Parking functions and triangulation of the associahedron

Jean-Louis Loday

Abstract. We show that a minimal triangulation of the associahedron (Stasheff polytope) of dimension n is made of $(n+1)^{n-1}$ simplices. We construct a natural bijection with the set of parking functions from a new interpretation of parking functions in terms of shuffles.

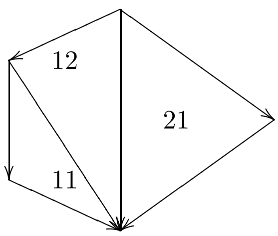
Introduction

The Stasheff polytope, also known as the associahedron, is a polytope which comes naturally with a poset structure on the set of vertices (Tamari poset), hence a natural orientation on each edge. We decompose this polytope into a union of oriented simplices, the orientation being compatible with the poset structure. This construction defines the associahedron as the geometric realization of a simplicial set. In dimension n the number of (non-degenerate) simplices is $(n+1)^{n-1}$.

A parking function is a permutation of a sequence of integers $i_1 \leq \cdots \leq i_n$ such that $1 \leq i_k \leq k$ for any k. For fixed n the number of parking functions is $(n+1)^{n-1}$. We show that the set PF_n of parking functions of length n admits the following inductive description:

$$PF_n = \bigcup_{\substack{p+q=n-1 \\ p\geq 0, q\geq 0}} \{1, \dots, p+1\} \times Sh(p,q) \times PF_p \times PF_q$$

where $Sh(p,q)$ is the set of (p,q)-shuffles. From this bijection we deduce a natural bijection between the top dimensional simplices of the associahedron and the parking functions.



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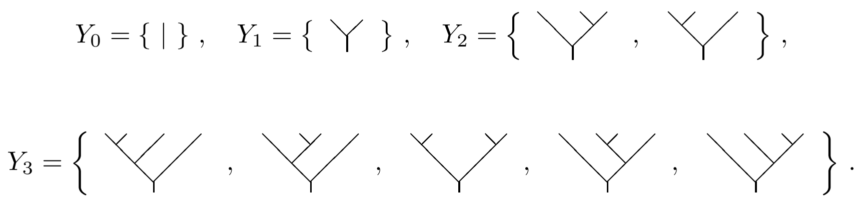
In the last section we investigate a similar triangulation of the permutohedron.

Thanks to André Joyal for mentioning to me Abel's formula during the Street's fest.

1. Associahedron

1.1. Planar trees, Stasheff complex. The associahedron can be constructed as a cellular complex as follows (cf. for instance [**BV**]).

Let Y_n be the set of planar binary rooted trees with n internal vertices:



Observe that $t \in Y_{n+1}$ has n internal edges. For each $t \in Y_{n+1}$ we take a copy of the cube I^n (where $I = [0,1]$ is the interval) which we denote by I^n_t. Then the associahedron of dimension n is the quotient

$$\mathcal{K}^n := \bigsqcup_t I^n_t / \sim$$

where the equivalence relation is as follows. We think of an element $\tau = (t; \lambda_1, \dots, \lambda_n) \in I^n_t$ as a tree of type t where the λ_i's are the lengths of the internal edges. If some of the λ_i's are 0, then the geometric tree determined by τ is not binary anymore (since some of its internal edges have been shrinked to a point). We denote the new tree by $\bar{\tau}$. For instance, if none of the λ_i's is zero, then $\bar{\tau} = t$; if all the λ_i's are zero, then the tree $\bar{\tau}$ is the corolla (only one vertex). The equivalence relation $\tau \sim \tau'$ is defined by the following two conditions:

- $\bar{\tau} = \bar{\tau'}$,
- the lengths of the nonzero-length edges of τ are the same as those of τ'.

Hence $\mathcal{K}^n$ is obtained as a cubical realization:

$\mathcal{K}^1$, $\mathcal{K}^2$

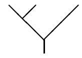
Since a cube can be decomposed into simplices, we can get a simplicial decomposition of $\mathcal{K}^n$. However our aim is to construct a minimal simplicialization. It was shown by Stasheff in [**Sta**] that $\mathcal{K}^n$ is homeomorphic to a ball. This is clear from the above construction. In fact this Stasheff complex can be realized as a polytope (cf. [**Lee**],[**GKZ**],[**SS**],[**Lod2**]). One way to construct it is recalled in the following section (taken from [**Lod2**]).

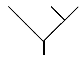
1.2. Associahedron. To any tree $t \in Y_n, n > 0$, we associate a point $M(t) \in \mathbb{R}^n$ with integral coordinates as follows. Let us number the leaves of t from left to right by $0, 1, \ldots, n$. So one can number the vertices from 1 to n (the vertex number i is in between the leaves $i-1$ and i). We consider the subtree generated by the ith vertex. Let a_i be the number of offspring leaves on the left side of the vertex i and let b_i be the number of offspring leaves on the right side. Observe that these numbers depend only on the subtree determined by the vertex i. We define:

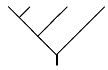
$$M(t) := (a_1 b_1, \ldots, a_i b_i, \ldots, a_n b_n) \in \mathbb{R}^n \ .$$

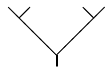
In low dimension we get:

$M($  $) = (1 \times 1) = (1),$

$M($  $) = (1 \times 1, 2 \times 1) = (1, 2),$

$M($  $) = (1 \times 2, 1 \times 1) = (2, 1),$

$M\Big($  $\Big) = (1 \times 1, 2 \times 1, 3 \times 1) = (1, 2, 3),$

$M\Big($  $\Big) = (1 \times 1, 2 \times 2, 1 \times 1) = (1, 4, 1).$

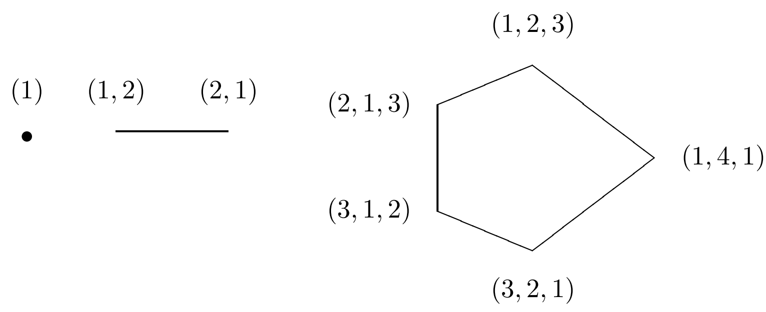
The planar binary trees with n internal vertices are in bijection with the parenthesizings of the word $x_0 x_1 \cdots x_{n+1}$. For the tree corresponding to the parenthesizing $(((x_0x_1)x_2)(x_3x_4))$ one gets the following point $(1\times 1, 2\times 1, 3\times 2, 1\times 1) = (1, 2, 6, 1)$.

Denote by H_n the hyperplane of $\mathbb{R}^n$ whose equation is

$$x_1 + \cdots + x_n = \frac{n(n+1)}{2}.$$

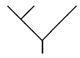
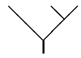
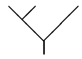
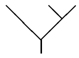
One can show that for any tree $t \in Y_n$ the point $M(t)$ belongs to the hyperplane H_n.

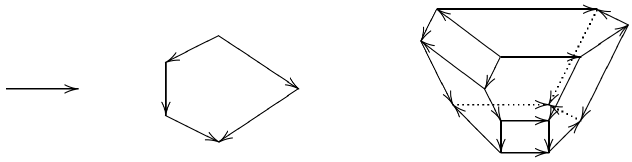
Then by [**Lod2**] the *associahedron* or *Stasheff polytope* $\mathcal{K}^{n-1}$ is the convex hull of the points $M(t)$ in the hyperplane H_n for $t \in Y_n$.



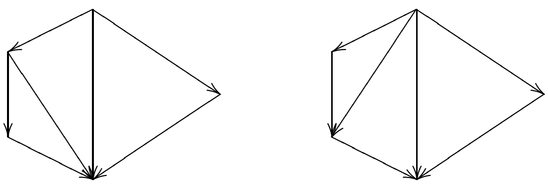
$\mathcal{K}^0$ $\quad$ $\mathcal{K}^1$ $\quad$ $\mathcal{K}^2$

An interesting variation of this construction has been performed by Hohlweg and Lange in [**HL**].

1.3. Order structure. Let us recall that, on the set Y_n, there is a partial order known as the Tamari order. It is induced by the order  $\to$  on Y_2 as follows. There is a covering relation $t \to t'$ between two elements of Y_n if t' can be obtained from t by replacing locally a subtree of the form  by a subtree of the form  . In low dimension the covering relations induce the following order on the vertices:

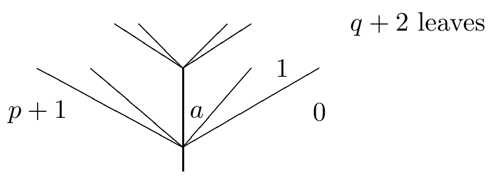


Our aim is to triangulate (we mean simplicialize) the associahedron $\mathcal{K}^n$ by (oriented) n-simplices, so that the oriented edges of the simplices are coherent with the Tamari order. We observe immediately that there are two choices for $\mathcal{K}^2$:



We choose the first one and we will construct a triangulation for $\mathcal{K}^n$ consistent with this choice.

1.4. Boundary of $\mathcal{K}^n$. The cells of the associahedron are in bijection with the planar rooted trees (see for instance [**LR2**]). The vertices correspond to the binary trees and the big cell corresponds to the corolla. The boundary of $\mathcal{K}^n$, denoted $\partial\mathcal{K}^n$, is made of cells of the form $\mathcal{K}^p \times \mathcal{K}^q$, $p + q = n - 1$. They are in bijection with the trees $\gamma(a; p, q)$ with two vertices:



Here $p + 2$ is the number of outgoing edges at the root, $q + 2$ is the number of outgoing edges (leaves) at the other vertex, and a is the index of the only edge which is not a leaf, so $0 \leq a \leq p + 1$. By convention we index the edges at the root from 0 to $p + 1$ from right to left.

Let us denote by S (like South pole) the vertex of $\mathcal{K}^n$ with coordinates $(n, n-1, \ldots, 1)$, which corresponds to the right comb.

PROPOSITION 1.1. *The associahedron $\mathcal{K}^n$ is the cone with vertex S over the union of the cells $\gamma(a;p,q), a \geq 1$, in $\partial\mathcal{K}^n$.*

Proof. Since $\mathcal{K}^n$ is a ball, $\partial\mathcal{K}^n$ is a sphere. The n-cells of $\partial\mathcal{K}^n$ which contain S are such that $a = 0$, because the tree of such a cell is obtained from the right comb (by collapsing $n-2$ edges). The other ones, for which $a \geq 1$, form a ball of dimension $n-1$ and obviously $\mathcal{K}^n$ can be viewed as a cone with vertex S over this $(n-1)$-dimensional ball. □

THEOREM 1.2. *The associahedron $\mathcal{K}^n$ can be constructed out of $\mathcal{K}^{n-1}$ as follows:*

(a) start with $\mathcal{K}^{n-1}$,

(b) "fatten" $\mathcal{K}^{n-1}$ by replacing its boundary faces $\gamma(a-1;p-1,q)$ (of the form $\mathcal{K}^{p-1} \times \mathcal{K}^q$), $p+q = n-1$, by $\gamma(a;p,q)$ (of the form $\mathcal{K}^p \times \mathcal{K}^q$),

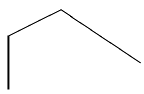
(c) take the cone over the fat-$\mathcal{K}^{n-1}$.

Proof. From Proposition 1.1 it suffices to check that, in $\mathcal{K}^n$, the union of the faces $\gamma(a;p,q)$, $a \geq 1$, is precisely fat-$\mathcal{K}^{n-1}$. Indeed, $\gamma(1;0,n-1)$ corresponds to $\mathcal{K}^{n-1}$ and all the other faces $\gamma(a;p,q)$, $a \geq 1, p \geq 1$, come from the cells $\gamma(a-1;p-1,q)$ of $\partial\mathcal{K}^{n-1}$ by fattening. □

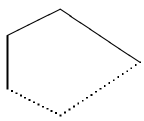
1.5. Examples. $n = 1$:

– $\mathcal{K}^1$

– fattened $\mathcal{K}^1$

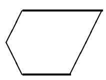


– Cone over fat-$\mathcal{K}^1$ $= \mathcal{K}^2$

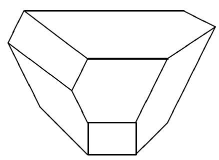


Example $n = 2$:

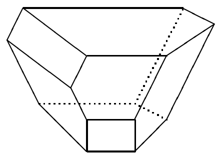
– $\mathcal{K}^2$



– fattened $\mathcal{K}^2$



– Cone over fat-$\mathcal{K}^2 = \mathcal{K}^3$

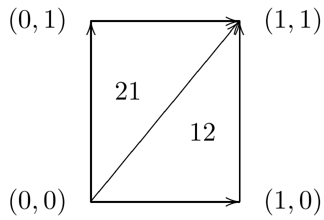


2. Triangulation of the associahedron

From the construction of $\mathcal{K}^n$ out of $\mathcal{K}^{n-1}$ performed in the preceding section, it is clear that one can triangulate $\mathcal{K}^n$ by induction.

2.1. Product of simplices. Let us recall that, if Δ^n denotes the standard simplex, then, in the triangulation of $\Delta^p \times \Delta^q$, the simplices are indexed by the (p, q)-shuffles.

For instance the triangulation of $\Delta^1 \times \Delta^1$ is:



The triangulation of $\Delta^2 \times \Delta^1$ (a prism) is made of three tetrahedrons:

shuffle	vertices			
123	$(0, 0)$,	$(1, 0)$,	$(2, 0)$,	$(2, 1)$
132	$(0, 0)$,	$(1, 0)$,	$(1, 1)$,	$(2, 1)$
312	$(0, 0)$,	$(0, 1)$,	$(1, 1)$,	$(2, 1)$

Here (i, j) stands for the point of $\Delta^2 \times \Delta^1$ which is the ith vertex of Δ^2 times the jth vertex of Δ^1.

THEOREM 2.1. *The associahedron $\mathcal{K}^n$ admits a triangulation by $(n+1)^{n-1}$ simplices, whose orientation is compatible with the Tamari order on the set of vertices. An n-simplex of this triangulation is a cone over an $(n-1)$-simplex determined by the following choices. Choose either*

- an $(n-1)$-simplex of $\mathcal{K}^{n-1}$, or

- in the fattened cell $\gamma(a; p, q)$ isomorphic to $\mathcal{K}^p \times \mathcal{K}^q$ choose a p-simplex of $\mathcal{K}^p$, a q-simplex of $\mathcal{K}^q$ and a (p, q)-shuffle.

Proof. The proof of the second assertion follows from Theorem 1.2 and the fact that the triangulation of $\Delta^p \times \Delta^q$ is indexed by the (p, q)-shuffles.

From this description of the triangulation we can count the number d_n of top dimensional simplices of $\mathcal{K}^n$ by induction. Let us suppose that $d_p = (p+1)^{p-1}$ for $p < n$, and recall that the number of (p, q)-shuffles is the binomial coefficient $\binom{p+q}{q}$. If p is fixed, then a can take the values $1, \ldots, p+1$. Hence there are $p+1$ cells of the form $\mathcal{K}^p \times \mathcal{K}^q, p+q=n-1$. We get

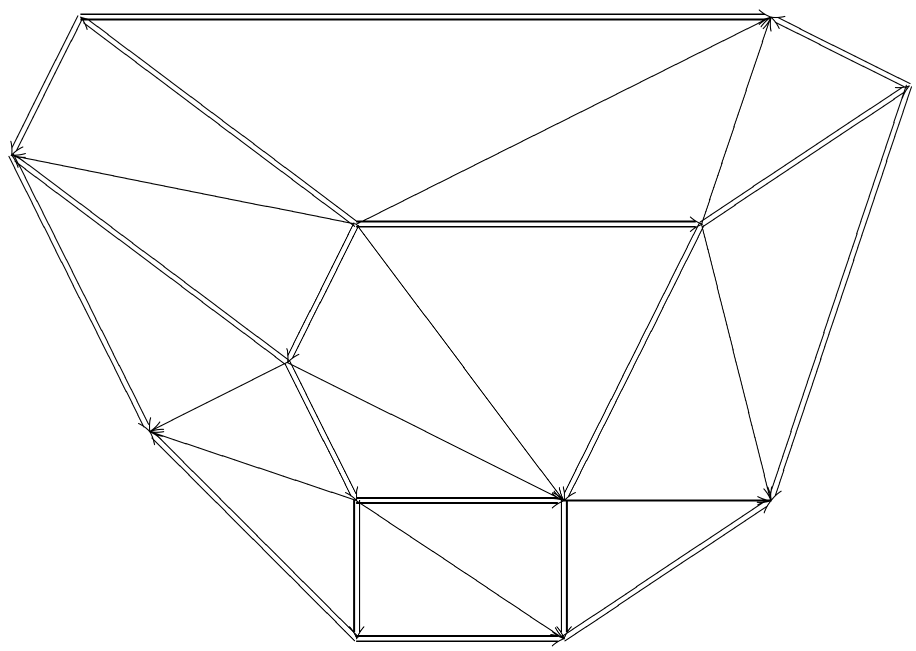
$$\begin{aligned} d_n &= \sum_{p=0}^{n-1}(p+1)\binom{n-1}{p} d_p d_{n-p-1} \\ &= \sum_{p=0}^{n-1}\binom{n-1}{p}(p+1)^p(n-p)^{n-p-2} \\ &= (n+1)^{n-1}. \end{aligned}$$

The last equality is a particular case of Abel's formula, cf. [**R**],

$$x^{-1}(x+y+n)^n = \sum_{k=0}^{n}\binom{n}{k}(x+k)^{k-1}(y+n-k)^{n-k}$$

for $x=y=1$ and $k=n-p$. Observe that, at each step of the construction of $\mathcal{K}^n$ out of $\mathcal{K}^{n-1}$, the orientation of the simplices coincides with the orientation of the edges given by the Tamari poset order. □

Example: triangulation of fat-$\mathcal{K}^2$ giving rise to the triangulation (by tetrahedrons) of $\mathcal{K}^3$:



3. Parking functions

A *parking function* is a sequence of integers $(i_1, \ldots, i_n)$ such that the associated ordered sequence $j_1 \leq \ldots \leq j_n$ satisfies the following conditions: $1 \leq j_k \leq k$ for any k. For instance there is only one parking function of length one : (1), there are three parking functions of length two: $(1,2),(2,1),(1,1)$. There are sixteen parking functions of length three: the permutations of $(1,2,3),(1,1,3),(1,2,2),(1,1,2),(1,1,1)$

(remark that 6+3+3+3+1=16). It is well-known, cf. for instance [**NT**], that there are $(n+1)^{n-1}$ parking functions of length n. We denote by PF_n the set of parking functions of length n. We denote by $Sh(p,q)$ the set of permutations which are (p,q)-shuffles.

THEOREM 3.1. *For any n there is a bijection*

$$\pi : \bigcup_{\substack{p+q=n-1\\ p\geq 0, q\geq 0}} \{1,\ldots,p+1\}\times Sh(p,q)\times PF_p\times PF_q \longrightarrow PF_n$$

given by $\pi(a,\theta;f,g) = (a,\theta_*(f_1,\ldots,f_p,p+1+g_1,\ldots,p+1+g_q))$.

Proof. Let $a\in\{1,\ldots,p+1\}, \theta\in Sh(p,q), f\in PF_p$ and $g\in PF_q$.

Let us first show that the sequence

$$x := (a,\theta_*(f_1,\ldots,f_p,p+1+g_1,\ldots,p+1+g_q))$$

is a parking function. Let $(\phi_1,\ldots,\phi_p)$, resp. $(\psi_1,\ldots,\psi_q)$, be the sequence f, resp. g, put in increasing order. In the sequence x put in increasing order we first find the sequence f with the number a inserted, then the sequence ψ. Since the sequence ϕ has p elements and $1\leq a\leq p+1$ the expected inequality is true for a. It is also true for all the elements of ϕ since ϕ_j is either at the place j or at the place $j+1$. The expected inequality is true for all the elements of the sequence $p+1+\psi$ since $p+1+\psi_j$ is at the place $p+1+j$. Hence x is a parking function.

Let us now construct a map in the other direction. Let

$$\underline{a} = (a = a_1, a_2, \ldots, a_n)$$

be a parking function, referred to as the original sequence. Let $\underline{x} = (x_1,\ldots,x_n)$ be the ordered sequence where $x_j = a$. Let k be the smallest integer such that $k > j$ and $x_k = k$. Then we put $p+2 = k$. It may happen that there is no such integer. In that case we put $p+2 = n+1$, that is $p+1 = n$ and so $q = 0$. With these choices there exists $\sigma\in S_{k-2}, \sigma'\in S_{n-k+1}$ and $\theta\in Sh(k-2,n-k+1)$ such that $(a,\theta\circ(\sigma\times\sigma')(x_1,\cdots,x_{j-1},x_j,\cdots,x_k,\cdots,x_n))$ is the original sequence $\underline{a}$.

Example: $\underline{a} = (3,6,1,7,2,1,3,6)$. Then we get $\underline{x} = (1,1,2,\underline{3},3,6,6,7)$ (where a has been underlined), $j = 4$ and $k = 6$, therefore $a = 3, p = 4, q = 3$. The two parking functions are $(1,2,1,3)$ and $(1,2,1)$ and the $(4,3)$-shuffle is the permutation whose action on $u_1u_2u_3u_4v_1v_2v_3$ gives $v_1u_1v_2u_2u_3u_4v_3$.

Hence we have constructed a map

$$PF_n \longrightarrow \bigcup_{\substack{p+q=n-1\\ p\geq 0, q\geq 0}} \{1,\ldots,p+1\}\times Sh(p,q)\times PF_p\times PF_q.$$

In order to show that it is the inverse of the previous map, it is sufficient to verify that our algorithm gives $k-2=p$ when we start with a parking function of the form $(a,\mathrm{sh}(\mathrm{pf}(1,\cdots,p),\mathrm{pf}(p+2,\cdots,p+1+q))$. First, in the ordered sequence of a parking function the first element is always 1, hence $p+2$ is the smallest element in $\mathrm{pf}(p+2,\cdots,p+1+q)$, or $q=0$ and $p=n+1$. Second, we know that $a\leq p+1$, so in the ordered sequence $a = x_j$ appears before $p+2=k$, hence $p+2$ is at the place $p+2$, whence $k>j$ and we are done. □

3.1. Remark. As a Corollary we get from Abel's formula (cf. the proof of Theorem 2.1) the well-known result:

$$\#PF_n = (n+1)^{n-1}.$$

3.2. Examples.

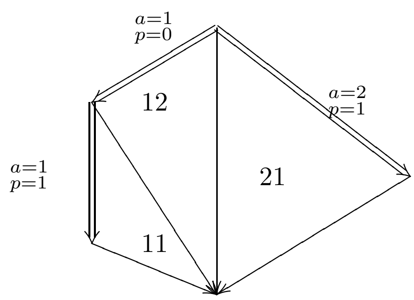
n	a	p	q	parking functions		
1	1	0	0	(1)		
2	1	1	0	(1,1)		
	1	0	1	(1,2)		
	2	1	0	(2,1)		
3	1	2	0	(1,1,1)	(1,1,2)	(1,2,1)
	1	1	1	(1,1,3)	(1,3,1)	
	1	0	2	(1,2,2)	(1,2,3)	(1,3,2)
	2	2	0	(2,1,1)	(2,1,2)	(2,2,1)
	2	1	1	(2,1,3)	(2,3,1)	
	3	2	0	(3,1,1)	(3,1,2)	(3,2,1)

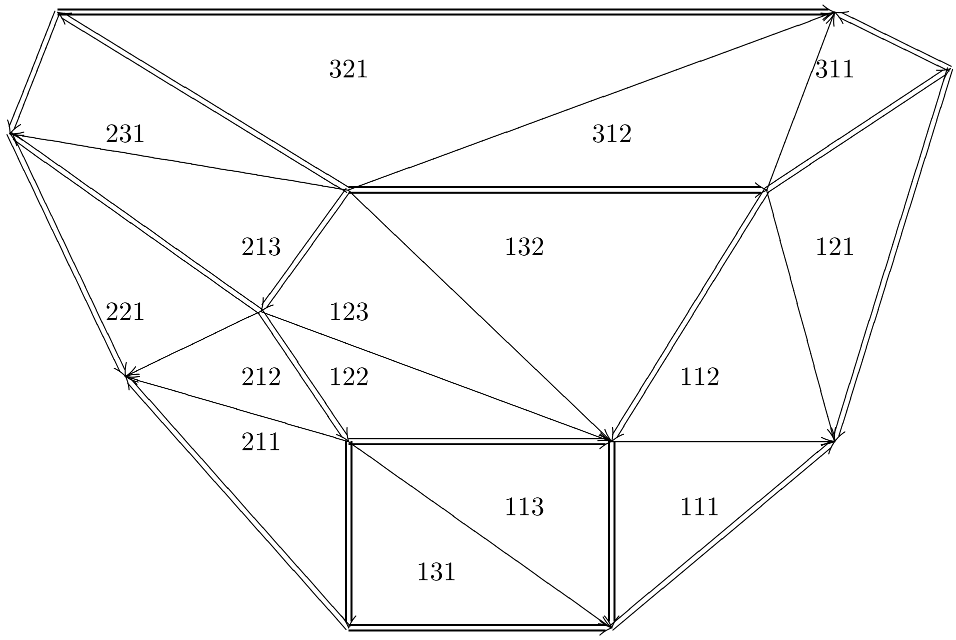
In the following statement we use Theorem 2.1.

THEOREM 3.2. *Let σ be a simplex of $\mathcal{K}^n$ determined either by*
– a simplex ω of $\mathcal{K}^{n-1}$, or by
– a triple (a,p,q) (determining a face $\mathcal{K}^p \times \mathcal{K}^q$), where $1 \le a \le p+1$, $p+q = n-1$, and a simplex α of $\mathcal{K}^p$, a simplex β of $\mathcal{K}^q$ and a (p,q)-shuffle θ.
The map Φ_n, which assigns to σ the parking function
$\Phi_n(\sigma) := \big(1, 1+\Phi_{n-1}(\theta)\big)$ *in the first case,*
$\Phi_n(\sigma) := \big(a, \theta_*(\Phi_p(\alpha), p+1+\Phi_q(\beta))\big)$ *in the second one,*
is a bijection from the set of n-simplices of $\mathcal{K}^n$ to the set PF_n of parking functions.

Proof. We work by induction on n. For $n=1$ there is no choice: $\Phi(\text{1-cell}) = (1)$. In the description of the triangulation of $\mathcal{K}^n$ given in Theorem 2.1 we have constructed a bijection between the set of n-simplices of $\mathcal{K}^n$ and the set $\bigcup_{\substack{p+q=n-1\\ p\ge 0, q\ge 0}} \{1,\dots,p+1\} \times Sh(p,q) \times PF_p \times PF_q$. By Theorem 3.1 this set is in bijection with PF_n. It is immediate to check that the composite of the two bijections is the map Φ_n described in the statement of the Theorem. □

3.3. Examples.





3.4. Simplicial associahedron. Another way of stating Theorem 2.1 and Theorem 3.2 is to say that we have constructed a simplicial set $K_{\cdot}^{n}$ such that $K_0^n = Y_n$, $K_n^n/\{\text{degenerate elements}\} = PF_n$, and such that the geometric realization is $|K_{\cdot}^{n}| = \mathcal{K}^n$. Let $c(p,q)$ be the number of non-degenerate simplices in K_p^q. So $c(p,0) = c_{p+1}$ is the Catalan number ($c_n = \frac{1}{n+1}\binom{2n}{n}$) and $c(p,p) = (p+1)^{p-1}$. Is there a close formula for the series $\sum_{p\geq 0, q\geq 0} c(p,q)\ x^p y^q$?

3.5. A property of the Catalan sets and layers of the associahedron. The well-known formula

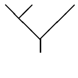
$$c_n = \sum_{\substack{k\geq 1\\ i_1+\cdots+i_k=n-k}} c_{i_1}\times\cdots\times c_{i_k}$$

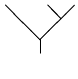
relating the Catalan numbers is a consequence of a bijection

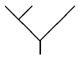
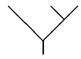
$$\phi : \bigcup_{\substack{k\geq 1\\ i_1+\cdots+i_k=n-k}} Y_{i_1}\times\cdots\times Y_{i_k} \longrightarrow Y_n$$

given by $(t_1,\cdots,t_k)\mapsto t_1\vee(t_2\vee(\cdots\vee(t_k\vee|)\cdots))$. For instance:

$(|)$ $\mapsto$  ,

() $\mapsto$  ,

$(|,|)$ $\mapsto$  .

This bijection induces a decomposition of Y_n as $Y_n = \bigcup_k Y_n^{(k)}$. On the geometric construction of $\mathcal{K}^n$ out of $\mathcal{K}^{n-1}$ it can be seen as follows. Let us say that a vertex of $\mathcal{K}^n$ lies in the k-th layer if, in the tree coding this vertex, the numbers of internal vertices on the segment going from the root to the rightmost leaf is k. For instance $k = 1$ for  , and $k = 2$ for  . In the construction of $\mathcal{K}^n$ out of $\mathcal{K}^{n-1}$ the vertices of $\mathcal{K}^{n-1}$ lie in layer 1. By induction the set of vertices of the fat-$\mathcal{K}^{n-1}$ is stratified by layers. Vertices in layer k and $k+1$ are, either not related, or related by an edge. The last layer (number $n+1$) contains only the South pole (right comb).

It is immediate to check that this decomposition into layers of the set Y_n of vertices of $\mathcal{K}^n$ corresponds exactly to the decomposition $Y_n = \bigcup_k Y_n^{(k)}$ induced by ϕ.

Algebraically the bijection ϕ induces an isomorphism

$$T\big(\bigoplus_{n>0} \mathbb{Z}[Y_{n-1}]\big) \longrightarrow \bigoplus_{n\geq 0} \mathbb{Z}[Y_n],$$

where $T(V)$ is the tensor module over V. Under this isomorphism the image of the tree $t \in Y_{n-1}$ is the tree $t \vee | \in Y_n$ (and the image of $1 \in \mathbb{Z}$ is the tree $|$). This isomorphism is going to play a role in the Poincaré-Birkhoff-Witt isomorphism for the triple of operads (As, OU, Mag), where As is the operad of associative algebras, Mag is the operad of magmatic algebras. The operad OU of *OverUnder* algebras is determined by two associative operations x/y (over) and $x\backslash y$ (under) satisfying $(x/y)\backslash z = x/(y\backslash z)$. The free OU-algebra on one generator is $\bigoplus_{n\geq 0} \mathbb{Z}[Y_n]$ where the operation x/y consists in grafting the root of x on the leftmost leaf of y and the operation $x\backslash y$ consists in grafting the root of y on the rightmost leaf of x. The proof will be part of a forthcoming paper.

3.6. Associahedron and cube. It is well-known that for the cube I^n the triangulation is indexed by the permutations, elements of the symmetric group S_n. In [**Lod2**] we observed that the associahedron is contained in a certain cube. Locally around the North Pole (vertex with coordinates $(1, 2, \ldots, n)$), the triangulation of the cube and the triangulation of the associahedron coincide. Our indexing of the simplices of the associahedron is such that the two indexings also coincide.

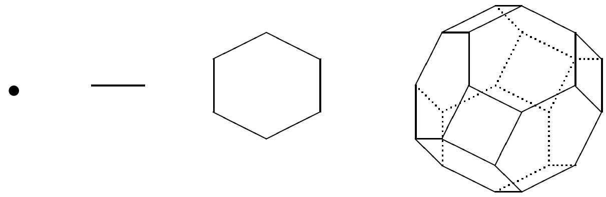
3.7. Other relationship between parking functions and associahedron. There are other links between parking functions and associahedron which are treated in [**PS**] by Pitman and Stanley, in [**Po**] by Postnikov and by Hivert (personal communication). It would be interesting to compare all of them.

3.8. Relationship with orientals. The drawings of triangulated associahedron appear (up to orientation) in the paper [**Str**] by Ross Street. See for instance p. 290. It woud be interesting to unravel a more precise relationship.

4. Triangulation of the permutohedron

In this section we briefly indicate how to simplicialize the permutohedron along the same line as the associahedron. So far we do not know of a nice combinatorial object playing the role of the parking functions.

4.1. Permutohedron. Let us recall that the *permutohedron* $\mathcal{P}^{n-1}$ is the convex hull of the points $M(\sigma) = (\sigma(1), \dots, \sigma(n)) \in \mathbb{R}^n$, where $\sigma \in S_n$ is a permutation.

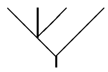


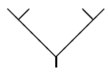
$\mathcal{P}^0$ $\mathcal{P}^1$ $\mathcal{P}^2$ $\mathcal{P}^3$

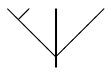
The weak Bruhat order on S_n is a partial order whose covering relations are in one to one correspondence with the edges of $\mathcal{P}^{n-1}$.

It is helpful to replace the permutations by *planar binary trees with levels.* See for instance [**LR1**] for a discussion of this framework. Under this remplacement the faces of the permutohedron $\mathcal{P}^n$ are labelled by the planar leveled trees with $n+2$ leaves which have only two levels. Let us denote by $[p]$ the totally ordered set $\{0, \dots, p\}$. A planar leveled tree with $n+2$ leaves and two levels is completely determined by the arity of the root, let us say $p+1$, and an ordered surjective map $f : [n+1] \to [p+1]$, $0 \le p \le n-1$. We denote the corresponding face by $\gamma(p; f)$.

Example of faces of $\mathcal{P}^2$:

$\gamma(0; f)$ =  where the image of f is (0,0,0,1),

$\gamma(0; f)$ =  where the image of f is (0,0,1,1),

$\gamma(1; f)$ =  where the image of f is (0,0,1,2).

The vertex $M(n+1, n, \dots, 1)$, whose corresponding tree is the right comb, is called the South pole. For a given p the face corresponding to the surjective map f_0, whose image is $(0, 1, \dots, p, p, \dots, p)$, contains the South pole. We denote by $SM(n, p)$ the set of ordered surjective maps from $[n+1]$ to $[p+1]$ minus the map f_0. For instance $SM(4, 1)$ has $10 - 1 = 9$ elements.

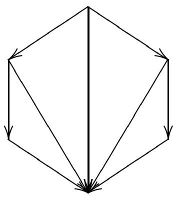
4.2. Triangulation of $\mathcal{P}^n$. We construct a simplicialization of $\mathcal{P}^n$ by induction as follows. For $n = 0$, the space $\mathcal{P}^0$ is a point (0-simplex). For $n = 1$ $\mathcal{P}^1$ is the interval (1-simplex). The permutohedron $\mathcal{P}^n$ is the cone over the South pole with basis the union of the faces which do not contain the South pole, that is the faces whose ordered surjective map $f : [n+1] \to [p+1]$ is not the map f_0 given by $(0, 1, \dots, p+1, p+1, \dots, p+1)$. Therefore an n-simplex of the triangulation of $\mathcal{P}^n$ is the cone over the South pole S with basis an $(n-1)$-simplex of the triangulation of the union of faces not containing S. Such an $(n-1)$-simplex is completely determined by the following choices

- a face $\gamma(p; f)$ not containing S (i.e. $f \neq f_0$),

- a top dimensional simplex in $\mathcal{P}^p$,
- a top dimensional simplex in $\mathcal{P}^{n-p-1}$,
- a shuffle in $Sh(p, n-p-1)$.

From this choice, it is clear, by induction, that the orientation of the simplices are compatible with the orientation of the edges induced by the weak Bruhat order.

Example: triangulation of $\mathcal{P}^2$:



In conclusion we have proved the following result.

THEOREM 4.1. *The set ZP_n of top dimensional simplices (n-simplices) of the permutohedron $\mathcal{P}^n$ satisfies the following recursive formula*

$$ZP_n = \bigcup_{p=0}^{n-1} SM(n,p) \times Sh(p, n-p-1) \times ZP_p \times ZP_{n-p-1},$$

where $SM(n,p) = \binom{n+1}{p+1} - 1$, and $Sh(p, n-p-1) = \binom{n-1}{p}$.

□

The number of top simplices is as follows in low dimension:

n	0	1	2	3	4	5	6	7	8
$\#ZP_n$	1	1	4	34	488	10512	316224	12649104	649094752

4.3. Permutohedron analogue of parking functions. It would be interesting to find a sequence of combinatorial objects analogous to the parking functions, whose number of elements is $\#ZP_n$, that is satisfythe combinatorial relation:

$$\#ZP_n = \sum_{p=0}^{n-1} \left(\binom{n+1}{p+1} - 1 \right) \times \binom{n-1}{p} \times \#ZP_p \times \#ZP_{n-p-1}$$

4.4. Appendix. Some integer sequences.

n	0	1	2	3	4	$\cdots$	n	$\cdots$
vertices of Δ^n	1	2	3	4	5	$\cdots$	$n+1$	$\cdots$
simplices of Δ^n	1	1	1	1	1	$\cdots$	1	$\cdots$
vertices of I^n	1	2	4	8	16	$\cdots$	2^n	$\cdots$
simplices of I^n	1	1	2	6	24	$\cdots$	$n!$	$\cdots$
vertices of $\mathcal{K}^n$	1	2	5	14	42	$\cdots$	$c_n = \frac{1}{n+1}\binom{2n}{n}$	$\cdots$
simplices of $\mathcal{K}^n$	1	1	3	16	125	$\cdots$	$(n+1)^{n-1}$	$\cdots$
vertices of $\mathcal{P}^n$	1	2	6	24	120	$\cdots$	$(n+1)!$	$\cdots$
simplices of $\mathcal{P}^n$	1	1	4	34	488	$\cdots$	??	$\cdots$

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Institut de Recherche Mathématique Avancée, CNRS et Université Louis Pasteur, 7 rue R. Descartes, 67084 Strasbourg Cedex, France

E-mail address: loday@math.u-strasbg.fr