Encyclopedia of types of algebras 2010

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This is a cornucopia of types of algebras with some of their properties from the operadic point of view.

Introduction

The following is a list of some types of algebras together with their properties under an operadic and homological point of view. In this version we restrict ourselves to types of algebras which are encoded by an algebraic operad, cf. [?].

We keep the information to one page per type and we provide one reference in full as Ariadne's thread (so that one can print only one page). More references are listed by the end of the paper. We work over a fixed field $\mathbb{K}$ though in many instances everything makes sense and holds over a commutative ground ring ($\mathbb{Z}$ for instance). The category of vector spaces over $\mathbb{K}$ is denoted by `Vect`. All tensor products are over $\mathbb{K}$ unless otherwise stated.

The items of a standard page (which is to be found at the end of this introduction) are as follows. Sometimes a given type appears under different names in the literature. The choice made in `Name` is, most of the time, the most common one (up to a few exceptions). The other possibilities appear under the item `Alternative`.

The presentation given in `Definit.` is the most common one (*Lie* excepted). When others are used in the literature they are given in `Alternative`. The item `oper.` gives the generating operations. The item `sym.` gives their symmetry properties, if any. The item `rel.` gives the relation(s). They are supposed to hold for any value of the variables $x, y, z, \ldots$. If, in the presentation, only binary operations appear, then the type is said to be *binary*. Analogously, there are *ternary, k-ary, multi-ary* types.

If, in the presentation, the relations involve only the composition of two operations at a time (hence 3 variables in the binary case, 5 variables in the ternary case), then the type is said to be *quadratic*.

For a given type of algebras $\mathcal{P}$ the category of $\mathcal{P}$-algebras is denoted by $\mathcal{P}$-alg. For each type there is defined a notion of *free algebra*. By definition the free algebra of type $\mathcal{P}$ over the vector space V is an algebra denoted by $\mathcal{P}(V)$ satisfying the following universal condition:

for any algebra A of type $\mathcal{P}$ and any linear map $\phi : V \to A$ there is a unique $\mathcal{P}$-algebra morphism $\tilde{\phi} : \mathcal{P}(V) \to A$ which lifts ϕ. In other words the forgetful functor $\mathcal{P}\text{-}\mathsf{alg} \longrightarrow \mathtt{Vect}$ admits a left adjoint $\mathcal{P} : \mathtt{Vect} \longrightarrow \mathcal{P}\text{-}\mathsf{alg}$. In all the cases mentioned here the relations involved in the presentation of the given type are (or can be made) multilinear. Hence the functor $\mathcal{P}(V)$ is of the form (at least in characteristic zero),

$$\mathcal{P}(V) = \bigoplus_{n\geq 1} \mathcal{P}(n) \otimes_{\mathbb{S}_n} V^{\otimes n} \ ,$$

where $\mathcal{P}(n)$ is some $\mathbb{S}_n$-module. The $\mathbb{S}_n$-module $\mathcal{P}(n)$ is called the space of n-ary operations since for any algebra A there is a map

$$\mathcal{P}(n) \otimes_{\mathbb{S}_n} A^{\otimes n} \to A.$$

The functor $\mathcal{P} : \mathtt{Vect} \to \mathtt{Vect}$ inherits a monoid structure from the properties of the free algebra. Hence there exist transformations of functors $\iota : \mathrm{Id} \to \mathcal{P}$ and $\gamma : \mathcal{P} \circ \mathcal{P} \to \mathcal{P}$ such that γ is associative and unital. The monoid $(\mathcal{P}, \gamma, \iota)$ is called a *symmetric operad*.

The symmetric operad $\mathcal{P}$ can also be described as a family of $\mathbb{S}_n$-modules $\mathcal{P}(n)$ together with maps

$$\gamma(i_1, \ldots, i_k) : \mathcal{P}(k) \otimes \mathcal{P}(i_1) \otimes \cdots \otimes \mathcal{P}(i_k) \longrightarrow \mathcal{P}(i_1 + \cdots + i_k)$$

satisfying some compatibility with the action of the symmetric group and satisfying the associativity property.

If $\mathbb{S}_n$ is acting freely on $\mathcal{P}(n)$, then $\mathcal{P}(n) = \mathcal{P}_n \otimes \mathbb{K}[\mathbb{S}_n]$ where $\mathcal{P}_n$ is some vector space, and $\mathbb{K}[\mathbb{S}_n]$ is the regular representation. If, moreover, the maps $\gamma(i_1, \ldots, i_n)$ are induced by maps

$$\gamma_{i_1,\ldots,i_k} : \mathcal{P}_k \otimes \mathcal{P}_{i_1} \otimes \cdots \otimes \mathcal{P}_{i_k} \longrightarrow \mathcal{P}_{i_1+\cdots+i_k},$$

then the operad $\mathcal{P}$ comes from a *nonsymmetric operad* (abbreviated ns operad), still denoted by $\mathcal{P}$ in general.

For more terminology and details about algebraic operads we refer to [?] or [?].

The generating series of the operad $\mathcal{P}$ is defined as

$$f^{\mathcal{P}}(t) := \sum_{n\geq 1} \frac{\dim \mathcal{P}(n)}{n!}\ t^n,$$

in the binary case. When dealing with a nonsymmetric operad it becomes

$$f^{\mathcal{P}}(t) := \sum_{n\geq 1} \dim \mathcal{P}_n\ t^n.$$

The Koszul duality theory of associative algebras has been extended to binary quadratic operads by Ginzburg and Kapranov, cf. [?], then to quadratic operads by Fresse, cf. [?]. A conceptual treatment of this theory, together with applications, is given in [?]. So, to any quadratic operad $\mathcal{P}$, there is associated a quadratic **Koszul dual operad** denoted $\mathcal{P}^!$. It is often a challenge to find a presentation of $\mathcal{P}^!$ out of a presentation of $\mathcal{P}$. One of the main results of the Koszul duality theory of operads is to show the existence of a natural differential map on the composite $\mathcal{P}^{!*}\circ\mathcal{P}$ giving rise to the *Koszul complex.* If it is acyclic, then $\mathcal{P}$ is said to be *Koszul*. One can show that, if $\mathcal{P}$ is Koszul, then so is $\mathcal{P}^!$. In this case the generating series are inverse to each other for composition, up to sign, that is:

$$f^{\mathcal{P}^!}(-f^{\mathcal{P}}(t)) = -t.$$

Recall that if $f^{\mathcal{P}^!}(t) = t + \sum_{n\geq 2} a_n t^n$ and $g^{\mathcal{P}}(t) = t + \sum_{n\geq 2} b_n t^n$, then

$$\begin{aligned} b_2 &= a_2,\\ b_3 &= -a_3 + 2a_2^2,\\ b_4 &= a_4 - 5a_3a_2 + 5a_2^3,\\ b_5 &= -a_5 + 3a_3^2 + 6a_2a_4 - 21a_2^2a_3 + 14a_2^4. \end{aligned}$$

In the k-ary case one introduces the skew-generating series

$$g^{\mathcal{P}}(t) := \sum_{n\geq 1} (-1)^k \frac{\dim \mathcal{P}((k-1)n+1)}{n!}\ t^{((k-1)n+1)}.$$

If the operad $\mathcal{P}$ is Koszul, then by [?] the following formula holds:

$$f^{\mathcal{P}^!}(-g^{\mathcal{P}}(t)) = -t.$$

The items `Free alg.`, `rep.` $\mathcal{P}(n)$ or $\mathcal{P}_n$, $\dim\mathcal{P}(n)$ or $\dim\mathcal{P}_n$, and `Gen.series` speak for themselves.

Koszulity of an operad implies the existence of a small chain complex to compute the (co)homology of a $\mathcal{P}$-algebra. When possible, the information on it is given in the item `Chain-cplx`. Moreover it permits us to construct the notion of $\mathcal{P}$-algebra up to homotopy, whose associated operad, which is a differential graded operad, is denoted by $\mathcal{P}_\infty$. The importance of this notion is due to the "Homotopy Transfer Theorem", see [?] section 10.3.

The item `Properties` lists the main features of the operad. *Set-theoretic* means that there is a set operad $\mathcal{P}_{Set}$ (monoid in the category of $\mathbb{S}$-Sets) such that $\mathcal{P} = \mathbb{K}[\mathcal{P}_{Set}]$. Usually this property can be read on the presentation of the operad: no algebraic sums. Quasi-regular means that $\mathcal{P}(n)$ is a sum of regular representations, but the operad is not necessarily coming from a ns operad.

In the item `Relationsh.` we list some of the ways to obtain this operad under some natural constructions like tensor product (Hadamard product) or Manin products (white $\circ$ or black $\bullet$), denoted $\square$ and $\blacksquare$ in the nonsymmetric framework, cf. [?] or [?] for instance. We also list some of the most common functors to other types of algebras. Keep in mind that a functor $\mathcal{P} \to \mathcal{Q}$ induces a functor $\mathcal{Q}\text{-alg} \to \mathcal{P}\text{-alg}$ on the categories of algebras.

Though we describe only algebras without unit, for some types there is a possibility of introducing an element 1 which is either a unit or a partial unit for some of the operations, see the discussion in [?]. We indicate it in the item `Unit`.

For binary operads the *opposite type* consists in defining new operations by $x \cdot y = yx$, etc. If the new type is isomorphic to the former one, then the operad is said to be *self-opposite*. When it is not the case, we mention whether the given type is called *right* or *left* in the item `Comment`.

In some cases the structure can be "integrated". For instance Lie algebras are integrated into Lie groups (Lie third problem). If so, we indicate it in the item `Comment`.

In the item `Ref.` we indicate a reference where information on the operad and/or on the (co)homology theory can be obtained. It is not necessarily the first paper in which this type of algebras first appeared. For the "three graces", that is the operads As, Com, Lie, the classical books by Cartan and Eilenberg "Homological Algebra" and by MacLane "Homology" are standard references.

Notation. We use the notation $\mathbb{S}_n$ for the symmetric group. Trees are very much in use in the description of operads. We use the following notation:

– PBT_n is the set of planar binary rooted trees with $n-1$ internal vertices (and hence n leaves). The number of elements in PBT_{n+1} is the Catalan number $c_n = \frac{1}{n+1}\binom{2n}{n}$.

–PT_n is the set of planar rooted trees with n leaves, whose vertices have valency greater than $1+2$ (one root, at least 2 inputs). So we have $PBT_n \subset PT_{n+1}$. The number of elements in PT_n is the super Catalan number, also called Schröder number, denoted C_n.

A planar binary rooted tree t is completely determined by its right part t^r and its left part t^l. More precisely t is the grafting of t^l and t^r: $t = t^l \vee t^r$.

Comments. Many thanks to Walter Moreira for setting up a software which computes the first dimensions of the operad from its presentation.

We remind the reader that we can replace the symmetric monoidal category `Vect` by many other symmetric monoidal categories. So there are notions of graded algebras, differential graded algebras, twisted algebras, and so forth. In the graded cases the Koszul sign rule is in order. Observe that there are also operads where the operations may have different degree (operad encoding Gerstenhaber algebras for instance).

We end this paper with a tableau of integer sequences appearing in this document.

This list of types of algebras is not as encyclopedic as the title suggests. We put only the types which are defined by a finite number of generating operations and whose relations are multilinear. You will not find the "restricted types" (like divided power algebras), nor bialgebras. Moreover we only put those which have been used some way or another. We plan to update this encyclopedia every now and then.

Please report any error or comment or possible addition to:

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Here is the list of the types included so far (with letter K indicating that they are Koszul dual to each other):

sample	As	self-dual
Com	Lie	K
$Pois$	none	self-dual
$Leib$	$Zinb$	K
$Dend$	$Dias$	K
$PreLie$	$Perm$	K
$Dipt$	$Dipt^!$	K
$2as$	$2as^!$	K
$Tridend$	$Trias$	K
$PostLie$	$ComTrias$	K
CTD	$CTD^!$	K
$L\text{-}dend$	$Ennea$	
$Gerst$	BV	
Mag	Nil_2	K
$ComMag$	$ComMag^!$	K
$Quad$	$Quad^!$	K
Dup	$Dup^!$	K
$As^{(2)}$	$As^{\langle 2\rangle}$	
$Lie\text{-}adm$	$PreLiePerm$	
$Altern$	$Param1rel$	
$MagFine$	$GenMag$	
NAP	$Moufang$	
$Malcev$	$Novikov$	
$DoubleLie$	$DiPreLie$	
$Akivis$	$Sabinin$	
$Jordan\ triples$	$t\text{-}As^{(3)}$	
$p\text{-}As^{(3)}$	LTS	
$Lie\text{-}Yamaguti$	$Comtrans$	
$Interchange$	$HyperCom$	
A_∞	C_∞	
L_∞	$Dend_\infty$	
$\mathcal{P}_\infty$	$Brace$	
MB	$n\text{-}Lie$	
$n\text{-}Leib$	$\mathcal{X}^\pm$	
your own		

An index is to be found at the end of the paper.

A page for personal notes (in fact, to ensure that an operad and its Koszul dual fit on opposite pages).

1. Type of algebras

algebra

`Name`	**Most common terminology**
`Notation`	our favorite notation for the operad (generic notation: $\mathcal{P}$)
`Def.oper.`	list of the generating operations
`sym.`	their symmetry if any
`rel.`	the relation(s)
`Free alg.`	the free algebra as a functor in V
`rep.` $\mathcal{P}(n)$	$\mathbb{S}_n$-representation $\mathcal{P}(n)$ and/or the space $\mathcal{P}_n$ if nonsymmetric
$\dim \mathcal{P}(n)$	the series (if close formula available), the list of the 7 first numbers beginning at $n = 1$
`Gen.series`	close formula for $f^{\mathcal{P}}(t) = \sum_{n\geq 1} \frac{\dim \mathcal{P}(n)}{n!} t^n$ when available
`Dual operad`	the Koszul dual operad
`Chain-cplx`	Explicitation of the chain complex, if not too complicated
`Properties`	among: nonsymmetric, binary, quadratic, set-theoretic, ternary, multi-ary, cubic, Koszul.
`Alternative`	alternative terminology, and/or notation, and/or presentation
`Relationsh.`	some of the relationships with other operads, either under some construction like symmetrizing, Hadamard product, Manin products, or under the existence of functors
`Unit`	whether one can assume the existence of a unit (or partial unit)
`Comment`	whatever needs to be said which does not fit into the other items
`Ref.`	a reference, usually dealing with the homology of the $\mathcal{P}$-algebras (not necessarily containing all the results of this page)

`Name`	**Associative algebra**associative
`Notation`	As (as nonsymmetric operad) Ass (as symmetric operad)
`Def.oper.`	xy, operadically: μ so that $\mu(x,y) = xy$
`sym.`	
`rel.`	$(xy)z = x(yz)$, operadically $\mu \circ_1 \mu = \mu \circ_2 \mu$ (associativity)
`Free alg.`	$As(V) = \overline{T}(V) = \oplus_{n\geq 1} V^{\otimes n}$ tensor algebra (noncommutative polynomials) $(x_1 \ldots x_p)(x_{p+1} \ldots x_{p+q}) = x_1 \ldots x_{p+q}$ (concatenation)
`rep.` $\mathcal{P}(n)$	$Ass(n) = \mathbb{K}[\mathbb{S}_n]$ (regular representation), $As_n = \mathbb{K}$
$\dim \mathcal{P}(n)$	$1, 2, 6, 24, 120, 720, 5040, \ldots, n!, \ldots$
`Gen.series`	$f^{As}(t) = \frac{t}{1-t}$
`Dual operad`	$As^! = As$
`Chain-cplx`	non-unital Hochschild complex, $C_n^{As}(A) = A^{\otimes n}$ $b'(a_1, \ldots, a_n) := \sum_{i=1}^{i=n-1} (-1)^{i-1}(a_1, \ldots, a_i a_{i+1}, \ldots, a_n)$ important variation: cyclic homology
`Properties`	ns, binary, quadratic, set-theoretic, Koszul, self-dual.
`Alternative`	associative algebra is often simply called *algebra*. Can be presented with commutative operation $x \cdot y := xy + yx$ and anti-symmetric operation $[x, y] = xy - yx$ satisfying $\begin{cases} [x \cdot y, z] = x \cdot [y, z] + [x, z] \cdot y, \\ (x \cdot y) \cdot z - x \cdot (y \cdot z) = [y, [x, z]] \ . \end{cases}$ (Livernet and Loday, unpublished)
`Relationsh.`	Ass-alg $\to$ Lie-alg, $[x, y] = xy - yx$, Com-alg $\to$ Ass-alg (inclusion), and many others
`Unit`	$1x = x = x1$
`Comment`	one the "three graces"

`Name`	**Commutative algebra**commutative
`Notation`	Com
`Def.oper.`	xy
`sym.`	$xy = yx$ (commutativity)
`rel.`	$(xy)z = x(yz)$ (associativity)
`Free alg.`	$Com(V) = \overline{S}(V)$ (polynomials) if $V = \mathbb{K}x_1 \oplus \cdots \oplus \mathbb{K}x_n$, then $\mathbb{K}1 \oplus Com(V) = \mathbb{K}[x_1, \dots, x_n]$
`rep.` $\mathcal{P}(n)$	$Com(n) = \mathbb{K}$ (trivial representation)
$\dim \mathcal{P}(n)$	$1, 1, 1, 1, 1, 1, 1, \dots, 1, \dots$
`Gen.series`	$f^{Com}(t) = \exp(t) - 1$
`Dual operad`	$Com^! = Lie$
`Chain-cplx`	Harrison complex in char. 0, André-Quillen cplx in general
`Properties`	binary, quadratic, set-theoretic, Koszul.
`Alternative`	sometimes called associative and commutative algebra other notation $Comm$
`Relationsh.`	Com-alg $\to$ Ass-alg, $Zinb$-alg $\to$ Com-alg
`Unit`	$1x = x = x1$
`Comment`	one of the "three graces"

Name	**Lie algebra**Lie
Notation	*Lie*
Def.oper.	$[x, y]$ (bracket)
sym.	$[x, y] = -[y, x]$ (anti-symmetry)
rel.	$[[x, y], z] = [x, [y, z]] + [[x, z], y]$ (Leibniz relation)
Free alg.	$Lie(V)$ = subspace of the tensor algebra $T(V)$ generated by V under the bracket
rep. $\mathcal{P}(n)$	$Lie(n) = \mathrm{Ind}_{C_n}^{\mathbb{S}_n}(\sqrt[n]{1})$
$\dim \mathcal{P}(n)$	$1, 1, 2, 6, 24, 120, 720, \ldots, (n-1)!, \ldots$
Gen.series	$f^{Lie}(t) = -\log(1-t)$
Dual operad	$Lie^! = Com$
Chain-cplx	Chevalley-Eilenberg complex $C_n^{Lie}(\mathfrak{g}) = \Lambda^n \mathfrak{g}$ $d(x_1 \wedge \ldots \wedge x_n) :=$ $\sum_{1 \leq i < j \leq n} (-1)^j (x_1 \wedge \ldots \wedge [x_i, x_j] \wedge \ldots \wedge \widehat{x_j} \wedge \ldots \wedge x_n)$
Properties	binary, quadratic, Koszul.
Alternative	The relation is more commonly written as the Jacobi identity: $[x, [y, z]] + [y, [z, x]] + [z, [x, y]] = 0$
Relationsh.	Ass-alg $\to$ Lie-alg, Lie-alg $\to$ $Leib$-alg, $PreLie$-alg $\to$ Lie-alg
Unit	no
Comment	one of the "three graces". Named after Sophus Lie. Integration: Lie groups.

`Name`	**Poisson algebra**Poisson
`Notation`	$Pois$
`Def.oper.`	$xy, \{x,y\}$
`sym.`	$xy = yx, \quad \{x,y\} = -\{y,x\}$
`rel.`	$\begin{cases} \{\{x,y\},z\} = \{x,\{y,z\}\} + \{\{x,z\},y\}, \\ \{xy,z\} = x\{y,z\} + \{x,z\}y, \\ (xy)z - x(yz) = 0 \ . \end{cases}$
`Free alg.`	$Pois(V) \cong \overline{T}(V)$ (tensor module, iso as Schur functors)
`rep.` $\mathcal{P}(n)$	$Pois(n) \cong \mathbb{K}[\mathbb{S}_n]$ (regular representation)
$\dim \mathcal{P}(n)$	$1, 2!, 3!, 4!, 5!, 6!, 7!, \ldots, n!, \ldots$
`Gen.series`	$f^{Pois}(t) = \frac{t}{1-t}$
`Dual operad`	$Pois^! = Pois$
`Chain-cplx`	Isomorphic to the total complex of a certain bicomplex constructed from the action of the Eulerian idempotents
`Properties`	binary, quadratic, quasi-regular, Koszul, self-dual.
`Alternative`	Can be presented with one operation $x * y$ with no symmetry satisfying the relation $(x*y)*z = x*(y*z) + \frac{1}{3}\big(+ x*(z*y) - z*(x*y) - y*(x*z) + y*(z*x)\big)$
`Relationsh.`	$Pois$-alg $\rightleftarrows$ Lie-alg, $Pois$-alg $\rightleftarrows$ Com-alg,
`Unit`	$1x = x = x1, [1,x] = 0 = [x,1]$
`Comment`	Named after Siméon Poisson.
`Ref.`	[?] B.Fresse, *Théorie des opérades de Koszul et homologie des algèbres de Poisson,* Ann.Math. Blaise Pascal 13 (2006), 237–312.

This page is inserted so that, in the following part, an operad and its dual appear on page $2n$ and $2n+1$ respectively. Since $Pois$ is self-dual there is no point to write a page about its dual.

Let us take the opportunity to mention that if A is a $\mathcal{P}$-algebra and B is a $\mathcal{P}^!$-algebra, then the tensor product $A \otimes B$ inherits naturally a structure of Lie algebra. If $\mathcal{P}$ is nonsymmetric, then so is $\mathcal{P}^!$, and $A \otimes B$ is in fact an associative algebra.

In some cases (like the Leibniz case for instance), $A \otimes B$ is a pre-Lie algebra.

`Name`	**Leibniz algebra**Leibniz
`Notation`	*Leib*
`Def.oper.`	$[x, y]$
`sym.`	
`rel.`	$[[x, y], z] = [x, [y, z]] + [[x, z], y]$ (Leibniz relation)
`Free alg.`	$Leib(V) \cong \overline{T}(V)$ (reduced tensor module, iso as Schur functors)
`rep.` $\mathcal{P}(n)$	$Leib(n) = \mathbb{K}[\mathbb{S}_n]$ (regular representation)
$\dim \mathcal{P}(n)$	1, 2!, 3!, 4!, 5!, 6!, 7!, ..., n!, ...
`Gen.series`	$f^{Leib}(t) = \frac{t}{1-t}$
`Dual operad`	$Leib^! = Zinb$
`Chain-cplx`	$C_n^{Leib}(\mathfrak{g}) = \mathfrak{g}^{\otimes n}$ $d(x_1, \ldots, x_n) = \sum_{1 \leq i < j \leq n} (-1)^j (x_1, \ldots, [x_i, x_j], \ldots, \widehat{x_j}, \ldots, x_n)$
`Properties`	binary, quadratic, quasi-regular, Koszul.
`Alternative`	Sometimes improperly called Loday algebra.
`Relationsh.`	$Leib = Perm \bigcirc Lie$ (Manin white product), see [?] Lie-alg $\to$ $Leib$-alg, $Dias$-alg $\to$ $Leib$-alg, $Dend$-alg $\to$ $Leib$-alg
`Unit`	no
`Comment`	Named after G.W. Leibniz. This is the *left* Leibniz algebra. The opposite type is called *right* Leibniz algebra. Integration : "coquecigrues" ! see for instance [?]
`Ref.`	[?] J.-L. Loday, *Une version non commutative des algèbres de Lie: les algèbres de Leibniz.* Enseign. Math. (2) 39 (1993), no. 3-4, 269–293.

Name	**Zinbiel algebra**Zinbiel
Notation	$Zinb$
Def.oper.	$x \cdot y$
sym.	
rel.	$(x \cdot y) \cdot z = x \cdot (y \cdot z) + x \cdot (z \cdot y)$ (Zinbiel relation)
Free alg.	$Zinb(V) = \overline{T}(V)$, $\cdot =$ halfshuffle $x_1 \dots x_p \cdot x_{p+1} \dots x_{p+q} = x_1 \mathrm{sh}_{p-1,q}(x_2 \dots x_p, x_{p+1} \dots x_{p+q})$
rep. $\mathcal{P}(n)$	$Zinb(n) \cong \mathbb{K}[\mathbb{S}_n]$ (regular representation)
$\dim \mathcal{P}(n)$	1, 2!, 3!, 4!, 5!, 6!, 7!, ..., n!, ...
Gen.series	$f^{Zinb}(t) = \frac{t}{1-t}$
Dual operad	$Zinb^! = Leib$
Chain-cplx	known, see Ref.
Properties	binary, quadratic, quasi-regular, Koszul.
Alternative	$Zinb = ComDend$ (commutative dendriform algebra), previously called *dual Leibniz algebra*.
Relationsh.	$Zinb$-alg $\to Com$-alg, $xy = x \cdot y + y \cdot x$, $Zinb$-alg $\to Dend$-alg, $x \prec y = x \cdot y = y \succ x$ $Zinb = PreLie \bullet Com$, see [?].
Unit	$1 \cdot x = 0, \quad x \cdot 1 = x$
Comment	symmetrization of the dot product gives, not only a commutative alg., but in fact a *divided power algebra*. Named after G.W. Zinbiel. This is right Zinbiel algebra.
Ref.	[?] J.-L. Loday, *Cup-product for Leibniz cohomology and dual Leibniz algebras.* Math. Scand. 77 (1995), no. 2, 189–196.

`Name`	**Dendriform algebra**dendriform
`Notation`	$Dend$
`Def.oper.`	$x \prec y,\ x \succ y$ (left and right operation)
`sym.`	
`rel.`	$\begin{cases} (x \prec y) \prec z = x \prec (y \prec z) + x \prec (y \succ z), \\ (x \succ y) \prec z = x \succ (y \prec z), \\ (x \prec y) \succ z + (x \succ y) \succ z = x \succ (y \succ z). \end{cases}$
`Free alg.`	$Dend(V) = \bigoplus_{n\geq 1} \mathbb{K}[PBT_{n+1}] \otimes V^{\otimes n}$, for pb trees s and t: $s \prec t := s^l \vee (s^r * t)$, and $s \succ t := (s * t^l) \vee t^r$ where $x * y := x \prec y + x \succ y$.
$\mathcal{P}_n$	$Dend_n = \mathbb{K}[PBT_{n+1}]$
$\dim \mathcal{P}_n$	$1, 2, 5, 14, 42, 132, 429, \ldots, c_n, \ldots$ where $c_n = \frac{1}{n+1}\binom{2n}{n}$ is the Catalan number
`Gen.series`	$f^{Dend}(t) = \frac{1-2t-\sqrt{1-4t}}{2t} = y, \quad y^2 - (1-2t)y + t = 0$
`Dual operad`	$Dend^! = Dias$
`Chain-cplx`	Isomorphic to the total complex of a certain explicit bicomplex
`Properties`	ns, binary, quadratic, Koszul.
`Alternative`	Handy to introduce $x * y := x \prec y + x \succ y$ which is associative.
`Relationsh.`	$Dend$-alg $\to As$-alg, $\quad x * y := x \prec y + y \succ x$, $Zinb$-alg $\to Dend$-alg, $\quad x \prec y := x \cdot y =: y \succ x$ $Dend$-alg $\to PreLie$-alg, $\quad x \circ y := x \prec y - y \succ x$ $Dend$-alg $\to Brace$-alg, see Ronco [?] $Dend = PreLie \bullet As$, see [?]
`Unit`	$1 \prec x = 0,\ x \prec 1 = x, \quad 1 \succ x = x,\ x \succ 1 = 0.$
`Comment`	dendro = tree in greek. There exist many variations.
`Ref.`	[?] J.-L. Loday, *Dialgebras,* Springer Lecture Notes in Math. 1763 (2001), 7-66.

`Name`	**Diassociative algebra**diassociative
`Notation`	$Dias$
`Def.oper.`	$x \dashv y,\ x \vdash y$ (left and right operation)
`sym.`	
`rel.`	$\begin{cases} (x \dashv y) \dashv z = x \dashv (y \dashv z), \\ (x \dashv y) \dashv z = x \dashv (y \vdash z), \\ (x \vdash y) \dashv z = x \vdash (y \dashv z), \\ (x \dashv y) \vdash z = x \vdash (y \vdash z), \\ (x \vdash y) \vdash z = x \vdash (y \vdash z). \end{cases}$
`Free alg.`	$Dias(V) = \bigoplus_{n\geq 1}(\underbrace{V^{\otimes n} \oplus \cdots \oplus V^{\otimes n}}_{n \text{ copies}})$ noncommutative polynomials with one variable marked
$\mathcal{P}_n$	$Dias_n = \mathbb{K}^n$
$\dim \mathcal{P}_n$	$1, 2, 3, 4, 5, 6, 7, \ldots, n, \ldots,$
`Gen.series`	$f^{Dias}(t) = \frac{t}{(1-t)^2}$
`Dual operad`	$Dias^! = Dend$
`Chain-cplx`	see Ref.
`Properties`	ns, binary, quadratic, set-theoretic, Koszul.
`Alternative`	previously called associative dialgebras or dialgebras.
`Relationsh.`	$As\text{-alg} \to Dias\text{-alg}, \quad x \dashv y := xy =: x \vdash y$ $Dias\text{-alg} \to Leib\text{-alg}, \quad [x, y] := x \dashv y - x \vdash y$ $Dias = Perm \bigcirc As = Perm \underset{\mathrm{H}}{\otimes} As$
`Unit`	Bar-unit: $x \dashv 1 = x = 1 \vdash x,\ 1 \dashv x = 0 = x \vdash 1$
`Comment`	
`Ref.`	[?] J.-L. Loday, *Dialgebras,* Springer Lecture Notes in Math. 1763 (2001), 7-66.

`Name`	**Pre-Lie algebra**pre-Lie
`Notation`	$PreLie$
`Def.oper.`	$\{x, y\}$
`sym.`	
`rel.`	$\{\{x, y\}, z\} - \{x, \{y, z\}\} = \{\{x, z\}, y\} - \{x, \{z, y\}\}$
`Free alg.`	$PreLie(V) = \{$rooted trees labeled by elements of $V\}$
`rep.` $\mathcal{P}(n)$	$PreLie(n) = \mathbb{K}[\{$rooted trees, vertices labeled by $1, \ldots, n\}]$
$\dim \mathcal{P}(n)$	$1, 2, 9, 64, 625, 1296, 117649, \ldots, n^{n-1}, \ldots$
`Gen.series`	$f^{PreLie}(t) = y$ which satisfies $y = t\ \exp(y)$
`Dual operad`	$PreLie^! = Perm$
`Chain-cplx`	see Ref.
`Properties`	binary, quadratic, Koszul.
`Alternative`	The relation is $as(x, y, z) = as(x, z, y)$.
`Relationsh.`	$PreLie$-alg $\to$ Lie-alg, $[x, y] := \{x, y\} - \{y, x\}$ $Dend$-alg $\to$ $PreLie$-alg, $\{x, y\} := x \prec y - y \succ x$ $Brace$-alg $\to$ $PreLie$-alg, forgetful functor
`Unit`	$\{1, x\} = x = \{x, 1\}$
`Comment`	This is *right* pre-Lie algebra, also called *right-symmetric algebra*, or *Vinberg algebra*Vinberg. The opposite type is left symmetric, first appeared in [?,?]. Symmetric brace algebra equivalent to pre-Lie algebra [?]
`Ref.`	[?] F. Chapoton, M. Livernet, *Pre-Lie algebras and the rooted trees operad.* Internat. Math. Res. Notices 2001, no. 8, 395–408.

Name	**Perm algebra**perm
Notation	$Perm$
Def.oper.	xy
sym.	
rel.	$(xy)z = x(yz) = x(zy)$
Free alg.	$Perm(V) = V \otimes S(V)$
rep. $\mathcal{P}(n)$	$Perm(n) = \mathbb{K}^n$
$\dim \mathcal{P}(n)$	$1, 2, 3, 4, 5, 6, 7, \ldots, n, \ldots$
Gen.series	$f^{Perm}(t) = t\exp(t)$
Dual operad	$Perm^! = PreLie$
Chain-cplx	
Properties	binary, quadratic, set-theoretic, Koszul.
Alternative	$Perm = ComDias$
Relationsh.	$Perm$-alg $\to$ Ass-alg Com-alg $\to$ $Perm$-alg, NAP-alg $\to$ $Perm$-alg, $Perm$-alg $\to$ $Dias$-alg
Unit	no, unless it is a commutative algebra
Comment	
Ref.	[?] F. Chapoton, *Un endofoncteur de la catégorie des opérades.* Dialgebras & related operads, 105–110, Lect. Notes in Math., 1763, Springer, 2001.

`Name`	**Dipterous algebra**dipterous
`Notation`	$Dipt$
`Def.oper.`	$x * y,\ x \prec y$
`sym.`	
`rel.`	$\begin{cases} (x * y) * z = x * (y * z) & \text{(associativity)} \\ (x \prec y) \prec z = x \prec (y * z) & \text{(dipterous relation)} \end{cases}$
`Free alg.`	$Dipt(V) = \bigoplus_{n\geq 1}(\mathbb{K}[PT_n] \oplus \mathbb{K}[PT_n]) \otimes V^{\otimes n}$, for $n = 1$ the two copies of PT_1 are identified
$\mathcal{P}_n$	$Dipt_n = \mathbb{K}[PT_n] \oplus \mathbb{K}[PT_n], n \geq 2$
$\dim \mathcal{P}_n$	$1, 2, 6, 22, 90, 394, 1806, \ldots, 2C_n, \ldots$ where C_n is the Schröder number: $\sum_{n\geq 1} C_n t^n = \frac{1+t-\sqrt{1-6t+t^2}}{4}$
`Gen.series`	$f^{Dipt}(t) = \frac{1-t-\sqrt{1-6t+t^2}}{2}$
`Dual operad`	$Dipt^!$
`Chain-cplx`	Isomorphic to the total complex of a certain explicit bicomplex
`Properties`	ns, binary, quadratic, Koszul.
`Alternative`	"diptère" in French
`Relationsh.`	$Dend$-alg $\to Dipt$-alg, $\quad x * y := x \prec y + y \succ x$, a variation: replace the dipterous relation by $(x \prec y) * z + (x * y) \prec z = x \prec (y * z) + x * (y \prec z)$ to get $Hoch$-algebras, see [?]. Same properties.
`Unit`	$1 \prec x = 0,\ x \prec 1 = x, \quad 1 * x = x = x * 1.$
`Comment`	dipterous = 2-fold in greek (free algebra has two planar trees copies)
`Ref.`	[?] J.-L. Loday, M. Ronco, *Algèbres de Hopf colibres*, C.R.Acad. Sci. Paris t. 337, Ser. I (2003), 153 -158.

Name	**Dual dipterous algebra**dual dipterous
Notation	$Dipt^!$
Def.oper.	$x \dashv y,\ x * y$
sym.	
rel.	$\begin{cases} (x*y)*z = x*(y*z), \\ (x \dashv y) \dashv z = x \dashv (y*z), \\ (x*y) \dashv z = 0, \\ (x \dashv y) * z = 0, \\ 0 = x*(y \dashv z) \\ 0 = x \dashv (y \dashv z). \end{cases}$
Free alg.	$Dipt^!(V) = \overline{T}(V) \oplus \overline{T}(V)$
$\mathcal{P}_n$	$Dipt^!_n = \mathbb{K}^2, n \geq 2$
$\dim \mathcal{P}_n$	$1, 2, 2, 2, 2, 2, 2, \ldots, 2, \ldots,$
Gen.series	$f^{Dipt^!}(t) = \frac{t+t^2}{1-t}$
Dual operad	$Dipt^{!!} = Dipt$
Chain-cplx	see Ref.
Properties	ns, binary, quadratic, Koszul.
Alternative	
Relationsh.	As-alg $\to Dipt^!$-alg, $\quad x \dashv y := xy =: x \vdash y$
Unit	no
Comment	
Ref.	[?] J.-L. Loday, M. Ronco, *Algèbres de Hopf colibres*, C.R.Acad. Sci. Paris t. 337, Ser. I (2003), 153 -158.

`Name`	**Two-associative algebra**two-associative
`Notation`	$2as$
`Def.oper.`	$x * y, \ x \cdot y,$
`sym.`	
`rel.`	$\begin{cases} (x * y) * z = x * (y * z), \\ (x \cdot y) \cdot z = x \cdot (y \cdot z). \end{cases}$
`Free alg.`	$2as(V) = \bigoplus_{n\geq 1}(\mathbb{K}[T_n] \oplus \mathbb{K}[T_n]) \otimes V^{\otimes n}$, where $T_n =$ planar trees for $n = 1$ the two copies of T_1 are identified
$\mathcal{P}_n$	$Dipt_n = \mathbb{K}[T_n] \oplus \mathbb{K}[T_n], n \geq 2$
$\dim \mathcal{P}_n$	$1, 2, 6, 22, 90, 394, 1806, \ldots, 2C_n, \ldots$ where C_n is the Schröder number: $\sum_{n\geq 1} C_n t^n = \frac{1+t-\sqrt{1-6t+t^2}}{4}$
`Gen.series`	$f^{2as}(t) = \frac{1-t-\sqrt{1-6t+t^2}}{2}$
`Dual operad`	$2as^!$
`Chain-cplx`	Isomorphic to the total complex of a certain explicit bicomplex
`Properties`	ns, binary, quadratic, set-theoretic, Koszul.
`Alternative`	
`Relationsh.`	$2as$-alg $\to Dup$-alg, $\quad 2as$-alg $\to B_\infty$-alg
`Unit`	$1 \cdot x = x = x \cdot 1, \quad 1 * x = x = x * 1.$
`Comment`	
`Ref.`	[?] J.-L. Loday, M. Ronco, *On the structure of cofree Hopf algebras*, J. reine angew. Math. 592 (2006), 123–155.

`Name`	**Dual 2-associative algebra**dual two associative
`Notation`	$2as^!$
`Def.oper.`	$x \cdot y,\ x * y$
`sym.`	
`rel.`	$\begin{cases} (x*y)*z = x*(y*z), \\ (x\cdot y)\cdot z = x\cdot(y\cdot z), \\ (x\cdot y)*z = 0, \\ (x*y)\cdot z = 0, \\ 0 = x*(y\cdot z) \\ 0 = x\cdot(y*z). \end{cases}$
`Free alg.`	$2as^!(V) = V \oplus \bigoplus_{n\geq 2} (V^{\otimes n} \oplus V^{\otimes n})$
$\mathcal{P}_n$	$Dipt^!_n = \mathbb{K} \oplus \mathbb{K}, n \geq 2.$
$\dim \mathcal{P}_n$	$1, 2, 2, 2, 2, 2, 2, \ldots, 2, \ldots$
`Gen.series`	$f^{2as^!}(t) = \frac{t+t^2}{1-t}$
`Dual operad`	$(2as^!)^! = 2as$
`Chain-cplx`	see Ref.
`Properties`	ns, binary, quadratic, Koszul.
`Alternative`	
`Relationsh.`	
`Unit`	no
`Comment`	
`Ref.`	[?] J.-L. Loday, M. Ronco, *On the structure of cofree Hopf algebras*, J. reine angew. Math. 592 (2006), 123–155.

Name	**Tridendriform algebra**tridendriform
Notation	$Tridend$
Def.oper.	$x \prec y,\ x \succ y,\ x \cdot y$
sym.	
rel.	$\begin{cases} (x \prec y) \prec z = x \prec (y * z)\ , \\ (x \succ y) \prec z = x \succ (y \prec z)\ , \\ (x * y) \succ z = x \succ (y \succ z)\ , \\ (x \succ y) \cdot z = x \succ (y \cdot z)\ , \\ (x \prec y) \cdot z = x \cdot (y \succ z)\ , \\ (x \cdot y) \prec z = x \cdot (y \prec z)\ , \\ (x \cdot y) \cdot z = x \cdot (y \cdot z)\ . \end{cases}$ where $x * y := x \prec y + x \succ y + x \cdot y$. One relation for each cell of the triangle.
Free alg.	Planar rooted trees with variables in between the leaves
$\mathcal{P}_n$	$Tridend_n = \mathbb{K}[PT_n]$
$\dim \mathcal{P}_n$	$1, 3, 11, 45, 197, 903, \ldots, C_n, \ldots$ where C_n is the Schröder (or super Catalan) number
Gen.series	$f^{Tridend}(t) = \frac{-1+3t+\sqrt{1-6t+t^2}}{4t}$
Dual operad	$Triend^! = Trias$
Chain-cplx	Isomorphic to the total complex of a certain explicit tricomplex
Properties	ns, binary, quadratic, Koszul.
Alternative	sometimes called dendriform trialgebra
Relationsh.	$Tridend$-alg $\to As$-alg, $x * y := x \prec y + y \succ x + x \cdot y$, $ComTridend$-alg $\to Tridend$-alg, $x \prec y := x \cdot y =: y \succ x$ $Tridend = PostLie \bullet Ass$
Unit	$1 \prec x = 0,\ x \prec 1 = x,\quad 1 \succ x = x,\ x \succ 1 = 0,$ $1 \cdot x = 0 = x \cdot 1.$
Comment	There exist several variations (see [?] for instance).
Ref.	[?] J.-L. Loday and M. Ronco, *Trialgebras and families of polytopes*, Contemp. Math. (AMS) 346 (2004), 369–398.

Name	**Triassociative algebra**triassociative
Notation	$Trias$
Def.oper.	$x \dashv y,\ x \vdash y,\ x \perp y$ (no symmetry)
rel.	$\begin{cases} (x \dashv y) \dashv z = x \dashv (y \dashv z), \\ (x \dashv y) \dashv z = x \dashv (y \vdash z), \\ (x \vdash y) \dashv z = x \vdash (y \dashv z), \\ (x \dashv y) \vdash z = x \vdash (y \vdash z), \\ (x \vdash y) \vdash z = x \vdash (y \vdash z), \\ (x \dashv y) \dashv z = x \dashv (y \perp z), \\ (x \perp y) \dashv z = x \perp (y \dashv z), \\ (x \dashv y) \perp z = x \perp (y \vdash z), \\ (x \vdash y) \perp z = x \vdash (y \perp z), \\ (x \perp y) \vdash z = x \vdash (y \vdash z), \\ (x \perp y) \perp z = x \perp (y \perp z). \end{cases}$ one relation for each cell of the pentagon.
Free alg.	noncommutative polynomials with several variables marked
$\mathcal{P}_n$	$Trias_n = \mathbb{K}^{2^n-1}$
$\dim \mathcal{P}_n$	$1, 3, 7, 15, 31, 63, 127, \ldots, (2^n - 1), \ldots$
Gen.series	$f^{Trias}(t) = \frac{t}{(1-t)(1-2t)}$
Dual operad	$Trias^! = Tridend$
Properties	ns, binary, quadratic, set-theoretic, Koszul.
Alternative	Also called *associative trialgebra*, or for short, *trialgebra*.
Relationsh.	As-alg $\to Trias$-alg, $x \dashv y = x \vdash y = x \perp y = xy$
Unit	Bar-unit: $x \dashv 1 = x = 1 \vdash x,\ 1 \dashv x = 0 = x \vdash 1,$ $1 \perp x = 0 = x \perp 1$
Comment	Relations easy to understand in terms of planar trees
Ref.	[?] J.-L. Loday and M. Ronco, *Trialgebras and families of polytopes*, Contemp. Math. (AMS) 346 (2004), 369–398.

Name	**PostLie algebra**post-Lie
Notation	$PostLie$
Def.oper.	$x \circ y,\ [x,y]$
sym.	$[x,y] = -[y,x]$
rel.	$[x,[y,z]] + [y,[z,x]] + [z,[x,y]] = 0$ $(x \circ y) \circ z - x \circ (y \circ z) - (x \circ z) \circ y + x \circ (z \circ y) = x \circ [y,z]$ $[x,y] \circ z = [x \circ z, y] + [x, y \circ z]$
Free alg.	$PostLie(V) \cong Lie(Mag(V))$
rep. $\mathcal{P}(n)$	
$\dim \mathcal{P}(n)$	$1, 3, 20, 210, 3024, \ldots$
Gen.series	$f(t) = -\log\left(\frac{1+\sqrt{1-4t}}{2}\right)$
Dual operad	$PostLie^! = ComTrias$
Chain-cplx	See Ref.
Properties	binary, quadratic, Koszul.
Alternative	
Relationsh.	$PostLie$-alg $\to$??-alg, $xy = x \circ y$??? $PostLie$-alg $\to Lie$-alg, $\{x,y\} = x \circ y - y \circ x + [x,y]$ $PreLie$-alg $\to PostLie$-alg, $[x,y] = 0$
Unit	
Comment	
Ref.	[?] Vallette B., *Homology of generalized partition posets*, J.Pure Appl. Algebra 208 (2007), no. 2, 699–725.

Name	**Commutative triassociative algebra**commutative triassociative
Notation	$ComTrias$
Def.oper.	$x \dashv y, x \perp y$
sym.	$x \perp y = y \perp x$
rel.	$\begin{cases} (x \dashv y) \dashv z = x \dashv (y \dashv z), \\ (x \dashv y) \dashv z = x \dashv (z \dashv y), \\ (x \dashv y) \dashv z = x \dashv (y \perp z), \\ (x \perp y) \dashv z = x \perp (y \dashv z), \\ (x \perp y) \perp z = x \perp (y \perp z). \end{cases}$
Free alg.	
rep. $\mathcal{P}(n)$	
$\dim \mathcal{P}(n)$	
Gen.series	$f(t) =$
Dual operad	$ComTrias^! = PostLie$
Chain-cplx	
Properties	binary, quadratic, Koszul.
Alternative	Triassociative with the following symmetry: $x \dashv y = y \vdash x$ and $x \perp y = y \perp x$
Relationsh.	$ComTrias$-alg $\to$ $Perm$-alg, forgetful functor $ComTrias$-alg $\to$ $Trias$-alg
Unit	$x \dashv 1 = x, 1 \dashv x = 0, 1 \perp x = 0$
Comment	
Ref.	[?] Vallette B., *Homology of generalized partition posets*, J.Pure Appl. Algebra 208 (2007), no. 2, 699–725.

Name **Commutative tridendriform algebra**commutative tridendriformCTD algebra

Notation CTD

Def.oper. $x \prec y,\ x \cdot y$

sym. $x \cdot y = y \cdot x$

rel.
$$\begin{cases} (x \prec y) \prec z = x \prec (y \prec z) + x \prec (z \prec y), \\ (x \cdot y) \prec z = x \cdot (y \prec z)\ , \\ (x \prec z) \cdot y = x \cdot (y \prec z)\ , \\ (x \prec z) \cdot y = (x \prec z) \cdot y\ , \\ (x \cdot y) \cdot z = x \cdot (y \cdot z)\ . \end{cases}$$

Free alg. $CTD(V)$ =quasi-shuffle algebra on $V\ = QSym(V)$

rep. $\mathcal{P}(n)$

$\dim \mathcal{P}(n)$ $1, 3, 13, 75, 541, 4683, \ldots$

Gen.series $f^{CTD}(t) = \frac{\exp(t)-1}{2-\exp(t)}$

Dual operad $CTD^! =$ see next page

Chain-cplx

Properties binary, quadratic, Koszul.

Alternative Handy to introduce $x * y := x \prec y + y \prec x + x \cdot y$ (assoc. and comm.) equivalently: tridendriform with symmetry: $x \prec y = y \succ x,\ x \cdot y = y \cdot x$

Relationsh. CTD-alg $\to Trident$-alg
CTD-alg $\to Com$-alg
CTD-alg $\to Zinb$-alg

Unit $1 \prec x = 0,\ x \prec 1 = x, \quad 1 \cdot x = 0 = x \cdot 1.$

Comment

Ref. [?] J.-L. Loday, *On the algebra of quasi-shuffles*, Manuscripta mathematica 123 (1), (2007), 79–93.

`Name`	**Dual CTD algebra**dual commutative tridendriformdual CTD
`Notation`	$CTD^!$
`Def.oper.`	$x \dashv y,\ [x,y]$
`sym.`	$[x,y] = -[y,x]$
`rel.`	to be done
`Free alg.`	
`rep.` $\mathcal{P}(n)$	
$\dim \mathcal{P}(n)$	$1, 3, 14, 90, 744, \dots$
`Gen.series`	$f^{CTD^!}(t) =$
`Dual operad`	$(CTD^!)^! = CTD$
`Chain-cplx`	
`Properties`	binary, quadratic, Koszul.
`Alternative`	
`Relationsh.`	$Trias$-alg $\to CTD^!$-alg Lie-alg $\to CTD^!$-alg $Leib$-alg $\to CTD^!$-alg
`Unit`	
`Comment`	
`Ref.`	

Name **L-dendriform algebra**L-dendriform

Notation *L-dend*

Def.oper. $x \triangleright y$ and $x \triangleleft y$
sym.
rel. $x \triangleright (y \triangleright z) - (x \bullet y) \triangleright z = y \triangleright (x \triangleright z) - (y \bullet x) \triangleright z$
$x \triangleright (y \triangleleft z) - (x \triangleright y) \triangleleft z = y \triangleleft (x \bullet z) - (y \triangleleft x) \triangleleft z$
where $x \bullet y := x \triangleright y + x \triangleleft y$.

Free alg.

rep. $\mathcal{P}(n)$

$\dim \mathcal{P}(n)$

Gen.series $f(t) =$

Dual operad

Chain-cplx

Properties $L\text{-}dend = PreLie \bullet PreLie$

Alternative

Relationsh. $Dend$-alg $\to$ $L\text{-}dend$-alg via $x \triangleright y := x \succ y$, $x \triangleleft y := x \prec y$
$L\text{-}dend$-alg $\to$ $PreLie$-alg, $(A, \triangleright, \triangleleft) \mapsto (A, \bullet)$
$preLie \bullet preLie = L\text{-}dend$

Unit

Comment Various variations like $L\text{-}quad$-alg, see [?,?]

Ref. [?] Bai, C.; Liu, L.; Ni, X. *Some results on L-dendriform algebras*, J. Geom. Phys. 60 (2010), no. 6-8, 940–950.

Name	**Ennea algebra**
Notation	*Ennea*
Def.oper.	9 binary operations
sym.	
rel.	49 relations, see Ref. (tridendriform splitting of As applied twice)
Free alg.	
$\mathcal{P}_n$	
$\dim \mathcal{P}_n$	$1, 9, 113, ?, ?, \ldots$
Gen.series	$f(t) =$
Dual operad	
Chain-cplx	
Properties	ns, binary, quadratic.
Alternative	
Relationsh.	several relations with tridendriform, for instance: $Ennea = Tridend \blacksquare Tridend$
Unit	
Comment	
Ref.	[?] Leroux, P., *Ennea-algebras*. J. Algebra 281 (2004), no. 1, 287–302.

`Name`	**Gerstenhaber algebra**Gerstenhaber
	underlying objects: graded vector spaces
`Notation`	*Gerst*
`Def.oper.`	$m =$ binary operation of degree 0, $c =$ binary operation of degree 1
`sym.`	m symmetric, c antisymmetric
`rel.`	$c \circ_1 c + (c \circ_1 c)^{(123)} + (c \circ_1 c)^{(321)} = 0$, $c \circ_1 m - m \circ_2 c - (m \circ_1 c)^{(23)} = 0$, $m \circ_1 m - m \circ_2 m = 0$.
`Free alg.`	
`rep.` $\mathcal{P}(n)$	
$\dim \mathcal{P}(n)$	
`Gen.series`	$f(t) =$
`Dual operad`	
`Chain-cplx`	
`Properties`	binary, quadratic, Koszul.
`Alternative`	
`Relationsh.`	
`Unit`	
`Comment`	To get the relations in terms of elements, do not forget to apply the Koszul sign rule. Last relation is associativity of m.
`Ref.`	[?] M. Gerstenhaber, *The cohomology structure of an associative ring*, Ann. of Math. (2), 78 (1963), 267–288.

`Name` **Batalin-Vilkovisky algebra**Batalin-Vilkovisky

underlying objects: graded vector spaces

`Notation` BV

`Def.oper.` Δ unary degree 1, m binary degree 0, c binary degree 1

`sym.` m symmetric, c antisymmetric

`rel.`
$m \circ_1 m - m \circ_2 m = 0,$
$\Delta^2 = 0,$
$c = \Delta \circ_1 m + m \circ_1 \Delta + m \circ_2 \Delta,$
$c \circ_1 c + (c \circ_1 c)^{(123)} + (c \circ_1 c)^{(321)} = 0,$
$c \circ_1 m - m \circ_2 c - (m \circ_1 c)^{(23)} = 0,$
$\Delta \circ_1 c + c \circ_1 \Delta + c \circ_2 \Delta = 0\ .$

`Free alg.`

`rep.` $\mathcal{P}(n)$

$\dim \mathcal{P}(n)$

`Gen.series` $f(t) =$

`Dual operad`

`Chain-cplx`

`Properties` unary and binary, inhomogeneous quadratic, Koszul.

`Alternative` Generated by Δ and m only.

`Relationsh.` BV-alg $\to$ $Gerst$-alg

`Unit`

`Comment` To get the relations in terms of elements, do not forget to apply the Koszul sign rule.

`Ref.` [?] J.-L. Koszul, *Crochet de Schouten-Nijenhuis et cohomologie*, Astérisque (1985), Numéro Hors Série, 257–271. See also [?].

`Name`	**Magmatic algebra**magmatic
`Notation`	Mag
`Def.oper.`	xy
`sym.`	
`rel.`	
`Free alg.`	basis: any parenthesizing of words, or planar binary trees, with product: $st = s \vee t$
$\mathcal{P}_n$	$Mag_n = \mathbb{K}[PBT_n]$ (planar binary trees with n leaves)
$\dim \mathcal{P}_n$	$1, 1, 2, 5, 14, 42, 132, \ldots, c_{n-1}, \ldots$ where $c_n = \frac{1}{n+1}\binom{2n}{n}$ (Catalan number)
`Gen.series`	$f^{Mag}(t) = (1/2)(1 - \sqrt{1-4t})$
`Dual operad`	$Mag^! = Nil_2$
`Chain-cplx`	
`Properties`	ns, binary, quadratic, set-theoretic, Koszul.
`Alternative`	most often called *nonassociative algebra.*
`Relationsh.`	many "inclusions" (all types of alg. with only one Gen. op.)
`Unit`	$1x = x = x1$
`Comment`	
`Ref.`	

Name	**2-Nilpotent algebra**nilpotent
Notation	Nil_2
Def.oper.	xy
sym.	
rel.	$(xy)z = 0 = x(yz)$
Free alg.	$Nil_2(V) = V \oplus V^{\otimes 2}$
$\mathcal{P}_n$	$(Nil_2)_2 = \mathbb{K},\ (Nil_2)_n = 0$ for $n \geq 3$
$\dim \mathcal{P}_n$	$1, 1, 0, 0, 0, 0, 0, \ldots$
Gen.series	$f^{\mathcal{P}}(t) = t + t^2$
Dual operad	${Nil_2}^! = Mag$
Chain-cplx	
Properties	ns, binary, quadratic, Koszul.
Alternative	
Relationsh.	
Unit	no
Comment	
Ref.	

Name	**Commutative magmatic algebra**magmatic
Notation	$ComMag$
Def.oper.	$x \cdot y$
sym.	$x \cdot y = y \cdot x$
rel.	none
Free alg.	any parenthesizing of commutative words
rep. $\mathcal{P}(n)$	$ComMag(n) = \mathbb{K}[shBT(n)]$ $shBT(n) = \{$shuffle binary trees with n leaves$\}$
$\dim \mathcal{P}(n)$	$1, 1, 3, 15, 105, \ldots, (2n-3)!!, \ldots,$ $\dim ComMag(n) = (2n-3)!! = 1 \times 3 \times \cdots \times (2n-3)$
Gen.series	$f^{ComMag}(t) = 1 - \sqrt{1-2t}$
Dual operad	$ComMag^!$-alg: $[x, y]$ antisymmetric, $[[x, y], z] = 0$.
Chain-cplx	
Properties	binary, quadratic, set-theoretic, Koszul.
Alternative	
Relationsh.	$ComMag \rightarrowtail PreLie, x \cdot y := \{x, y\} + \{y, x\}$, cf. [?]
Unit	$1x = x = x1$
Comment	
Ref.	

Name	**Anti-symmetric nilpotent algebra**anti-symmetric nilpotent
Notation	$ComMag^!$
Def.oper.	$[x,y]$
sym.	$[x,y] = -[y,x]$
rel.	$[[x,y],z] = 0$
Free alg.	$V \oplus \Lambda^2 V$
rep. $\mathcal{P}(n)$	
$\dim \mathcal{P}(n)$	$1, 1, 0, \ldots, 0, \ldots$
Gen.series	$f^{ComMag^!}(t) = t + \frac{t^2}{2}$
Dual operad	$ComMag$
Chain-cplx	
Properties	binary, quadratic, set-theoretic, Koszul.
Alternative	
Relationsh.	Lie-alg $\to ComMag^!$-alg
Unit	no
Comment	
Ref.	

`Name` **Quadri-algebra**quadri-

`Notation` $Quad$

`Def.oper.` $x \nwarrow y,\ x \nearrow y,\ x \searrow y,\ x \swarrow y$ called NW, NE, SE, SW oper.

`rel.`

$$(x \nwarrow y) \nwarrow z = x \nwarrow (y \star z) \qquad (x \nearrow y) \nwarrow z = x \nearrow (y \prec z) \qquad (x \wedge y) \nearrow z = x \nearrow (y \succ z)$$

$$(x \swarrow y) \nwarrow z = x \swarrow (y \wedge z) \qquad (x \searrow y) \nwarrow z = x \searrow (y \nwarrow z) \qquad (x \vee y) \nearrow z = x \searrow (y \nearrow z)$$

$$(x \prec y) \swarrow z = x \swarrow (y \vee z) \qquad (x \succ y) \swarrow z = x \searrow (y \swarrow z) \qquad (x \star y) \searrow z = x \searrow (y \searrow z)$$

where

$$\begin{aligned} x \succ y &:= x \nearrow y + x \searrow y, \quad x \prec y := x \nwarrow y + x \swarrow y \\ x \vee y &:= x \searrow y + x \swarrow y, \quad x \wedge y := x \nearrow y + x \nwarrow y \\ x \star y &:= x \searrow y + x \nearrow y + x \nwarrow y + x \swarrow y \\ &= x \succ y + x \prec y = x \vee y + x \wedge y \end{aligned}$$

`Free alg.`

$\mathcal{P}_n$

$\dim \mathcal{P}_n$ $1, 4, 23, 156, 1162, 9192, \ldots$

$$\dim \mathcal{P}_n = \frac{1}{n} \sum_{j=n}^{2n-1} \binom{3n}{n+1+j} \binom{j-1}{j-n}$$

`Gen.series` $f^{Quad}(t) =$

`Dual operad` $Quad^!$

`Properties` ns, binary, quadratic, Koszul.

`Alternative` $Quad = Dend \blacksquare Dend = PreLie \bullet Dend$

`Relationsh.` Related to dendriform in several ways

`Unit` partial unit (like in dendriform)

`Comment` There exist several variations like $\mathcal{P} \blacksquare \mathcal{P}$ (cf. [?,?])

`Ref.` [?] M. Aguiar, J.-L. Loday, *Quadri-algebras*, J. Pure Applied Algebra 191 (2004), 205–221.

`Name`	**Dual quadri-algebra**dual quadri-
`Notation`	$Quad^!$
`Def.oper.`	$x \nwarrow y,\ x \nearrow y,\ x \searrow y,\ x \swarrow y$
`sym.`	
`rel.`	To be done
`Free alg.`	
$\mathcal{P}_n$	
$\dim \mathcal{P}_n$	$1, 4, 9, 16, 25, \ldots, n^2, \ldots$
`Gen.series`	$f^{Quad^!}(t) = \frac{t(1+t)}{(1-t)^3}$
`Dual operad`	
`Chain-cplx`	
`Properties`	ns, binary, quadratic, Koszul.
`Alternative`	
`Relationsh.`	$Quad^! = Dias \square Dias = Perm \circ Dias$ (cf. [?])
`Unit`	
`Comment`	
`Ref.`	[?] B.Vallette, *Manin products, Koszul duality, Loday algebras and Deligne conjecture.* J. Reine Angew. Math. 620 (2008), 105–164.

`Name`	**Duplicial algebra**duplicialduplicial
`Notation`	Dup
`Def.oper.`	$x \prec y,\ x \succ y$
`sym.`	
`rel.`	$\begin{cases} (x \prec y) \prec z = x \prec (y \prec z), \\ (x \succ y) \prec z = x \succ (y \prec z), \\ (x \succ y) \succ z = x \succ (y \succ z). \end{cases}$
`Free alg.`	$Dup(V) = \oplus_{n\geq 1}\mathbb{K}[PBT_{n+1}] \otimes V^{\otimes n}$, for p.b. trees s and t: $x \succ y$ the *over* operation is grafting of x on the leftmost leaf of y; $x \prec y$ the *under* operation is grafting of y on the rightmost leaf of x
$\mathcal{P}_n$	$Dup_n = \mathbb{K}[PBT_{n+1}]$
$\dim \mathcal{P}_n$	$1, 2, 5, 14, 42, 132, 429, \ldots, c_n, \ldots$ where $c_n = \frac{1}{n+1}\binom{2n}{n}$ is the Catalan number
`Gen.series`	$f^{Dup}(t) = \frac{1-2t-\sqrt{1-4t}}{2t} = y, \quad y^2 - (1-2t)y + t = 0$
`Dual operad`	$Dup^!$
`Chain-cplx`	Isomorphic to the total complex of a certain explicit bicomplex
`Properties`	ns, binary, quadratic, set-theoretic, Koszul.
`Alternative`	
`Relationsh.`	$2as$-alg $\to$ Dup-alg $\to$ As^2-alg
`Unit`	
`Comment`	The associator of $xy := x \succ y - x \prec y$ is $as(x, y, z) = x \prec (y \succ z) - (x \prec y) \succ z$. Appeared first in [?]
`Ref.`	[?] J.-L. Loday, *Generalized bialgebras and triples of operads*, Astérisque (2008), no 320, x+116 p.

Name	**Dual duplicial algebra**dual duplicialdual duplicial
Notation	$Dup^!$
Def.oper.	$x \prec y,\ x \succ y$
sym.	
rel.	$\begin{cases} (x \prec y) \prec z = x \prec (y \prec z), \\ (x \prec y) \succ z = 0, \\ (x \succ y) \prec z = x \succ (y \prec z), \\ \qquad\qquad 0 = x \prec (y \succ z), \\ (x \succ y) \succ z = x \succ (y \succ z). \end{cases}$
Free alg.	$Dup^!(V) = \oplus_{n\geq 1}\ n\ V^{\otimes n}$ noncommutative polynomials with one variable marked
$\mathcal{P}_n$	$Dup^!_n = \mathbb{K}^n$
$\dim \mathcal{P}_n$	$1, 2, 3, 4, 5, 6, 7, \ldots, n, \ldots,$
Gen.series	$f^{Dup^!}(t) = \frac{t}{(1-t)^2}$
Dual operad	$Dup^{!!} = Dup$
Chain-cplx	see ref.
Properties	ns, binary, quadratic, Koszul.
Alternative	
Relationsh.	
Unit	
Comment	
Ref.	

`Name`	$As^{(2)}$**-algebra**$As^{(2)}$-algebra
`Notation`	$As^{(2)}$
`Def.oper.`	$x * y, x \cdot y$
`sym.`	
`rel.`	$(x \circ_1 y) \circ_2 z = x \circ_1 (y \circ_2 z)$ for $\circ_i = *$ or $\cdot$ (4 relations)
`Free alg.`	
$\mathcal{P}_n$	$As^{(2)}_n = \mathbb{K}[\{0,1\}^{n-1}]$
$\dim \mathcal{P}_n$	2^{n-1}
`Gen.series`	$f^{As^{(2)}}(t) = \frac{t}{1-2t}$
`Dual operad`	$As^{(2)^!} = As^{(2)}$
`Chain-cplx`	
`Properties`	ns, binary, quadratic, set-theoretic.
`Alternative`	
`Relationsh.`	$2as$-alg $\to$ Dup-alg $\to$ $As^{(2)}$-alg
`Unit`	
`Comment`	Variations: $As^{(k)}$
`Ref.`	

Name	$As^{\langle 2\rangle}$**-algebra**$As^{\langle 2\rangle}$-algebra
Notation	$As^{\langle 2\rangle}$, compatible products algebra
Def.oper.	$x * y, x \cdot y$
sym.	
rel.	$(x * y) * z = x * (y * z)$ $(x * y) \cdot z + (x \cdot y) * z = x * (y \cdot z) + x \cdot (y * z)$ $(x \cdot y) \cdot z = x \cdot (y \cdot z)$
Free alg.	similar to dendriform and duplicial
rep. $\mathcal{P}_n$	$As^{\langle 2\rangle}_n = \mathbb{K}[PBT_{n+1}]$
$\dim \mathcal{P}_n$	$c_n = \frac{1}{n+1}\binom{2n}{n}$
Gen.series	$f^{As^{\langle 2\rangle}}(t) = \frac{1-2t-\sqrt{1-4t}}{2t}$
Dual operad	An $(As^{\langle 2\rangle})^!$-algebra has 2 generating binary operations and 5 relations: $(x \circ_i y) \circ_j z = x \circ_i (y \circ_j z)$ and $(x * y) \cdot z + (x \cdot y) * z = x * (y \cdot z) + x \cdot (y * z)$
Chain-cplx	
Properties	ns, binary, quadratic, Koszul.
Alternative	
Relationsh.	$(As^{\langle 2\rangle})^!$-alg $\to As^{(2)}$-alg $\to As^{\langle 2\rangle}$-alg
Unit	
Comment	equivalently $\lambda\, x * y + \mu\, x \cdot y$ is associative for any λ, μ Variations: $As^{\langle k\rangle}$, $Hoch$-alg. Do not confuse $As^{(k)}$ and $As^{\langle k\rangle}$.
Ref.	[?] Dotsenko, V., *Compatible associative products and trees.* Algebra Number Theory 3 (2009), no. 5, 567–586. See also [?], [?].

Name	**Lie-admissible algebra**Lie-admissible
Notation	*Lie-adm*
Def.oper.	xy
sym.	
rel.	$[x,y]=xy-yx$ is a Lie bracket, that is $\sum_\sigma \mathrm{sgn}(\sigma)\sigma\big((xy)z-x(yz)\big)=0$
Free alg.	
$\mathcal{P}_n$	$Lie\text{-}adm(n)=?$
$\dim \mathcal{P}_n$	$1, 2, 11, ?, ?, \ldots$
Gen.series	$f^{Lie\text{-}adm}(t)=?$
Dual operad	$Lie\text{-}adm^!$
Chain-cplx	
Properties	ns, binary, quadratic, Koszul ??.
Alternative	
Relationsh.	As-alg $\to$ $PreLie$-alg $\to$ $Lie\text{-}adm$-alg
Unit	
Comment	
Ref.	

Name	**PreLiePerm algebra**PreLiePerm
Notation	$PreLiePerm$
Def.oper.	$x \prec y,\ x \succ y,\ x * y$
sym.	
rel.	$\begin{cases} (x \prec y) \prec z = x \prec (y * z), \\ (x \prec y) \prec z = x \prec (z * y), \\ (x \succ y) \prec z = x \succ (y \prec z), \\ (x \succ y) \prec z = x \succ (z \succ y), \\ (x * y) \succ z = x \succ (y \succ z), \end{cases}$
Free alg.	
rep. $\mathcal{P}(n)$	
$\dim \mathcal{P}(n)$	$1, 6,$
Gen.series	
Dual operad	$PermPreLie^! = Perm \circ PreLie = Perm \otimes PreLie$
Chain-cplx	
Properties	binary, quadratic, set-theoretic.
Alternative	
Relationsh.	$PreLiePerm = PreLie \bullet Perm$, see [?], p. 132. $Zinb$-alg $\to$ $PreLiePerm$-alg $\to$ $Dend$-alg
Unit	no
Comment	
Ref.	[?] B.Vallette, *Homology of generalized partition posets,* J. Pure Appl. Algebra 208 (2007), no. 2, 69–725.

Name	**Alternative algebra**alternative
Notation	*Altern*
Def.oper.	xy
sym.	
rel.	$(xy)z - x(yz) = -(yx)z + y(xz)$, $(xy)z - x(yz) = -(xz)y + x(zy)$,
Free alg.	
rep. $\mathcal{P}(n)$	
$\dim \mathcal{P}(n)$	$1, 2, 7, 32, 175, ??$
Gen.series	$f(t) =$
Dual operad	$(xy)z = x(yz)$, $xyz + yxz + zxy + xzy + yzx + zyx = 0$ $\dim Altern^!(n) = 1, 2, 5, 12, 15, \dots$
Chain-cplx	
Properties	binary, quadratic, nonKoszul [?].
Alternative	Equivalent presentation: the associator $as(x, y, z)$ is skew-symmetric: $\sigma \cdot as(x, y, z) = \mathrm{sgn}(\sigma) as(x, y, z)$
Relationsh.	Ass-alg $\to$ $Altern$-alg
Unit	$1x = x = x1$
Comment	The octonions are an example of alternative algebra Integration: Moufang loops $\dim \mathcal{P}(n)$ computed by W. Moreira
Ref.	[?] Shestakov, I. P., *Moufang loops and alternative algebras.* Proc. Amer. Math. Soc. 132 (2004), no. 2, 313–316.

`Name`	**Parametrized-one-relation algebra**parametrized-one-relation
`Notation`	$Param1rel$
`Def.oper.`	xy
`sym.`	none
`rel.`	$(xy)z = \sum_{\sigma\in\mathbb{S}_3} a_\sigma \sigma \cdot x(yz)$ where $a_\sigma \in \mathbb{K}$
`Free alg.`	
`rep.` $\mathcal{P}(n)$	
$\dim \mathcal{P}(n)$	
`Gen.series`	$f(t) =$
`Dual operad`	$x(yz) = \sum_{\sigma\in\mathbb{S}_3} \mathrm{sgn}(\sigma) a_\sigma \sigma^{-1} \cdot (xy)z$
`Chain-cplx`	
`Properties`	
`Alternative`	
`Relationsh.`	Many classical examples are particular case: $As, Leib$, $Zinb, Pois$
`Unit`	
`Comment`	Problem: for which families of parameters $\{a_\sigma\}_{\sigma\in\mathbb{S}_3}$ is the operad a Koszul operad ?
`Ref.`	

`Name`	**Magmatic-Fine algebra**magmatic-Fine
`Notation`	$MagFine$
`Def.oper.`	$(x_1, \ldots, x_n)_i^n$ for $1 \leq i \leq n-2, n \geq 3$
`sym.`	
`rel.`	
`Free alg.`	Described in terms of some coloured planar rooted trees
$\mathcal{P}_n$	vector space indexed as said above
$\dim \mathcal{P}_n$	$1, 0, 1, 2, 6, 18, 57, \ldots, F_{n-1}, \ldots$ where $F_n =$ Fine number
`Gen.series`	$f^{MagFine}(t) = \frac{1+2t-\sqrt{1-4t}}{2(2+t)}$
`Dual operad`	$MagFine^!$ same generating operations, any composition is trivial $\dim MagFine^!_n = n-2$ $f^{MagFine!}(t) = t + \frac{t^3}{(1-t)^2}$
`Chain-cplx`	
`Properties`	ns, multi-ary, quadratic, Koszul.
`Alternative`	
`Relationsh.`	
`Unit`	
`Comment`	
`Ref.`	[?] Holtkamp, R., Loday, J.-L., Ronco, M., *Coassociative magmatic bialgebras and the Fine numbers*, J.Alg.Comb. 28 (2008), 97–114.

`Name`	**Generic magmatic algebra**generic magmatic
`Notation`	$GenMag$
`Def.oper.`	a_n generating operations of arity n, $a_1 = 1$
`sym.`	
`rel.`	
`Free alg.`	
$\mathcal{P}_n$	based on trees
$\dim \mathcal{P}_n$	
`Gen.series`	$f^{GenMag}(t) = \sum_n b_n t^n$, $b_n =$ polynomial in $a_1, \ldots, a_n$
`Dual operad`	same generating operations, any composition is trivial $\dim GenMag^!_n = a_n$, $f^{GenMag^!}(t) = \sum_n a_n t^n$
`Chain-cplx`	
`Properties`	ns, multi-ary, quadratic, Koszul.
`Alternative`	
`Relationsh.`	For $MagFine^!$ $a_n = n - 2$.
`Unit`	
`Comment`	give a nice proof of the Lagrange inversion formula for a generic power series (computation of the polynomial b_n)
`Ref.`	[?] Loday, J.-L., *Inversion of integral series enumerating planar trees.* Séminaire lotharingien Comb. 53 (2005), exposé B53d, 16pp.

Name	**Nonassociative permutative algebra**nonassociative permutativeNAP-
Notation	NAP
Def.oper.	xy
sym.	
rel.	$(xy)z = (xz)y$
Free alg.	$NAP(V)$ can be described in terms of rooted trees
rep. $\mathcal{P}(n)$	$NAP(n) = PreLie(n)$ as $\mathbb{S}_n$-modules
$\dim \mathcal{P}(n)$	$1, 2, 9, 64, 625, \ldots, n^{n-1}, \ldots$
Gen.series	$f^{NAP}(t) = y$ which satisfies $y = t\ \exp(y)$
Dual operad	$NAP^!$
Chain-cplx	
Properties	binary, quadratic, set-theoretic, Koszul.
Alternative	
Relationsh.	$Perm$-alg $\to NAP$-alg
Unit	no
Comment	This is right NAP algebra
Ref.	[?] Livernet, M., *A rigidity theorem for pre-Lie algebras*, J. Pure Appl. Algebra 207 (2006), no. 1, 1–18.

`Name`	**Moufang algebra**Moufang
`Notation`	$Moufang$
`Def.oper.`	xy
`sym.`	
`rel.`	$x(yz) + z(yx) = (xy)z + (zy)x,$ $((xy)z)t + ((zy)x)t = x(y(zt)) + z(y(xt)),$ $t(x(yz) + z(yx)) = ((tx)y)z + ((tz)y)x,$ $(xy)(tz) + (zy)(tx) = (x(yt))z + (z(yt))x.$
`Free alg.`	
`rep.` $\mathcal{P}(n)$	
$\dim \mathcal{P}(n)$	$1, 2, 7, 40, ??$
`Gen.series`	$f(t) =$
`Dual operad`	
`Chain-cplx`	
`Properties`	binary.
`Alternative`	Relation may be written in terms of the Jacobiator
`Relationsh.`	$Altern$-alg $\xrightarrow{-}$ $Moufang$-alg $\to$ $NCJordan$-alg
`Unit`	
`Comment`	Integration: Moufang loops. From this presentation there is an obvious definition of "nonantisymmetric Malcev algebra".
`Ref.`	[?] Shestakov, I., Pérez-Izquierdo, J.M., *An envelope for Malcev algebras.* J. Alg. 272 (2004), 379–393.

Name	**Malcev algebra**Malcev
Notation	$Malcev$
Def.oper.	xy
sym.	$xy = -yx$
rel.	$((xy)z)t + (x(yz))t + x((yz)t) + x(y(zt))+$ $+((ty)z)x + (t(yz))x + t((yz)x) + t(y(zx)) =$ $(xy)(zt) + (ty)(zx)$
Free alg.	
rep. $\mathcal{P}(n)$	
$\dim \mathcal{P}(n)$	$1, 1, 3, 9, ??$
Gen.series	$f(t) =$
Dual operad	
Chain-cplx	
Properties	cubic.
Alternative	
Relationsh.	$Altern$-alg $\xrightarrow{-} Malcev$-alg, Lie-alg $\to Malcev$-alg
Unit	
Comment	
Ref.	[?] Shestakov, I., Pérez-Izquierdo, J.M., *An envelope for Malcev algebras.* J. Alg. 272 (2004), 379–393.

`Name`	**Novikov algebra**Novikov
`Notation`	$Novikov$
`Def.oper.`	xy
`sym.`	
`rel.`	$(xy)z - x(yz) = (xz)y - x(zy)$ $x(yz) = y(xz)$
`Free alg.`	
`rep.` $\mathcal{P}(n)$	
$\dim \mathcal{P}(n)$	$1, 2, ??$
`Gen.series`	$f(t) =$
`Dual operad`	
`Chain-cplx`	
`Properties`	binary, quadratic.
`Alternative`	Novikov is pre-Lie + $x(yz) = y(xz)$
`Relationsh.`	$Novikov$-alg $\to$ $PreLie$-alg
`Unit`	
`Comment`	
`Ref.`	

Name	**Double Lie algebra**double Lie
Notation	*DoubleLie*
Def.oper.	$[x, y], \{x, y\}$
sym.	$[x, y] = -[y, x], \{x, y\} = \{y, x\}$
rel.	Any linear combination is a Lie bracket
Free alg.	
rep. $\mathcal{P}(n)$	See Ref.
$\dim \mathcal{P}(n)$	
Gen.series	$f(t) =$
Dual operad	
Chain-cplx	
Properties	binary, quadratic.
Alternative	
Relationsh.	
Unit	
Comment	
Ref.	[?] Dotsenko V., Khoroshkin A., *Character formulas for the operad of two compatible brackets and for* the bihamiltonian operad, Functional Analysis and Its Applications, 41 (2007), no.1, 1-17.

Name	**DipreLie algebra**dipreLie
Notation	$DipreLie$
Def.oper.	$x \circ y,\ x \bullet y$
sym.	
rel.	$(x \circ y) \circ z - x \circ (y \circ z) = (x \circ z) \circ y - x \circ (z \circ y)$ $(x \bullet y) \bullet z - x \bullet (y \bullet z) = (x \bullet z) \bullet y - x \bullet (z \bullet y)$ $(x \circ y) \bullet z - x \circ (y \bullet z) = (x \bullet z) \circ y - x \bullet (z \circ y)$
Free alg.	
rep. $\mathcal{P}(n)$	
$\dim \mathcal{P}(n)$	
Gen.series	$f(t) =$
Dual operad	
Chain-cplx	
Properties	binary, quadratic.
Alternative	
Relationsh.	relationship with the Jacobian conjecture (T. Maszczsyk)
Unit	
Comment	
Ref.	T. Maszczsyk, unpublished.

Name	**Akivis algebra**Akivis
Notation	$Akivis$
Def.oper.	$[x, y], (x, y, z)$
sym.	$[x, y] = -[y, x]$
rel.	$[[x, y], z] + [[y, z], x] + [[z, x], y] =$ $(x, y, z) + (y, z, x) + (z, x, y) - (x, z, y) - (y, x, z) - (z, y, x)$ (Akivis relation)
Free alg.	
rep. $\mathcal{P}(n)$	
$\dim \mathcal{P}(n)$	$1, 1, 8, \ldots,$
Gen.series	$f(t) =$
Dual operad	
Chain-cplx	
Properties	binary and ternary, quadratic.
Alternative	relation also called "nonassociative Jacobi identity"
Relationsh.	$Akivis$-alg $\to$ $Sabinin$-alg, Mag-alg $\to$ $Akivis$-alg, $[x, y] = xy - yx$, $(x, y, z) = (xy)z - x(yz)$
Unit	
Comment	
Ref.	[?] Akivis, M. A. *The local algebras of a multidimensional three-web.* (Russian) Sibirsk. Mat. ?. 17 (1976), no. 1, 5-11, 237. See also [?,?]

`Name`	**Sabinin algebra**Sabinin
`Notation`	*Sabinin*
`Def.oper.`	$\langle x_1, \dots, x_m; y, z\rangle, m \geq 0$ $\Phi(x_1, \dots, x_m; y_1, \dots, y_n), \quad m \geq 1, n \geq 2,$
`sym.`	$\langle x_1, \dots, x_m; y, z\rangle = -\langle x_1, \dots, x_m; z, y\rangle$, and for $\omega \in \mathbb{S}_m, \theta \in \mathbb{S}_n$ $\Phi(x_1, \dots, x_m; y_1, \dots, y_n) = \Phi(\omega(x_1, \dots, x_m); \theta(y_1, \dots, y_n)),$
`rel.`	$\langle x_1, \dots, x_r, u, v, x_{r+1}, \dots, x_m; y, z\rangle - \langle x_1, \dots, x_r, v, u, x_{r+1}, \dots, x_m; y, z\rangle$ $+ \sum_{k=0}^{r} \sum_{\sigma} \langle x_{\sigma(1)}, \dots, x_{\sigma(k)}; \langle x_{\sigma(k+1)}, \dots, x_{\sigma(r)}; u, v\rangle, x_{r+1}, \dots, x_m; y, z\rangle$ where σ is a $(k, r-k)$-shuffle $K_{u,v,w}\big[\langle x_1, \dots, x_r; y, z\rangle +$ $\sum_{k=0}^{r} \sum_{\sigma} \langle x_{\sigma(1)}, \dots, x_{\sigma(k)}; \langle x_{\sigma(k+1)}, \dots, x_{\sigma(r)}; v, w\rangle, u\rangle\big] = 0$ where $K_{u,v,w}$ is the sum over all cyclic permutations
`Free alg.`	
`rep.` $\mathcal{P}(n)$	
$\dim \mathcal{P}(n)$	$1, 1, 8, 78, 1104, \dots$
`Gen.series`	$f^{Sab}(t) = \log(1 + (1/2)(1 - \sqrt{1-4t}))$
`Dual operad`	
`Chain-cplx`	
`Properties`	quadratic, ns.
`Alternative`	There exists a more compact form of the relations which uses the tensor algebra over the Sabinin algebra
`Relationsh.`	Mag-alg $\longrightarrow$ $Sabinin$-alg, $\langle y, z\rangle = yz - zy, \langle x; y, z\rangle =??$ $Akivis$-alg $\longrightarrow$ $Sabinin$-alg, $\langle y, z\rangle = -[y, z]$, $\langle x; y, z\rangle = (x, z, y) - (x, y, z), \langle x_1, \dots, x_m; y, z\rangle = 0, m \geq 2$
`Unit`	
`Comment`	Integration: local analytic loops
`Ref.`	[?] D. Pérez-Izquierdo, *Algebras, hyperalgebras, nonassociative bialgebras and loops.* Adv. in Maths 208 (2007), 834–876.

`Name`	**Jordan triple systems**Jordan triples
`Notation`	$Jordantriples$ or JT
`Def.oper.`	(xyz) or (x,y,z)
`sym.`	$(xyz) = (zyx)$
`rel.`	$(xy(ztu)) = ((xyz)tu) - (z(txy)u) + (zt(xyu))$
`Free alg.`	
`rep.` $\mathcal{P}(2n-1)$	
$\dim \mathcal{P}(2n-1)$	$1, 3, 50, ??$
`Gen.series`	$f(t) =$
`Dual operad`	
`Chain-cplx`	
`Properties`	ternary, quadratic, Koszul ?.
`Alternative`	
`Relationsh.`	
`Unit`	
`Comment`	Remark that the quadratic relation, as written here, has a Leibniz flavor. Dimension computed by Walter Moreira
`Ref.`	[?] E. Neher, Jordan triple systems by the grid approach, Lecture Notes in Mathematics, 1280, Springer-Verlag, 1987.

Name	**Totally associative ternary algebra**totally associative ternary
Notation	$t\text{-}As^{\langle 3\rangle}$
Def.oper.	(xyz)
sym.	
rel.	$((xyz)uv) = (x(yzu)v) = (xy(zuv))$
Free alg.	
$\mathcal{P}_{2n-1}$	
$\dim \mathcal{P}_{2n-1}$	$1, 1, 1, \dots, 1, \dots$
Gen.series	$f^{t\text{-}As^3}(t) = \frac{t}{1-t^2}$
Dual operad	$t\text{-}As^{\langle 3\rangle\,!} = p\text{-}As^{\langle 3\rangle}$
Chain-cplx	
Properties	ns, ternary, quadratic, set-theoretic, Koszul ?.
Alternative	
Relationsh.	As-alg $\to t\text{-}As^{\langle 3\rangle}$
Unit	
Comment	
Ref.	[?] Gnedbaye, A.V., *Opérades des algèbres* $(k+1)$*-aires.* Operads: Proceedings of Renaissance Conferences, 83–113, Contemp. Math., 202, Amer. Math. Soc., Providence, RI, 1997.

`Name`	**Partially associative ternary algebra**partially associative ternary
`Notation`	$p\text{-}As^{\langle 3\rangle}$
`Def.oper.`	(xyz)
`sym.`	
`rel.`	$((xyz)uv) + (x(yzu)v) + (xy(zuv)) = 0$
`Free alg.`	
$\mathcal{P}_{2n-1}$	
$\dim \mathcal{P}_{2n-1}$	
`Gen.series`	$f(t) =$
`Dual operad`	$p\text{-}As^{\langle 3\rangle\ !} = t\text{-}As^{\langle 3\rangle}$
`Chain-cplx`	
`Properties`	ns, ternary, quadratic, Koszul ?.
`Alternative`	
`Relationsh.`	
`Unit`	
`Comment`	
`Ref.`	[?] Gnedbaye, A.V., *Opérades des algèbres* $(k+1)$*-aires.* Operads: Proceedings of Renaissance Conferences, 83–113, Contemp. Math., 202, Amer. Math. Soc., Providence, RI, 1997.

Name	**Lie triple systems**Lie triple systems
Notation	LTS
Def.oper.	$[xyz]$
sym.	$[xyz] = -[yxz]$, $[xyz] + [yzx] + [zxy] = 0$.
rel.	$[xy[ztu]] = [[xyz]tu] - [z[txy]u] + [zt[xyu]]$
Free alg.	
rep. $\mathcal{P}(n)$	
$\dim \mathcal{P}(n)$	
Gen.series	$f(t) =$
Dual operad	
Chain-cplx	
Properties	ternary, quadratic, Koszul ?.
Alternative	the relation admits many different versions due to the symmetry
Relationsh.	Lie-alg $\to$ LTS-alg, $[xyz] = [[xy]z]$
Comment	Appreciate the Leibniz presentation Integration: symmetric spaces
Ref.	[?] Loos O., Symmetric spaces. I. General theory. W. A. Benjamin, Inc., New York-Amsterdam (1969) viii+198 pp.

Name **Lie-Yamaguti algebra**Lie-Yamaguti

Notation $Lie-Yamaguti$ or LY

Def.oper. $x \cdot y, \quad [x,y,z]$

sym. $x \cdot y = -y \cdot x, \quad [x,y,z] = -[y,x,z]$

$[x,y,z] + [y,z,x] + [z,x,y] + (x \cdot y) \cdot z + (y \cdot z) \cdot x$
$+(z \cdot x) \cdot y = 0$

rel. $\sum_{cyclic}[x \cdot y, z, t] = 0$

$[x,y,u \cdot v] = u \cdot [x,y,v] + [x,y,u] \cdot v$

$[x,y,[z,t,u]] = [[x,y,z],t,u] - [z,[t,x,y],u] + [z,t,[x,y,u]]$

Free alg.

rep. $\mathcal{P}(n)$

$\dim \mathcal{P}(n)$

Gen.series $f(t) =$

Dual operad

Chain-cplx

Properties binary and ternary, quadratic, Koszul ?.

Alternative Generalized Lie triple systems

Relationsh. $LTS\text{-alg} \to LY\text{-alg}, x \cdot y = 0, [x,y,z] = [xyz]$

Unit

Comment

Ref. [?] R. Kinyon, M. Weinstein, A., *Leibniz algebras, Courant algebroids, and multiplications on reductive homogeneous spaces.* Amer. J. Math. 123 (2001), no. 3, 525–550.

Name **Comtrans algebras**

Notation *Comtrans*

Def.oper. $[x,y,z]$ and $\langle x,y,z\rangle$ (2 ternary operations)

sym. $[x,y,z]+[y,x,z]=0,$
$\langle x,y,z\rangle+\langle y,z,x\rangle+\langle z,x,y\rangle=0,$
$[x,y,z]+[z,y,x]=\langle x,y,z\rangle+\langle z,y,x\rangle.$

rel.

Free alg.

rep. $\mathcal{P}_{2n-1}$

$\dim\mathcal{P}_{2n-1}$

Gen.series $f(t)=$

Dual operad

Chain-cplx

Properties ternary.

Alternative

Relationsh.

Unit

Comment

Ref. [?] J.D.H. Smith, *Multilinear algebras and Lie's Theorem for formal n-loops*, Arch. Math. 51 (1988), 169–177.

`Name`	**Interchange algebra**interchangeIT-
`Notation`	$Interchange$
`Def.oper.`	$x \cdot y, \quad x * y$
`sym.`	
`rel.`	$(x \cdot y) * (z \cdot t) = (x * z) \cdot (y * t)$
`Free alg.`	
`rep.` $\mathcal{P}(n)$	
$\dim \mathcal{P}(n)$	
`Gen.series`	$f(t) =$
`Dual operad`	
`Chain-cplx`	
`Properties`	binary, cubic, set-theoretic.
`Alternative`	
`Relationsh.`	Strongly related with the notions of 2-category and 2-group
`Unit`	if a unit for both, then $* = \cdot$ and they are commutative (Eckmann-Hilton trick)
`Comment`	Many variations depending on the hypotheses on $*$ and $\cdot$ Most common $\cdot$ and $*$ are associative.
`Ref.`	

Name	**Hypercommutative algebra**hypercommutative
	underlying objects: graded vector spaces
Notation	$HyperCom$
Def.oper.	$(x_1, \ldots, x_n)$ n-ary operation of degree $2(n-2)$ for $n \geq 2$
sym.	totally symmetric
rel.	$\sum_{S_1 \sqcup S_2 = \{1,\ldots,n\}} ((a, b, x_{S_1}), c, x_{S_2}) =$ $\sum_{S_1 \sqcup S_2 = \{1,\ldots,n\}} (-1)^{\|c\|\|x_{S_1}\|} (a, (b, x_{S_1}, c), x_{S_2})$, for any $n \geq 0$.
Free alg.	
rep. $\mathcal{P}(n)$	
$\dim \mathcal{P}(n)$	
Gen.series	$f(t) =$
Dual operad	Gravity algebra, see Ref below
Chain-cplx	
Properties	
Alternative	
Relationsh.	
Unit	
Comment	
Ref.	[?] E. Getzler, *Operads and moduli spaces of genus* 0 *Riemann surfaces*, The moduli space of curves (Texel Island, 1994), Progr. Math., 129, (1995), 199–230.

`Name`	**Associative algebra up to homotopy** A_∞
	operad with underlying space in `dgVect`
`Notation`	A_∞
`Def.oper.`	m_n for $n \geq 2$ (operation of arity n and degree $n-2$)
`sym.`	
`rel.`	$\partial(m_n) = \sum_{\substack{n=p+q+r \\ k=p+1+r \\ k>1, q>1}} (-1)^{p+qr} m_k \circ (\mathrm{id}^{\otimes p} \otimes m_q \otimes \mathrm{id}^{\otimes r})$.
`Free alg.`	
$\mathcal{P}_n$	$(A_\infty)_n = \mathbb{K}[PT_n]$ isomorphic to $C_\bullet(\mathcal{K}^{n-2})$ as chain complex where $\mathcal{K}^n$ is the Stasheff polytope of dimension n
$\dim \mathcal{P}_n$	
`Gen.series`	$f(t) =$
`Dual operad`	
`Chain-cplx`	
`Properties`	ns, multi-ary, quadratic, minimal model for As
`Alternative`	Cobar construction on $As^{¡}$: $A_\infty = As_\infty := \Omega As^{¡}$
`Relationsh.`	Many, see the literature.
`Unit`	Good question !
`Comment`	There are two levels of morphisms between A_∞-algebras: the morphisms and the ∞-morphisms, see [?] for instance.
`Ref.`	[?] J. Stasheff, *Homotopy associativity of H-spaces. I, II.* TAMS 108 (1963), 275-292 ; ibid. 108 (1963), 293312.

`Name`	**Commutative algebra up to homotopy** C_∞
	operad with underlying space in `dgVect`
`Notation`	C_∞
`Def.oper.`	m_n for $n \geq 2$ (operation of arity n and degree $n-2$) which vanishes on the sum of $(p, n-p)$-shuffles, $1 \leq p \leq n-1$.
`sym.`	
`rel.`	$\partial(m_n) = \sum_{\substack{n=p+q+r \\ k=p+1+r \\ k>1, q>1}} (-1)^{p+qr} m_k \circ (\mathrm{id}^{\otimes p} \otimes m_q \otimes \mathrm{id}^{\otimes r})$.
`Free alg.`	
$\mathcal{P}(n)$	
$\dim \mathcal{P}(n)$	
`Gen.series`	$f(t) =$
`Dual operad`	
`Chain-cplx`	
`Properties`	multi-ary, quadratic, minimal model for Com
`Alternative`	Cobar construction on $Com^{\text{¡}}$: $C_\infty = Com_\infty := \Omega Com^{\text{¡}}$
`Relationsh.`	
`Unit`	
`Comment`	
`Ref.`	[?] T. Kadeishvili, *The category of differential coalgebras and the category of $A(\infty)$-algebras.* Proc. Tbilisi Math.Inst. 77 (1985), 50-70.

Name **Lie algebra up to homotopy** L_∞

operad with underlying space in `dgVect`

Notation L_∞

Def.oper. ℓ_n, n-ary operation of degree $n-2$, for all $n \geq 2$

sym.

rel. $\partial_A(\ell_n) = \sum_{\substack{p+q=n+1 \\ p,q>1}} \sum_{\sigma \in Sh_{p,q}^{-1}} \mathrm{sgn}(\sigma)(-1)^{(p-1)q}(\ell_p \circ_1 \ell_q)^\sigma$,

Free alg.

rep. $\mathcal{P}(n)$

$\dim \mathcal{P}(n)$

Gen.series $f(t) =$

Dual operad

Chain-cplx

Properties multi-ary, quadratic, minimal model for *Lie*

Alternative Cobar construction on $Lie^{¡}$: $L_\infty = Lie_\infty := \Omega Lie^{¡}$

Relationsh.

Unit

Comment

Ref. [?] V. Hinich, and V. Schechtman, *Homotopy Lie algebras.* I. M. Gel'fand Seminar, 128, Adv. Soviet Math., 16, Part 2, Amer. Math. Soc., Providence, RI, 1993.

`Name` **Dendriform algebra up to homotopy** $Dend_\infty$

operad with underlying space in `dgVect`

`Notation` $Dend_\infty$

`Def.oper.` $m_{n,i}$ is an n-ary operation, $1 \leq i \leq n$, for all $n \geq 2$

`sym.` none

`rel.` $\partial(m_{n,i}) = \sum(-1)^{p+qr} m_{p+1+r,\ell}(\underbrace{\mathrm{id}, \cdots, \mathrm{id}}_{p}, m_{q,j}, \underbrace{\mathrm{id}, \cdots, \mathrm{id}}_{r})$

sum extended to all the quintuples p, q, r, ℓ, j satisfying:
$p \geq 0, q \geq 2, r \geq 0, p+q+r = n, 1 \leq \ell \leq p+1+q, 1 \leq j \leq q$
and either one of the following:
$i = q + \ell$, when $1 \leq p+1 \leq \ell - 1$,
$i = \ell - 1 + j$, when $p + 1 = \ell$,
$i = \ell$, when $\ell + 1 \leq p + 1$.

`Free alg.`

`rep.` $\mathcal{P}_n$

$\dim \mathcal{P}_n$

`Gen.series` $f(t) =$

`Dual operad`

`Chain-cplx`

`Properties` ns, multi-ary, quadratic, minimal model for *Dend*

`Alternative`

`Relationsh.`

`Unit`

`Comment`

`Ref.` See for instance [?]

`Name`	**$\mathcal{P}$-algebra up to homotopy**$\mathcal{P}_\infty$
	operad with underlying space in `dgVect`
`Notation`	$\mathcal{P}_\infty$
	the operad $\mathcal{P}$ is supposed to be quadratic and Koszul
`Def.oper.` `sym.` `rel.`	
`Free alg.`	
`rep.` $\mathcal{P}(n)$	
$\dim \mathcal{P}(n)$	
`Gen.series`	$f(t) =$
`Dual operad`	
`Chain-cplx`	
`Properties`	quadratic, ns if $\mathcal{P}(n) = \mathcal{P}_n \otimes \mathbb{K}[\mathbb{S}_n]$.
`Alternative`	Cobar construction on $\mathcal{P}^{¡}$: $\mathcal{P}_\infty := \Omega\ \mathcal{P}^{¡}$
`Relationsh.`	
`Unit`	
`Comment`	
`Ref.`	See for instance [?]

Name	**Brace algebra**brace
Notation	$Brace$
Def.oper.	$\{x_0; x_1, \ldots, x_n\}$ for $n \geq 0$
sym.	$\{x; \emptyset\} = x$
rel.	$\{\{x; y_1, \ldots, y_n\}; z_1, \ldots, z_m\} =$ $\sum\{x; \ldots, \{y_1; \ldots\}, \ldots \quad \ldots, \{y_n; \ldots, \}, \ldots\}.$ the dots are filled with the variables z_i's (in order).
Free alg.	
rep. $\mathcal{P}(n)$	$Brace(n) = \mathbb{K}[PBT_{n+1}] \otimes \mathbb{K}[\mathbb{S}_n]$
$\dim \mathcal{P}(n)$	$1, 1 \times 2!, 2 \times 3!, 5 \times 4!, 14 \times 5!, 42 \times 6!, 132 \times 7!, \ldots,$ $c_{n-1} \times n!, \ldots$
Gen.series	$f(t) =$
Dual operad	
Chain-cplx	
Properties	multi-ary, quadratic, quasi-regular.
Alternative	
Relationsh.	$Brace$-alg $\to MB$-alg $Brace$-alg $\to PreLie$-alg, ($\{-;-\}$ is a pre-Lie product) If A is a brace algebra, then $T^c(A)$ is a cofree Hopf algebra
Unit	
Comment	There exists a notion of brace algebra with differential useful in algebraic topology
Ref.	[?] Ronco, M. *Eulerian idempotents and Milnor-Moore theorem for certain non-cocommutative Hopf algebras.* J. Algebra 254 (2002), no. 1, 152–172.

Name	**Multi-brace algebra$\mathbf{B}_\infty$**
Notation	MB
Def.oper.	$(x_1, \ldots, x_p; y_1, \ldots, y_q\}$ for $p \geq 1, q \geq 1$
sym.	
rel.	$\mathcal{R}_{ijk}$, see Ref.
Free alg.	
rep. $\mathcal{P}(n)$	$MB(n) = (\mathbb{K}[PT_n] \oplus \mathbb{K}[PT_n]) \otimes \mathbb{K}[\mathbb{S}_n], n \geq 2$
$\dim \mathcal{P}(n)$	$1, 1 \times 2!, 6 \times 3!, 22 \times 4!, 90 \times 5!, ?? \times 6!, ??? \times 7!, \ldots,$ $2C_n \times n!, \ldots$, $C_n =$ Schröder number (super Catalan)
Gen.series	$f(t) =$
Dual operad	
Chain-cplx	
Properties	multi-ary, quadratic, quasi-regular.
Alternative	Used to be denoted by B_∞ or $\mathbf{B}_\infty$ confusing notation with respect to algebras up to homotopy
Relationsh.	$Brace$-alg $\to MB$-alg $Brace$-alg $\to PreLie$-alg, $(\{-;-\}$ is a pre-Lie product) If A is a brace algebra, then $T^c(A)$ is a cofree Hopf algebra (and vice-versa)
Unit	
Comment	There exists a notion of MB-infinity algebra with differentials useful in algebraic topology (and called B_∞-algebra)
Ref.	[?] Loday, J.-L., and Ronco, M. *On the structure of cofree Hopf algebras* J. reine angew. Math. 592 (2006) 123–155.

Name n-**Lie algebras**n-Lie

Notation n-*Lie*

Def.oper. $[x_1, \dots, x_n]$ (n-ary operation)
sym. fully anti-symmetric
rel. Leibniz type identity, see next page

Free alg.

rep. $\mathcal{P}(n)$

$\dim \mathcal{P}(n)$

Gen.series $f(t) =$

Dual operad

Chain-cplx

Properties n-ary

Alternative

Relationsh.

Unit

Comment

Ref. [?] Filippov, V. T. n-Lie algebras. (Russian) Sibirsk. Mat. Zh. 26 (1985), no. 6, 126140, 191. English translation: Siberian Math.J. 26 (1985), no. 6, 879–891.

Name	**n-Leibniz algebras**n-Leibniz
Notation	n-*Leib*
Def.oper.	$[x_1, \ldots, x_n]$ (n-ary operation)
sym.	
rel.	$[[x_1, \ldots, x_n], y_1, \ldots, y_{n-1}] = \sum_{i=1}^{n} [x_1, \ldots, x_{i-1}, [x_i, y_1, \ldots, y_{n-1}], x_{i+1}, \ldots, x_n]$
Free alg.	
rep. $\mathcal{P}(n)$	
$\dim \mathcal{P}(n)$	
Gen.series	$f(t) =$
Dual operad	
Chain-cplx	
Properties	n-ary
Alternative	
Relationsh.	
Unit	
Comment	
Ref.	[?] Casas, J. M.; Loday, J.-L.; Pirashvili, T., *Leibniz n-algebras*. Forum Math. 14 (2002), no. 2, 189–207.

`Name` **$\mathcal{X}^{\pm}$-algebra**$\mathcal{X}^{\pm}$-algebra

`Notation` $\mathcal{X}^{\pm}$-alg

`Def.oper.` $x \nwarrow y,\ x \nearrow y,\ x \searrow y,\ x \swarrow y.$

`sym.`

`rel.`

$$
\begin{array}{l}
(\nwarrow)\nwarrow = \nwarrow(\nwarrow) + \nwarrow(\swarrow),\ (\swarrow)\nwarrow = \swarrow(\nwarrow),\ (\nwarrow)\swarrow + (\swarrow)\swarrow = \swarrow(\swarrow),\\
(\nwarrow)\nwarrow = \nwarrow(\searrow) + \nwarrow(\nearrow),\ (\swarrow)\nwarrow = \swarrow(\nearrow),\ (\nwarrow)\swarrow + (\swarrow)\swarrow = \swarrow(\searrow),\\
(\nearrow)\nwarrow = \nearrow(\nwarrow) + \nearrow(\swarrow),\ (\searrow)\nwarrow = \searrow(\nwarrow),\ (\nearrow)\swarrow + (\searrow)\swarrow = \searrow(\swarrow),\\
(\nwarrow)\nearrow = \nearrow(\nearrow) + \nearrow(\searrow),\ (\swarrow)\nearrow = \searrow(\nearrow),\ (\nwarrow)\searrow + (\swarrow)\searrow = \searrow(\searrow),\\
(\nearrow)\nearrow = \nearrow(\nearrow) + \nearrow(\searrow),\ (\searrow)\nearrow = \searrow(\nearrow),\ (\nearrow)\searrow + (\searrow)\searrow = \searrow(\searrow).
\end{array}
$$

$$(\nearrow)\searrow - (\nwarrow)\searrow = +\nwarrow(\swarrow) - \nwarrow(\searrow)\ , \qquad (16+)$$
$$(\nearrow)\searrow - (\nwarrow)\searrow = -\nwarrow(\swarrow) + \nwarrow(\searrow)\ . \qquad (16-)$$

`Free alg.`

$\mathcal{P}_n$

$\dim \mathcal{P}_n$

`Gen.series` $f(t) =$

`Dual operad` Both sef-dual.

`Chain-cplx`

`Properties` ns, binary, quadratic.

`Alternative`

`Relationsh.` *Dend* $\blacksquare$ *Dias* $\twoheadrightarrow \mathcal{X}^{\pm} \twoheadrightarrow$ *Dend* $\square$ *Dias*
Fit into the "operadic butterfly" diagram, see the reference.

`Unit`

`Comment`

`Ref.` [?] Loday J.-L., *Completing the operadic butterfly*, Georgian Math Journal 13 (2006), no 4. 741–749.

`Name`	*put your own type of* **algebras**
`Notation`	
`Def.oper.`	
`sym.`	
`rel.`	
`Free alg.`	
`rep.` $\mathcal{P}(n)$	
$\dim \mathcal{P}(n)$	
`Gen.series`	$f(t) =$
`Dual operad`	
`Chain-cplx`	
`Properties`	
`Alternative`	
`Relationsh.`	
`Unit`	
`Comment`	
`Ref.`	

Integer sequences which appear in this paper, up to some shift and up to multiplication by $n!$ or $(n-1)!$.

1	1	1	1	1	$\cdots$	1	$\cdots$	Com, As
1	2	0	0	0	$\cdots$	0	$\ldots$	Nil
1	2	2	2	2	$\cdots$	2	$\cdots$	$Dual2as$
1	2	3	4	5	$\cdots$	n	$\cdots$	$Dias, Perm$
1	2	5	12	15	$\cdots$	??	$\cdots$	$Altern^!$
1	2	5	14	42	$\ldots$	c_{n-1}	$\ldots$	$Mag, Dend, brace, Dup$
1	2	6	18	57	$\ldots$	f_{n+2}	$\ldots$	$MagFine$
1	2	6	22	90	$\ldots$	$2C_n$	$\ldots$	$Dipt, 2as, brace$
1	2	6	24	120	$\cdots$	$n!$	$\cdots$	$As, Lie, Leib, Zinb$
1	2	7			$\cdots$	??	$\ldots$	$Lie\text{-}adm$
1	2	7	32	175	$\cdots$	??	$\ldots$	$Altern$
1	2	7	40		$\cdots$	??	$\ldots$	$Moufang$
1	2	9	64	625	$\ldots$	n^{n-1}	$\ldots$	$PreLie, NAP$
1	2	10	26	76	$\ldots$	??	$\ldots$	$Parastat$
1	3	7	15	31	$\ldots$	(2^n-1)	$\ldots$	$Trias$
1	3	9			$\cdots$	??	$\cdots$	$Malcev$
1	3	11	45	197	$\ldots$	C_n	$\ldots$	$TriDend$
1	3	13	75	541	$\cdots$	??	$\ldots$	CTD
1	3	16	125	6^5	$\cdots$	$(n+1)^{n-1}$	$\cdots$	$Park$
1	3	20	210	3024	$\cdots$	$a(n)$	$\cdots$	$PostLie$
1	3	50			$\cdots$	??	$\cdots$	$Jordan\ triples$
1	4	9	16	25	$\ldots$	$n^2n!$	$\ldots$	$Quadri^!$
1	4	23	156	1162	$\cdots$	??	$\ldots$	$Quadri$
1	4	23	181		$\cdots$	??	$\ldots$	see $PreLie$
1	4	27	256		$\cdots$	n^n	$\cdots$	
1	8				$\cdots$	??	$\ldots$	$Akivis$
1	8	78	1104		$\cdots$	??	$\ldots$	$Sabinin$

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