the relationship with the Stasheff polytopes.

2.1 Dendriform trialgebras. By definition a *dendriform trialgebra* is a vector space D equipped with three binary operations :

\prec called *left*, \succ called *right*, \cdot called *middle*,

satisfying the following relations :

$$\begin{cases} (x \prec y) \prec z = x \prec (y * z) , \\ (x \succ y) \prec z = x \succ (y \prec z) , \\ (x * y) \succ z = x \succ (y \succ z) , \end{cases}$$

$$\begin{cases} (x \succ y) \cdot z = x \succ (y \cdot z) , \\ (x \prec y) \cdot z = x \cdot (y \succ z) , \\ (x \cdot y) \prec z = x \cdot (y \prec z) , \end{cases}$$

$$\{ (x \cdot y) \cdot z = x \cdot (y \cdot z) ,$$

where $x * y := x \prec y + x \succ y + x \cdot y$.

2.2 Lemma. *The operation $*$ is associative.*

Proof. It suffices to add up all the relations to observe that on the right side we get $(x * y) * z$ and on the left side $x * (y * z)$. Whence the assertion. \square

In other words, a dendriform trialgebra is an associative algebra for which the associative operation is the sum of three operations and the associative relation splits into 7 relations.

We denote by **Tridend** the category of dendriform trialgebras and by *Tridend* the associated operad. By the preceding lemma, there is a well-defined functor:

$$\mathbf{Tridend} \rightarrow \mathbf{As} ,$$

where **As** is the category of (nonunital) associative algebras.

Observe that the operad *Tridend* does not come from a set operad because the operation $*$ needs a sum to be defined. However there is a property which is close to it. This is discussed and exploited in [L3].

2.3 Examples of dendriform trialgebras.

- (a) If D is a dendriform trialgebra, then the $n \times n$ -matrices over D still form a dendriform trialgebra.
- (b) If the operation \cdot is taken to be trivial (i.e $x \cdot y = 0$ for any $x, y \in D$), then $x * y = x \prec y + x \succ y$, and we get simply a dendriform dialgebra as defined in [L1, L2] (two generating operations and 3 relations). This defines a functor

$$\mathbf{Didend} \rightarrow \mathbf{Tridend}.$$

- (c) *Quasi-shuffles.* Let A be an associative algebra (not necessarily unital) whose product is denoted by \bullet . On the tensor module

$$T(A) = K.1 \oplus \bar{T}(A) = K.1 \oplus A \oplus A^{\otimes 2} \oplus \dots$$

we define by induction the three operations \prec, \succ and \cdot as follows:

$$\begin{aligned} a\omega \prec b\theta &:= a(\omega * b\theta), \\ a\omega \succ b\theta &:= b(a\omega * \theta), \\ a\omega \cdot b\theta &:= (a \bullet b)(\omega * \theta), \end{aligned}$$

where $x*y := x \prec y + x \succ y + x \cdot y$ and 1 is a unit for $*$. In these formulas a, b are elements of A and ω, θ are tensors. It is straightforward to check that these formulas define a structure of dendriform trialgebra on $\bar{T}(A)$. As an associative algebra it is called the algebra of quasi-shuffles.

If the \bullet product is 0, then we get the shuffle algebra over the vector space A .

If $A = K$, then we get the algebra of quasi-symmetric functions, cf. [C, formula 94], [H].

(d) *The partition example.* For $n \geq 1$ a *partition of n of length r* is a surjective map $f : \{1, \dots, n\} \rightarrow \{1, \dots, r\}$. Let $\Pi_{n,r}$ be the set of partitions of n of length r and $\Pi_n = \coprod_{r=1}^n \Pi_{n,r}$. Observe that $\Pi_{n,n}$ is the symmetric group S_n , and that $\Pi_{n,1}$ has only one element. In fact the elements of $\Pi_{n,r}$ are in bijection with the $(n-r)$ -cells of the permutohedron of dimension $n-1$.

Given $f \in \Pi_{n,r}$ and $g \in \Pi_{m,k}$ we denote by $f \times g \in \Pi_{n+m,r+k}$ the concatenation of f and g :

$$f \times g := \begin{cases} f(i) & \text{for } 1 \leq i \leq n, \\ g(i-n) + r & \text{for } n+1 \leq i \leq n+m. \end{cases}$$

An element $f \in \Pi_n$ is said to be a $(p, n-p)$ -*shuffle* for $0 \leq p \leq n$, if $f(1) \leq \dots \leq f(p)$ and $f(p+1) \leq \dots \leq f(n)$. We denote by $SH(p, n-p)$ the subset of Π_n consisting of $(p, n-p)$ -shuffles. The set $SH(p, n-p)$ is the disjoint union of the following three subsets:

$$SH^{\succ}(p, n-p) := \{f \in SH(p, n-p) \mid f(n) = r \text{ and } f(p) < r \text{ for } f \in \Pi_{n,r}\}$$

$$SH^{\prec}(p, n-p) := \{f \in SH(p, n-p) \mid f(n) < r \text{ and } f(p) = r \text{ for } f \in \Pi_{n,r}\}$$

$$SH^{\cdot}(p, n-p) := \{f \in SH(p, n-p) \mid f(n) = r \text{ and } f(p) = r \text{ for } f \in \Pi_{n,r}\}$$

Let $K[\Pi_{\infty}]$ be the space spanned by Π_{∞} . For $f \in \Pi_{n,r}$ and $g \in \Pi_{m,k}$ the composition $f \circ g$ is a well-defined element of $\Pi_{m,r}$ provided that $n = k$. We extend it to all cases by putting $f \circ g = 0 \in K[\Pi_{\infty}]$ otherwise. Observe that for $m = r$ this is the product in the symmetric group.

We define the three operations \succ, \prec and \cdot on $K[\Pi_{\infty}]$ as follows:

$$f \prec g := \sum_{h \in SH^{\prec}(n,k)} h \circ (f \times g)$$

$$f \succ g := \sum_{h \in SH^{\succ}(n,k)} h \circ (f \times g)$$

$$f \cdot g := \sum_{h \in SH(n,k)} h \circ (f \times g)$$

for $f \in \Pi_{n,r}$ and $g \in \Pi_{m,k}$. It is straightforward to check that $(K[\Pi_\infty], \prec, \succ, \cdot)$ is a dendriform trialgebra.

2.4 Planar trees. We denote by T_n the set of planar rooted trees with $n + 1$ leaves, $n \geq 0$ (and one root) such that the valence of each internal vertex is at least 2. Here are the first of them:

$$T_0 = \{\{\}\}, \quad T_1 = \{\vee\}, \quad T_2 = \{\vee\vee, \vee\vee; \vee\vee\}$$

$$T_3 = \{\vee\vee\vee, \vee\vee\vee; \vee\vee\vee, \vee\vee\vee; \vee\vee\vee; \vee\vee\vee, \vee\vee\vee; \vee\vee\vee, \vee\vee\vee; \vee\vee\vee; \vee\vee\vee\}$$

The integer n is called the *degree* of $t \in T_n$. Recall that the number of elements in T_n is the so-called *super Catalan number* C_n :

n	1	2	3	4	5
C_n	1	3	11	45	197

The generating series is $\sum_{n \geq 1} C_n x^n = \frac{(1-3x) - \sqrt{1-6x+x^2}}{4x}$.

The set T_n is the disjoint union of the sets $T_{n,k}$ made of the planar trees which have $n - k + 1$ internal vertices. For instance $T_{n,1}$ is made of the planar binary trees, and its cardinality is the Catalan number $(2n)!/n!(n+1)!$. On the other extreme the set $T_{n,n}$ has only one element, which is the planar tree with one vertex. It is sometimes called a *corolla*. So we have

$$T_n = T_{n,1} \cup \dots \cup T_{n,n}.$$

By convention $T_0 = T_{0,0}$.

The grafting of $k + 1$ planar trees $x^{(0)}, \dots, x^{(k)}$ is a planar tree denoted $x^{(0)} \vee \dots \vee x^{(k)}$ obtained by joining the $k + 1$ roots to a new vertex and adding a new root. Any planar tree can be uniquely obtained as $x = x^{(0)} \vee \dots \vee x^{(k)}$, where $k + 1$ is the valence of the lowest vertex. We will use the uniqueness of this decomposition in the construction of a dendriform trialgebra structure on planar trees. Observe that the degree of $x^{(i)}$ is strictly smaller than the degree of x .

2.5 Free dendriform trialgebra. The free dendriform trialgebra over the vector space V is a dendriform trialgebra $Tridend(V)$ equipped with a map $V \rightarrow Tridend(V)$ which satisfies the classical

universal property, cf. 1.5. In the following theorem we make it explicit in terms of planar trees.

2.6 Theorem. *The free dendriform trialgebra on one generator is*

$$\mathit{Tridend}(K) = \bigoplus_{n \geq 1} K[T_n],$$

where T_n is the set of planar trees with $(n + 1)$ leaves.

The binary operations are given on $T_p \times T_q$ by the recursive formulas:

$$\begin{aligned} x \prec y &= x^{(0)} \vee \cdots \vee (x^{(k)} * y) , \\ x \cdot y &= x^{(0)} \vee \cdots \vee (x^{(k)} * y^{(0)}) \vee \cdots \vee y^{(\ell)} , \\ x \succ y &= (x * y^{(0)}) \vee \cdots \vee y^{(\ell)} , \end{aligned}$$

where $x = x^{(0)} \vee \cdots \vee x^{(k)} \in T_p$ and $y = y^{(0)} \vee \cdots \vee y^{(\ell)} \in T_q$. As before $x * y := x \prec y + x \cdot y + x \succ y$ and $| \in T_0$ is a unit for $*$.

Proof. It follows from the following two lemmas. In the first one we prove that $(\bigoplus_{n \geq 1} K[T_n]; \prec, \succ, \cdot)$ is a dendriform trialgebra generated by the tree \vee . As a consequence there is a unique dendriform trialgebra morphism $\mathit{Tridend}(K) \rightarrow \bigoplus_{n \geq 1} K[T_n]$ which sends the generator x of $\mathit{Tridend}(K)$ to $\vee \in T_1$. In order to prove that this (surjective) map is an isomorphism, we construct explicitly its inverse in the second lemma. \square

2.7 Lemma. *The binary operations \prec, \succ and \cdot defined on $\bigoplus_{n \geq 1} K[T_n]$ in theorem 2.6 satisfy the axioms of 2.1.*

Proof. The proof is straightforward by induction on the degree. Let us show for instance that

$$(x \prec y) \prec z = x \prec (y * z).$$

For $x = x^{(0)} \vee \cdots \vee x^{(k)}$ one has

$$\begin{aligned} (x \prec y) \prec z &= (x^{(0)} \vee \cdots \vee (x^{(k)} * y)) \prec z \\ &= x^{(0)} \vee \cdots \vee ((x^{(k)} * y) * z). \end{aligned}$$

On the other hand one has

$$x \prec (y * z) = x^{(0)} \vee \cdots \vee (x^{(k)} * (y * z)).$$

Since the degree of $x^{(k)}$ is strictly smaller than the degree of x , we can assume that all the relations are fulfilled for $x^{(k)}, y$ and z . In particular the associativity relation $(x^{(k)} * y) * z = x^{(k)} * (y * z)$ holds. Therefore one gets $(x \prec y) \prec z = x \prec (y * z)$ as expected.

All the other formulas are proved similarly. \square

2.8 Lemma. *Let us denote by u the generator of the free dendri-form trialgebra $\text{Tridend}(K)$. The map $\alpha : \bigoplus_{n \geq 0} K[T_n] \rightarrow K.1 \oplus \text{Tridend}(K)$ defined inductively by*

$$\alpha(|) := 1, \quad \alpha(x^{(0)} \vee x^{(1)}) := \alpha(x^{(0)}) \succ u \prec \alpha(x^{(1)}),$$

and

$$\alpha(x^{(0)} \vee \dots \vee x^{(k)}) := (\alpha(x^{(0)}) \succ u) \cdot \alpha(x^{(1)} \vee \dots \vee x^{(k-1)}) \cdot (u \prec \alpha(x^{(k)}))$$

for $k \geq 2$ is a morphism of dendri-form trialgebras when restricted to $\bigoplus_{n \geq 1} K[T_n]$.

Proof. Since it may happen that $x^{(0)} = |$, (resp. $x^{(k)} = |$), we need to specify that $1 \succ z = z = z \prec 1$. Similarly it may happen that, when $k = 2$, one has $x^{(1)} = |$. So we need to specify that

$$\alpha(x^{(0)} \vee | \vee x^{(1)}) := (\alpha(x^{(0)}) \succ u) \cdot (u \prec \alpha(x^{(2)})).$$

For instance, one has $\alpha(\bigvee) = \alpha(1 \vee 1 \vee 1) = (1 \succ u) \cdot (u \prec 1) = u \cdot u$. We want to show that

$$\alpha(x \prec y) = \alpha(x) \prec \alpha(y), \quad \alpha(x \succ y) = \alpha(x) \succ \alpha(y), \quad \alpha(x \cdot y) = \alpha(x) \cdot \alpha(y)$$

for any $x \in T_p, y \in T_q, z \in T_r$. We check the first equality, the checking of the others is similar.

Let us first check the case $x = x^{(0)} \vee x^{(1)}$. On one hand we have

$$\begin{aligned} \alpha(x \prec y) &= \alpha(x^{(0)} \vee (x^{(1)} * y)) \\ &= \alpha(x^{(0)}) \succ u \prec (\alpha(x^{(1)} * y)) \\ &= \alpha(x^{(0)}) \succ u \prec (\alpha(x^{(1)}) * \alpha(y)), \text{ by induction.} \end{aligned}$$

On the other hand we have

$$\begin{aligned} \alpha(x) \prec \alpha(y) &= \alpha(x^{(0)} \vee x^{(1)}) \prec \alpha(y) \\ &= (\alpha(x^{(0)}) \succ u \prec \alpha(x^{(1)})) \prec \alpha(y) \quad \text{by relation (2),} \\ &= \alpha(x^{(0)}) \succ u \prec (\alpha(x^{(1)}) * \alpha(y)) \quad \text{by relation (1).} \end{aligned}$$

Therefore one gets $\alpha(x \prec y) = \alpha(x) \prec \alpha(y)$ as expected.

Let us now suppose that $x = x^{(0)} \vee \dots \vee x^{(k)}$ with $k \geq 2$. On one hand we have

$$\begin{aligned} \alpha(x \prec y) &= \alpha(x^{(0)} \vee \dots \vee (x^{(k)} * y)) \\ &= (\alpha(x^{(0)}) \succ u) \cdot \alpha(x^{(1)} \vee \dots \vee x^{(k-1)}) \cdot (u \prec \alpha(x^{(k)} * y)) \\ &= (\alpha(x^{(0)}) \succ u) \cdot \alpha(x^{(1)} \vee \dots \vee x^{(k-1)}) \cdot (u \prec (\alpha(x^{(k)}) * \alpha(y))) \end{aligned}$$

On the other hand we have

$$\begin{aligned} \alpha(x) \prec \alpha(y) &= (\alpha(x^{(0)}) \succ u) \cdot \alpha(x^{(1)} \vee \dots \vee x^{(k-1)}) \cdot (u \prec \alpha(x^{(k)})) \prec \alpha(y) \\ &= (\alpha(x^{(0)}) \succ u) \cdot \alpha(x^{(1)} \vee \dots \vee x^{(k-1)}) \cdot ((u \prec \alpha(x^{(k)})) \prec \alpha(y)) \\ &= (\alpha(x^{(0)}) \succ u) \cdot \alpha(x^{(1)} \vee \dots \vee x^{(k-1)}) \cdot (u \prec (\alpha(x^{(k)}) * \alpha(y))) \end{aligned}$$

by relations (6) and (1), whence the result.

If $x = x^{(0)} \vee | \vee x^{(1)}$, then the proof is similar and uses also the relations (6) and (1) of 2.1. \square

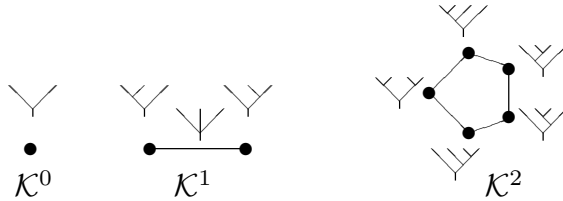
2.9 Corollary. *The free dendriform trialgebra $\text{Tridend}(V)$ on the vector space V is*

$$\text{Tridend}(V) = \bigoplus_{n \geq 1} K[T_n] \otimes V^{\otimes n},$$

and the operations are induced by the operations on $\bigoplus_{n \geq 1} K[T_n]$ and concatenation. \square

Proof. Follows from Theorem 2.6 by the same argument as in Corollary 1.7. \square

2.10 The family of Stasheff polytopes. Let \mathcal{K}^n be the Stasheff polytope (alias associahedron, also denoted K_{n+2}) of dimension n , cf. [St]. The cells of \mathcal{K}^{n-1} are in one-to-one correspondence with the planar trees with n leaves. More precisely the set $T_{n,k}$ of planar trees with n leaves and $n-k+1$ vertices labels the cells of dimension $k-1$ of \mathcal{K}^{n-1} . In particular the planar binary trees are in 1-1 correspondence with the vertices.



2.11 The operad of dendriform trialgebras. The operad $Tridend$ comes from a non- Σ operad and so is completely determined by the free dendriform trialgebra on one generator. The filtration on T_n (cf. 2.4) is compatible with the three operations (cf. [L3, section 9.9]). In particular there is a functor from the category of dendriform dialgebras to the category of dendriform trialgebras (take $x \cdot y = 0$). The operad of dendriform dialgebras involves only the planar *binary* trees.

Since the operad $Tridend$ is filtered, we can build the graded associated operad $gr\ Tridend$ as follows:

$$gr\ Tridend(n) := F_n Tridend / F_{n-1} Tridend.$$

It is clear that the 0th part of the graded operad is the operad of dendriform dialgebras.

From the bijection between the cells of the Stasheff polytopes and the planar trees, and the description of the free dendriform trialgebra given in Theorem 2.6 it follows that the generating series of the family of Stasheff polytopes is equal to the generating series of the (filtered) operad $Tridend$:

$$f_t^{\mathcal{K}}(x) = f_t^{Tridend}(x).$$

One way of keeping track of the filtration is to introduce a type of algebra depending on a parameter $q \in K$ as follows. In the relations 1 and 3 of 2.1 we replace the occurrences of $a \cdot b$ (where a and b are x, y or z) by $q(a \cdot b)$. When $q = 1$ this is the dendriform trialgebra. When $q = 0$, this is (almost) the case treated by Chapoton in [Ch]. Almost because he is working with graded vector spaces and modify the grading for the operation \cdot , like when passing from Poisson algebras to Gerstenhaber algebras.

2.12 Remark. The Stasheff polytopes form an operad (cf. [St]), which encodes the associative algebras up to homotopy (A_∞ -algebras). But, in this case the Stasheff polytope \mathcal{K}^n is put in dimension $n + 2$ while for the operad $Tridend$ it is put in dimension $n + 1$. In other words, in the A_∞ -algebra case a cell of the Stasheff polytope \mathcal{K}^n encodes an operation on $n + 2$ variables, though in the dendriform case it encodes an operation on $n + 1$ variables. So they are completely different operads.

2.13 Hopf algebra structure. By using the method given in [L4] one can equip the free dendriform algebra with a unit and then construct a cooperation Δ , which makes $(K.1 \oplus Tridend(V), *, \Delta)$ into a

Hopf algebra. Its primitive part can be studied along the same lines as in [R1] and [R2], which are dealing with dendriform dialgebras.

3. Homology and Koszul duality. In [G-K] Ginzburg and Kapranov have extended the notion of Koszul duality to binary quadratic operads. Both operads *Trias* and *Tridend* are binary and quadratic, hence we can apply this theory here. In particular we can construct the chain complex of an associative trialgebra (resp. of a dendriform trialgebra), and also the Koszul complex of these operads.

3.1 Theorem. *The operad *Trias* of associative trialgebras is dual to the operad *Tridend* of dendriform trialgebras :*

$$Trias^! = Tridend \quad \text{and} \quad Tridend^! = Trias.$$

Proof. Let us compute the Koszul dual of *Trias*. Since we are dealing with operads coming from non- Σ operads, that is $\mathcal{P}(n) = \mathcal{P}_n \otimes K[S_n]$ we can forget about the action of the symmetric group and work with \mathcal{P}_n . The space of generating operations is $Trias_2 = K[P_2] = K \dashv \oplus K \vdash \oplus K \perp$. The space of operations that one can perform on three variables is $K[P_2 \times P_2] \oplus K[P_2 \times P_2]$. This is the part of degree 3 of the free non- Σ operad generated by $K[P_2]$. The operad *Trias* is completely determined by some subspace $R \subset K[P_2 \times P_2] \oplus K[P_2 \times P_2]$. Let us denote by $(\circ_1)\circ_2$ (resp. $\circ_1(\circ_2)$) the basis vectors of the first (resp. second) summand $K[P_2 \times P_2]$. Then R is generated by the 11 vectors of the form $(\circ_1)\circ_2 - \circ_1(\circ_2)$ obtained from the 11 relations of definition of associative trialgebras (cf. 1.1).

Let us identify the dual of $K[P_2]$ with itself by identifying a basis vector with its dual. Then the dual operad $Trias^!$ is completely determined by $R^\perp \subset K[P_2 \times P_2] \oplus K[P_2 \times P_2]$, where R^\perp is the orthogonal space of R under the quadratic form $\begin{pmatrix} \text{Id} & 0 \\ 0 & -\text{Id} \end{pmatrix}$ (cf. [G-K]).

We claim that, under the identification $\dashv = \prec$, $\vdash = \succ$, $\perp = \cdot$, the space R^\perp is the space R' generated by the vectors obtained from the 7 relations of definition of dendriform trialgebras (cf. 2.1). Indeed, since $\dim K[P_2 \times P_2] \oplus K[P_2 \times P_2] = 18$, $\dim R = 11$ and $\dim R' = 7$, it is sufficient to prove that $\langle v, w \rangle = 0$ for any basis vector v of R and any basis vector w of R' . This is a straightforward checking. We verify this equality in one case, the others are similar. Let $v = (\dashv) \dashv - \dashv(\dashv)$ which we identified with $(\prec) \prec - \prec(\prec)$. We get

$$\langle v, (x \prec y) \prec z - x \prec(y * z) \rangle = 1 - 1 = 0,$$

$$\begin{aligned}
\langle v, (x \succ y) \prec z - x \succ (y \prec z) \rangle &= 0, \\
\langle v, (x * y) \succ z - x \succ (y \succ z) \rangle &= 0, \\
\langle v, (x \succ y) \cdot z - x \succ (y \cdot z) \rangle &= 0, \\
\langle v, (x \prec y) \cdot z - x \cdot (y \succ z) \rangle &= 0, \\
\langle v, (x \cdot y) \prec z - x \cdot (y \prec z) \rangle &= 0, \\
\langle v, (x \cdot y) \cdot z - x \cdot (y \cdot z) \rangle &= 0.
\end{aligned}$$

Hence the dual of the operad *Trias* is the operad *Tridend*. \square

3.2 Trialgebras versus dialgebras. There is a functor from the category of associative trialgebras to the category of associative dialgebras (cf. 1.2), that is a map from the operad *Dias* to the operad *Trias*. This is a map of binary quadratic operads. Its dual is a map from *Tridend* to *Didend*, which gives the functor from the category of dendriform dialgebras to the category of dendriform trialgebras described in 2.3.b. Similarly the dual of the functor $\mathbf{As} \rightarrow \mathbf{Trias}$ is the functor $\mathbf{Tridend} \rightarrow \mathbf{As}$.

3.3 Homology of associative trialgebras. Ginzburg and Kapranov's theory of algebraic operads shows that there is a well-defined chain complex for any algebra A over the binary quadratic operad \mathcal{P} , constructed out of the dual operad $\mathcal{P}^!$ as follows.

The chain complex of the \mathcal{P} -algebra A is $C_n^{\mathcal{P}}(A) = \mathcal{P}^!(n)^* \otimes_{S_n} A^{\otimes n}$ in dimension n and the differential d agrees, in low dimension, with the \mathcal{P} -algebra structure of A

$$\gamma_A(2) : \mathcal{P}(2) \otimes A^{\otimes 2} \rightarrow A$$

under the identification $\mathcal{P}^!(2)^* \cong \mathcal{P}(2)$.

In fact d is characterized by this condition plus the fact that on the cofree $\mathcal{P}^!$ -coalgebra $C_*^{\mathcal{P}}(A) = \mathcal{P}^{!*}(A)$ it is a graded coderivation.

3.4 Proposition. *The chain complex of an associative trialgebra A is given by*

$$C_n^{\mathbf{Trias}}(A) = K[T_n] \otimes A^{\otimes n}, \quad d = \sum_{i=1}^{i=n-1} (-1)^i d_i,$$

where $d_i(t; a_1, \dots, a_n) = (d_i(t); a_1, \dots, a_i \circ_i^t a_{i+1}, \dots, a_n)$, and $d_i(t)$ is the tree obtained from t by deleting the i th leaf and where \circ_i^t is given by

$$\circ_i^t = \begin{cases} \dashv & \text{if the } i\text{th leaf of } t \text{ is left oriented,} \\ \vdash & \text{if the } i\text{th leaf of } t \text{ is right oriented,} \\ \perp & \text{if the } i\text{th leaf of } t \text{ is a middle leaf.} \end{cases}$$

Observe that at a given vertex of a tree there is only one left leaf, one right leaf, but there may be none or several middle leaves. Here are examples for d_1 :

$$\begin{aligned} (\swarrow \searrow ; x, y) &\mapsto (\swarrow \searrow ; x \dashv y) \\ (\swarrow \searrow ; x, y) &\mapsto (\swarrow \searrow ; x \vdash y) \\ (\swarrow \searrow ; x, y) &\mapsto (\swarrow \searrow ; x \perp y). \end{aligned}$$

Proof. First observe that this is a chain complex since the operators d_i satisfy the presimplicial relations

$$d_i d_j = d_{j-1} d_i \text{ for } i < j.$$

Indeed, this relation is either immediate (when i and j are far apart), or it is a consequence of the axioms of associative trialgebras when $j = i + 1$. It suffices to check the case $n = 3$. Let us check explicitly that $d_1 d_1(t; x, y, z) = d_2 d_1(t; x, y, z)$ in the particular case $t = \swarrow \searrow$. The left-hand side gives

$$(\swarrow \searrow ; x, y, z) \mapsto (\swarrow \searrow ; x \perp y, z) \mapsto (\swarrow \searrow ; (x \perp y) \vdash z).$$

The right-hand side gives

$$(\swarrow \searrow ; x, y, z) \mapsto (\swarrow \searrow ; x, y \vdash z) \mapsto (\swarrow \searrow ; x \vdash (y \vdash z)).$$

These two elements are equal by relation 10 in 1.1.

By Theorems 3.1 and 2.6 Ginzburg and Kapranov theory gives, as expected,

$$C_n^{Trias}(A) = K[T_n] \otimes A^{\otimes n}.$$

It is clear from 1.1 that d agrees with the *Trias*-algebra structure of A in low dimension. Since d is completely explicit, the coderivation property, which determines d in the construction of Ginzburg and Kapranov, is immediate to check. \square

3.5 Proposition. *The chain complex of a dendriform trialgebra A is given by*

$$C_n^{Tridend}(A) = K[P_n] \otimes A^{\otimes n}, \quad d = \sum_{i=1}^{i=n-1} (-1)^i d_i,$$

where $d_i(X; a_1, \dots, a_n) = (d_i(X); a_1, \dots, a_i \circ_i^X a_{i+1}, \dots, a_n)$, and $d_i(X)$ is the image of X under the map $d_i : [n] \rightarrow [n-1]$ given by

$$d_i(\underline{r}) = \begin{cases} r-1 & \text{if } i \leq r, \\ r & \text{if } i \geq r+1. \end{cases}$$

and where \circ_i^X is given by

$$\circ_i^X = \begin{cases} \cdot & \text{if } i-1 \in X \text{ and } i \in X, \\ \succ & \text{if } i-1 \notin X \text{ and } i \in X, \\ \prec & \text{if } i-1 \in X \text{ and } i \notin X, \\ * & \text{if } i-1 \notin X \text{ and } i \notin X. \end{cases}$$

Proof. Again, observe that this is a chain complex since the operators d_i satisfy the presimplicial relations

$$d_i d_j = d_{j-1} d_i \text{ for } i < j.$$

Indeed, this relation is either immediate (when i and j are far apart), or it is a consequence of the axioms of dendriform trialgebras when $j = i+1$. It suffices to check the case $n = 3$. We actually do the computation in one particular case, the others are similar:

$$\begin{aligned} d_1 d_2(\{0, 2\}, a_1, a_2, a_3) &= d_1(\{0, 1\}; a_1, a_2 \prec a_3) = (\{0\}; a_1 \cdot (a_2 \succ a_3)), \\ d_1 d_1(\{0, 2\}, a_1, a_2, a_3) &= d_1(\{0, 1\}; a_1 \succ a_2, a_3) = (\{0\}; (a_1 \prec a_2) \cdot a_3), \end{aligned}$$

These two elements are equal by the fifth relation in 2.1. \square

4. Acyclicity of the Koszul complex. By definition the Koszul complex associated to the operad $Trias$ is the differential functor $Tridend \circ Trias$ from Vect to Vect . We will show that it is quasi-isomorphic to the identity functor. Equivalently we have the

4.1 Theorem. *The homology of the free associative trialgebra on V is*

$$H_n^{Trias}(Trias(V)) = \begin{cases} V & \text{if } n=1, \\ 0 & \text{otherwise.} \end{cases}$$

4.2 Corollary. *The operads $Trias$ and $Tridend$ are Koszul operads.*

4.3 Corollary. *The homology of the free dendriform trialgebra on V is:*

$$H_n^{Tridend}(Tridend(V)) = \begin{cases} V & \text{if } n=1, \\ 0 & \text{otherwise.} \end{cases}$$

4.4 Corollary. *Let $f_t^{\mathcal{K}}(x)$ be the generating series of the Stasheff polytope (i.e. of the planar rooted trees), as defined in 1.11. Then one has*

$$f_t^{\mathcal{K}}(x) = \frac{-(1 + (2 + t)x) + \sqrt{1 + 2(2 + t)x + t^2x^2}}{2(1 + t)x}.$$

Proof of the Corollaries. By Ginzburg and Kapranov theory [G-K] the first two Corollaries follow from the vanishing of the homology of the free associative trialgebra.

The last Corollary follows from the functional equation relating the two operads and the computation of the generating series for the associative trialgebra operad (cf. 1.11). \square

Proof of Theorem 4.1.

The acyclicity of the augmented complex $C_*^{Trias}(Trias(V))$ is proved in several steps as follows.

1. We show that it is sufficient to treat the case $V = K$.
2. The chain complex $C_*^{Trias}(Trias(K))$ splits into the direct sum of chain complexes $C_*(u)$, one for each element u in P_m , $m \geq 1$.
3. The chain complex $C_*(u)$ is shown to be the cell complex of a simplicial set $X(u)$.
4. The space $X(u)$ is shown to be the join of spaces $X(v)$ for certain particular elements v in P_m .
5. The spaces $X(v)$ are shown to be contractible by constructing a series of retractions by deformation.

1. *First step.* Recall from Corollary 1.7 that $Trias(V) = \bigoplus_{n \geq 1} K[P_n] \otimes V^{\otimes n}$. Therefore one has

$$\begin{aligned} C_j^{Trias}(Trias(V)) &= K[T_j] \otimes \left(\bigoplus_{n \geq 1} K[P_n] \otimes V^{\otimes n} \right)^{\otimes j} \\ &= K[T_j] \otimes \bigoplus_{m \geq 1} \left(\bigoplus_{n_1 + \dots + n_j = m} K[P_{n_1} \times \dots \times P_{n_j}] \right) \otimes V^{\otimes m}. \end{aligned}$$

Since d is homogeneous in V , the complex C_*^{Trias} splits into the direct sum of subcomplexes, one for each $m \geq 1$. This subcomplex is in fact of finite length and, up to tensoring by $V^{\otimes m}$, is of the following form:

$$\begin{aligned} C_*(P_m) : \quad & 0 \rightarrow K[T_m \times P_1 \times \dots \times P_1] \rightarrow \dots \\ & \rightarrow \bigoplus_{n_1 + \dots + n_j = m} K[T_j \times P_{n_1} \times \dots \times P_{n_j}] \rightarrow \dots \rightarrow K[T_1 \times P_m]. \end{aligned}$$

Recall that P_1 and T_1 have only one element. The case $m = 1$ gives the subcomplex of length 0 reduced to V . This shows that $H_1^{Trias}(Trias(V))$ contains V as expected.

For $m \geq 2$, the differential is simply the differential of $C_*(P_m)$ tensored by the identity of $V^{\otimes m}$, hence it is sufficient to prove the acyclicity of $C_*(P_m)$ to prove the theorem.

2. *Second step.* The chain complex $C_*(P_m)$ can still be split into the direct sum of smaller complexes indexed by the elements u of P_m . Indeed, let $\alpha := (t; u_1, \dots, u_j) \in T_j \times P_{n_1} \times \dots \times P_{n_j}$ be a basis element. Under applying $j - 1$ face operators successively to α , we get an element $(\bigvee; u) \in T_1 \times P_m$ which does not depend on the choice of the face operators because of the simplicial relations (cf. 3.4). Considering t as an operation on m variables for associative trialgebras, u is nothing but the result of the evaluation of t on (x, \dots, x) , cf. 1.1. Fixing u , let $C_*(u)$ be the chain subcomplex linearly generated by the elements α whose image is $(\bigvee; u) \in T_1 \times P_m$. It is clear that $C_*(P_m)$ is the direct sum of the chain complexes $C_*(u)$, $u \in P_m$.

Observe that $C_*(u)$ is of simplicial type, that is, its boundary is of the form $d = -\sum_{i=1}^{i=n-1} (-1)^i d_i$.

3. *Third step.* We fix $u \in P_m$. At this point it is helpful to modify slightly our indexing of the faces and have them to run from 0 to $n - 2$ rather than from 1 to $n - 1$. We also shift the indexing of the complex $C_*(u)$ by 1, putting $K[T_1 \times P_m]$ in dimension -1 . For any generator α of $C_*(u)$ the faces $d_i(\alpha)$, $0 \leq i \leq n - 2$, are still generators of $C_*(u)$. Hence $C_*(u)$ is the normalized augmented complex of an augmented simplicial set that we denote by $X(u)$. The nondegenerate simplices of $X(u)$ are the linear generators α of $C_*(u)$. The top dimensional ones are of the form $(t; x, \dots, x) \in X(u)_{m-2}$ where $t = t_0 \vee \dots \vee t_k \in T_q$. The integer k is the number of decorations (Cech signs) appearing in u . We denote by $T_{\{u\}}$ this subset of T_m . At the other end the augmentation set is $X(u)_{-1} = T_1 \times \{u\}$ (one element). The geometric realization of $X(u)$ is the amalgamation of simplices Δ^{m-2} (one for each $t \in T_{\{u\}}$) under the following rule:

if $d_{i_k} \dots d_{i_1}(t; x, \dots, x) = d_{i_k} \dots d_{i_1}(t'; x, \dots, x)$ for some $m - 2 \geq i_k \geq \dots \geq i_1 \geq 0$, then we identify the corresponding (oriented) faces of the simplices t and t' . Observe that under this rule a vertex of type i is identified only with a vertex of type i .

4. *Fourth step.* Let us first recall the join construction of augmented simplicial sets (cf. for instance [E-P]). An augmented simplicial set is

a simplicial set X , together with a set X_{-1} and a map $d_0 : X_0 \rightarrow X_{-1}$ satisfying $d_1 d_0 = d_0 d_0$. The join of two augmented simplicial sets X and Y is $Z = X * Y$, defined by $Z_n = \bigsqcup_{p+q=n-1} X_p \times Y_q$. The faces are

$$\begin{aligned} d_i(x, y) &= (d_i x, y) \text{ for } 0 \leq i \leq p, \\ d_i(x, y) &= (x, d_{i-p-1} y) \text{ for } p+1 \leq i \leq p+1+q, \end{aligned}$$

and similarly for the degeneracies. The geometric realization of the simplicial join is the topological join

$$X * Y = X \times I \times Y / \{(x, 0, y) \sim (x', 0, y), (x, 1, y) \sim (x, 0, y')\}.$$

In particular one has $\Delta^p * \Delta^q = \Delta^{p+q+1}$.

Let $u = x \cdots x \check{x} x \cdots x \check{x} x \cdots x \in P_m$. By direct inspection we see that $X(u)$ is the simplicial join of the simplicial sets

$$X(x \cdots \check{x}), X(\check{x} \cdots x \cdots \check{x}), \dots, X(\check{x} \cdots x \cdots \check{x}), X(\check{x} \cdots x).$$

The point is that there are only one Cech signs at the extreme locations. Hence it is sufficient to show the contractibility of $X(u)$ in the cases $u = \check{x} \cdots x \cdots \check{x}$ and $u = \check{x} \cdots x$.

5. *Fifth step:* the case $u = \check{x} \cdots x \cdots \check{x}$ or $u = \check{x} \cdots x \in P_m$. We treat in detail the case $u = \check{x} \cdots x$, the other one is similar.

Since $u = \check{x} \cdots x$ the trees t in $T_{\{u\}}$ are of the form

Hence the 0-cell $(d_0)^{m-2}(t; x, \dots, x) = (\bigvee; \check{x} x \cdots x, x)$ is the same for all $t \in T_{\{u\}}$. We denote this vertex by P . In other words, in the amalgamation of the $(m-2)$ -simplices $(t; x, \dots, x)$ giving $X(u)$, all the vertices of type $m-2$ get identified to P .

We will show that there exists a sequence of retractions by deformation

$$X(u) = X(u)^{\langle m-2 \rangle} \rightarrow \dots \rightarrow X(u)^{\langle k \rangle} \xrightarrow{\phi_k} \dots \rightarrow X(u)^{\langle 0 \rangle} = P.$$

The simplicial set $X(u)^{\langle k \rangle}$ is a subsimplicial set of $X(u)$ determined by its nondegenerate k simplices. It is defined inductively as follows. We suppose that $X(u)^{\langle k \rangle}$ has been defined (the induction process begins with $k = m-2$) and we determine $X(u)^{\langle k-1 \rangle}$. On $X(u)^{\langle k \rangle}$ we introduce the equivalence relation generated by: $\alpha \sim \beta$ if either $d_k \alpha = d_k \beta$ or $d_{k-1} \alpha = d_{k-1} \beta$. Then in each equivalence class we pick an element, say α_0 . By definition $X(u)^{\langle k-1 \rangle}$ is made of the elements $d_{k-1} \alpha_0$, one for each equivalence class.

The map ϕ_k is defined by $\phi_k(\alpha) = s_{k-1}d_{k-1}\alpha_0$. On the geometric realization the map ϕ_k consists in collapsing each k -simplex α to its last face (the edge relating the vertices $k - 1$ and k collapses to a point), and then embedding this face into $X(u)$ as $d_{k-1}\alpha_0$. All the collapsing are coherent, and so assemble to give a collapsing of $X(u)^{\langle k \rangle}$ to $X(u)^{\langle k-1 \rangle}$, because one can verify that for each vertex of type $k - 1$ in $X(u)^{\langle k \rangle}$ there is only one edge to the edge relating it to the vertex of type k , that is P .

Here is an illustration for $m = 4$, $u = \check{x}xxx$ and the planar binary trees.

	d_0	d_1	d_2	
$a =$	$(\begin{array}{c} \diagdown \diagup \\ \diagdown \diagup \\ \diagdown \diagup \\ \diagdown \diagup \end{array}; x, x, x, x)$	$(\begin{array}{c} \diagdown \diagup \\ \diagdown \diagup \\ \diagdown \diagup \\ \diagdown \diagup \end{array}; \check{x}x, x, x)$	$(\begin{array}{c} \diagdown \diagup \\ \diagdown \diagup \\ \diagdown \diagup \\ \diagdown \diagup \end{array}; x, x\check{x}, x)$	$(\begin{array}{c} \diagdown \diagup \\ \diagdown \diagup \\ \diagdown \diagup \\ \diagdown \diagup \end{array}; x, x, x\check{x})$
$b =$	$(\begin{array}{c} \diagdown \diagup \\ \diagdown \diagup \\ \diagdown \diagup \\ \diagdown \diagup \end{array}; x, x, x, x)$	$(\begin{array}{c} \diagdown \diagup \\ \diagdown \diagup \\ \diagdown \diagup \\ \diagdown \diagup \end{array}; \check{x}x, x, x)$	$(\begin{array}{c} \diagdown \diagup \\ \diagdown \diagup \\ \diagdown \diagup \\ \diagdown \diagup \end{array}; x, \check{x}x, x)$	$(\begin{array}{c} \diagdown \diagup \\ \diagdown \diagup \\ \diagdown \diagup \\ \diagdown \diagup \end{array}; x, x, x\check{x})$
$c =$	$(\begin{array}{c} \diagdown \diagup \\ \diagdown \diagup \\ \diagdown \diagup \\ \diagdown \diagup \end{array}; x, x, x, x)$	$(\begin{array}{c} \diagdown \diagup \\ \diagdown \diagup \\ \diagdown \diagup \\ \diagdown \diagup \end{array}; \check{x}x, x, x)$	$(\begin{array}{c} \diagdown \diagup \\ \diagdown \diagup \\ \diagdown \diagup \\ \diagdown \diagup \end{array}; x, \check{x}x, x)$	$(\begin{array}{c} \diagdown \diagup \\ \diagdown \diagup \\ \diagdown \diagup \\ \diagdown \diagup \end{array}; x, x, \check{x}x)$
$d =$	$(\begin{array}{c} \diagdown \diagup \\ \diagdown \diagup \\ \diagdown \diagup \\ \diagdown \diagup \end{array}; x, x, x, x)$	$(\begin{array}{c} \diagdown \diagup \\ \diagdown \diagup \\ \diagdown \diagup \\ \diagdown \diagup \end{array}; \check{x}x, x, x)$	$(\begin{array}{c} \diagdown \diagup \\ \diagdown \diagup \\ \diagdown \diagup \\ \diagdown \diagup \end{array}; x, \check{x}x, x)$	$(\begin{array}{c} \diagdown \diagup \\ \diagdown \diagup \\ \diagdown \diagup \\ \diagdown \diagup \end{array}; x, x, \check{x}x)$
$e =$	$(\begin{array}{c} \diagdown \diagup \\ \diagdown \diagup \\ \diagdown \diagup \\ \diagdown \diagup \end{array}; x, x, x, x)$	$(\begin{array}{c} \diagdown \diagup \\ \diagdown \diagup \\ \diagdown \diagup \\ \diagdown \diagup \end{array}; \check{x}x, x, x)$	$(\begin{array}{c} \diagdown \diagup \\ \diagdown \diagup \\ \diagdown \diagup \\ \diagdown \diagup \end{array}; x, x\check{x}, x)$	$(\begin{array}{c} \diagdown \diagup \\ \diagdown \diagup \\ \diagdown \diagup \\ \diagdown \diagup \end{array}; x, x, \check{x}x)$

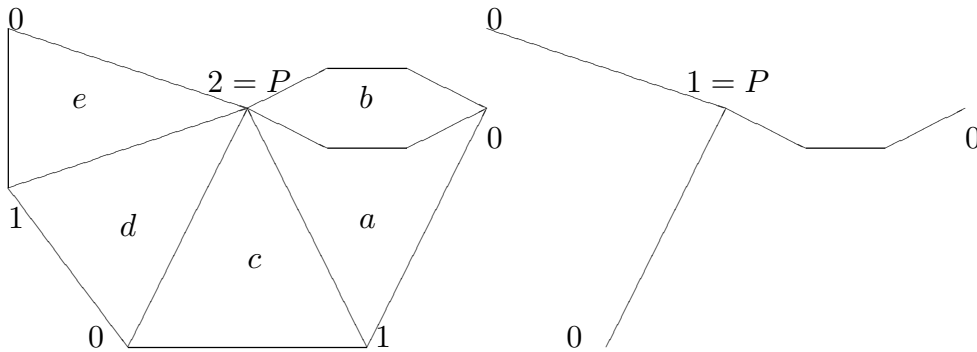
Hence the simplices a, b, c, d, e of type Δ^2 are amalgamated under the following rules:

$$d_0(a) = d_0(b) = d_0(c), \quad d_0(d) = d_0(e), \quad d_1(c) = d_1(d), \quad d_2(a) = d_2(b).$$

The first two spaces of the sequence (binary case)

$$X(\check{x}xxx) = X(\check{x}xxx)^{\langle 2 \rangle} \twoheadrightarrow X(\check{x}xxx)^{\langle 1 \rangle} \twoheadrightarrow X(\check{x}xxx)^{\langle 0 \rangle} = P$$

are shown below:



In the planar tree case $X(u)^{\langle 2 \rangle}$ is made of eleven 2-simplices, $X(u)^{\langle 1 \rangle}$ is made of seven 1-simplices and $X(u)^{\langle 0 \rangle}$ is made of one 0-simplex (namely P).

Since each map ϕ_k is a retraction by deformation, the space $X(u) = X(u)^{\langle m-2 \rangle}$ has the same homotopy type as $X(u)^{\langle 0 \rangle} = P$ hence it is contractible. \square

5. Cubical trialgebras and hypercubes.

One can also associate a type of trialgebras to the family of hypercubes. Once the correct relations are found the proof follows the same pattern as in the previous sections. It turns out that the associated operad is self-dual, so the generating series, which is $f_t^I(x) = \frac{-x}{1+(2+t)x}$, is its own inverse, a fact which is immediate to check: $f_t^I(f_t^I(x)) = x$.

5.1 Definition. A *cubical trialgebra* is a vector space A equipped with 3 binary operations : \dashv called *left*, \vdash called *right* and \perp called *middle*, satisfying the following 9 relations :

$$(x \circ_1 y) \circ_2 z = x \circ_1 (y \circ_2 z)$$

where \circ_1 and \circ_2 are either \dashv or \vdash or \perp .

We obtain the definition of a *cubical dialgebra* by restricting ourself to the first two operations (this structure has been considered earlier by B. Richter [Ri]). We denote by **Tricub** and **Dicub** the associated categories of algebras. There is an obvious functor

$$\mathbf{As} \rightarrow \mathbf{Tricub}$$

consisting in putting $x \dashv y = x \vdash y = x \perp y = xy$.

Let Q_n be the set of cells of the hypercube I^n , where I is the interval $[-1, 1]$. Alternatively Q_n can be described as $\{-1, 0, +1\}^n$ or $\{\dashv, \perp, \vdash\}^n$. Obviously Q_n is graded by the dimension of the cells (resp. the numbers of 0's or \perp signs).

5.2 Proposition. *The free cubical trialgebra on one generator, $\text{Tricub}(K) = \bigoplus_{n \geq 1} \text{Tricub}(n)$ is such that $\text{Tricub}(n) = K[Q_{n-1}]$ with operations:*

$$\begin{aligned} a \dashv b &= (a, -1, b) \in Q_{p+q-1}, \\ a \perp b &= (a, 0, b) \in Q_{p+q-1}, \\ a \vdash b &= (a, +1, b) \in Q_{p+q-1}, \end{aligned}$$

for $a \in Q_{p-1}$ and $b \in Q_{q-1}$.

5.3 Theorem. *The operad \mathbf{Tricub} is self-dual.*

(Observe that $18 - 9 = 9$.) □

5.4 Cubical trialgebras and associative algebras. By Koszul duality the functor $\mathbf{As} \rightarrow \mathbf{Tricub}$ gives a functor $\mathbf{Tricub} \rightarrow \mathbf{As}$ since both operads are self-dual. It is immediately seen that it is given by putting $x * y := x \dashv y + x \vdash y + x \perp y$. So a cubical trialgebra is an associative algebra for which the associative operation is the sum of three operations and the associative relation splits into 9 relations.

5.5 Proposition. *The homology of a cubical trialgebra A is given by the following chain complex $C_n^{\mathbf{Tricub}}(A)$:*

$$C_n^{\mathbf{Tricub}}(A) = K[Q_n] \otimes A^{\otimes n}, \quad d = - \sum_{i=1}^{i=n-1} (-1)^i d_i,$$

where $d_i(X; a_1, \dots, a_n) = (d_i(X); a_1, \dots, a_i \circ_i^X a_{i+1}, \dots, a_n)$, and the element $d_i(X)$ is obtained from X by deleting the i th coordinate X_i , and the operation \circ_i^X is given by

$$\circ_i^X = \begin{cases} \dashv & \text{if } X_i = -1, \\ \perp & \text{if } X_i = 0, \\ \vdash & \text{if } X_i = +1. \end{cases}$$

5.6 Theorem. *Let $\mathbf{Tricub}(V)$ be the free cubical trialgebra on V . Its homology is*

$$H_n^{\mathbf{Tricub}}(\mathbf{Tricub}(V)) = \begin{cases} V & \text{if } n=1, \\ 0 & \text{otherwise.} \end{cases}$$

5.7 Corollary. *The operad \mathbf{Tricub} is a Koszul operad.*

Proof. The same arguments as in the proof of theorem 4.1 lead to the chain complex

$$\begin{aligned} 0 \rightarrow K[Q_m \times Q_1 \times \dots \times Q_1] \rightarrow \dots \\ \rightarrow \bigoplus_{n_1 + \dots + n_j = m} K[Q_j \times Q_{n_1} \times \dots \times Q_{n_j}] \rightarrow \dots \rightarrow K[Q_1 \times Q_m]. \end{aligned}$$

This complex is the direct sum of complexes $C_*(u)$, one for each generator u of Q_m . By direct inspection we see that $C_*(u)$ is nothing

but the normalized augmented chain complex of the standard simplex Δ^{m-1} , hence it is acyclic. □

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