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Parastatistics Algebra and Super Semistandard Young Tableaux

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Abstract

We consider the parastatistics algebra with both parabosonic and parafermionic operators and show that the states in the universal parastatistics Fock space are in bijection with the Super Semistandard Young Tableaux (SSYT). Using deformation of the parastatistics algebra we get a monoid structure on SSYT which is a super version of the plactic monoid.

1 Parastatistics Fock Spaces

The quantum description of a system of identical particles is based on the canonical commutation or anticommutation relations between the creation and the annihilation operators

$$[a_i, a_j^\dagger]_\mp = \delta_{ij} \qquad [a_i, a_j]_\mp = 0 \qquad [a_i^\dagger, a_j^\dagger]_\mp = 0 \tag{1}$$

according to the statistics, Bose-Einstein or Fermi-Einstein, respectively. The Fock space is a representation $\mathcal{F}$ of the canonical creation-annihilation algebra built on a unique vacuum state $|0\rangle$ (killed by all annihilation modes a_i)

$$\mathcal{F} = \bigoplus_{r\geq 0} \bigoplus_{i_1 \ldots i_r} a_{i_1}^\dagger \ldots a_{i_r}^\dagger |0\rangle \ , \qquad\qquad a_i|0\rangle = 0$$

where the second sum is subject to the condition $i_1 \leq i_2 \leq \ldots \leq i_r$ for Bose and $i_1 < i_2 < \ldots < i_r$ for Fermi operators. In other words we deal with the symmetric or the exterior algebra of the creation modes, respectively.

A scheme of quantization which generalizes the canonical quantization was introduced by H.S. Green [1]. In this scheme instead of the canonical (anti)commutation relations the creation and the annihilation operators satisfy the trilinear parastatistical relations taken with the upper (lower) sign

$$\begin{aligned} [[a_i^\dagger, a_j]_\pm, a_k^\dagger]_- &= 2\delta_{jk} a_i^\dagger & \qquad [[a_i^\dagger, a_j]_\pm, a_k]_- &= -2\delta_{ik} a_j \\ [[a_i^\dagger, a_j^\dagger]_\pm, a_k^\dagger]_- &= 0 & \qquad [[a_i, a_j]_\pm, a_k]_- &= 0 \end{aligned} \tag{2}$$

and the quantum particles are called parabosons (parafermions). The Bose statistics is a particular instance of the more general parabosonic statistics, and the Fermi statistics is an instance of the parafermi statistics because the bilinear canonical relations imply the trilinear parastatistics ones, (1)⇒(2).

More generally, for a system including both parabosons and parafermions the *parastatistics relations* define the superalgebra with relations

$$\begin{array}{rclcrcl} [\![[\![a_i^\dagger, a_j]\!], a_k^\dagger]\!] &=& 2\delta_{jk}a_i^\dagger & \qquad & [\![a_k, [\![a_j^\dagger, a_i]\!]]\!] &=& 2\delta_{jk}a_i \\ [\![[\![a_i^\dagger, a_j^\dagger]\!], a_k^\dagger]\!] &=& 0 & & [\![a_k, [\![a_j, a_i]\!]]\!] &=& 0 \end{array} \tag{3}$$

where $[\![x, y]\!] := xy - (-1)^{\hat{x}\hat{y}}yx$ is a Lie superbracket, the parabosonic operators are odd, and the parafermionic are even generators (note that here the grading is the opposite to the usual one in which bose are even and fermi are odd).

Theorem 1.1 *(Palev[2]) The creation-annihilation superalgebra* (3) *with* m *parafermionic and* n *parabosonic degrees of freeedom is isomorphic to the the orthosymplectic superalgebra* $\mathfrak{osp}_{1+2m|2n}$.

This theorem allows us to define the parastatistics Fock space as a special $U(\mathfrak{osp}_{1+2m|2n})$-representation.

Definition 1.2 *The representation of the universal enveloping algebra* $U(\mathfrak{osp}_{1+2m|2n})$ *built on a unique vacuum space* $|0\rangle$ *such that*

$$a_i\,|0\rangle = 0 \qquad [\![a_i, a_j^\dagger]\!]\,|0\rangle = p\delta_{ij}\,|0\rangle \tag{4}$$

will be referred to as parastatistics Fock space $\mathcal{F}(m|n;p)$ *of the creation-annihilation algebra (3) with* m *parafermions and* n *parabosons. The number* p *is called the order of the parastatistics.*

Note that for $p = 1$ the parastatistics Fock space $PS(m|n;p)$ is the ordinary Fock space $\mathcal{F}$ of a system with m fermions and n bosons.

How the parastatistics Fock spaces are constructed? Let us denote by V the vector superspace of dimension $m|n$ spanned by the even ($\hat{i} = 0$) parafermionic and odd ($\hat{i} = 1$) parabosonic creation operators $V = V_0 \oplus V_1 \cong \mathbb{C}^{m|n}$ and we suppose $V_0 = \bigoplus_{i=1}^m \mathbb{C}a_i^\dagger$ and $V_1 = \bigoplus_{i=m+1}^{m+n} \mathbb{C}a_i^\dagger := \bigoplus_{i=\bar{1}}^{\bar{n}} \mathbb{C}a_i^\dagger$.

The Lie superalgebra $\mathcal{L}$ closed from the creation parastatistics modes $a_i^\dagger$ is 2-nilpotent in view of the relation $[\![[\![a_i^\dagger, a_j^\dagger]\!], a_k^\dagger]\!] = 0$, cf. (3), thus for the Lie superalgebra $\mathcal{L}$ we have

$$\mathcal{L} = V \oplus [\![V, V]\!].$$

Definition 1.3 *The creation parastatistics algebra* $PS(V)$ *is the universal enveloping algebra of the Lie superalgebra* $\mathcal{L}$, $PS(V) := U(\mathcal{L})$.

Therefore from the Poincaré-Birkhof-Witt theorem for Lie superalgebras we get

$$PS(V) = U(\mathcal{L}) \cong S(V) \otimes S([\![V, V]\!]) \tag{5}$$

where $S(A)$ is the symmetric superalgebra generated from A (see below).

Lemma 1.4 *The elements $E_{ij} = \frac{1}{2}[\![a_i^\dagger, a_j]\!]$ of the creation-annihilation algebra (3) satisfy*

$$[\![E_{ij}, E_{kl}]\!] = E_{il}\delta_{jk} - (-1)^{(\hat{i}-\hat{j})(\hat{k}-\hat{l})} E_{jk}\delta_{il}$$

i.e. they close the general linear Lie superalgebra $\mathfrak{gl}_{m|n}$. The superspace V is a fundamental representation of the superalgebra $\mathfrak{gl}_{m|n}$, $E_{ij}a_k^\dagger = \delta_{jk}a_i^\dagger$.

The algebra $\mathfrak{gl}_{m|n}$ can be extended to the parabolic subalgebra

$$\mathcal{P} = span\{\, [\![a_i^\dagger, a_j]\!], a_i, [\![a_i, a_j]\!]\ ;\ i, j = 1, \ldots, m+n\}$$

thus we have the chain of inclusions $\mathfrak{gl}_{m|n} \subset \mathcal{P} \subset \mathfrak{osp}_{1+2m|2n}$. The subalgebra $\mathcal{P}$ acts trivially on the vacuum space $\mathbb{C}|0\rangle$ hence the parastatistics Fock space $\mathcal{F}(m|n;p)$ is the induced module

$$\mathcal{F}(m|n;p) = \mathrm{Ind}_{\mathcal{P}}^{\mathfrak{osp}_{1+2m|2n}} \mathbb{C}|0\rangle$$

The inclusion $\mathfrak{gl}_{m|n} \subset \mathfrak{osp}_{1+2m|2n}$ implies that the space $\mathcal{F}(m|n;p)$ has decomposition into irreducible representations of $\mathfrak{gl}_{m|n}$.

The creation parastatistics algebra $PS(V)$ is universal in the following sense; the parastatistics Fock space $\mathcal{F}(m|n;p)$ of order p is isomorphic to its quotient

$$\mathcal{F}(m|n;p) \cong PS(V)/M(V,p)$$

An exhaustive study of the parabosonic Fock space $\mathcal{F}(0|n;p)$ of parastatistics order p was done in [3] (see also their contribution in this volume).

The creation parastatistics algebra $PS(V)$ is interesting in its own, it provides a $U(\mathfrak{gl}_{m|n})$-model for the polynomial $U(\mathfrak{gl}_{m|n})$-modules indexed by Young diagrams, that is, in its decomposition into $U(\mathfrak{gl}_{m|n})$-irreducibles all tensor representations appear once and exactly once. Having this in mind one can index the states in the parastatistics Fock space by super semistandard Young tableaux with entries $\{1, \ldots m, \bar{1}, \ldots, \bar{n}\}$ (see below).

2 Tensor $U(\mathfrak{gl}_{m|n})$-modules and Super Young Tableaux

By definition the creation parastatistics algebra $PS(V)$ is the factor of the tensor algebra $T(V)$ by the ideal $I(V)$ generated by the double supercommutators[1]

$$PS(V) = T(V)/I(V) \qquad I(V) = ([\![V, [\![V, V]\!]_\otimes]\!]_\otimes) \tag{6}$$

thus $PS(V)$ is a $U(\mathfrak{gl}_{m|n})$-representation as a factor of two $U(\mathfrak{gl}_{m|n})$-representations. In general, representations of a Lie superalgebra need not be completely reducible. However, the tensor powers $V^{\otimes r}$ of the vector representation V are completely reducible $U(\mathfrak{gl}_{m|n})$-modules. The $U(\mathfrak{gl}_{m|n})$-irreducible subrepresentation of $V^{\otimes r}$ are indexed by Young diagrams (or partitions), i.e., in the same vein as the representations of the symmetric group.

The very reason for this parallel stems from the double centralizing property of the superalgebra action and the sign permutation action of $\mathfrak{S}_r$ in $\mathrm{End}(V^{\otimes r})$.

[1] Degreewise for $r \geq 3$ one has $I_r(V) = \sum_{i+j+3=r} V^{\otimes i} \otimes [\![V, [\![V, V]\!]_\otimes]\!]_\otimes \otimes V^{\otimes j}$.

Theorem 2.1 *(The Schur-Weyl duality [4]) Let the $\mathfrak{gl}_{m|n}$-action ρ on $V^{\otimes r}$ be*

$$\rho(X)(a^\dagger_{i_1}\otimes\ldots\otimes a^\dagger_{i_r}) := \sum_k (-1)^{p_k(X)} a^\dagger_{i_1}\otimes\ldots(X a^\dagger_{i_k})\ldots\otimes a^\dagger_{i_r}, \quad X\in\mathfrak{gl}_{m|n}$$

where $p_k(X) = \hat{X}\sum_{j=1}^{k-1}\hat{i}_j$. Let the sign permutation action σ on $V^{\otimes r}$ be

$$(a^\dagger_{i_1}\otimes\ldots\otimes a^\dagger_{i_r})\,\sigma(\tau) := \epsilon(\tau, I)\, a^\dagger_{\tau(i_1)}\otimes\ldots\otimes a^\dagger_{\tau(i_r)}, \qquad \tau\in\mathfrak{S}_r$$

where $\epsilon(\tau, I) = \pm 1$ is the parity of the odd-odd (paraboson) exchanges. The actions ρ and σ of the generators are extended by linearity. The algebras $\sigma(\mathbb{C}[\mathfrak{S}_r])$ and $\rho(U(\mathfrak{gl}_{m|n}))$ are centralizers to each other in $\mathrm{End}(V^{\otimes r})$.

Thus the superalgebra modules are determined from those of $\mathfrak{S}_r$. An irreducible $\mathfrak{S}_r$-modules S^λ defines an irreducible $U(\mathfrak{gl}_{m|n})$-module V^λ through the Schur functor

$$V^\lambda := V^{\otimes r}\otimes_{\mathfrak{S}_r} S^\lambda$$

where $\mathfrak{S}_r$ acts on $V^{\otimes r}$ by the sign permutation action σ. For instance the r-th degree of the *symmetric algebra* $S(V)$ is the Schur module attached to a single row diagram with r cells

$$S(V) = \oplus_{r\geq 0} S^r V, \qquad S^r V := V^{(r)} \quad (S^0 V := \mathbb{C}).$$

The Schur functor is surjective, i.e., for some λ the corresponding Schur modules V^λ are trivial, $V^\lambda \equiv 0$. The subset Γ of Young diagrams such that V^λ are nontrivial is described by the Hook theorem.

Theorem 2.2 *(Berele and Regev [4]) The image $\sigma(\mathbb{C}[\mathfrak{S}_r]) = \bigoplus_{\lambda\in\Gamma} A^\lambda$ of the sign representation σ in $\mathrm{End}(V^{\otimes r})$ for the $m|n$-dimensional vector representation V is labelled by the subset Γ of diagrams with r cells included in a hook of arm-height m and leg-width n, $H(m,n;r) = \{\lambda\vdash r | \lambda_j \leq n \text{ if } j > m\}$,*

$$\sigma(\mathbb{C}[\mathfrak{S}_r]) \cong \bigoplus_{\lambda\in H(m,n;r)} S^\lambda\,. \tag{7}$$

Definition 2.3 *Let us consider the alphabet of the indices of the basis of the superspace $V = V_0\oplus V_1$, it is an ordered*[1] *signed alphabet (letters have $\mathbb{Z}_2$-degree, $\hat{i}\in\{0,1\}$). Super Semistandard Young Tableau is a filling of the Young diagram with indices increasing on rows and columns with possible repetitions of the even ($\hat{i}=0$) indices on rows and of the odd ($\hat{i}=1$) indices on columns.*

The basis of a Schur module V^λ is indexed by the SSYT of shape $\lambda\in H(m|n)$.

[1]We choose an order $1<\ldots<m<\bar{1}<\ldots<\bar{n}$ induced from the order on $\mathbb{N}$ ($\bar{i}\equiv i+m$).

3 Decomposition of the $U(\mathfrak{gl}_{m|n})$-module $PS(V)$

Lemma 3.1 *The double supercommutator subspace $I_3(V) = [\![V, [\![V, V]\!]_\otimes]\!]_\otimes \subset V^{\otimes 3}$ is an irreducible Schur module*

$$V^{(2,1)} = I_3(V) = V^{\otimes 3} \otimes_{\mathfrak{S}_3} \mathbb{C}[\mathfrak{S}_3]e$$

arising as the Schur functor image of the $\mathfrak{S}_3$-module $S^{(2,1)} = \mathbb{C}[\mathfrak{S}_3]e$, where e stands for the Eulerian idempotent [5]

$$e = \frac{1}{3}\left(123 - \frac{1}{2}(231 + 213 + 132 + 312) + 321\right).$$

Proof. The cyclic permutation of $I_3(V)$ vanishes due to the super Jacobi idenity

$$[\![x, [\![y, z]\!]]\!] + (-1)^{\hat{x}\hat{y}+\hat{x}\hat{z}}[\![y, [\![z, x]\!]]\!] + (-1)^{\hat{x}\hat{z}+\hat{y}\hat{z}}[\![z, [\![x, y]\!]]\!] = 0\,, \qquad x, y, z \in V$$

thus $I_3(V) \cap V^{(1^3)} = 0 = I_3(V) \cap V^{(3)}$ and, counting the dimensions, we conclude that $I_3(V) = V^{(2,1)}$. For the $\mathfrak{S}_3$-representation the Jacobi identity implies $S^{(2,1)} = \mathrm{Ind}_{\mathbb{Z}_3}^{\mathfrak{S}_3} \mathbb{1}$. The rest is a direct calculation. □

Theorem 3.2 *Let V be the $m|n$-dimensional super space. In the decomposition of the $U(\mathfrak{gl}_{m|n})$-module $PS(V)$ into irreducibles each $U(\mathfrak{gl}_{m|n})$-module V^λ, $\lambda \in H(m, n)$ appears once and exactly once $PS(V) \cong \bigoplus_{\lambda \in H(m,n)} V^\lambda$.*

Proof. Let us consider first the case of an m-dimensional even space $V = V_0$. The left hand side of the Schur formula

$$\prod_{i=1}^{m} \frac{1}{1 - x_i} \prod_{1 \le i < j \le m} \frac{1}{1 - x_i x_j} = \sum_\lambda s_\lambda(x)$$

is the character of the $U(\mathfrak{gl}_m)$-module $PS(V) \cong S(V) \otimes S([V, V])$ in view of the Poincaré-Birkhoff-Witt theorem. Then the sum of the Schur polynomials $s_\lambda(x)$ (which are characters of the irreducible $U(\mathfrak{gl}_m)$-modules V^λ) on the right hand side implies $PS(V) \cong \bigoplus_\lambda V^\lambda = \bigoplus_{\lambda \in H(m,0)} V^\lambda$ for $V = V_0$ where the sum on λ runs on the Young diagrams with no more than m rows, $\lambda_{m+1} = 0$. Thus all nontrivial $U(\mathfrak{gl}_m)$-modules modules are present in $PS(V_0)$.

Lemma 3.3 *Let us have $\mathfrak{S} = \bigoplus_{r \ge 0} \mathfrak{S}_r$. The decomposition of the $\mathfrak{S}$-module $PS = \bigoplus_{r \ge 0} PS(r)$ contains each irreducible finite dimensional $\mathbb{C}[\mathfrak{S}_r]$-module S^λ, $r \ge 0$, exactly once $PS = \bigoplus_\lambda S^\lambda$.*

Proof of the lemma. We have $PS(V_0) = \bigoplus_{r \ge 0} PS_r(V_0)$. Let us denote by $PS(r)$ the multilinear part of $PS(V_0)$ of the r-homogeneous Schur functor $PS_r(V_0)$ for even space V_0 of dimension r. The space $PS(r)$ is a reducible $\mathfrak{S}_r$-module and from the decomposition of $PS(V_0)$ follows $PS(r) \cong \bigoplus_{\lambda \vdash r} S^\lambda$. The statement of the lemma follows by induction on the dimension r.

Now let us take V to be a $m|n$-dimensional space. It is enough to apply the Schur functor PS to the superspace V. The nontrivial $U(\mathfrak{gl}_{m|n})$-modules V^λ are labelled by Young diagrams within the (m,n)-hook and all these appear exactly once. Since $V^\lambda \equiv 0$ iff $\lambda \notin H(m,n)$ we get $PS(V) \cong \bigoplus_{\lambda\in H(m,n)} V^\lambda$. □

The $U(\mathfrak{gl}_{m|n})$ character of the Schur module V^λ , $\lambda \in H(m,n)$ of the $m|n$-dimensional superspace space is the hook Schur function [4]

$$hs_\lambda(x_1,\ldots,x_{m+n}) = \sum_{\mu\subset\lambda} s_\mu(x_1,\ldots,x_m)s_{\lambda'/\mu'}(x_{m+1},\ldots,x_{m+n}).$$

From the Poincaré-Birkhoff-Witt theorem $PS(V) = S(V)\otimes S([\![V,V]\!])$ and from theorem 3.2 we get two ways of writing the $U(\mathfrak{gl}_{m|n})$-character of $PS(V)$.

Corollary 3.4 *The hook generalization of the Schur identity reads*

$$\frac{\prod_{i<j\,,\,\hat{i}\neq\hat{j}}(1+x_ix_j)}{\prod_i(1-x_i)\prod_{i<j\,,\,\hat{i}=\hat{j}}(1-x_ix_j)} = \sum_\lambda hs_\lambda(x_1,\ldots,x_{m+n}) \tag{8}$$

where $hs_\lambda(x)$ stands for the Hook Schur function of m even and n odd variables.

4 Deformed para-Fock Space and Superplactic Algebra

The parastatistics algebras (3) of creation and annihilation operators allow for q-deformations introduced by Palev [6]. The idea is to replace the universal enveloping algebra (UEA) $U(\mathfrak{osp}_{1+2m|2n})$ by the quantum UEA $U_q(\mathfrak{osp}_{1+2m|2n})$ written in an alternative form, with a system of relations between generators corresponding to the parastatistics creation and annihilation operators. We are going to describe the deformation $\mathcal{PS}(V)$ of the creation parastatistics algebra $PS(V)$. The space $\mathcal{PS}(V)$ is naturally a $U_q(\mathfrak{gl}_{m|n})$-module and instead of working with the $U_q(\mathfrak{osp}_{1+2m|2n})$ relations we choose another approach based on the q-Schur modules and the Hecke algebra. Our aim is to extract from $\mathcal{PS}(V)$ a combinatorial algebra having as elements the super semistandard Young tableaux.

The Hecke algebra H_r is generated by $g_1,\ldots,g_{r-1}$ with relations

$$\begin{array}{rcll} g_ig_{i+1}g_i &=& g_{i+1}g_ig_{i+1} & i=1,\ldots,r-1\\ g_ig_j &=& g_jg_i & |i-j|\geq 2\\ g_i^2 &=& 1+(q-q^{-1})g_i & i=1,\ldots,r-1 \end{array} \tag{9}$$

The specialization $q=1$ yields the Coxeter relations of the symmetric group $\mathfrak{S}_r$. Thus H_r is a deformation of the symmetric group $\mathfrak{S}_r$ and the elements of $H_r(q)$ are indexed by permutations $\sigma\in\mathfrak{S}_r$.

The Schur-Weyl duality between the action of the superalgebra $U(\mathfrak{gl}_{m|n})$ and the sign permutation action of $\mathfrak{S}_r$ has its "quantum" counterpart. Let now V be $m|n$-dimensional superspace over $\mathbb{C}(q)$, $V=\oplus_{i=1}^{m+n}\mathbb{C}(q)a_i^\dagger$. The sign permutation action of $H_r(q)$ is given by the formula

$$(a_{i_1}^\dagger\otimes\ldots\otimes a_{i_r}^\dagger)\sigma_q(g_s) = \sum_{j_s,j_{s+1}} a_{i_1}^\dagger\otimes\ldots\otimes a_{j_s}^\dagger\otimes a_{j_{s+1}}^\dagger\otimes\ldots\otimes a_{i_r}^\dagger\hat{R}_{i_si_{s+1}}^{j_sj_{s+1}}$$

where $\hat{R}$ is an R-matrix corresponding to the quantum group $GL_q(m|n)$

$$\hat{R}^{i\,j}_{k\,l} = (-1)^{\hat{i}\hat{j}} q^{(-1)^{\hat{i}}\delta_{ij}}\, \delta^i_l \delta^j_k + (q - q^{-1})\, \theta_{ji}\, \delta^i_k \delta^j_l \tag{10}$$

with $\theta_{ij} = 1$ for $i > j$ and $\theta_{ij} = 0$ for $i \leq j$ (note that at $q = 1$ we get the sign permutation action). The matrix $\hat{R}$ satisfy the Yang-Baxter equation $\hat{R}_{12}\hat{R}_{23}\hat{R}_{12} = \hat{R}_{23}\hat{R}_{12}\hat{R}_{23}$ and the Hecke relation $\hat{R}^2 = \mathbb{1} + (q-q^{-1})\hat{R}$ which guarantees that σ_q is a representation of $H_r(q)$. The action ρ of the generators of $U_q(\mathfrak{gl}_{m|n})$ is the same as in Theorem 2.1.

Theorem 4.1 *(Quantum Schur-Weyl duality [7]) The algebras $\sigma_q(H_r(q))$ and $\rho(U_q(\mathfrak{gl}_{m|n}))$ are centralizers to each other in* $\mathrm{End}(V^{\otimes r})$.

Due to this duality the $U_q(\mathfrak{gl}_{m|n})$-modules are determined from those of $H_r(q)$. An irreducible $H_r(q)$-modules $\mathcal{H}^\lambda$ defines an irreducible $U_q(\mathfrak{gl}_{m|n})$-module V^λ through the q-Schur functor

$$V^\lambda(q) := V^{\otimes r} \otimes_{H_r(q)} \mathcal{H}^\lambda$$

where $H_r(q)$ acts on $V^{\otimes r}$ by the sign permutation action σ_q. The irreducible $U_q(\mathfrak{gl}_{m|n})$-modules are labelled again by Young diagrams, and their bases by SSYT in full parallel with the $U(\mathfrak{gl}_{m|n})$-modules.

Definition 4.2 *Let the deformed creation parastatistics algebra $\mathcal{PS}(V)$ be*

$$\mathcal{PS}(V) := T(V)/(\mathcal{I}_3(V)) \qquad \textit{with} \qquad \mathcal{I}_3(V) := V^{\otimes 3} \otimes_{H_3(q)} \mathcal{H}^{(2,1)}$$

where the $H_3(q)$-module $\mathcal{H}^{(2,1)} = H_3(q)e(q)$ is defined by the idempotent

$$\begin{aligned} e(q) \;&:=\; \frac{1}{[3]}\left(T_{123} - \frac{1}{2}(T_{231} + T_{213} + T_{132} + T_{312}) + T_{321}\right) \\ &+\; \frac{q-q^{-1}}{2[3]}\,(T_{213} - T_{312} - T_{231} + T_{132})\,. \end{aligned}$$

Note that $e(q)$ is a deformation of the Eulerian idempotent e, $e(1) = e$. Moreover if we denote by ω the maximal element in $H_3(q)$, $\omega = g_1 g_2 g_1$ then the symmetry $\omega e(q) = e(q)$ fixes the idempotent $e(q)$ completely. The explicit form of the deformation of the subspace $[\![V, [\![V, V]\!]]\!]$ is given by the following

Proposition 4.3 *The subspace $\mathcal{I}_3(V)$ is an irreducible $U_q(\mathfrak{gl}_{m|n})$-module $\mathcal{I}_3(V) = \bigoplus_{\substack{i_1 i_2\\ i_3}\in SSYT} \mathbb{C}(q)\ \Gamma^{i_1 i_2}_{i_3} \cong V^{(2,1)}$ spanned by the following elements*

$$\begin{aligned}
\Gamma^{i_1 i_3}_{i_2} &:= [\![a^\dagger_{i_2}, [\![a^\dagger_{i_3}, a^\dagger_{i_1}]\!]]\!]_{q^{-2}} + q^{-1}\ [\![a^\dagger_{i_3}, [\![a^\dagger_{i_1}, a^\dagger_{i_2}]\!]]\!] && i_1 < i_2 < i_3\,, \\
\Gamma^{i_1 i_2}_{i_3} &:= [\![[\![a^\dagger_{i_3}, a^\dagger_{i_1}]\!], a^\dagger_{i_2}]\!]_{q^{-2}} + q^{-1}\ [\![[\![a^\dagger_{i_2}, a^\dagger_{i_3}]\!], a^\dagger_{i_1}]\!] && i_1 < i_2 < i_3\,, \\
\Gamma^{i_1 i_2}_{i_2} &:= [\![[\![a^\dagger_{i_1}, a^\dagger_{i_2}]\!], a^\dagger_{i_2}]\!]_{q^{-1}} && i_1 < i_2,\ \hat{i}_2 = 1\,, \\
\Gamma^{i_1 i_2}_{i_2} &:= [\![a^\dagger_{i_2}, [\![a^\dagger_{i_1}, a^\dagger_{i_2}]\!]]\!]_{q^{-1}} && i_1 < i_2,\ \hat{i}_2 = 0\,, \\
\Gamma^{i_2 i_2}_{i_3} &:= [\![a^\dagger_{i_2}, [\![a^\dagger_{i_2}, a^\dagger_{i_3}]\!]]\!]_{q^{-1}} && i_2 < i_3,\ \hat{i}_2 = 1\,, \\
\Gamma^{i_2 i_3}_{i_2} &:= [\![[\![a^\dagger_{i_2}, a^\dagger_{i_3}]\!], a^\dagger_{i_2}]\!]_{q^{-1}} && i_2 < i_3,\ \hat{i}_2 = 0\,.
\end{aligned}$$

By construction the $U_q(\mathfrak{gl}_{m|n})$-module $\mathcal{PS}(V)$ has the same decomposition into irreducibles as $PS(V)$, i.e., $\mathcal{PS}(V) \cong \bigoplus_{\lambda \in H(m,n)} V^\lambda(q)$ and the basis in each module $V^\lambda(q)$ is indexed by SSYT of shape λ. Thus we have a bijection between the states in $\mathcal{PS}(V) = \mathcal{PS}(m|n)$ and SSYT in the (m,n)-hook. The formal limit of the $\mathcal{PS}(V)$ deformation parameter $q \to \infty$ is particulary interesting.

Corollary 4.4 *The algebra $\mathcal{PS}(V)$ at $q^{-1} = 0$ has the relations $(x, y, z \in V)$*

$$xzy = (-1)^{\hat{x}\hat{z}} zxy\,, \qquad (x \le y < z\,,\ \hat{y} = 0) \quad or \quad (x < y \le z\,,\ \hat{y} = 1)$$
$$yxz = (-1)^{\hat{x}\hat{z}} yzx\,, \qquad (x < y \le z\,,\ \hat{y} = 0) \quad or \quad (x \le y < z\,,\ \hat{y} = 1)$$

which are $\mathbb{Z}_2$-graded version of the Knuth relations, thus this superalgebra is the superfication $Plac_{\mathbb{Z}_2}(V)$ of the algebra of the plactic monoid [8].

We can fix reading rules attaching to any $SSYT$ a representative word in $Plac_{\mathbb{Z}_2}(V)$ written with letters of the signed alphabet $V = V_0 \oplus V_1$ of the creation parastatistics operators (or better just of their indices $\{1, \ldots m, \bar{1}, \ldots, \bar{n}\}$). To each state in $PS(V)$ and $\mathcal{PS}(V)$ corresponds a single word in $Plac_{\mathbb{Z}_2}(V)$ (modulo the super-Knuth relations). Putting the words of two SSYT one next to the other we get the word of a new SSYT, i.e., we have a structure of monoid (forgetting about the sign factor $(-1)^{\hat{x}\hat{z}}$). The super-Knuth relations obtained in the work [9] are the same as ours up to this factor depending on the $\mathbb{Z}_2$-grading.

More details will be given in the forthcoming paper [10].

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