Math. Z. 221, 41–47 (1996)

Mathematische Zeitschrift

Künneth-style formula for the homology of Leibniz algebras

Jean-Louis Loday

Institut de Recherche Mathématique Avancée, Université Louis Pasteur et C.N.R.S., 7, rue René-Descartes, F-67084 Strasbourg Cedex, France

Received 9 December 1993; in final form 7 June 1994

For any Lie algebra $\mathfrak{g}$ over a field k its homology $H_*(\mathfrak{g})$ is the homology of a certain complex $(\Lambda\,\mathfrak{g}, d)$ where $\Lambda\,\mathfrak{g}$ is the exterior module over $\mathfrak{g}$ (*cf.* [K]). In this framework the Künneth formula reads as follows. For any Lie algebras $\mathfrak{g}$ and $\mathfrak{g}'$ there is a canonical isomorphism of graded vector spaces

$$H_*(\mathfrak{g} \oplus \mathfrak{g}') \cong H_*(\mathfrak{g}) \otimes H_*(\mathfrak{g}').$$

There is another type of homology theory for Lie algebras : the Leibniz homology $HL_*(\mathfrak{g})$, defined as the homology of a complex $(T\,\mathfrak{g}, d)$ where $T\,\mathfrak{g}$ is the tensor module over $\mathfrak{g}$ and the differential is a lifting of the preceding differential. In fact this theory is defined for a larger class of algebras : the Leibniz algebras (see 1 below). The aim of this paper is to compute $HL_*(\mathfrak{g} \oplus \mathfrak{g}')$ in terms of $HL_*(\mathfrak{g})$ and $HL_*(\mathfrak{g}')$. The result is as follows. For any Leibniz algebras $\mathfrak{g}$ and $\mathfrak{g}'$ over the field k, there is a canonical isomorphism of graded vector spaces

$$HL_*(\mathfrak{g} \oplus \mathfrak{g}') \cong HL_*(\mathfrak{g}) * HL_*(\mathfrak{g}').$$

In this formula $*$ is a sort of non-commutative tensor product for graded modules (see 2 below). It satisfies $T(V \oplus W) \cong TV * TW$.

The proof of this Künneth-style formula is based on an algebraic version of *Chen's iterated integrals*. More information on Leibniz algebras can be found in [L].

Throughout the paper k denotes a field. We adopt the notation

$$[f, g] = f \circ g - (-1)^{|f|\cdot|g|} g \circ f$$

for maps f and g of degree $|f|$ and $|g|$ respectively. For any vector space V the tensor module TV is the infinite direct sum

$$TV = k \oplus V \oplus V^{\otimes 2} \oplus \ldots \oplus V^{\otimes n} \oplus \ldots \quad .$$

1 Leibniz algebras and homology

By definition a *Leibniz algebra* is a k-vector space $\mathfrak{g}$ equipped with a bilinear map

$$[-,-]:\mathfrak{g}\times\mathfrak{g}\longrightarrow\mathfrak{g},$$

called the bracket, satisfying the (right) Leibniz relation

$$(L)\qquad [x,[y,z]]=[[x,y],z]-[[x,z],y]\ ,\ \text{for all } x,y,z\in\mathfrak{g}\,.$$

Let $T\,\mathfrak{g}$ be the tensor module over $\mathfrak{g}$. It is a graded module with $T^0\,\mathfrak{g}=k$ and $T^n\,\mathfrak{g}=\mathfrak{g}^{\otimes n}$ for $n\geq 1$. If there is no ambiguity the element $x_1\otimes\ldots\otimes x_n$ of $\mathfrak{g}^{\otimes n}$ is denoted by $x_1x_2\ldots x_n$. Let us define a right action of $\mathfrak{g}$ on $T\,\mathfrak{g}$ by

$$(1.1)\qquad [x_1x_2\ldots x_n,x]=\sum_{i=1}^{n}x_1\ldots x_{i-1}[x_i,x]x_{i+1}\ldots x_n,$$

where $x_1,\ldots,x_n,x\in\mathfrak{g}$.

By definition the map $d=d^{\mathfrak{g}}:\mathfrak{g}^{\otimes n}\to\mathfrak{g}^{\otimes n-1}$ is 0 for $n\leq 1$ and defined inductively by the formula

$$(1.2)\qquad d(\omega x)=(d\omega)x+(-1)^{|\omega|+1}[\omega,x].$$

It is a consequence of the Leibniz relation that $d^2=0$ (*cf.* [L-P]). Therefore $(T\,\mathfrak{g},d)$ is a well-defined complex whose homology is a graded vector space denoted $HL_*(\mathfrak{g})$. For instance $HL_0(\mathfrak{g})=k$ and $HL_1(\mathfrak{g})=\mathfrak{g}\,/[\mathfrak{g},\mathfrak{g}]$.

2 Non-commutative tensor product of $\mathbb{N}$-graded modules

Let R and S be two unital associative algebras. Their *coproduct* (sometimes called free product) in the category of unital associative algebras is denoted $R*S$. It is made out of the words $r_1s_1r_2s_2\ldots$, for $r_i\in R$ and $s_i\in S$.

Let M and N be two $\mathbb{N}$-graded modules such that $M_0=k=N_0$. Then $M=k\oplus\overline{M}$ can be considered as a unital associative algebra by decreeing that $\overline{M}=\bigoplus_{i>0}M_i$ is a square-zero ideal (and similarly for N). Then $M*N$ is also an $\mathbb{N}$-graded module, which we call the *non-commutative tensor product* of M and N :

$$M*N=k\ \oplus\ \overline{M}\ \oplus\ \overline{N}\ \oplus\ \overline{M}\otimes\overline{N}\ \oplus\ \overline{N}\otimes\overline{M}\ \oplus\ \ldots\ .$$

Explicitly one gets

$$(M*N)_0=k$$
$$(M*N)_1=M_1\oplus N_1$$
$$(M*N)_2=M_2\oplus(M_1\otimes N_1)\oplus(N_1\otimes M_1)\oplus N_2.$$

More generally $(M*N)_n$ is the sum of the 2^n components

$$X_{i_1} \otimes Y_{i_2} \otimes X_{i_3} \otimes Y_{i_4} \otimes \ldots$$

where $X = M$ and $Y = N$, or $X = N$ and $Y = M$, and where $i_j \geq 1$, $\sum_j i_j = n$.

Note that for two vector spaces V and W there is a canonical isomorphism $T(V \oplus W) \cong TV * TW$. One gets for instance

$$(V \oplus W)^{\otimes 2} = (V \oplus W) \otimes (V \oplus W) \cong (V \otimes V) \oplus (V \otimes W) \oplus (W \otimes V) \oplus (W \otimes W).$$

3 Theorem. – *For any Leibniz algebras $\mathfrak{g}$ and $\mathfrak{g}'$ over k there is a canonical isomorphism of graded modules*

$$HL_*(\mathfrak{g} \oplus \mathfrak{g}') \cong HL_*(\mathfrak{g}) * HL_*(\mathfrak{g}').$$

Note that, as usual, the sum $\mathfrak{g} \oplus \mathfrak{g}'$ is equipped with the Leibniz product

$$[x + x', y + y'] = [x, y] + [x', y'],$$

for $x, y \in \mathfrak{g}$ and $x', y' \in \mathfrak{g}'$.

The rest of the paper is devoted to the proof of this theorem, except for the last section where we propose a generalization of this result.

4 Plan of the proof

On the graded module $T\,\mathfrak{g} * T\,\mathfrak{g}'$ there is defined a boundary map $d^{\mathfrak{g}} * 1 + 1 * d^{\mathfrak{g}'}$. However the canonical isomorphism of graded modules $T(\mathfrak{g} \oplus \mathfrak{g}') \cong T\,\mathfrak{g} * T\,\mathfrak{g}'$ is *not* a complex morphism when one puts the Leibniz boundary map $d^{\mathfrak{g} \oplus \mathfrak{g}'}$ on $T(\mathfrak{g} \oplus \mathfrak{g}')$.

Our strategy consists, essentially, in constructing (by induction) another isomorphism which will make $d^{\mathfrak{g} \oplus \mathfrak{g}'}$ and $d^{\mathfrak{g}} * 1 + 1 * d^{\mathfrak{g}'}$ commute.

Let $C\,\mathfrak{g}$ be the submodule of $T(\mathfrak{g} \oplus \mathfrak{g}') = T\,\mathfrak{g} * T\,\mathfrak{g}'$ made of the vector spaces $\mathfrak{g}^{\otimes n} \otimes \mathfrak{g}'^{\otimes m} \otimes \mathfrak{g}^{\otimes p} \otimes \ldots$ such that $n \geq 1$ and $m \geq 1$.

From the definition of $d = d^{\mathfrak{g} \oplus \mathfrak{g}'}$ it is clear that $C\,\mathfrak{g}$ is a subcomplex of $(T(\mathfrak{g} \oplus \mathfrak{g}'), d)$. In fact $(T(\mathfrak{g} \oplus \mathfrak{g}'), d)$ splits as a direct sum of complexes

$$(k, 0) \oplus (\overline{T}\,\mathfrak{g}, d^{\mathfrak{g}}) \oplus (C\,\mathfrak{g}, d) \oplus (C\,\mathfrak{g}', d) \oplus (\overline{T}\,\mathfrak{g}', d^{\mathfrak{g}'}).$$

Proposition 5 will show that $C\,\mathfrak{g}$ is isomorphic to the tensor product of complexes $\overline{T}\,\mathfrak{g} \otimes (C\,\mathfrak{g}' \oplus \overline{T}\,\mathfrak{g}')$, and similarly with $\mathfrak{g}$ and $\mathfrak{g}'$ exchanged. Therefore $T(\mathfrak{g} \oplus \mathfrak{g}')$ is isomorphic to the sum of tensor products of complexes

$$k \;\oplus\; \overline{T}\mathfrak{g} \;\oplus \overline{T}\,\mathfrak{g}' \;\oplus \overline{T}\,\mathfrak{g} \otimes \overline{T}\,\mathfrak{g}' \;\oplus \overline{T}\,\mathfrak{g}' \otimes \overline{T}\,\mathfrak{g} \ldots.$$

Applying the classical Künneth formula ends the proof of theorem 3.

5 Proposition. – *There exists a commutative diagram*

$$\begin{array}{ccc} C\,\mathfrak{g} & \xrightarrow{h} & \overline{T}\,\mathfrak{g} \otimes (C\,\mathfrak{g}' \oplus \overline{T}\,\mathfrak{g}') \\ d\downarrow & & \downarrow d\otimes 1 + 1\otimes d \\ C\,\mathfrak{g} & \xrightarrow{h} & \overline{T}\,\mathfrak{g} \otimes (C\,\mathfrak{g}' \oplus \overline{T}\,\mathfrak{g}'). \end{array}$$

in which d stands for the Leibniz boundary map (or some restriction of it) and h is an isomorphism.

Proof. Note first that $C\,\mathfrak{g}$ and $\overline{T}\,\mathfrak{g}\otimes(C\,\mathfrak{g}'\oplus\overline{T}\,\mathfrak{g}')$ can be identified as graded modules. The generic summand is of the form $\mathfrak{g}^{\otimes n}\otimes\mathfrak{g}'^{\otimes m}\otimes\mathfrak{g}^{\otimes p}\otimes\ldots$, with $n\geq 1$ and $m\geq 1$. Under this identification one has

$$d = d\otimes 1 + 1\otimes d + \mathrm{ad},$$

where the morphism ad is given by 0 on $\mathfrak{g}^{\otimes n}\otimes\mathfrak{g}'^{\otimes m}$ and by

$$\begin{aligned} &\mathrm{ad}(v_1\ldots v_n \otimes w_1\ldots w_m \otimes v_{n+1}\ldots) \\ &= \sum_{1\leq i\leq n<j} (-1)^{|v_j|} v_1\ldots[v_i,v_j]\ldots\otimes w_1\ldots w_m\otimes v_{n+1}\ldots\hat{v}_j\ldots \end{aligned}$$

on $\mathfrak{g}^{\otimes n}\otimes\mathfrak{g}'^{\otimes m}\otimes\mathfrak{g}^{\otimes p}\otimes\ldots$. Here $|v_j|$ denotes the place of v_j in the tensor product, for instance $|v_{n+1}| = n+m+1$.

We are going to construct degree 0 morphisms

$$h^{\langle k\rangle} : C\,\mathfrak{g} \longrightarrow \overline{T}\,\mathfrak{g}\otimes(C\,\mathfrak{g}'\oplus\overline{T}\,\mathfrak{g}'),$$

for $k\geq 0$, such that

a) $h^{\langle k_1\rangle}\circ\ldots\circ h^{\langle k_p\rangle}(x) = 0$ if $k_1+\ldots+k_p$ is large enough,

b) $[1\otimes d, h^{\langle k\rangle}] + [d\otimes 1, h^{\langle k+1\rangle}] = h^{\langle k\rangle}\,\mathrm{ad}$, for $k\geq 0$.

Put $h^{\langle 0\rangle} := id$ and $h := id + h^+ = \sum_{k\geq 0} h^{\langle k\rangle}$. By property a) h is well-defined and h^+ is nilpotent. So h is an isomorphism. Since $[d\otimes 1, h^{\langle 0\rangle}] = [d\otimes 1, id] = 0$, property b) implies

$$[1\otimes d, h] + [d\otimes 1, h] = h\circ\mathrm{ad},$$

that is,

$$(d\otimes 1 + 1\otimes d)\circ h = h\circ(d\otimes 1 + 1\otimes d + \mathrm{ad}).$$

6 Construction of $h^{\langle k\rangle}$

Let

$$x = v_1\ldots v_n\otimes w_1\ldots w_m\otimes v_{n+1}\ldots v_{n+p}\otimes\ldots\otimes\ldots$$

be an element of $\mathfrak{g}^{\otimes n} \otimes \mathfrak{g}'^{\otimes m} \otimes \mathfrak{g}^{\otimes p} \otimes \ldots \subset C\,\mathfrak{g}$. The last factor of V in x is denoted v_r (so $r \geq n$). Recall that $n \geq 1$ and $m \geq 1$. By definition $h^{<k>}$ is 0 if $r - n < k$ and otherwise it is given by

$$h^{\langle k\rangle}(x) = \sum_{n<i_1<\ldots<i_k} \pm v_1 \ldots v_n v_{i_1} \ldots v_{i_k} \otimes w_1 \ldots w_m \otimes \ldots \hat{v}_{i_1} \ldots \hat{v}_{i_k} \ldots$$

where the sum is extended over all k-tuples of indices $\{i_1, \ldots, i_k\}$ in $\{n+1, \ldots, r\}$. The sign is the signature of the permutation

$$v_1 \ldots v_n w_1 \ldots w_m v_{n+1} \ldots v_{n+p} \ldots \mapsto v_1 \ldots v_n v_{i_1} \ldots v_{i_k} w_1 \ldots w_m \ldots \hat{v}_{i_1} \ldots \hat{v}_{i_k} \ldots .$$

This is the analogue of Chen's iterated integral.

7 Lemma. – *For any x as above $h^{\langle k_1\rangle} \circ \ldots \circ h^{\langle k_s\rangle}(x) = 0$ as soon as $k_1 + \ldots + k_s > r - n$.*

Proof. By definition $h^{\langle k\rangle}(x) \in \mathfrak{g}^{\otimes n+k} \otimes \mathfrak{g}'^{\otimes m} \otimes \ldots$, so $h^{\langle k\rangle}$ diminishes the value of $r - n$ by k, so ultimately one gets 0 for the image of x.

8 Lemma. – $h^{\langle k\rangle}\,\mathrm{ad} = [1 \otimes d, h^{\langle k\rangle}] + [d \otimes 1, h^{\langle k+1\rangle}]$.

Proof. For $k = 0$, $h^{\langle 0\rangle} = id$, so we only need to prove that $\mathrm{ad} = [d \otimes 1, h^{\langle 1\rangle}]$. Put $\omega = v_1 \ldots v_n$. Since $d(\omega v) = (d\omega)v + (-1)^{n+1}[\omega, v]$, it comes

$$[d \otimes 1, h^{\langle 1\rangle}](x) = \sum_{n<i\leq r} \pm[\omega, v_i] \otimes w_1 \ldots \hat{v}_i \ldots = \mathrm{ad}(x).$$

More generally let us write $x = \omega \otimes \Omega$ (with $\omega \in \mathfrak{g}^{\otimes n}$) and

$$(d \otimes 1) \circ h^{\langle k+1\rangle}(x) = Z_1 + Z_2 + Z_3$$

according to the decomposition

$$\begin{aligned} d(\omega v_{i_0} \ldots v_{i_k}) = (d\omega) v_{i_0} \ldots v_{i_k} & \\ & + \sum_{j=0}^{k} (-1)^{n+1+j} [\omega, v_{i_j}] v_{i_0} \ldots \hat{v}_{i_j} \ldots v_{i_k} \\ & + (-1)^n \omega d(v_{i_0} \ldots v_{i_k}) . \end{aligned}$$

It is clear that the element $Z_1 = \sum (d\omega) v_{i_0} \ldots v_{i_k} \otimes \Omega(\hat{v}_{i_0}, \ldots, \hat{v}_{i_k})$, where $\Omega(\hat{v}_{i_0}, \ldots, \hat{v}_{i_k})$ means Ω with $v_{i_0}, \ldots, v_{i_k}$ removed, is precisely $h^{\langle k+1\rangle}(d \otimes 1)(x)$.

The element Z_2 is equal to $h^{\langle k\rangle}\,\mathrm{ad}(x)$ because

$$\begin{aligned} h^{\langle k\rangle}\,\mathrm{ad}(x) &= h^{\langle k\rangle}\Big(\sum_j \pm[\omega, v_j] \otimes \Omega(\hat{v}_j)\Big) \\ &= \sum_{i_1<\ldots<i_k} \sum_j \pm[\omega, v_j] v_{i_1} \ldots v_{i_k} \otimes \Omega(\hat{v}_{i_1}, \ldots, \hat{v}_j, \ldots, \hat{v}_{i_k}) . \end{aligned}$$

Let us now compute $[1\otimes d, h^{\langle k\rangle}](x)$. First one has $h^{\langle k\rangle}\circ(1\otimes d)(x) = h^{\langle k\rangle}(\omega\otimes d\Omega)$. In $d\Omega$ there are two kinds of k-tuples : those which contain some bracket $[v_i, v_j]$ and those which do not. The second ones give precisely $(1\otimes d)h^{\langle k\rangle}(x)$ and therefore

$$[1\otimes d, h^{\langle k\rangle}](x) = \sum \pm\omega v_{i_1}...[v_{i_a}, v_{i_b}]...v_{i_k}\otimes \Omega(\hat{v}_{i_1},...,\hat{v}_{i_a},...,\hat{v}_{i_b},...,\hat{v}_{i_k}).$$

This element is precisely Z_3. This ends up the proof of the formula.

9 Conclusion remark

In [L-P] it was shown that

$$H_*(\mathfrak{g}, k) \cong \mathrm{Tor}_*^{UL(\mathfrak{g})}(U(\mathfrak{g}_{Lie}), k),$$

where $UL(\mathfrak{g})$ is the universal enveloping algebra of the Leibniz algebra $\mathfrak{g}$, and $U(\mathfrak{g}_{Lie})$ is the universal enveloping algebra of the Lie algebra $\mathfrak{g}_{Lie}$ associated to $\mathfrak{g}$. Note that $U(\mathfrak{g}_{Lie})$ is a quotient of $UL(\mathfrak{g})$.

By putting $A = UL(\mathfrak{g}), A/I = U(\mathfrak{g}_{Lie})$ and similarly for $\mathfrak{g}'$, theorem 3 can be written

$$(10)\qquad \mathrm{Tor}_*^{A\otimes A'/I\otimes I'}(A/I\otimes A'/I', k) \cong \mathrm{Tor}_*^{A}(A/I, k) * \mathrm{Tor}_*^{A'}(A'/I', k).$$

Indeed one has $(\mathfrak{g}\oplus\mathfrak{g}')_{Lie} = \mathfrak{g}_{Lie}\oplus\mathfrak{g}'_{Lie}$, so

$$U((\mathfrak{g}\oplus\mathfrak{g}')_{Lie}) = U(\mathfrak{g}_{Lie})\otimes U(\mathfrak{g}'_{Lie}) = A/I\otimes A'/I'$$

and

$$\begin{aligned} UL(\mathfrak{g}\oplus\mathfrak{g}') &= (k\oplus\mathfrak{g}_{Lie}\oplus\mathfrak{g}'_{Lie})\otimes U(\mathfrak{g}_{Lie}\oplus\mathfrak{g}'_{Lie}) \\ &= (k\oplus\mathfrak{g}_{Lie}\oplus\mathfrak{g}'_{Lie})\otimes U(\mathfrak{g}_{Lie})\otimes U(\mathfrak{g}'_{Lie}) \\ &= (k\oplus\mathfrak{g}_{Lie})\otimes U(\mathfrak{g}_{Lie})\otimes(k\oplus\mathfrak{g}'_{Lie})\otimes U(\mathfrak{g}'_{Lie})\ /\ (\mathfrak{g}_{Lie}\otimes\mathfrak{g}'_{Lie}) \\ &= A\otimes A'/I\otimes I'. \end{aligned}$$

Obviously formula (10) is valid in other cases (for instance $A = k[x], I = (x), A' = k[x'], I = (x')$). It would be interesting to know the most general hypotheses under which formula (10) is valid.

Acknowledgement. Thanks are due to J.-M. Oudom for detecting an error in the first draft of this paper.

References

[K] J.-L. Koszul, *Homologie et cohomologie des algèbres de Lie*, Bull. Soc. Math. France **78** (1950), 65–127.

[L] J.-L. Loday, *Une version non commutative des algèbres de Lie : les algèbres de Leibniz*, L'Enseignement Mathématique **39** (1993), 269–293.

[L-P] J.-L. Loday and T. Pirashvili, *Universal enveloping algebras of Leibniz algebras and (co)homology*, Math. Ann. **296** (1993), 139–158.