Advances in Mathematics **139**, 293–309 (1998)
Article No. AI981759

Hopf Algebra of the Planar Binary Trees

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Received October 27, 1997; accepted April 30, 1998

Let k be a field and let S_n be the symmetric group. The group algebra $k[S_n]$ contains the *Solomon descent algebra*, which is of dimension 2^{n-1}. In [MR] Malvenuto and Reutenauer construct a graded Hopf algebra structure on

$$k[S_\infty] := \bigoplus_{n \geqslant 0} k[S_n]$$

so that the sum of the Solomon descent algebras

$$Sol_\infty := \bigoplus_{n \geqslant 0} Sol_n$$

is a sub-Hopf algebra.

There is a basis Q_n of Sol_n such that the composite map $k[Q_n] \simeq Sol_n \to k[S_n]$ has the following property: its linear dual $k[S_n] \to k[Q_n]$ is induced by a set-theoretic map $S_n \twoheadrightarrow Q_n$.

In between S_n and Q_n there is an intermediate set Y_n, which is made of *planar binary trees with n vertices*. For instance, for $n = 3$ the number of elements in S_3, Y_3, and Q_3 is 6, 5 and 4 respectively. The projection from S_n to Q_n is the composite of two maps

$$S_n \xrightarrow{\psi} Y_n \xrightarrow{\phi} Q_n.$$

0001-8708/98 $25.00

The main result of this paper is to show that the graded vector space

$$k[Y_\infty] := \bigoplus_{n \geqslant 0} k[Y_n]$$

may be equipped with a structure of graded Hopf algebra, such that ϕ and ψ induce Hopf algebra homomorphisms.

The associative product on $k[Y_\infty]$ turns out to be the sum of two operations, which makes the augmentation ideal into a "dual-dialgebra," as introduced in [Lod2]. In fact, it is the free dual-dialgebra on one generator. We also show that, as an associative algebra, $k[Y_\infty]$ is free.

We give a simple description of the two maps ψ and ϕ by interpreting a permutation as a "planar binary tree with levels." The map ψ consists in forgetting the levels, and the map ϕ consists in looking at the orientation of the leaves. These maps happen to be the restriction to the vertices of homotopy equivalences of different cellular decompositions of the spheres: the permutohedron, the associahedron and the hypercube.

Contents. In the first section we recall the Hopf algebra structure on $k[S_\infty]$. In the second section we introduce the interpretation of a permutation as a planar binary tree with levels (also called increasing tree in the literature), and we describe the inclusion map $\psi^*: k[Y_n] \to k[S_n]$. In the third section we show that $k[Y_\infty]$ is a sub-Hopf algebra of $k[S_\infty]$ and we relate the associative structure of $k[Y_\infty]$ to the dual-dialgebras by using the grafting operation on trees. In the fourth section we deal with the Solomon descent algebra and prove the results referred to above. In the last section we comment on the relationship with cellular decompositions of the sphere.

Convention. In this paper the ground field is denoted by k.

1. HOPF ALGEBRA OVER THE SYMMETRIC GROUPS.

Let S_n be the symmetric group acting on the set $\{1, ..., n\}$. By convention S_0 is the trivial group with one element denoted by 1. The image of $\{1, ..., n\}$ under the permutation σ is denoted by $(\sigma(1), ..., \sigma(n))$. Recall that an (n, m)-shuffle is a permutation $\sigma \in S_{n+m}$ such that

$$\sigma(1) < \sigma(2) < \cdots < \sigma(n) \qquad \text{and} \qquad \sigma(n+1) < \sigma(n+2) < \cdots < \sigma(n+m).$$

The product in the group algebra $k[S_n]$ is denoted either by $\cdot$ or by concatenation of the elements. The unit, which is the trivial permutation, is denoted by 1_n. The *disjoint* union of the groups S_n for $n \geqslant 0$ is denoted by S_∞, and so

$$k[S_\infty] := \bigoplus_{n \geqslant 0} k[S_n].$$

1.1. *Product on* $k[S_\infty]$. We define a graded product on $k[S_\infty]$ as follows. For $\sigma \in S_n$ and $\tau \in S_m$ we denote by $\sigma \times \tau$ the permutation in S_{n+m} which consists in letting σ act on the first n variables and let τ act on the last m variables. The sum of all the (n, m)-shuffles is denoted by $sh_{n,m} \in k[S_{n+m}]$. By definition we put

$$x * y := sh_{n,m} \cdot (x \times y) \in k[S_{n+m}], \qquad \text{for} \quad x \in k[S_n],\ y \in k[S_m].$$

In the particular case $n = 0$, or $m = 0$ one has $1 * y = y$ and $x * 1 = x$. From the classical properties of the shuffles it follows that the operation $*$ is associative.

1.2. *Coproduct on* $k[S_\infty]$. In order to define the graded coproduct

$$\Delta\colon k[S_\infty] \to k[S_\infty] \otimes k[S_\infty]$$

it is sufficient to define it on a permutation $\sigma \in S_n$. It takes the form

$$\Delta(\sigma) = \sum_{i=0}^{i=n} \sigma(i) \otimes \sigma'_{(n-i)},$$

where the permutations $\sigma_{(i)}$ and $\sigma'_{(n-i)}$ are given by the following result.

1.3. Lemma. *For any permutation* $\sigma \in S_n$ *and any integer* i, $1 \leqslant i \leqslant n$, *there exist a unique* $(i, n-i)$*-shuffle* $\omega \in S_n$ *and unique permutations* $\sigma_{(i)} \in S_i$ *and* $\sigma'_{(n-i)} \in S_{n-i}$ *verifying*

$$\sigma = (\sigma_{(i)} \times \sigma'_{(n-i)}) \cdot \omega^{-1}.$$

Proof. Given $\sigma \in S_n$ there exist unique order preserving bijections

$$\alpha_i\colon \{1 < \cdots < i\} \to \{\sigma^{-1}(1), ..., \sigma^{-1}(i)\}$$

$$\alpha'_{n-i}\colon \{1 < \cdots < n-i\} \to \{\sigma^{-1}(i+1), ..., \sigma^{-1}(n)\}$$

where the order considered on

$$\{\sigma^{-1}(1), ..., \sigma^{-1}(i)\} \qquad \text{and on} \quad \{\sigma^{-1}(i+1), ..., \sigma^{-1}(n)\}$$

is the one induced by the usual order of the natural numbers.

Define $\sigma_{(i)} := (\sigma \cdot \alpha_i(1), ..., \sigma \cdot \alpha_i(i))$ and $\sigma'_{(n-i)} := (\sigma \cdot \alpha'_{n-i}(1) - i, ..., \sigma \cdot \alpha'_{n-i}(n-i) - i)$.

One has $\sigma \cdot \omega = \sigma_{(i)} \times \sigma'_{(n-i)}$, where

$$\omega(j) = \begin{cases} \alpha_i(j) & \text{for } 1 \leqslant j \leqslant i \\ \alpha'_{n-i}(j-i) & \text{for } i+1 \leqslant j \leqslant n. \end{cases}$$

Since α_i and α'_{n-i} are order preserving bijections, ω is a $(i, n-i)$-shuffle. Uniqueness is proved by direct inspection. ∎

This coproduct has the following description on permutation matrices. For instance, let

$$\begin{pmatrix} & & & 1 & \\ & 1 & & & \\ 1 & & & & \\ & & & & 1 \\ & & 1 & & \end{pmatrix}$$

be the matrix associated to the permutation $\sigma = (32514)$. Select the i columns whose single nontrivial entry is on one of the first i rows. Rearrange the columns so that these i columns stay in the same order and become the first i columns. The new matrix appears as a direct sum of an $i \times i$-matrix and a $j \times j$-matrix $(i + j = n)$. For $i = 3$ we get

$$\begin{pmatrix} & & 1 & & \\ & 1 & & & \\ 1 & & & & \\ & & & & 1 \\ & & & 1 & \end{pmatrix}.$$

These two permutation matrices determine $\sigma_{(i)}$ and $\sigma'_{(j)}$ respectively. In our example:

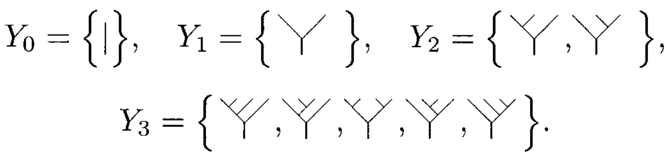
$$\Delta(32514) = 1 \otimes (32514) + (1) \otimes (2143) + (21) \otimes (132)$$
$$+ (321) \otimes (21) + (3214) \otimes (1) + (32514) \otimes 1.$$

It is immediate from this description that the coproduct is coassociative. Observe that $\Delta(1_n) = \sum_i 1_i \otimes 1_{n-i}$.

1.4. THEOREM [MR]. *The graded vector space $k[S_\infty]$, equipped with the product $*$ and the coproduct Δ, is a connected graded Hopf algebra.*

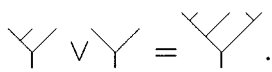
2. PERMUTATIONS AND PLANAR BINARY TREES

2.1. *Planar binary trees.* A *planar tree* is an oriented graph drawn on a plane, with only one root. It is *binary* when any vertex is trivalent (one root and two leaves). We denote by Y_n the set of planar binary trees with $n+1$ leaves, that is, n interior vertices.

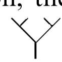


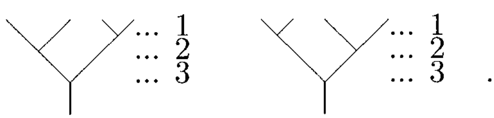
An element T in Y_n is called an n-tree for short, and n is called the degree of T. When necessary we label the leaves of an n-tree by 0, ..., n from left to right. The number of elements in Y_n is the Catalan number $c_n = (2n)!/n!\,(n+1)!$.

The *grafting* of a p-tree T_1 and a q-tree T_2 is the $(p+q+1)$-tree $T_1 \vee T_2$ obtained by joining the roots of T_1 and T_2 and create a new root. For instance



For any tree T there are unique trees T_1 and T_2 such that $T = T_1 \vee T_2$. It is clear that any tree in Y_n can be obtained from | by successive graftings.

2.2. *Planar binary trees with levels and permutations.* Here is a variant of the planar binary tree. By definition a *planar binary tree with levels* is a planar binary tree, in Y_n for instance, such that each internal vertex has a "level", that is a number in $\{1, ..., n\}$. It is required that the levels respects the partial order of the tree. They are also called *increasing trees* in the literature. By convention, the level of the vertex attached to the root is n. For instance the 3-tree  gives rise to two different trees with levels:



We denote by $\tilde{Y}_n$ the set of planar binary trees with n levels.

To any planar binary tree with levels one can associate a permutation as follows: the image of i is the level of the vertex which lies in between the leaves $i-1$ and i (numbered from left to right). For instance the permutations associated to the two preceding trees are

$$(231) \qquad \text{and} \qquad (132),$$

respectively.

The following result is well-known (cf. for instance [Han]).

2.3. PROPOSITION. *The map $\tilde{Y}_n \to S_n$ is a bijection between planar binary trees with levels and permutations.*

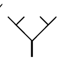
Proof. We know how to construct a permutation out of a planar binary tree with levels. Let us now show, by induction, how to construct a planar binary tree with levels out of a permutation. The permutation $\sigma \in S_n$ gives rise to two sequences of integers: the sequence before n and the sequence after n in $(\sigma(1)\,\sigma(2) \cdots \sigma(n))$. Observe that one of them may be empty. By relabelling the integers in each sequence, they form a permutation. For instance (341625) gives the two sequences (341) and (25), which, after relabelling, give (231) and (21). By induction these two permutations give rise to two trees with levels. The grafting of the two trees gives the tree of σ. The original permutation is used to determine the levels of each vertex.

The two constructions are inverse to each other. ∎

2.4. *The inclusion map $\psi^*: k[Y_n] \to k[S_n]$.* We denote by

$$\psi: S_n \twoheadrightarrow Y_n$$

the map which is the composition of the bijection $S_n \simeq \tilde{Y}_n$ with the forgetful map $\tilde{Y}_n \twoheadrightarrow Y_n$. The induced linear map $\psi: k[S_n] \twoheadrightarrow k[Y_n]$ has a linear dual $\psi^*: [Y_n] \to k[S_n]$ obtained by identifying each basis with its own dual. Observe that the inclusion ψ^* is not induced by a set map. For instance

ψ^* () $= (231) + (132).$

More generally, for any $T \in Y_n$, define the set

$$Z_T := \{\sigma \in S_n \mid \psi(\sigma) = T\}.$$

One has $\psi^*(T) = \sum_{\sigma \in Z_T} \sigma$.

2.5. LEMMA. *Let $T = T_1 \vee T_2$, where $T_1 \in Y_n$ and $T_2 \in Y_m$. Then, $\sigma \in Z_T$ if and only if $\sigma = \gamma(\alpha_1 \times (1) \times \alpha_2)$ for some $(n+1, m)$-shuffle γ such that*

$\gamma(n+1)=n+m+1$, *some* $\alpha_1 \in Z_{T_1}$, *and some* $\alpha_2 \in Z_{T_2}$. *Moreover, such a triple* $(\alpha, \gamma_1, \gamma_2)$ *is unique.*

Proof. This assertion is clear by direct inspection on the tree with levels associated to σ. ∎

3. HOPF ALGEBRA OVER THE SETS OF PLANAR BINARY TREES

The inclusion maps $\psi^*: [Y_n] \to k[S_n]$, $n \geq 0$, constructed in the previous section, give rise to a graded linear map $\psi^*: k[Y_\infty] \to k[S_\infty]$.

3.1. THEOREM. *The image of the inclusion map* $\psi^*: k[Y_\infty] \to k[S_\infty]$ *is a sub Hopf algebra of* $k[S_\infty]$. *So,* $k[Y_\infty]$ *inherits a structure of Hopf algebra (in particular of an associative algebra).*

Proof. It is sufficient to check that this image is stable under the product $*$ and the coproduct Δ. These two facts are consequences of the following propositions which have their own interest. They relate the grafting operation with the product and with the coproduct, respectively. ∎

Recall that any tree T (except $|$) can be uniquely written as $T_1 \vee T_2$, where $\vee$ is the grafting operation on trees.

3.2. PROPOSITION. *The product* $*$ *restricted to* $k[Y_\infty] \otimes k[Y_\infty]$ *satisfies the formulas*

$$T * T' = T_1 \vee (T_2 * T') + (T * T'_1) \vee T'_2,$$

where $T = T_1 \vee T_2$ *and* $T' = T'_1 \vee T'_2$ *and*

$$| * T = T = T * |.$$

Hence $*$ *is internal in* $k[Y_\infty]$.

Proof. There are two kinds of (n, m)-shuffles σ: those for which $\sigma(n) = n+m$, and those for which $\sigma(n+m) = n+m$. According to this dichotomy, the product $T * T'$ can be written

$$T * T' = \sum_{\alpha \in Z_T,\, \beta \in Z_{T'}} \left(\sum_{\substack{\sigma = (n,m)-\text{shuffle} \\ \sigma(n) = n+m}} \sigma \cdot (\alpha \times \beta) + \sum_{\substack{\sigma = (n,m)-\text{shuffle} \\ \sigma(n+m) = n+m}} \sigma \cdot (\alpha \times \beta) \right).$$

We will prove that the first summand is equal to $T_1 \vee (T_2 * T')$. A similar argument shows that the second summand is equal to $(T * T'_1) \vee T'_2$.

From Lemma 2.5 one has

$$\sum_{\alpha \in Z_T,\, \beta \in Z_{T'}} \sum_{\substack{\sigma = (n, m)-\text{shuffle} \\ \sigma(n) = n+m}} \sigma \cdot (\alpha \times \beta)$$

$$= \sum_{\alpha_1 \in Z_{T_1},\, \alpha_2 \in Z_{T_2},\, \beta \in Z_{T'}} \sum_{\tau \in \Omega(n, m, i)} \tau \cdot (\alpha_1 \times (1) \times \alpha_2 \times \beta),$$

where

$$\Omega(n, m, i) := \{\tau = \sigma \cdot (\delta \times 1_m) \in S_{n+m} \mid \sigma \text{ is a}(n, m)-\text{shuffle},$$
$$\delta \text{ is a}(i, n-i)-\text{shuffle, such that } \sigma(n) = n+m \text{ and } \delta(i) = n\}.$$

By using the description of Z_T given in Section 2, it is immediate to check that

$$T_1 \vee (T_2 * T') = \sum_{\alpha_1 \in Z_{T_1},\, \alpha_2 \in Z_{T_2},\, \beta \in Z_{T'}} \sum_{\tau \in \Theta(n, m, i)} \tau \cdot (\alpha_1 \times (1) \times \alpha_2 \times \beta),$$

where

$$\Theta(n, m, i) := \{\tau = \rho \cdot (1_{S_i} \times \omega) \in S_{n+m} \mid \rho \text{ is a}(i, n+m-i)-\text{shuffle},$$
$$\omega \text{ is a}(n-i, m)-\text{shuffle, such that } \rho(i) = n+m\}.$$

It suffices, now, to prove that $\Omega(n, m, i) = \Theta(n, m, i)$.

Given $\sigma \cdot (\delta \times 1_m) \in \Omega(n, m, i)$, one has $\sigma \cdot \delta(1) < \cdots < \sigma \cdot \delta(i)$. So there exists a unique $(i, n+m, i)$-shuffle $\rho \in S_{n+m}$ such that

$$\rho(j) = \sigma \cdot \delta(j), \qquad \text{for} \quad 1 \leqslant j \leqslant i.$$

Observe that $\rho(i) = \sigma \cdot \delta(i) = \sigma(n) = n+m$.

Now, one has

$$\rho^{-1} \cdot \sigma \cdot (\delta \times 1_m) = 1_{S_i} \times \omega,$$

for some $\omega \in S_{n+m-i}$. We need to prove that ω is a $(n-i, m)$-shuffle.

Since $\{\delta(i+1) < \cdots < \delta(n)\} \subseteq \{1, ..., n\}$, one has, from the definition of ρ,

$$\{\sigma \cdot \delta(i+1) < \cdots < \sigma \cdot \delta(n)\} \subseteq \{\rho(i+1) < \cdots < \rho(n+m)\}.$$

Since ρ is a $(i, n+m-i)$-shuffle and $\omega(j) = \rho^{-1} \cdot \sigma \cdot \delta(j+i)$, for $1 \leqslant j \leqslant n-i$, one gets

$$\omega(1) < \cdots < \omega(n-i).$$

Again, $\{\sigma(n+1) < \cdots < \sigma(n+m)\} \subseteq \{\rho(i+1) < \cdots < \rho(n+m)\}$ and $\omega(j) = \rho^{-1} \cdot \sigma(j+i)$, for $n-i+1 \leqslant j \leqslant n+m-i$, imply that

$$\omega(n-i+1) < \cdots < \omega(n+m-i).$$

Conversely, let $\rho(1_{S_i} \times \omega) \in S_{n+m}$ be an element of $\Theta(n, m, i)$.

Define $\sigma \in S_{n+m}$ to be the unique (n, m)-shuffle such that $\sigma(n+j) := \rho(\omega(n+j-i)+i)$, for $1 \leqslant j \leqslant m$. It is clear that $\sigma(n) = n+m$.

Define $\delta \in S_n$ by

$$\delta(j) := \begin{cases} \sigma^{-1} \cdot \rho(j) & \text{for} \quad 1 \leqslant j \leqslant i \\ \sigma^{-1} \cdot \rho(\omega(j-i)+i) & \text{for} \quad i+1 \leqslant j \leqslant n. \end{cases}$$

Now, the inclusion $\{\rho(1) < \cdots < \rho(i)\} \subseteq \{\sigma(1) < \cdots < \sigma(n)\}$ and the fact that σ is a (n, m)-shuffle imply that

$$\delta(1) < \cdots < \delta(i).$$

Moreover, from $\rho(i) = n+m$ and $\sigma(n) = n+m$, one gets $\delta(i) = n$.

For $i+1 \leqslant j \leqslant n$, the inclusions

$$\{\rho(\omega(1)+i) < \cdots < \rho(\omega(n-i)+i\} \subseteq \{\rho(i+1) < \cdots < \rho(n)\}$$
$$\subset \{\sigma(1) < \cdots < \sigma(n)\},$$

and the fact that σ is a (n, m)-shuffle, imply

$$\delta(i+1) < \cdots < \delta(n).$$

So, $\Omega(n, m, i) = \Theta(n, m, i)$, and the proof is done. ∎

3.3. PROPOSITION. *Let $T = T_1 \vee T_2 \in Y_{n+m+1}$ be a tree, with $T_1 \in Y_n$ and $T_2 \in Y_m$. The restriction of the coproduct Δ to $k[Y_\infty]$ satisfies the relation*

$$\Delta(T) = \sum_{j,k} (T_{1(j)} * T_{2(k)}) \otimes (T'_{1(n-j)} \vee T'_{2(m-k)}) + T \otimes |,$$

where $\Delta(T_1) = \sum_j T_{1(j)} \otimes T'_{1(n-j)}$ and $\Delta(T_2) = \sum_k T_{2(k)} \otimes T'_{2(m-k)}$.

Hence Δ is internal in $k[Y_\infty]$.

Proof. Let σ be an element of Z_T. We want to compute $\sigma_{(r)} \otimes \sigma'_{(n+m+1-r)}$, for $0 \leqslant r \leqslant n+m+1$.

For $r = n+m+1$, it is immediate to check that

$$\sigma_{(n+m+1)} \otimes \sigma'_{(0)} = T \otimes |.$$

Suppose now that $0 \leqslant r \leqslant n+m$. From the description of Z_T, there exist unique $\alpha \in Z_{T_1}$, $\beta \in Z_{T_2}$ and $\rho \in S_{n+m+1}$ such that ρ is a $(n+1, m)$-shuffle with $\rho(n+1) = n+m+1$, and $\sigma = \rho(\alpha \times (1) \times \beta)$.

Denote by l the element $N(\rho, r)$, that is, l is the number of elements of $\{1, ..., r\} \cap \{\rho(1), ..., \rho(n+1)\}$. By Lemma 1.3,

$$\rho = (\rho_{(r)} \times \rho'_{(n+m+1-v)}) \cdot \omega^{-1},$$

where

$$\omega^{-1}(j) = \begin{cases} j & \text{for } \ 1 \leqslant j \leqslant l \\ & \quad\ \text{and } \ n+r-l+2 \leqslant j \leqslant n+m+1 \\ j+r-l & \text{for } \ l+1 \leqslant j \leqslant n+1 \\ j-(n+1-l) & \text{for } \ n+2 \leqslant j \leqslant n+r-l+1 \end{cases}$$

Note that $\rho(n+1) = n+m+1$ implies $\rho'_{(n+m+1-r)}(n+1-l) = n+m+1$.

Again, from Lemma 1.3, there exist a $(l, n-l)$-shuffle $\delta \in S_n$ and a $(r-l, m-r+l)$-shuffle $\lambda \in S_m$ such that

$$\alpha = (\alpha_{(l)} \times \alpha'_{(n-l)}) \cdot \delta^{-1},$$

and

$$\beta = (\beta_{(r-l)} \times \beta'_{(m-r+l)}) \cdot \lambda^{-1}.$$

It is straightforward to check that

$$\begin{aligned} &\omega^{-1} \cdot ((\alpha_{(l)} \times \alpha'_{(n-l)} \times (1) \times \beta_{(r-l)} \times \beta'_{(m-r+l)}) \\ &\quad = (\alpha_{(l)} \times \beta_{(r-l)} \times \alpha'_{(n-l)} \times (1) \times \beta'_{(m-r+l)}) \cdot \omega^{-1}. \end{aligned}$$

The element $(\delta \times 1_n \times \lambda) \cdot \omega$ is a $(r, n+m-r)$-shuffle, so

$$\begin{aligned} &\sigma_{(r)} \otimes \sigma'_{(n+m+1-r)} \\ &\quad = \rho_{(r)} \cdot (\alpha_{(l)} \times \beta_{(r-l)}) \times \rho'_{(n+m+1-r)} \cdot (\alpha'_{(n-l)} \times (1) \times \beta'_{(m-r+l)}). \end{aligned}$$

The result follows from the uniqueness of the decomposition of elements of Z_T, from $\rho \cdot (\alpha_{(l)} \times \beta_{(r-l)}) \in (T_{1(l)} * T_{2(r-l)})$ and from

$$\rho'_{(n+m+1-r)} \cdot (\alpha'_{(n-l)} \times (1) \times \beta'_{(m-r+l)}) \in (T'_{1(n-l)} \vee T'_{2(m-r+l)}). \quad \blacksquare$$

3.4. *Decomposition of the associative operation.* Let us denote by $\overline{k[Y_\infty]}$ the augmentation ideal of $k[Y_\infty]$, that is $\overline{k[Y_\infty]} := \bigoplus_{n \geqslant 1} k[Y_n]$. By Proposition 3.2 the product of two elements of $k[Y_\infty]$ is the sum of two well-defined elements. Let us introduce these two operations $\prec$ and $\succ$, called *left* and *right product*, respectively,

$$T \prec T' := T_1 \vee (T_2 * T'), \tag{3.4.1}$$

$$T \succ T' := (T * T'_1) \vee T'_2, \tag{3.4.2}$$

where $T = T_1 \vee T_2$ and $T' = T'_1 \vee T'_2$. These operations are extended to $\overline{k[Y_\infty]}$ by linearity. By Proposition 3.2 we have

$$T * T' = T \prec T' + T \succ T'.$$

3.5. THEOREM. *The left and right products on $\overline{k[Y_\infty]}$ satisfy the following relations*:

$$\text{(i)}\quad (a \prec b) \prec c = a \prec (b \prec c) + a \prec (b \succ c)$$

$$\text{(ii)}\quad a \succ (b \prec c) = (a \succ b) \prec c$$

$$\text{(iii)}\quad a \succ (b \succ c) = (a \prec b) \succ c + (a \succ b) \succ c$$

for any $a, b, c \in \overline{k[Y_\infty]}$.

In other words $\overline{k[Y_\infty]}$, *equipped with* $\succ$ *and* $\prec$, *is a dual-dialgebra* (*in the sense of* [Lod2).

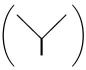
Proof. Let $T = T_1 \vee T_2$, $T' = T'_1 \vee T'_2$ and $T'' = T''_1 \vee T''_2$ be planar binary trees. The following equalities follow from the definitions of the operations and the associativity of $*$:

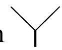
$$\begin{aligned} \text{(i)}\quad (T \prec T') \prec T'' &= (T_1 \vee (T_2 * T')) \prec T'' \\ &= T_1 \vee (T_2 * T' * T'') \\ &= T \prec (T' \prec T'') + T \prec (T' \succ T''). \end{aligned}$$

$$\begin{aligned} \text{(ii)}\quad T \succ (T' \prec T'') &= T \succ (T'_1 \vee (T'_2 * T'')) = (T * T'_1) \vee (T'_2 * T'') \\ &= ((T * T'_1) \vee T'_2) \prec T'' = (T \succ T') \prec T''. \end{aligned}$$

$$\begin{aligned} \text{(iii)}\quad T \succ (T' \succ T'') &= T \succ ((T' * T''_1) \vee T''_2) \\ &= (T * T' * T''_1) \vee T''_2 \\ &= (T \prec T') \succ T'' + (T \succ T') \succ T''. \quad \blacksquare \end{aligned}$$

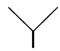
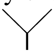
3.6. *Dual-dialgebra.* By definition a *dual-dialgebra* is a vector space equipped with two binary operations $\prec$ and $\succ$ satisfying the relations (i), (ii) and (iii) of Theorem 3.5. This notion appeared in [Lod2] as the Koszul dual notion of another type of algebras with two operations: the dialgebras (cf. loc. cit.). It is a general fact that, for any dual dialgebra, the product $x * y := x \prec y + x \succ y$ is associative (immediate checking). We may be more precise about the dual-dialgebra structure of $\overline{k[Y_\infty]}$:

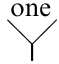
3.7. PROPOSITION. *The dual dialgebra* $\overline{k[Y_\infty]}$ *is free on one generator* .

Proof. If T is different from , then formulas (3.4.1) and (3.4.2) imply that T can be written either as

$$T = T_1 \vee T_2 \in k[Y_n], \deg T_1 \geqslant 1, \deg T_2 \geqslant 1,$$

$$\text{or} \quad T = (T_1 \vee |) \prec T_2, \deg T_1 \geqslant 1, \deg T_2 \geqslant 1.$$

By induction it is now clear that any tree can be obtained from  by the operations $\succ$ and $\prec$. Hence the dual-dialgebra $\overline{k[Y_\infty]}$ is spanned by .

Denote by $F_{Dias^!}(k)$ the free dual-dialgebra spanned by one element z. Since $\overline{k[Y_\infty]}$ is a dual-dialgebra, there is one (and only one) morphism from $F_{Dias^!}(k)$ to $\overline{k[Y_\infty]}$ which sends z to . This map is surjective by the preceding remark. Since $k[Y_n]$ and $F_{Dias^!}(k)_n$ are of the same dimension c_n (cf. [Lod2]), this map is an isomorphism. ∎

In [PR] Poirier and Reutenauer show that, as an associative algebra, $k[S_\infty]$ is free. The same is true for $k[Y_\infty]$ as shown below.

3.8. THEOREM. *The Hopf algebra* $k[Y_\infty]$ *is the free associative algebra (with 1) spanned by the set* $W_\infty := \bigcup_{n \geqslant 1} W_n$, *where* $W_n := \{T = | \vee T' \in Y_n$, *for* $T' \in Y_{n-1}\}$.

Proof. Since $W_\infty \subset Y_\infty$, there is a well-defined algebra morphism

$$\alpha\colon T(k[W_\infty]) \to k[Y_\infty].$$

Let us first show that α is surjective. Let $T = T_1 \vee T_2 \in Y_n$, with $T_1 \in Y_r$ and $T_2 \in Y_{n-1-r}$. If $T_1 = |$, then $T \in W_n$. Now, let $T_1 = T_{11} \vee T_{12} \in Y_r$, with $r > 0$. One has

$$T\colon T_1 \vee T_2 = T_1 * (| \vee T_2) - T_{11} \vee (T_{12} * T_2).$$

Since $T_{11} \in Y_j$, with $j < r$, using a recursive argument on r, T_1 and $T_{11} \vee (T_{12} * T_2)$ are in the algebra spanned by W_∞, which implies that T is in the algebra spanned by W_∞.

The above argument shows that $k[Y_\infty]$ is spanned, as an associative k-algebra, by W_∞, so α is surjective.

Let us show that α is in fact an isomorphism. Let $y=\sum_{n\geqslant 0} c_n x^n$ be the generating function of the Catalan numbers. It is well-known that y satisfies the following functional equation: $xy^2-y+1=0$. The Poincaré series of $k[W_\infty]$ is $f_W(x)=\sum_{n\geqslant 1} c_{n-1}x^n=xy$. On the other hand the Poincaré series of $T(k[W_\infty])$ is

$$f_{TW}(x)=\sum_{n\geqslant 0} f_{W^n}(x)=\sum_{n\geqslant 0} f_w(x)^n=(1-f_W(x))^{-1}=(1-xy)^{-1}.$$

By the functional equation, we get $f_{TW}(x)=y$ and so $\dim T(k[W_\infty])_n=c_n$.

The map α: $T(k[W_\infty])_n\to k[Y_n]$ is surjective and both vector spaces have the same dimension. Hence it is an isomorphism. ∎

Observe that $\dim k[W_n]=c_{n-1}$ and $\dim T(k[W_\infty])_n=c_n$.

4. SOLOMON DESCENT ALGEBRA AND HOPF STRUCTURE

4.1. *The Solomon descent algebra.* By definition, a permutation $\sigma\in S_n$ has a *descent* at i, $1\leqslant i\leqslant n-1$, if $\sigma(i)>\sigma(i+1)$. For instance 1_n has no descent, and $(2\ 3\ 1)$ has a descent at 2. For any subset J of $\{1, ..., n-1\}$, let X^n_J be the set of permutations in S_n which have no descent in J. Define

$$x_J:=\sum_{\sigma\in X^n{}_J}\sigma\in k[S_n].$$

Louis Solomon proved in [Sol] that, for a fixed n, the elements x^n_J generate a subalgebra of $k[S_n]$ of dimension 2^{n-1}. In fact, they form a linear basis of this algebra, called the *Solomon descent algebra*. We denote it by Sol_n (instead of Σ_n used sometimes in the literature). Let Sol_∞ denote the graded vector space $\bigoplus_{n\geqslant 0} Sol_n$.

The following result is due to C. Reutenauer.

4.2. THEOREM [Reu]. *For any $x_J\in Sol_n$ and $x_K\in Sol_m$ one has*

$$x_J * x_K:=sh_{n,m}\cdot(x_J\times x_K)=x_{J\cup K},$$

where $J\cup K$ denotes the subset of $\{1, ..., n+m-1\}$ defined by

$$i\in J\cup K \quad \text{if} \quad i\in J \quad \text{or if} \quad i-n\in K. \tag{1}$$

Moreover, there exists a unique coproduct Δ on Sol_∞ such that:

$$\text{(i)}\quad \Delta(x_{\{1,\,...,\,n-1\}}) = \sum_{i=0}^{n} x_{\{1,\,...,\,i-1\}} \otimes x_{\{1,\,...,\,n-i-1\}},$$

(ii) $(Sol_\infty, *, \Delta)$ *is a Hopf algebra over* k.

4.3. Proposition [MR]. *The Solomon algebra Sol_∞ is a sub-Hopf algebra of $k[S_\infty]$.*

Proof. Formula (2) of Theorem 4.2 shows that Sol_∞ is closed under the product $*$.

Note that $x_{\{1,\,...,\,n-1\}} \in Sol_n$ is the identity element 1_n of S_n, $n \geqslant 1$. From the definition of Δ on $k[S_\infty]$ one has

$$\Delta(1_n) = \sum_{i=0}^{n} 1_i \otimes 1_{n-i}.$$

Hence the two coproducts coincide on Sol_∞. ∎

4.4. *Hypercube and trees.* Let $Q_n := \{+1, -1\}^{n-1}$ be the set of vertices of the hypercube. There is a map ϕ: $Y_n \twoheadrightarrow Q_n$, which is defined as follows: $\phi(y) = (\varepsilon_1, ..., \varepsilon_{n-1})$, where ε_i is $+1$ when the ith leaf of y is right oriented, and -1 when it is left oriented. Observe that we take into account only the interior leaves (numbered from 1 to $n-1$) of y, since the orientation of the two extreme ones does not depend on y.

The induced linear map $\phi: k[Y_n] \twoheadrightarrow k[Q_n]$ has a linear dual $\phi^*: k[Q_n] \to k[Y_n]$ obtained by identifying each basis with its own dual. Observe that the inclusion ϕ^* is not induced by a set-theoretic map. For instance

$\phi^*(-1,+1) =$  $+$  .

The composite map $\phi \circ \psi$: $S_n \to Q_n$ has the following interpretation. The ith entry of $\phi \circ \psi(\sigma)$ is $+1$ if σ has no descent at i, and is -1 if σ has a descent at i.

4.5. Theorem. *The image of $k[Q_\infty]$ in $k[S_\infty]$ under $(\phi \circ \psi)^*$ is the Solomon descent algebra Sol_∞.*

Proof. For $n \geqslant 1$ and $J \subseteq \{1, ..., n-1\}$, define

$$Y_J^n := \{\sigma \in S_n \mid \sigma \text{ has a descent at } i \text{ if and only if } i \notin J\}.$$

The elements $y_J := \sum_{\sigma \in Y^n_J} \sigma$, for $J \subseteq \{1, ..., n-1\}$, form a basis of Sol_n. Moreover, one has

$$x_J = \sum_{J \subseteq K} y_K \qquad \text{and} \qquad y_J = \sum_{J \subseteq K} (-1)^{|K| - |J|}\, x_K.$$

For any element $\varepsilon = (\varepsilon_1, ..., \varepsilon_{n-1})$ of Q_n define $J(\varepsilon) \subseteq \{1, ..., n-1\}$ by

$$i \in J(\varepsilon) \qquad \text{iff} \quad \varepsilon_i = -1.$$

It is clear that the image of ε under $(\phi \circ \psi)^*$ is $y_{J(\varepsilon)}$. So $k[Q_n] = Sol_n$, for $n \geqslant 1$, which ends the proof. ∎

4.6. *Associative structure of* $k[Q_\infty]$. Theorem 4.5 states that $k[Q_\infty]$ is a sub-Hopf algebra of $k[Y_\infty]$ and $k[S_\infty]$. A straightforward calculation shows that the product $*$ restricted to $k[Q_\infty]$ satisfies the formula

$$\begin{aligned}(\varepsilon_1, ..., \varepsilon_{n-1}) &* (\delta_1, ..., \delta_{m-1}) \\ &= (\varepsilon_1, ..., \varepsilon_{n-1}, +1, \delta_1, ..., \delta_{m-1}) + (\varepsilon_1, ..., \varepsilon_{n-1}, -1, \delta_1, ..., \delta_{m-1}).\end{aligned}$$

4.7. *Remarks.* The Hopf algebra Sol_∞ appears also in [GKLLRT] under the name "noncommutative symmetric functions" and the notation *Sym*. It is a cocommutative Hopf algebra, isomorphic to $T(\bar{T}(k))$, where $\bar{T}(k)$ is the reduced tensor coalgebra over k and $T(\bar{T}(k))$ the tensor algebra over it.

The Hopf algebra Sol_∞ contains a graded commutative Hopf algebra (of dimension n in degree n). It is linearly generated in Sol_n by the "Eulerian idempotents" (cf. [Lod1]).

The relationship between the product structure of the Hopf algebra Sol_∞ (sometimes called "outer product") and the product structure of the Solomon algebra Sol_n (sometimes called "inner product") has been investigated in [Ron].

Observe that the two Hopf algebras $k[Y_\infty]$ and $k[S_\infty]$ are not commutative, nor cocommutative. However $k[S_\infty]$ is an associative algebra with involution. The involution on $k[S_n]$ is conjugation by $\omega = (1\ n)(2\ n-1) \cdots$.

5. CELLULAR DECOMPOSITIONS OF THE SPHERES

The maps

$$S_n \xrightarrow{\psi} Y_n \xrightarrow{\phi} Q_n$$

can be seen as the restriction, to the set of vertices, of cellular maps between different cellular decompositions of the sphere S^{n-2}, as follows.

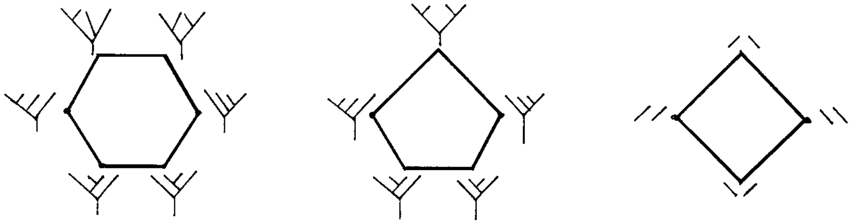


FIG. 1. $n=3$, cellular decompositions of S^1.

The *permutohedron* P_n (cf. [Mil]) is the convex envelope of the $n!$ vertices $(\sigma^{-1}(1), ..., \sigma^{-1}(n)) \in \mathbf{R}^n$, for permutations $\sigma \in S_n$. It is a cellular complex whose barycentric subdivision is the geometric realization of the partition lattice (cf. [Han]), that is the poset $\mathscr{P}_n$ of partitions of $\{1, ..., n\}$ (excluding the trivial one). The vertices of $\mathscr{P}_n$ are precisely the initial elements of the poset.

The *associahedron* K_n (cf. [Sta]) is a cellular complex whose barycentric subdivision is the geometric realization of the poset of parenthesizing n variables. This poset is, of course, the same as the poset $\mathscr{K}_n$ of planar trees with $n+1$ leaves (from which we exclude the tree with only one vertex). The vertices of $\mathscr{K}_n$ are in bijection with the set of planar binary trees Y_n, which are precisely the initial elements of the poset.

The *hypercube* C_n is the convex envelope of the 2^{n-1} elements of Q_n viewed as vertices $(\varepsilon_1, ..., \varepsilon_{n-1}) \in \mathbf{R}^{n-1}$ (recall that $\varepsilon_i = +1$ or -1). It is a cellular complex whose barycentric subdivision is the geometric realization of the poset $\mathscr{C}_n = \{-1, 0, +1\}^{n-1} - (0, ..., 0)$. The partial order is as follows: $(x_1, ..., x_{n-1}) < (y_1, ..., y_{n-1})$ if and only if, for any i, $y_i = 0$ whenever $x_i \neq y_i$. The vertices of C_n are precisely the initial elements of the poset.

There are obvious equivalence relations on $\mathscr{P}_n$ and on $\mathscr{K}_n$ which define poset morphisms

$$\mathscr{P}_n \xrightarrow{\psi} \mathscr{K}_n \xrightarrow{\phi} \mathscr{C}_n$$

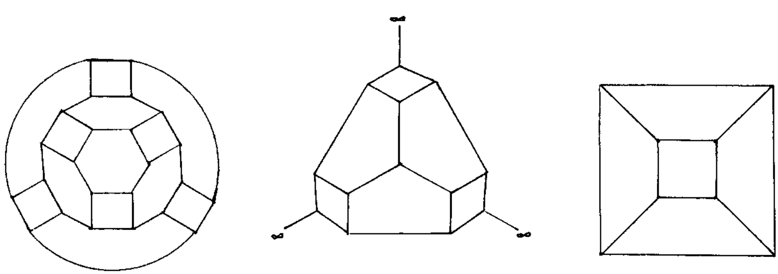


FIG. 2. $n=4$, cellular decompositions of S^2 (stereographic projections).

extending the maps ψ and ϕ on the initial elements (cf. for instance [Ton] for ϕ). The geometric realizations of these maps are retract by deformation of the sphere S^{n-2}.

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