ON THE CONTINUITY OF THE FINITE BLOCH-KATO COHOMOLOGY

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Abstract. Let K_0 be an unramified, complete discrete valuation field of mixed characteristics $(0,p)$ with perfect residue field. We consider two finite, free $\mathbb{Z}_p$-representations of G_{K_0}, T_1 and T_2, such that $T_i \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$, for $i = 1, 2$, are crystalline representations with Hodge-Tate weights between 0 and $r \leq p-2$.

Let K be a totally ramified extension of degree e of K_0. Supposing that $p \geq 3$ and $e(r-1) \leq p-1$, we prove that for every integer $n \geq 1$ and $i = 1, 2$, the inclusion $H^1_{\mathrm{fin}}(K, T_i)/p^n H^1_{\mathrm{fin}}(K, T_i) \hookrightarrow H^1(K, T_i/p^n T_i)$ of the finite Bloch-Kato cohomology into the Galois cohomology is functorial with respect to morphisms as $\mathbb{Z}/p^n\mathbb{Z}[G_{K_0}]$-modules from T_1/p^nT_1 to T_2/p^nT_2. In the appendix we give a related result for $p = 2$.

Contents

1. Introduction

Let K denote a complete discrete valuation field of mixed characteristics $(0,p)$ with perfect residue field k. We choose an algebraic closure $\overline{K}$ of K and denote by G_K the

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Galois group of $\overline{K}$ over K. For a p-adic representation V of G_K, the finite cohomology group $H^1_{\text{fin}}(K,V)$ is defined as

$$H^1_{\text{fin}}(K,V) = \ker(H^1(K,V) \to H^1(K, V \otimes_{\mathbb{Q}_p} \mathrm{B}_{\text{cris}}))$$

where $H^1(K,-)$ denotes the first continuous Galois cohomology functor and B_{cris} is the ring of crystalline periods. If V is a crystalline representation then the cohomology classes in $H^1_{\text{fin}}(K,V)$ represent classes of extensions

$$0 \to V \to E \to \mathbb{Q}_p \to 0$$

which are crystalline, i.e. such that E is a crystalline G_K-representation. If $T \subset V$ is a $\mathbb{Z}_p$-lattice stable by G_K, the group $H^1_{\text{fin}}(K,T)$ is defined as the fiber product of the diagram

$$\begin{array}{ccc} & & H^1(K,T) \\ & & \downarrow \eta \\ H^1_{\text{fin}}(K,V) & \hookrightarrow & H^1(K,V) \end{array}$$

where η is the map induced by the injection $T \subset V$. By definition the classes in $H^1_{\text{fin}}(K,T)$ correspond to classes of extensions of T by $\mathbb{Z}_p$ which are crystalline (after extending scalars to $\mathbb{Q}_p$). The canonical map $H^1_{\text{fin}}(K,T) \to H^1(K,T)$ is injective and identifies $H^1_{\text{fin}}(K,T)$ to $\eta^{-1}(H^1_{\text{fin}}(K,V))$; on the other hand the map $H^1_{\text{fin}}(K,T) \to H^1_{\text{fin}}(K,V)$ is not injective in general ($H^1_{\text{fin}}(K,T)$ contains all the p-torsion of $H^1(K,T)$).

The main relevance of the groups $H^1_{\text{fin}}(K,V)$ comes from global arithmetic. Suppose that F is a number field and that M is a p-adic representation of $G_F := \mathrm{Gal}(\overline{F}/F)$. Then the Selmer group of M is the subgroup of the continuous cohomology group $H^1(F,M)$ of classes which satisfy certain local restrictions, in particular the classes in the Selmer group restricted to the decomposition group at v of G_F, for places v of F dividing p, lie in $H^1_{\text{fin}}(F_v, M)$.

If the p-adic representation M as above is the p-adic realization of a motive over a number field F, then the family of p-Selmer groups over the cyclotomic tower of F (or some other $\mathbb{Z}_p$-tower) encodes important arithmetic information about the motive, in particular its algebraic p-adic L-functions are defined in terms of it. Ralph Greenberg was the first to ask the question : suppose we have two motives which are congruent modulo a power of p. What can one say about the two families of p-Selmer groups?

For example in [GV00] and [GIP07] one studies the case of two elliptic curves defined over $\mathbb{Q}$ (or even modular forms of weight two) and one obtains: if the elliptic curves are both ordinary or both supersingular at p and they are congruent modulo p, then under certain further hypothesis one shows that the families of non-primitive Selmer groups attached to these elliptic curves are also congruent modulo p. It follows, using results of K. Kato that if the main conjecture holds for one of the elliptic curves then it holds for the other.

Of course in order to study the behaviour of Selmer groups with respect to congruences, one has to first investigate the behaviour of the local conditions defining the Selmer groups with respect to the congruences, in other words to ask whether $H^1_{\text{fin}}(K,T)$ varies p-adically continuosly with T (where now K is a p-adic field and T a $\mathbb{Z}_p$-representation of G_K as at the beginning of this introduction). This is the main task of this article.

We will now be more precise and consider two crystalline $\mathbb{Z}_p$-representations of G_K, T_1 and T_2 which are isomorphic modulo p^n (we say that T_1 and T_2 are congruent modulo p^n). Consider a class $\xi \in H^1(K, T_1/p^nT_1)$ which is the reduction modulo p^n of a class coming from $H^1_{\text{fin}}(K, T_1)$. We would like to know if the image of ξ via the isomorphism between $H^1(K, T_1/p^nT_1)$ and $H^1(K, T_2/p^nT_2)$ is the reduction of a class in $H^1_{\text{fin}}(K, T_2)$. This could be reformulated as follows: given an isomorphism $\iota\colon T_1/p^nT_1 \to T_2/p^nT_2$ of $\mathbb{Z}/p^n\mathbb{Z}[G_K]$-modules, is there an isomorphism $\tilde{\iota}$ of $\mathbb{Z}/p^n\mathbb{Z}$-modules making the following diagram commutative?

$$\begin{array}{ccc}
H^1_{\text{fin}}(K,T_1)/p^nH^1_{\text{fin}}(K,T_1) & \longrightarrow & H^1(K,T_1/p^nT_1) \\
{\scriptstyle \exists ?}\ \downarrow\ {\scriptstyle \tilde{\iota}} & & {\scriptstyle \cong}\ \downarrow\ {\scriptstyle H^1(K,\iota)} \\
H^1_{\text{fin}}(K,T_2)/p^nH^1_{\text{fin}}(K,T_2) & \longrightarrow & H^1(K,T_2/p^nT_2)
\end{array}$$

The maps $H^1_{\text{fin}}(K,T_i)/p^nH^1_{\text{fin}}(K,T_i) \to H^1(K,T_i/p^nT_i)$, for $i = 1, 2$, in the diagram above are induced by the injections $H^1_{\text{fin}}(K,T_i) \hookrightarrow H^1(K,T_i)$ and $H^1(K,T_i)/p^nH^1(K,T_i) \hookrightarrow H^1(K,T_i/p^nT_i)$. We recall that they are injective, cf. Lemma 4.1.1, and so this reformulation of the question is only apparently more general. However it makes sense to ask more generally whether $H^1_{\text{fin}}(K,-)/p^nH^1_{\text{fin}}(K,-)$ defines a subfunctor of $T/p^nT \mapsto H^1(K,T/p^nT)$.

It is known that these questions do not have positive answers in general. For example let $T_1 = \mathbb{Z}_p(1)$, $T_2 = \mathbb{Z}_p(p)$, $K = \mathbb{Q}_p$ and $p \geq 3$. We have clearly $T_1/pT_1 = \mathbb{F}_p(1) \cong T_2/pT_2$ and Kummer theory implies that $H^1(\mathbb{Q}_p, \mathbb{F}_p(1)) = \mathbb{Q}_p^*/(\mathbb{Q}_p^*)^p$, which is a two dimensional $\mathbb{F}_p$-vector space. Simple cohomological arguments show that the $\mathbb{F}_p$-vector spaces $H^1_{\text{fin}}(\mathbb{Q}_p,T_1)/pH^1_{\text{fin}}(\mathbb{Q}_p,T_1)$ and $H^1_{\text{fin}}(\mathbb{Q}_p,T_2)/pH^1_{\text{fin}}(\mathbb{Q}_p,T_2)$ have both codimension one in $H^1(\mathbb{Q}_p, \mathbb{F}_p(1))$. However, one can show that they are orthogonal with respect to cup-product and they correspond to sub-spaces of $\mathbb{Q}_p^*/(\mathbb{Q}_p^*)^p$ spanned respectively by p and $p+1$. This example shows also that our problem cannot be solved by using only arguments based on the lengthes of the $\mathbb{Z}/p^n\mathbb{Z}$-modules $H^1_{\text{fin}}(K,T_i)/p^nH^1_{\text{fin}}(K,T_i)$.

Another example, due to R. Greenberg, shows that the equivariance of the morphism $\iota\colon T_1/p^nT_1 \to T_2/p^nT_2$ with respect to G_K is too weak in general. Consider two ordinary elliptic curves E_1 and E_2 over K, assume that K contains the coordinates of the p-torsion points $E_1[p]$ and $E_2[p]$ and that it has finite residue field. For $i = 1, 2$, let $T_i = T_p(E_i)$, where T_p denotes the p-adic Tate module. A result of Coates and Grinberg (cf. [CG96]) states that $H^1_{\text{fin}}(K,T_i)$ is equal to the image of $H^1(K, T(\hat{E}_i)) \hookrightarrow H^1(K,T_i)$, where $\hat{E}_i$ is the formal group of E_i (which has height 1 by ordinarity). By the hypothesis on K, the reduction $T_i/pT_i = E_i[p]$ is a two dimensional $\mathbb{F}_p$-vector space, trivial as G_K-module, in particular T_1 and T_2 are congruent modulo p. So any isomorphism $\iota\colon T_1/pT_1 \to T_2/pT_2$ not sending the image of $T_p(\hat{E}_1)/pT_p(\hat{E}_1)$ to $T_p(\hat{E}_2)/pT_p(\hat{E}_2)$ gives a negative answer to our question.

1.1. **The main result.**

1.1.1. We use the notations of the previous section and denote by K_0 the maximal unramified extension of $\mathbb{Q}_p$ in K, by $e := [K : K_0]$ the absolute ramification degree of K and by G_{K_0} the absolute Galois group of K_0.

Let $n \geq 1$ be an integer, $\mathcal{W} \subset \mathbb{Z}$ a subset and Z a $\mathbb{Z}/p^n\mathbb{Z}$-module of finite type endowed with a continuous action of G_K. We say that Z is crystalline with Hodge-Tate weights in $\mathcal{W}$ if there exists an exact sequence of $\mathbb{Z}_p[G_K]$-modules

$$0 \longrightarrow T' \longrightarrow T \longrightarrow Z \longrightarrow 0$$

where T is a crystalline $\mathbb{Z}_p$-representation with Hodge-Tate weights in $\mathcal{W}$.

We denote $\mathrm{Rep}_{\mathbb{Q}_p}(G_K)^{\mathcal{W}}_{\mathrm{cris}}$ (resp. $\mathrm{Rep}_{\mathbb{Z}_p}(G_K)^{\mathcal{W}}_{\mathrm{cris}}$, resp. $\mathrm{Rep}_{\mathbb{Z}/p^n\mathbb{Z}}(G_K)^{\mathcal{W}}_{\mathrm{cris}}$) the category of crystalline $\mathbb{Q}_p$-representations (resp. $\mathbb{Z}_p$-representations, resp. $\mathbb{Z}/p^n\mathbb{Z}$-representations) with Hodge-Tate weights in $\mathcal{W}$. For any integers $a \leq b$, we denote $[a, b]$ the set of integers i, such that $a \leq i \leq b$.

Let denote by $\mathrm{Mod}(\mathbb{Z}/p^n\mathbb{Z})$ the category of $\mathbb{Z}/p^n\mathbb{Z}$-modules.

1.1.2. To address the problem of continuity for the Bloch Kato finite cohomology we proceed as follows. First assume $p \geq 3$. For every integer $0 \leq r \leq p-2$, we construct explicitly a functor

$$H^1_r(K, -) \colon \mathrm{Rep}_{\mathbb{Z}/p^n\mathbb{Z}}(G_{K_0})^{[0,r]}_{\mathrm{crys}} \to \mathrm{Mod}(\mathbb{Z}/p^n\mathbb{Z}),$$

endowed with a morphism of functors $t^{\infty}_{\mathrm{st}} \colon H^1_r(K, -) \to H^1(K, -)$, such that for every crystalline $\mathbb{Z}_p$-representations T of G_{K_0} with Hodge-Tate weights in $[0, r]$, we have the canonical factorisation:

$$\begin{array}{ccc} H^1_{\mathrm{fin}}(K,T)/p^n H^1_{\mathrm{fin}}(K,T) & \hookrightarrow & H^1(K, T/p^nT) \\ & {\scriptstyle \varepsilon} \searrow \quad \nearrow {\scriptstyle t^{\infty}_{\mathrm{st}}} & \\ & H^1_r(K, T/p^nT) & \end{array}$$

For any crystalline $\mathbb{Z}_p$-representations T of G_{K_0}, denotes by $r_{\max}$ or $r_{\max}(T)$ the largest Hodge-Tate weight of T.

The following statement which is proved as Theorem 4.1.6 in section §4.1 constitutes the main result of this article.

Theorem *Let $p \geq 3$ be a prime integer and we fix an integer r with $0 \leq r \leq p-2$. We denote by T a crystalline $\mathbb{Z}_p$-representations of G_{K_0} with Hodge-Tate weights in $[0, r]$ and assume $e(r_{\max} - 1) \leq p - 1$. Then for every integer $n \geq 1$, $H^1_{\mathrm{fin}}(K, T)/p^n H^1_{\mathrm{fin}}(K, T)$ is isomorphic to $H^1_r(K, T/p^nT)$ via ε.*

Remark 1.1.3.

1. We point out that if $r_{\max} \leq 1$ the condition on the ramification index e of K is empty.
2. The hypothesis $e(r_{\max} - 1) \leq p - 1$ is necessary, see Proposition 4.2.3.

(3) For every r such that $r_{\max} \leq r \leq p-2$, we have $H^1_r(K, T/p^nT) = H^1_{r_{\max}}(K, T/p^nT)$ canonically. It is a consequence of the theorem but actually it follows already by the construction of H^1_r, see Remark 4.1.3.

In particular if $e(r-1) \leq p-1$, $H^1_r(K, -)$ is a subfunctor of $H^1(K, -)$; in this case set

$$H^1_{\mathrm{fin}}(K, -) := H^1_r(K, -). \tag{1.1.3.1}$$

The next statement which is proved as Corollary 4.1.12 in section §4.1 states the hypothesis under which the correspondence $T/p^nT \mapsto H^1_{\mathrm{fin}}(K,T)/p^nH^1_{\mathrm{fin}}(K,T)$ is well-defined and functorial (we have indeed $H^1_{\mathrm{fin}}(K,T)/p^nH^1_{\mathrm{fin}}(K,T) = H^1_{\mathrm{fin}}(K, T/p^nT)$).

Corollary *Let $p \geq 3$ and let T_1 and T_2 be two crystalline $\mathbb{Z}_p$-representations of G_{K_0} with Hodge-Tate weights in $[0, r] \subseteq [0, p-2]$ and assume $e(r-1) \leq p-1$. Then for every morphism (resp. isomorphism) $\iota \colon T_1/p^nT_1 \to T_2/p^nT_2$ of $\mathbb{Z}/p^n\mathbb{Z}[G_{K_0}]$-modules, there exists a morphism (resp. isomorphism) $\tilde{\iota}$ of $\mathbb{Z}/p^n\mathbb{Z}$-modules making the following diagram commutative.*

$$\begin{array}{ccc} H^1_{\mathrm{fin}}(K,T_1)/p^nH^1_{\mathrm{fin}}(K,T_1) & \hookrightarrow & H^1(K, T_1/p^nT_1) \\ \downarrow \tilde{\iota} & & \downarrow H^1(K,\iota) \\ H^1_{\mathrm{fin}}(K,T_2)/p^nH^1_{\mathrm{fin}}(K,T_2) & \hookrightarrow & H^1(K, T_2/p^nT_2) \end{array}$$

Remark 1.1.4.

(1) Some cases of Corollary 4.1.12 were already known:
   (a) The case where $e = 1$ and the representations T_1 and T_2 have Hodge-Tate weights in $[0, p-2]$. In this case the result follows from [BK90, Lemmas 4.4 and 4.5] using Fontaine-Lafaille theory.
   (b) In the case where the Hodge-Tate weights of T_1 and T_2 are in $[0, 1]$, under the hypothesis $(T_1/p^nT_1)^{G_{K_0}} = (T_1/p^nT_1)^{G_K}$ and $p \geq 3$, the result was proved in [GIP07].
   (c) The case of representations coming from Barsotti-Tate groups have been treated by Nekovar [Nek11, A.2.6] using flat cohomology; if $p \geq 3$ this implies, by a result of Kisin [Kis06, (2.2.6-7)], the case where representations have Hodge-Tate weights in $[0, 1]$.
(2) Under the stronger hypothesis $er \leq p-2$ we can prove a more general functoriality, see Corollary 4.2.1.
(3) The hypothesis $e(r-1) \leq p-1$ is not necessary in Corollary 4.1.12. For example, assume that k is finite and let T be a crystalline $\mathbb{Z}_p$-representation of rank one and Hodge-Tate weight $r \geq 2$. We have $H^1_{\mathrm{fin}}(K,T) = H^1(K,T)$, with no restrictions on e (cf. [BK90, Example 3.9]); by Tate duality it follows $H^1_{\mathrm{fin}}(K,T)/p^nH^1_{\mathrm{fin}}(K,T) = H^1(K, T/p^n)$ and the functoriality is obvious.
(4) If the representations have Hodge Tate weights in $[0, r] \subseteq [0, p-2]$, but not in $[1, r]$, then we do not know if the hypothesis $e(r-1) \leq p-1$ is necessary in Corollary 4.1.12. In particular, we do not know if Corollary 4.1.12 is true even in the following particular case: the representations T_1 and T_2 are irreducible of rank

2 with Hodge-Tate weights exactly 0 and $p-2$, with $p \geq 7$ and $e > 1$ (or $p = 5$ and $e > 2$).

(5) In Appendix A we deal with the case of characteristic $p = 2$ which is essentially independent from the rest of the paper.

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1.2. **Notation and conventions.**

1.2.1. We denote by $\mathcal{O}_{\overline{K}}$ (resp. $\mathcal{O}_K$) the ring of integers of $\overline{K}$ (resp. K).

For any ring A, we denote by $W(A)$ the ring of Witt vectors with coefficients in A. When $A = k$, we write simply $W = W(k)$. We recall that we have denoted by K_0 the field of fractions of W and let σ denote the arithmetic Frobenius of W (and K_0), i.e. the isomorphism lifting the p-power map on k. We set for every $n \geq 0$, $W_n = W/p^nW$.

We denote by $\mathbb{N}$ the set of integers $i \geq 0$.

2. Strongly divisible modules

In the following, except for the appendix, we assume $p > 2$.

2.1. **The ring S.** In this section we recall a definition of G. Faltings in [Fal99, §2], see also [Bre02] and [BM02].

2.1.1. Let π be a uniformizer of K and denote by $E(u)$ its minimal monic polynomial. It is an Eisenstein polynomial with coefficents in W of degree e.

We denote by $\mathrm{ev}_\pi \colon W[u] \to \mathcal{O}_K$ the W-algebra homomorphism sending u to π. Its kernel is generated by $E(u)$. We let S denote the p-adic completion of the divided power hull of $W[u]$ with respect to the ideal $\ker(\mathrm{ev}_\pi)$, on which we have divided powers compatible with the canonical divided powers of the ideal $pW[u]$. The ring S can be identified to the sub-ring of $\mathbb{Q}_p[\![u]\!]$ of series $\sum_{i\in\mathbb{N}} a_i \frac{u^i}{q(i)!}$ such that the a_i's belong to W, the sequence $(a_i)_{i\geq 0}$ converges to 0 for i going to $+\infty$ and $q(i)$ is the quotient of the Euclidean division of i by e.

2.1.2. The ring S is endowed with the following extra structures.

(1) A Frobenius homomorphism $\varphi \colon S \to S$, which is the unique continuous σ-semi-linear morphism sending u to u^p.

(2) A decreasing filtration $(\mathrm{Fil}^i S)_{i\in\mathbb{N}}$ on S, defined as follows: for every integer $i \geq 0$, $\mathrm{Fil}^i S$ is the the p-adic closure in S of the ideal generated by $\frac{E(u)^j}{j!}$, for all $j \geq i$.

We have: for all $0 \leq i \leq p-1$, $\varphi(\mathrm{Fil}^i S) \subseteq p^i S$; for such an integer i, put $\varphi_i = p^{-i}\varphi|_{\mathrm{Fil}^i S}$.

Finally, we set $c_1 = \varphi_1(E(u))$: it is a unit in S.

2.1.3. We denote by $\Omega^1_{\log} = \widehat{\Omega}^1_{S/W}(\log)$ (resp. $\Omega^1 = \widehat{\Omega}^1_{S/W}$) the module of continuous logarithmic differential (resp. regular) 1-forms of S over W and $d\colon S \to \Omega^1 \subset \Omega^1_{\log}$ the canonical differential. We have $\Omega^1 = Sdu$ and $\Omega^1_{\log} = Su^{-1}du$.

We denote again $\varphi\colon \Omega^1_{\log} \to \Omega^1_{\log}$ the φ-semi-linear homomorphism, induced by the Frobenius on S via the universal property of the differentials forms. It maps $u^{-1}du$ to $pu^{-1}du$ and we put $\varphi_1 = p^{-1}\varphi$. The submodule Ω^1 is stable under φ and φ_1. Clearly we have $d\varphi = p\varphi d$. In the literature, it is also common to introduce the continuous W-derivation $N\colon S \to S$ sending u to $-u$.

2.2. **Strongly divisible lattices over S.** (Following [Bre02] and [Liu08]).

Let r be an integer such that $0 \leq r \leq p-2$.

2.2.1. We first define the category $'\mathrm{Mod}(S)^r_{\log}$. Its objects are 4-uples $(M, \mathrm{Fil}^r M, \varphi_r, \nabla_M)$, where:

(1) M is an S-module;
(2) $\mathrm{Fil}^r M$ is a sub-S-module of M containing $(\mathrm{Fil}^r S)M$;
(3) φ_r is a φ-semi-linear map $\varphi_r\colon \mathrm{Fil}^r M \to M$, such that for all s in $\mathrm{Fil}^r S$ and x in M we have $\varphi_r(sx) = c_1^{-r}\varphi_r(s)\varphi_r(E(u)^r x)$;
(4) $\nabla_M\colon M \to M \otimes_S \Omega^1_{\log}$ is a logarithmic connexion satisfying $E(u)\nabla_M(\mathrm{Fil}^r M) \subseteq \mathrm{Fil}^r M \otimes_S \Omega^1_{\log}$ such that the following diagram commutes.

$$\begin{array}{ccc} \mathrm{Fil}^r M & \xrightarrow{E(u)\nabla_M} & \mathrm{Fil}^r M \otimes_S \Omega^1_{\log} \\ \big\downarrow{\scriptstyle \varphi_r} & & \big\downarrow{\scriptstyle \varphi_r \otimes \varphi_1} \\ M & \xrightarrow{c_1 \nabla_M} & M \otimes_S \Omega^1_{\log} \end{array}$$

When there is no risk of confusion, we will simply denote $\nabla = \nabla_M$.

The morphisms in this category are defined to be the S-linear maps preserving the submodule Fil^r and commuting with φ_r and ∇.

We remark that the integer r is fixed in the definition of the category $'\mathrm{Mod}(S)^r_{\log}$, and thus the datum of $\mathrm{Fil}^r M$ in the 4-uple $(M, \mathrm{Fil}^r M, \varphi_r, \nabla_M)$ refers to a *single* sub-module of M.

If no confusion is arising, we will sometimes denote an object $(M, \mathrm{Fil}^r M, \varphi_r, \nabla)$ only by its underlying module M.

The category $'\mathrm{Mod}(S)^r_{\log}$ is $\mathbb{Z}_p$-linear. A short sequence $0 \to M' \to M \to M'' \to 0$ of objects in $'\mathrm{Mod}(S)^r_{\log}$ is said *exact*, if the induced sequences $0 \to M' \to M \to M'' \to 0$ and $0 \to \mathrm{Fil}^r M' \to \mathrm{Fil}^r M \to \mathrm{Fil}^r M'' \to 0$ are short exact sequences of S-modules.

2.2.2. We denote $'\mathrm{Mod}(S)^r$ the full subcategory of $'\mathrm{Mod}(S)^r_{\log}$ of objects whose connection ∇ is regular. In the literature it is common to use the derivation $N_M = \nabla_M(-u\frac{d}{du})$ instead of the connection ∇: it is called the *monodromy operator* and the regularity condition is equivalent to $N(M) \subset uN(M)$.

2.2.3. *Free objects.* We denote $\mathrm{Mod}(S)^r$ (resp. $\mathrm{Mod}(S)^r_{\log}$) the full subcategory of $'\mathrm{Mod}(S)^r$ (resp. $'\mathrm{Mod}(S)^r_{\log}$) whose objects satisfy:

(1) M is free of finite type as an S-module;
(2) the S-module $M/\mathrm{Fil}^r M$ has no p-torsion;
(3) the image of φ_r generates M as an S-module.

We call $\mathrm{Mod}(S)^r$ (resp. $\mathrm{Mod}(S)^r_{\log}$) the category of *strongly divisible lattices* (resp. *logarithmic strongly divisible lattices*) over S of weight r .

2.2.4. *Torsion objects.* We denote $\mathrm{ModFI}(S)^r$ the full subcategory of $'\mathrm{Mod}(S)^r$ whose objects satisfy:

(1) M is isomorphic to $\bigoplus_{i\in I} S/p^{n_i}S$ as an S-module, where I is a finite set and n_i are positive integers;
(2) the image of φ_r generates M over S.

For every $n \geq 1$, we denote $\mathrm{ModFI}(S/p^nS)^r$ the full subcategory of $\mathrm{ModFI}(S)^r$ of objects annihilated by p^n, as S-modules.

2.3. **Functors towards Galois representations.**

2.3.1. We denote by R the projective limit $\varprojlim \mathcal{O}_{\overline{K}}/p\mathcal{O}_{\overline{K}}$, where the transition maps are given by the absolute Frobenii $x \mapsto x^p$. We denote by A_{cris} the $W(R)$-algebra of crystalline periods defined by Fontaine in [Fon94, 2.2.3].

Let $\underline{\pi} = (\pi^{(n)})_{n\in\mathbb{N}}$ be a compatible sequence of p^n-roots of π, i.e. every $\pi^{(n)}$ belongs to $\overline{K}$, $\pi^{(0)} = \pi$ and $\pi^{(n+1)^p} = \pi^{(n)}$. We denote by $[\underline{\pi}] \in W(R) \subset A_{\mathrm{cris}}$ its Teichmüller representative, and, for any g in G_K, we put $\epsilon(g) = g([\underline{\pi}])/[\underline{\pi}]$ in A_{cris}.

2.3.2. We recall the definition of a ring introduced by Kato in [Kat88, §3], and denoted afterwards $\widehat{A}_{\mathrm{st}}$, cf. [Bre97] and [Bre99] for more details.

Let $A_{\mathrm{cris}}\langle x\rangle$ be the divided power hull of the polynomial algebra $A_{\mathrm{cris}}[x]$ compatible with the canonical divided powers of the ideal $pA_{\mathrm{cris}}[x]$; for $j \geq 0$, we denote by $\gamma_j(x)$ the j-th divided power of x. The ring $\widehat{A}_{\mathrm{st}}$ is the p-adic completion of $A_{\mathrm{cris}}\langle x\rangle$. It is endowed with the following structures:

(1) A filtration: for any integer $i \geq 0$,

$$\mathrm{Fil}^i \widehat{A}_{\mathrm{st}} = \left\{ \sum_{n=0}^{+\infty} a_n\gamma_n(x) \,\middle|\, a_n \in A_{\mathrm{cris}}, \lim_{n\to+\infty} a_n = 0, \forall n \leq i, a_n \in \mathrm{Fil}^{i-n} A_{\mathrm{cris}} \right\}.$$

We note that for every $n \geq 0$, we have $\mathrm{Fil}^i \widehat{A}_{\mathrm{st}} \cap p^n \widehat{A}_{\mathrm{st}} = p^n \mathrm{Fil}^i \widehat{A}_{\mathrm{st}}$.

(2) A Frobenius φ, extending that of A_{cris} by mapping x to $(1+x)^p - 1$.

(3) A G_K-action extending the action on A_{cris}, defined as follow: for any g in G_K, $g(x) = \epsilon(g)x + \epsilon(g) - 1$.
(4) An S-algebra structure given by the monomorphism $S \hookrightarrow \widehat{A}_{\text{st}}$, $u \mapsto [\underline{\pi}](1+x)^{-1}$. It identifies S to $(\widehat{A}_{\text{st}})^{G_K}$ and it is compatible with all the others structures.
(5) A p-adically continuous connection $\nabla\colon \widehat{A}_{\text{st}} \to \widehat{A}_{\text{st}} \otimes_S \Omega^1_{\log}$, satisfying $\nabla(A_{\text{cris}}) = 0$ and $\nabla(x) = -(1+x) \otimes u^{-1}du$.

Remark that for every $0 \leq r \leq p-2$, the datum $(\widehat{A}_{\text{st}}, \mathrm{Fil}^r \widehat{A}_{\text{st}}, p^{-r}\varphi|_{\mathrm{Fil}^r \widehat{A}_{\text{st}}}, \nabla)$ belongs to $'\mathrm{Mod}(S)^r_{\log}$.

2.3.3. We denote by $T^*_{\text{st}}\colon \mathrm{Mod}(S)^r_{\log} \to \mathrm{Rep}_{\mathbb{Z}_p}(G_K)^{[0,r]}_{\text{st}}$, the contravariant functor defined as follows:
$$T^*_{\text{st}}(M) = \mathrm{Hom}_{'\mathrm{Mod}(S)^r_{\log}}\big(M, \widehat{A}_{\text{st}}\big),$$
with the G_K-action induced by that on $\widehat{A}_{\text{st}}$.

The functor $\mathbb{D}^r_{\text{free}}\colon \mathrm{Rep}_{\mathbb{Z}_p}(G_K)^{[0,r]}_{\text{st}} \to \mathrm{Rep}_{\mathbb{Z}_p}(G_K)^{[0,r]}_{\text{st}}$, sending a $\mathbb{Z}_p$-representation T to its r-twisted $\mathbb{Z}_p$-linear dual $\mathrm{Hom}_{\mathbb{Z}_p}\big(T, \mathbb{Z}_p\big)(r)$, is an anti-equivalence of categories because, by definition, the objects in $\mathrm{Rep}_{\mathbb{Z}_p}(G_K)^{[0,r]}_{\text{st}}$ are free as $\mathbb{Z}_p$-modules. We denote $T_{\text{st}}\colon \mathrm{Mod}(S)^r_{\log} \to \mathrm{Rep}_{\mathbb{Z}_p}(G_K)^{[0,r]}_{\text{st}}$ the covariant functor which is the composition of T^*_{st} with $\mathbb{D}^r_{\text{free}}$.

In [Bre02, Conj. 2.2.6-(1)] Breuil conjectured that the functor T^*_{st} is an anti-equivalence[1] of categories and this has been proved by T. Liu.

Theorem 2.3.4. [Liu08, Th.2.3.5] *For $0 \leq r \leq p-2$, the functor $M \mapsto T^*_{\text{st}}(M)$ establishes an anti-equivalence between the category of logarithmic strongly divisible lattices of weight r and the category of semi-stable $\mathbb{Z}_p$-representations of G_K with Hodge-Tate weights in $[0,r]$.*

Corollary 2.3.5. *If M is in $\mathrm{Mod}(S)^r$, then $T^*_{\text{st}}(M)$ is crystalline and the restriction T^*_{st} to $\mathrm{Mod}(S)^r$ is an equivalence of categories with $\mathrm{Rep}_{\mathbb{Z}_p}(G_K)^{[0,r]}_{\text{cris}}$.*

Proof. By a result of Breuil [Bre97] the monodromy operator of $D_{\text{st}}(T^*_{\text{st}}(M) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p)$ is the residue at zero of the connection of $M \otimes_W K_0$: the corollary follows. □

Example 2.3.6. Let $0 \leq r \leq p-2$ be an integer.
(1) The datum $(S, \mathrm{Fil}^r S, \varphi_r, d)$ is an object of $\mathrm{Mod}(S)^r$ that we denote shortly by $\mathbb{1}(r)$.
(2) The datum (S, S, φ, d) is an object of $\mathrm{Mod}(S)^r$ that we denote by $\mathbb{1}$.

We have $T_{\text{st}}(\mathbb{1}) = T^*_{\text{st}}(\mathbb{1}(r)) = \mathbb{Z}_p$ and $T_{\text{st}}(\mathbb{1}(r)) = T^*_{\text{st}}(\mathbb{1}) = \mathbb{Z}_p(r)$.

2.3.7. We denote $\widehat{A}^\infty_{\text{st}} = \widehat{A}_{\text{st}} \otimes_W (K_0/W)$ and endow it with the induced structures. It is an object of $'\mathrm{Mod}(S)^r_{\log}$ in a straightforward way. We consider the contravariant functor $T^{\infty,*}_{\text{st}}\colon \mathrm{ModFI}(S/p^nS)^r \to \mathrm{Rep}_{\mathbb{Z}/p^n\mathbb{Z}}(G_K)$ defined by
$$T^{\infty,*}_{\text{st}}(M) = \mathrm{Hom}_{'\mathrm{Mod}(S)^r_{\log}}\big(M, \widehat{A}^\infty_{\text{st}}\big).$$

[1]The notations for T^*_{st} and T_{st} are exchanged in [Liu08].

We denote $\mathbb{D}^r_{\mathrm{tor}}\colon \mathrm{Rep}_{\mathbb{Z}/p^n\mathbb{Z}}(G_K) \to \mathrm{Rep}_{\mathbb{Z}/p^n\mathbb{Z}}(G_K)$ the Pontryagin duality twisted by r, which is the functor sending T to $\mathrm{Hom}_{\mathbb{Z}_p}\big(T, \mathbb{Q}_p/\mathbb{Z}_p\big)(r)$. It is an anti-equivalence of categories and we have $\mathbb{D}^r_{\mathrm{tor}}(T) = \mathrm{Hom}_{\mathbb{Z}/p^n\mathbb{Z}}\big(T, \mathbb{Z}/p^n\mathbb{Z}\big)(r)$ because T is annihilated by p^n. We set

$$T^{\infty}_{\mathrm{st}}(M) = \mathbb{D}^r_{\mathrm{tor}}(T^{\infty,*}_{\mathrm{st}}(M)).$$

By [Bre99, 2.3.1.1–2.3.1.3] the functor $T^{\infty,*}_{\mathrm{st}}$ is exact and so the functor T^{∞}_{st} is also exact. In general the functor T^{∞}_{st} is not full, but if $er \le p-2$ it is fully faithful, cf. [Car06, Théorème 1.0.4].

For every integer $n \ge 1$, we have the following commutative diagram of functors

$$\begin{array}{ccccc} T_{\mathrm{st}}\colon \mathrm{Mod}(S)^r & \xrightarrow{T^*_{\mathrm{st}}} & \mathrm{Rep}_{\mathbb{Z}_p}(G_K)^{[0,r]}_{\mathrm{cris}} & \xrightarrow{\mathbb{D}^r_{\mathrm{free}}} & \mathrm{Rep}_{\mathbb{Z}_p}(G_K)^{[0,r]}_{\mathrm{cris}} \\ \downarrow & & \downarrow & & \downarrow \\ T^{\infty}_{\mathrm{st}}\colon \mathrm{ModFI}(S/p^nS)^r & \xrightarrow{T^{\infty,*}_{\mathrm{st}}} & \mathrm{Rep}_{\mathbb{Z}/p^n\mathbb{Z}}(G_K) & \xrightarrow{\mathbb{D}^r_{\mathrm{tor}}} & \mathrm{Rep}_{\mathbb{Z}/p^n\mathbb{Z}}(G_K) \end{array} \tag{2.3.7.1}$$

where the vertical functors are reduction modulo p^n: indeed by [CL09, 1.2.2] we have

$$0 \to T^*_{\mathrm{st}}(M) \xrightarrow{p^n} T^*_{\mathrm{st}}(M) \to T^{\infty,*}_{\mathrm{st}}(M/p^nM) \to 0, \tag{2.3.7.2}$$

which gives the commutativity of the left square; the commutativity of the right square is obvious because the representations in $\mathrm{Rep}_{\mathbb{Z}_p}(G_K)^{[0,r]}_{\mathrm{cris}}$ are free $\mathbb{Z}_p$-modules of finite type.

3. Extensions of crystalline representations

3.1. **The Fontaine-Lafaille theory.** We recall some definitions and results of Fontaine and Lafaille, see [FL82], [Wa97, §2] and [BK90, §4].

Let $0 \le r \le p-2$ be an integer.

3.1.1. Let $\mathrm{MF}^{[0,r]}_W$ be the category whose objects $(D, \mathrm{Fil}^i D, \varphi_i, i \in \mathbb{Z})$ are the following data:
(1) a W-module of finite type D;
(2) a decreasing filtration $(\mathrm{Fil}^i D)_{i\in\mathbb{Z}}$ by direct summands, satisfying $\mathrm{Fil}^{r+1}D = 0$ and $\mathrm{Fil}^0 D = D$;
(3) for all i, a σ-linear map $\varphi_i\colon \mathrm{Fil}^i D \to D$, such that $\varphi_i|_{\mathrm{Fil}^{i+1}} = p\varphi_{i+1}$ and

$$\sum_{i=0}^{r} \varphi_i(\mathrm{Fil}^i D) = D.$$

The morphisms of $\mathrm{MF}^{[0,r]}_W$ are W-linear maps preserving the filtrations and commuting with the φ_i's, for all i.

3.1.2. We say that an object D of $\mathrm{MF}^{[0,r]}_W$ is a *strongly divisible lattice* if it is free as W-module. We denote $\mathrm{MF}^{[0,r]}_{W,\mathrm{free}}$ the full subcategory of $\mathrm{MF}^{[0,r]}_W$ of strongly divisible lattices. We denote $\mathrm{MF}^{[0,r]}_{W_n}$ the full subcategory of $\mathrm{MF}^{[0,r]}_W$ whose objects are finite lenght W_n-modules.

3.1.3. The Fontaine-Lafaille theory gives a commutative diagram of functors

$$\begin{array}{ccc} \mathrm{MF}^{[0,r]}_{W,\mathrm{free}} & \underset{\sim}{\overset{T^*_{\mathrm{cris}}}{\longrightarrow}} & \mathrm{Rep}_{\mathbb{Z}_p}(G_{K_0})^{[0,r]}_{\mathrm{cris}} \\ & \underset{\sim}{\overset{D^*_{\mathrm{cris}}}{\longleftarrow}} & \\ \downarrow & & \downarrow \\ \mathrm{MF}^{[0,r]}_{W_n} & \underset{\sim}{\overset{T^{\infty,*}_{\mathrm{cris}}}{\longrightarrow}} & \mathrm{Rep}_{\mathbb{Z}/p^n\mathbb{Z}}(G_{K_0})^{[0,r]}_{\mathrm{cris}} \\ & \underset{\sim}{\overset{D^{\infty,*}_{\mathrm{cris}}}{\longleftarrow}} & \end{array}$$

where the vertical functors are reduction modulo p^n; the functors T^*_{cris} and $T^{\infty,*}_{\mathrm{cris}}$ are (exact) equivalences of categories defined by

$$T^*_{\mathrm{cris}} = \mathrm{Hom}_{W,\mathrm{Fil}^{\cdot},\varphi_{\cdot}}\big(-, A_{\mathrm{cris}}\big) \text{ and } T^{\infty,*}_{\mathrm{cris}} = \mathrm{Hom}_{W,\mathrm{Fil}^{\cdot},\varphi_{\cdot}}\big(-, A_{\mathrm{cris}}/p^n A_{\mathrm{cris}}\big);$$

and the quasi-inverses of T^*_{cris} and $T^{\infty,*}_{\mathrm{cris}}$ are given respectively by

$$D^*_{\mathrm{cris}} = \mathrm{Hom}_{\mathbb{Z}_p[G_{K_0}]}\big(-, A_{\mathrm{cris}}\big) \text{ and } D^{\infty,*}_{\mathrm{cris}} = \mathrm{Hom}_{\mathbb{Z}/p^n\mathbb{Z}[G_{K_0}]}\big(-, A_{\mathrm{cris}}/p^n A_{\mathrm{cris}}\big).$$

We will need also covariant versions of these equivalences, defined by composing with $\mathbb{D}^r_{\mathrm{free}}$ or $\mathbb{D}^r_{\mathrm{tor}}$ respectively:

$$\begin{aligned} T_{\mathrm{cris}} &:= \mathbb{D}^r_{\mathrm{free}} \circ T^*_{\mathrm{cris}} \colon D \mapsto \mathrm{Fil}^r(A_{\mathrm{cris}} \otimes_W D)^{\varphi_r = \mathrm{Id}}, \\ T^{\infty}_{\mathrm{cris}} &:= \mathbb{D}^r_{\mathrm{tor}} \circ T^{\infty,*}_{\mathrm{cris}} \colon D \mapsto \mathrm{Fil}^r(A_{\mathrm{cris}}/p^n A_{\mathrm{cris}} \otimes_{W_n} D)^{\varphi_r = \mathrm{Id}}, \\ D_{\mathrm{cris}} &:= D^*_{\mathrm{cris}} \circ \mathbb{D}^r_{\mathrm{free}} \colon T \mapsto (A_{\mathrm{cris}} \otimes_W T(-r))^{G_{K_0}}, \\ D^{\infty}_{\mathrm{cris}} &:= D^{\infty,*}_{\mathrm{cris}} \circ \mathbb{D}^r_{\mathrm{tor}} \colon T \mapsto (A_{\mathrm{cris}}/p^n A_{\mathrm{cris}} \otimes_{W_n} T(-r))^{G_{K_0}}. \end{aligned}$$

These covariant equivalences are compatible with reduction modulo p^n as in the diagram above for contravariant ones. In particular we will need the following: let T be in $\mathrm{Rep}_{\mathbb{Z}_p}(G_{K_0})^{[0,r]}_{\mathrm{cris}}$. To the short exact sequence $0 \to T \overset{p^n\cdot}{\to} T \to T/p^nT \to 0$ we associate the sequence of filtered modules

$$0 \to D_{\mathrm{cris}}(T) \overset{p^n\cdot}{\to} D_{\mathrm{cris}}(T) \to D^{\infty}_{\mathrm{cris}}(T/p^nT) \to 0 \qquad (3.1.3.1)$$

which is exact in the sense that, for every $i \in \mathbb{Z}$, the induced sequence

$$0 \to \mathrm{Fil}^i D_{\mathrm{cris}}(T) \overset{p^n\cdot}{\to} \mathrm{Fil}^i D_{\mathrm{cris}}(T) \to \mathrm{Fil}^i D^{\infty}_{\mathrm{cris}}(T/p^nT) \to 0$$

is a short exact sequence of W-modules, cf. [Wa97, 2.2.3.1].

Remark 3.1.4. The covariant functors D_{cris} and $D^{\infty}_{\mathrm{cris}}$ depend on the choice of r, unlike their controvariant variants D^*_{cris} and $D^{\infty,*}_{\mathrm{cris}}$. When we need to be precise we shall denoted them by $D^{[0,r]}_{\mathrm{cris}} := D^*_{\mathrm{cris}} \circ \mathbb{D}^r_{\mathrm{free}}$ and $D^{\infty,[0,r]}_{\mathrm{cris}} := D^*_{\mathrm{cris}} \circ \mathbb{D}^r_{\mathrm{tor}}$. For any integers $0 \le r_1 \le r_2 \le p-2$, and any representation $T \in \mathrm{Rep}_{\mathbb{Z}_p}(G_{K_0})^{[0,r_1]}_{\mathrm{crys}}$, the strongly divisible lattices $D^{[0,r_1]}_{\mathrm{cris}}(T)$ and $D^{[0,r_2]}_{\mathrm{cris}}(T)$ have the same underlying W-module but have filtrations (and Frobenii) shifted as follow: for all $i \in \mathbb{Z}$,

$$\mathrm{Fil}^i(D^{[0,r_2]}_{\mathrm{cris}}(T)) = \mathrm{Fil}^{i+(r_1-r_2)}(D^{[0,r_1]}_{\mathrm{cris}}(T)) \quad \text{and} \quad \varphi^{[0,r_2]}_i = \varphi^{[0,r_1]}_{i+(r_1-r_2)}.$$

In particular the breaks of the filtration of $D_{\mathrm{cris}}^{[0,r_2]}(T)$ are between r_2-r_1 and r_2. The same formula is true for $D_{\mathrm{cris}}^{\infty,[0,r]}$.

3.2. **Relation with strongly divisible lattices over S.**

3.2.1. Let $(D, \mathrm{Fil}^i D, \varphi_i, i \in \mathbb{Z})$ be an object of $\mathrm{MF}_W^{[0,r]}$. We define an object

$$M(D) = (D \otimes_W S, \mathrm{Fil}^r M(D), \varphi_r, \nabla)$$

of $'\mathrm{Mod}(S)^r$ as follow:

$$\mathrm{Fil}^r M(D) = \sum_{i=0}^{r} \mathrm{Fil}^i D \otimes_W \mathrm{Fil}^{r-i} S \subset D \otimes_W S, \tag{3.2.1.1}$$

$\varphi_r = \sum_{i=0}^r \varphi_i \otimes_W \varphi_{r-i}$ and $\nabla = \mathrm{Id}_D \otimes d$, i.e. $\nabla(m \otimes_W \alpha) = m \otimes_W d(\alpha)$. By using the fact that S belongs to $'\mathrm{Mod}(S)^r$, it is a straightforward exercise to verify that $M(D)$ belongs to $'\mathrm{Mod}(S)^r$; it is also clear that $D \mapsto M(D)$ is functorial.

By the definitions of the categories $\mathrm{MF}_W^{[0,r]}$, $\mathrm{MF}_{W_n}^{[0,r]}$, $\mathrm{Mod}(S)^r$ and $\mathrm{ModFI}(S/p^nS)^r$, we can prove that:

- if D is free over W, then $M(D)$ belongs to $\mathrm{Mod}(S)^r$;
- if D belongs to $\mathrm{MF}_{W_n}^{[0,r]}$, then $M(D)$ is belongs to $\mathrm{ModFI}(S/p^nS)^r$;
- we have $M(D)/p^nM(D) = M(D/p^nD)$.

The only non trivial property to check is that $\varphi_r(\mathrm{Fil}^r M(D))$ generates $M(D)$ as an S-module. Using (3.2.1.1), it is enough to check that every element of $M(D) = D \otimes_W S$ of the form $v \otimes_W 1$ is in the image of $\sum_{i=0}^r \varphi_i \otimes_W \varphi_{r-i}$. By hypothesis $v = \sum_{i=0}^r \varphi_i(f_i)$, for some $f_i \in \mathrm{Fil}^i D$. Put $m_i = f_i \otimes_W E(u)^{r-i} \in \mathrm{Fil}^i D \otimes_W \mathrm{Fil}^{r-i} S$. Since $\varphi_{r-i}(E(u)^{r-i}) = c_1^{r-i}$ in S we have

$$\sum_{i=0}^{r} \varphi_r(m_i) c_1^{-(r-i)} = \sum_{i=0}^{r} \varphi_i(f_i) \otimes_W \varphi_{r-i}(E(u)^{r-i}) c_1^{-(r-i)} = \sum_{i=0}^{r} \varphi_i(f_i) \otimes_W 1 = v \otimes_W 1.$$

Proposition 3.2.2. (1) *Let D be in $\mathrm{MF}_{W,\mathrm{free}}^{[0,r]}$. We have a natural isomorphism of crystalline $\mathbb{Z}_p$-representations of G_K*

$$T_{\mathrm{st}}(M(D)) \cong (T_{\mathrm{cris}}(D))_{|G_K}.$$

(2) *Let D be in $\mathrm{MF}_{W_n}^{[0,r]}$. We have a natural isomorphism of crystalline $\mathbb{Z}/p^n\mathbb{Z}$-representations of G_K*

$$T_{\mathrm{st}}^{\infty}(M(D)) \cong (T_{\mathrm{cris}}^{\infty}(D))_{|G_K}.$$

Proof. The proofs of (1) and (2) are analogue: let us prove (2) and leave (1) to the reader. By composing with the dual functor $\mathbb{D}_{\mathrm{tor}}^r = \mathrm{Hom}_{\mathbb{Z}_p}\big(-, \mathbb{Z}/p^n\mathbb{Z}\big)(r)$ we are reduced to prove the analogous statements on contravariant functors. Since $A_{\mathrm{cris}}/p^nA_{\mathrm{cris}} = (\widehat{A}_{\mathrm{st}}/p^n\widehat{A}_{\mathrm{st}})^{\nabla=0}$, cf. [Bre98, Lemme 3.1.2.3], we have

$$\begin{aligned} T^{\infty,*}_{\mathrm{cris}}(D) &= \mathrm{Hom}_{W,\mathrm{Fil}^\cdot,\varphi_\cdot}\big(D, A_{\mathrm{cris}}/p^n A_{\mathrm{cris}}\big) \\ &= \mathrm{Hom}_{W,\mathrm{Fil}^\cdot,\varphi_\cdot}\big(D, (\widehat{A}_{\mathrm{st}}/p^n \widehat{A}_{\mathrm{st}})^{\nabla=0}\big) \\ &= \mathrm{Hom}_{W,\mathrm{Fil}^\cdot,\varphi_\cdot,\nabla}\big(D, \widehat{A}_{\mathrm{st}}/p^n \widehat{A}_{\mathrm{st}}\big) \\ &= \mathrm{Hom}_{S,\mathrm{Fil}^\cdot,\varphi_\cdot,\nabla}\big(D\otimes_W S, \widehat{A}_{\mathrm{st}}/p^n \widehat{A}_{\mathrm{st}}\big) \\ &\subseteq \mathrm{Hom}_{'\mathrm{Mod}(S)^r_{\log}}\big(D\otimes_W S, \widehat{A}_{\mathrm{st}}/p^n \widehat{A}_{\mathrm{st}}\big) = T^*_{\mathrm{st}}(M(D))\,. \end{aligned}$$

Let us prove that the inclusion in the last line is an equality. Let $f\colon D\otimes_W S\to \widehat{A}_{\mathrm{st}}/p^n\widehat{A}_{\mathrm{st}}$ be in $T^*_{\mathrm{st}}(M(D))$. *A priori* we have $f(\mathrm{Fil}^i M(D))\subseteq \mathrm{Fil}^i(\widehat{A}_{\mathrm{st}}/p^n\widehat{A}_{\mathrm{st}})$ and $f\varphi_i=\varphi_i f$ only for the last step of the filtration: $i=r$; let us prove this is true for any i. Let d be in $\mathrm{Fil}^i D$, for $i\in[0,r]$. For every $\alpha\in\mathrm{Fil}^{r-i}S$ and $d\in D$, we have $\alpha f(d\otimes 1)=f(d\otimes\alpha)\in \mathrm{Fil}^r(\widehat{A}_{\mathrm{st}}/p^n\widehat{A}_{\mathrm{st}})=\mathrm{Fil}^r\widehat{A}_{\mathrm{st}}/p^n\mathrm{Fil}^r\widehat{A}_{\mathrm{st}}$. In particular for $\alpha=E(u)^{r-i}$, we get

$$E([\underline{\pi}](1+x)^{-1})^{r-i}f(d\otimes 1)\in \mathrm{Fil}^r\widehat{A}_{\mathrm{st}}/p^n\mathrm{Fil}^r\widehat{A}_{\mathrm{st}}.$$

We can re-write it as $E([\underline{\pi}])^{r-i}f(d\otimes 1)+x\beta$, with $\beta\in\widehat{A}_{\mathrm{st}}$. Since $f(d\otimes 1)$ belongs to $A_{\mathrm{cris}}/p^nA_{\mathrm{cris}}$, we get $E([\underline{\pi}])^{r-i}f(d\otimes 1)\in\mathrm{Fil}^r(A_{\mathrm{cris}}/p^nA_{\mathrm{cris}})$. Considering that $E([\pi])$ is a generator of $\ker(\theta\colon \mathrm{W}(R)\to\mathcal{O}_{\mathbb{C}_p})$, we conclude that $f(d\otimes 1)$ belongs to $\mathrm{Fil}^i(A_{\mathrm{cris}}/p^nA_{\mathrm{cris}})$.

Let us check the commutation of f with φ_i, for $i\in[0,r]$. For every $\alpha\in\mathrm{Fil}^{r-i}S$ and $d\in\mathrm{Fil}^i D$, we have

$$\begin{aligned} \varphi_{r-i}(\alpha)f(\varphi_i(d)\otimes 1) &= f(\varphi_i(d)\otimes\varphi_{r-i}(\alpha)) = f(\varphi_r(d\otimes\alpha)) = \varphi_r(f(d\otimes\alpha)) \\ &= \varphi_r(\alpha f(d\otimes 1)) = \varphi(\alpha)\varphi_r(f(d\otimes 1)) = \varphi_{r-i}(\alpha)\varphi_i(f(d\otimes 1)). \end{aligned}$$

By taking $\alpha=E(u)^{r-i}$, we get $\varphi_{r-i}(\alpha)=c_1^{r-i}$, which is invertible; thus we have, for all $i\in[0,r]$ and $d\in\mathrm{Fil}^iD$, $f(\varphi_i(d\otimes 1))=\varphi_i(f(d\otimes 1))$. This finishes the proof. □

3.3. **A double complex computing extensions.** The goal of this subsection is to compute the groups of extensions of strongly divisible modules over S by using explicit complexes.

3.3.1. For $(M,\mathrm{Fil}^rM,\varphi_r,\nabla)$ in $'\mathrm{Mod}(S)^r$ and $0\le r\le p-2$ we set $\mathrm{Fil}^{r-1}M=\{m\in M\,|\,E(u)m\in\mathrm{Fil}^rM\}$ and define $\varphi_{r-1}\colon\mathrm{Fil}^{r-1}M\to M$ by putting, for $m\in\mathrm{Fil}^{r-1}M$,

$$\varphi_{r-1}(m)=c_1^{-1}\varphi_r(E(u)m),$$

where $c_1=\varphi_1(E(u))$, cf. (2.1.2). Analogously, we define $\varphi\colon M\to M$ by $\varphi(m)=c_1^{-r}\varphi_r(E(u)^rm)$.

We will need later the following.

Lemma 3.3.2. (1) *Let $f\in S$ and $j\ge 1$ be an integer. If $E(u)f\in\mathrm{Fil}^jS$ then $f\in\mathrm{Fil}^{j-1}S$.*

(2) *Let $f\in S/p^nS$ and $1\le j\le p-1$ be an integer. If $E(u)f\in\mathrm{Fil}^j(S/p^nS)$, then $f\in\mathrm{Fil}^{j-1}(S/p^nS)$.*

(3) *Let D be either in $\mathrm{MF}^{[0,r]}_{W,\mathrm{free}}$ or in $\mathrm{MF}^{[0,r]}_{W_n}$. Set $M = M(D)$. If $r = 0$ we have $\mathrm{Fil}^{-1}M = M$ and $\varphi_{-1} = p\varphi$; if $r \geq 1$ we have*

$$\mathrm{Fil}^{r-1}M = \sum_{i=0}^{r-1} \mathrm{Fil}^i D \otimes_W \mathrm{Fil}^{r-1-i} S \subset D \otimes_W S$$

and $\varphi_{r-1} = \sum_{i=0}^{r-1} \varphi_i^D \otimes \varphi_{r-1-i} \colon \mathrm{Fil}^{r-1}M \to M$.

(4) *Let $0 \to D_1 \to D_2 \to D_3 \to 0$ be a short exact sequence in $\mathrm{MF}^{[0,r]}_W$. Then the sequence $0 \to \mathrm{Fil}^{r-1}M(D_1) \to \mathrm{Fil}^{r-1}M(D_2) \to \mathrm{Fil}^{r-1}M(D_3) \to 0$ induced by functoriality is exact.*

Proof. Let us prove (1) and (2). Every element $f \in S$ (resp. S/p^n) can be written as $f = \sum_{i\geq 0} a_i(u)E(u)^{[i]}$, where $E(u)^{[i]}$ denotes the divided power of $E(u)$ (*i.e.* $i!E(u)^{[i]} = E(u)^i$) and $a_i(u)$ are polynomials in $W[u]$ of degree smaller than e, converging to zero (for i going to infinity). We have $E(u)f = \sum_{i\geq 0} a_i(u)(i+1)E(u)^{[i+1]}$. If $E(u)f$ belongs to $\mathrm{Fil}^j S$ (resp. $\mathrm{Fil}^j(S/p^nS)$) then $a_i(u)(i+1) = 0$ for $i \leq j-1$, which implies $a_i(u) = 0$ (when $f \in S/p^nS$, $i+1 \leq j \leq p-1$ by hypothesis, so that $i+1$ is invertible).

Let us prove (3). The statement for $r = 0$ is clear so let us assume $r \geq 1$. Let assume first D in $\mathrm{MF}^{[0,r]}_{W,\mathrm{free}}$. We recall that D is filtered free, and let $(e_i)_{1\leq i\leq d}$ be a base adapted to the filtration; *i.e.* for every $i \in [0,r]$, $(e_j)_{1\leq j\leq \mathrm{rk}\,\mathrm{Fil}^i D}$ is a base of $\mathrm{Fil}^i D$ (cf. [Wa97, 2.2.2]). Put

$$^*\mathrm{Fil}^{r-1}M = \sum_{i=0}^{r-1} \mathrm{Fil}^i D \otimes_W \mathrm{Fil}^{r-1-i} S \subseteq D \otimes_W S.$$

The inclusion $^*\mathrm{Fil}^{r-1}M \subseteq \mathrm{Fil}^{r-1}M$ is obvious so let us prove the converse. Let $x \in \mathrm{Fil}^{r-1}M$, we may assume $x = e_s \otimes f$, for some $f \in S$ and $1 \leq s \leq d$. By hypothesis we have $E(u)x = e_s \otimes E(u)f \in \mathrm{Fil}^r(M) = \sum_{i=0}^r \mathrm{Fil}^i D \otimes_W \mathrm{Fil}^{r-i} S \subseteq D \otimes_W S$. We can write

$$\begin{aligned} E(u)x = e_s \otimes_W E(u)f &= \sum_{i=0}^{r} m_i \otimes_W g_i \qquad (m_i \in \mathrm{Fil}^i D,\ g_i \in \mathrm{Fil}^{r-i} S) \\ &= \sum_{i=0}^{r} \sum_{j=1}^{\mathrm{rk}\,\mathrm{Fil}^i D} (e_j \otimes_W h_{j,i}) \qquad (h_{j,i} \in \mathrm{Fil}^{r-i} S) \\ &= \sum_{j=1}^{d} e_j \quad \otimes_W \sum_{\substack{0\leq i\leq r \\ \mathrm{rk}\,\mathrm{Fil}^i D \geq j}} h_{j,i}. \end{aligned}$$

Therefore $E(u)f = \sum_i h_{s,i}$, where the sum is taken over all $i \in [0,r]$ such that $\mathrm{rk}\,\mathrm{Fil}^i D \geq s$. If we denote by m the maximum of the integers i such that $\mathrm{rk}\,\mathrm{Fil}^i D \geq s$ (or equivalently $e_s \in \mathrm{Fil}^i D$), then $E(u)f$ belongs to $\mathrm{Fil}^{r-m} S$. By (1) we get $f \in \mathrm{Fil}^{r-m-1} S$ and $x = e_s \otimes f \in \mathrm{Fil}^m D \otimes \mathrm{Fil}^{r-m-1} S \subseteq {}^*\mathrm{Fil}^{r-1}M$.

To finish the prove of (3) for $D \in \mathrm{MF}^{[0,r]}_{W,\mathrm{free}}$, let us compute the Frobenius:

$$\varphi_{r-1}(x) = c_1^{-1}\varphi_r(E(u)x) = c_1^{-1}\varphi_r(e_s \otimes E(u)f) = \\ c_1^{-1}\varphi_m(e_l) \otimes \varphi_{r-m}(E(u)f) = \varphi_m(e_s) \otimes \varphi_{r-1-m}(f).$$

The proof of (3) for $D \in \mathrm{MF}^{[0,r]}_{W_n}$ is similar because D admits a lifting $\tilde{D} \in \mathrm{MF}^{[0,r]}_{W,\mathrm{free}}$: so $M = M(D) = \tilde{D} \otimes_W S/p^nS$ and we use (2) instead of (1). Finally (4) follows from (3) and the fact that for all i,

$$0 \to \mathrm{Fil}^i D_1 \to \mathrm{Fil}^i D_2 \to \mathrm{Fil}^i D_3 \to 0$$

is exact (cf. [Wa97, 2.2.3.1]). □

Definition 3.3.3. For $(M, \mathrm{Fil}^r M, \varphi_r, \nabla)$ in $'\mathrm{Mod}(S)^r$, we denote $\mathrm{Fil}^r C^{\bullet,\bullet}(M)$ the double complex of $\mathbb{Z}_p$-modules

$\mathrm{Fil}^r C^{\bullet,\bullet}(M)$:

$$\begin{array}{ccccccc} & & 0 & & 0 & & \\ & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \mathrm{Fil}^r M & \xrightarrow{\nabla} & \mathrm{Fil}^{r-1} M \otimes_S \Omega^1 & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \varphi_r - \mathrm{Id}} & & \downarrow{\scriptstyle (\varphi_{r-1}\otimes\varphi_1)-\mathrm{Id}} & & \\ 0 & \longrightarrow & M & \xrightarrow{\nabla} & M \otimes_S \Omega^1 & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \\ & & 0 & & 0 & & \end{array}$$

where the module $\mathrm{Fil}^r M$ is the $(0,0)$-entry of the complex, and the square is commutative by (2.2.1)-(4) and the definition of φ_{r-1}, cf. (3.3.1). The simple complex attached to $\mathrm{Fil}^r C^{\bullet,\bullet}(M)$ is

$\mathrm{Fil}^r C^{\bullet}(M)$: $\quad 0 \to \mathrm{Fil}^r M \xrightarrow{\gamma^0} M \oplus (\mathrm{Fil}^{r-1} M \otimes_S \Omega^1) \xrightarrow{\gamma^1} M \otimes_S \Omega^1 \to 0,$

where $\gamma^0(m) = (\varphi_r(m) - m, \nabla_M(m))$ and for $a \in M$, $\omega \in \mathrm{Fil}^{r-1} M \otimes_S \Omega^1$,

$$\gamma^1(a, \omega) = \nabla_M(a) + \omega - \varphi_{r-1} \otimes \varphi_1(\omega).$$

By construction $H^i(\mathrm{Fil}^r C^{\bullet}(M)) = 0$ for $i \neq 0, 1, 2$ and

$$H^0(\mathrm{Fil}^r C^{\bullet}(M)) \cong \mathrm{Hom}_{'\mathrm{Mod}(S)^r}(\mathbb{1}, M).$$

Proposition 3.3.4. *We have the following functorial isomorphisms:*

(1) *if M is in $\mathrm{ModFI}(S/p^nS)^r$, then $H^1(\mathrm{Fil}^r C^{\bullet}(M)) \cong \mathrm{Ext}^1_{\mathrm{ModFI}(S/p^nS)^r}(\mathbb{1}/p^n\mathbb{1}, M)$;*

(2) *if M is in $\mathrm{Mod}(S)^r$, then $H^1(\mathrm{Fil}^r C^{\bullet}(M)) \cong \mathrm{Ext}^1_{\mathrm{Mod}(S)^r}(\mathbb{1}, M)$;*

where Ext^1 denotes the group of Yoneda extensions.

Proof. The proofs of (1) and (2) are analogous. We will prove them by computing explicitly the group of Yoneda extensions and showing that it coincides with $H^1(\mathrm{Fil}^r C^{\bullet}(M))$. An alternative approach would have been to work in the bigger category $'\mathrm{Mod}(S)^r$ and prove that $H^1(\mathrm{Fil}^r C^\bullet(-))$ is an *effaçable* functor. The problem is that the category $'\mathrm{Mod}(S)^r$ is not abelian in general and we do not know if there are enough injectives.

The proof takes the rest of this section. Let us start by fixing a convention in order to treat cases (1) and (2) at the same time: for $n \in \mathbb{N} \cup \{\infty\}$, we set

$$\mathbb{1}_n = \begin{cases} \mathbb{1}/p^n\mathbb{1} & \text{if } n \in \mathbb{N}; \\ \mathbb{1} & \text{if } n = \infty; \end{cases} \qquad S_n = \begin{cases} S/p^nS & \text{if } n \in \mathbb{N}; \\ S & \text{if } n = \infty; \end{cases}$$

$$\mathrm{Mod}_n = \begin{cases} \mathrm{ModFI}(S/p^nS)^r & \text{if } n \in \mathbb{N}; \\ \mathrm{Mod}(S)^r & \text{if } n = \infty. \end{cases}$$

Let M be in Mod_n, for some $n \in \mathbb{N} \cup \{\infty\}$, we denote respectively by φ_r^M and ∇_M the Frobenius and the connection of M.

Let $[X]$ be a class in $\mathrm{Ext}^1_{\mathrm{Mod}_n}(\mathbb{1}_n, M)$, represented by a couple of extensions

$$\begin{array}{ccccccccc} 0 & \longrightarrow & M & \longrightarrow & X & \longrightarrow & S_n & \longrightarrow & 0\,; \\ & & \uparrow & & \uparrow & & \| & & \\ 0 & \longrightarrow & \mathrm{Fil}^r M & \longrightarrow & \mathrm{Fil}^r X & \longrightarrow & S_n & \longrightarrow & 0. \end{array}$$

These extensions clearly split as extensions of S_n-modules (indeed if $n \in \mathbb{N}$, $p^nX = 0$ by hypothesis); therefore we can write

$$X = (M \oplus S_n, \mathrm{Fil}^r M \oplus S_n, \varphi_r^X, \nabla_X).$$

Let us describe more precisely the Frobenius $\varphi_r^X : \mathrm{Fil}^r M_n \oplus S_n \to M \oplus S_n$ and the regular connection $\nabla_X : M \oplus S_n \to (M \oplus S_n) \otimes_{S_n} S_n du$.

We have $\varphi_r^X(0,1) = (a,1)$, for a unique $a \in M$. For all $(m,s) \in \mathrm{Fil}^r M \oplus S_n$, we get

$$\begin{aligned} \varphi_r^X(m,s) &= (\varphi_r^M(m),0) + \varphi(s)\varphi_r^X(0,1) = (\varphi_r^M(m),0) + \varphi(s)(a,1) \\ &= (\varphi_r^M(m) + \varphi(s)a, \varphi(s)). \end{aligned} \tag{3.3.4.1}$$

We have $\nabla_X(0,1) = (g,0) \otimes du$, for a unique $g \in M$. For all $s \in S_n$, we get

$$\begin{aligned} \nabla_X(0,s) = s\nabla_X(0,1) + (0,1) \otimes ds &= s(g,0) \otimes du + (0, \tfrac{ds}{du}) \otimes du \\ &= (sg, \tfrac{ds}{du}) \otimes du\,; \end{aligned}$$

thus, for all $(m,s) \in M \oplus S_n$, we have

$$\nabla_X(m,s) = (\nabla_M(\tfrac{d}{du})(m) + sg, \tfrac{ds}{du}) \otimes du.$$

The elements a, g satisfy some compatibility conditions corresponding to those satisfied by φ_r^X and ∇_X (cf. 2.2.1, 2.2.3 and 2.2.4): let us write them explicitly.

Condition 2.2.1-(3): for all α in $\mathrm{Fil}^r S$ and x in X we have $\varphi_r^X(\alpha x) = c_1^{-r}\varphi_r(\alpha)\varphi_r^X(E(u)^r x)$. For $x = (m,s)$, by using (3.3.4.1), we get

$$\begin{aligned} &c_1^{-r}\varphi_r(\alpha)\varphi_r^X(E(u)^r x) = \\ &c_1^{-r}\varphi_r(\alpha) \cdot (\varphi_r^M(E(u)^r m) + \varphi(E(u)^r s)a,\ \varphi(E(u)^r s)) = \\ &(c_1^{-r}\varphi_r(\alpha)\varphi_r^M(E(u)^r m) + c_1^{-r}\varphi_r(\alpha)\varphi(E(u)^r s)a,\ c_1^{-r}\varphi_r(\alpha)\varphi(E(u)^r s)). \end{aligned}$$

By using $\varphi(E(u)^r) = \varphi(E(u))^r = (p\varphi_1(E(u)))^r = p^r c_1^r$, this is equal to

$$\begin{aligned}&(\varphi_r^M(\alpha m) + c_1^{-r}\varphi_r(\alpha)p^r c_1^r \varphi(s)a,\, \varphi(\alpha s)) =\\&(\varphi_r^M(\alpha m) + \varphi_r(\alpha)p^r \varphi(s)a,\, \varphi(\alpha s)) =\\&(\varphi_r^M(\alpha m) + \varphi(\alpha s)a, \varphi(\alpha s)) = \varphi_r^X(\alpha x),\end{aligned}$$

which gives no-condition on (a, g).

Condition 2.2.1-(4): the regular connection $\nabla_X \colon X \to X \otimes_S \Omega^1$ satisfies the Griffith transversality

$$E(u)\nabla_X(\mathrm{Fil}^r X) \subseteq \mathrm{Fil}^r X \otimes_S \Omega^1$$

and the following diagram commutes.

$$\begin{array}{ccc}\mathrm{Fil}^r X & \xrightarrow{\ \varphi_r^X\ } & X\\ {\scriptstyle E(u)\nabla_X}\downarrow & & \downarrow{\scriptstyle c_1\nabla_X}\\ \mathrm{Fil}^r X \otimes_S \Omega^1 & \xrightarrow{\ \varphi_r^X\otimes\varphi_1\ } & X \otimes_S \Omega^1\end{array}$$

The first condition explicitly gives $E(u)\nabla_X(0,1) = (E(u)g, 0) \otimes du \in \mathrm{Fil}^r X \otimes \Omega^1$, which is

$$g \in \mathrm{Fil}^{r-1} M, \tag{3.3.4.2}$$

cf. 3.3.1 for the definition of $\mathrm{Fil}^{r-1}M$.

For the second one let us compute $\varphi_r^X \otimes \varphi_1(E(u)\nabla_X(0,1)) = \varphi_r^X \otimes \varphi_1((E(u)g, 0) \otimes du) = (\varphi_r^M(E(u)g), 0) \otimes \varphi_1(du)$ and $c_1\nabla_X(\varphi_r^X(0,1)) = c_1\nabla_X(a, 1) = c_1(\nabla_M(a) + g \otimes du, 0)$. Thus the condition is

$$\nabla_M(a) = c_1^{-1}\varphi_r^M(E(u)g) \otimes \varphi_1(du) - g \otimes du$$

or equivalently

$$\nabla_M(a) = (\varphi_{r-1}^M \otimes \varphi_1)(g \otimes du) - g \otimes du. \tag{3.3.4.3}$$

The last requirement is that the image of φ_r^X generates X, which is always fulfilled because φ_r^M generates M and $\varphi_r^X(0,1) = (a, 1)$.

We have constructed a surjective map $Z^1(\mathrm{Fil}^r C^\bullet(M)) \to \mathrm{Ext}^1_{\mathrm{Mod}_n}(\mathbb{1}_n, M)$, $(a, g \otimes du) \mapsto [X]$. The extension X is split if and only if there exists $m \in \mathrm{Fil}^r M$, such that $m - \varphi_r(m) = a$ and $\nabla_M(-m) = g \otimes du$. Indeed, the extension

$$0 \to M \to X \to \mathbb{1}_n \to 0$$

is split if and only there is a section $\mathfrak{s} \colon \mathbb{1}_n \to X$. Giving such a section is equivalent to give $\mathfrak{s}(1) = (m, 1)$, with m satisfying the conditions above. Therefore, the map $(a, g) \mapsto [X]$ induces a bijection of set

$$H^1(\mathrm{Fil}^r C^\bullet(M)) \xrightarrow{\sim} \mathrm{Ext}^1_{\mathrm{Mod}_n}(\mathbb{1}/p^n\mathbb{1}, M).$$

To finish we have to prove that this map is a homorphism of groups. Let $[X^{(1)}]$ and $[X^{(2)}]$ be extensions associated to couples $(a^{(1)}, g^{(1)})$ and $(a^{(2)}, g^{(2)})$ respectively. We want to show that the element $(a^{(1)} + a^{(2)}, g^{(1)} + g^{(2)})$ corresponds to an extension representing the Baer sum $[X^{(1)}] + [X^{(2)}]$. This verification is straightforward: it works essentially because φ^M and ∇_M are additive. We leave the details to the reader. □

Question 3.3.5. *Are there functorial isomorphisms:*

(1) *for M in* $\mathrm{ModFI}(S/p^nS)^r$, $H^2(\mathrm{Fil}^r C^\bullet(M)) \cong \mathrm{Ext}^2_{\mathrm{ModFI}(S/p^nS)^r}(\mathbb{1}/p^n\mathbb{1}, M)$*?*
(2) *for M in* $\mathrm{Mod}(S)^r$, $H^2(\mathrm{Fil}^r C^\bullet(M)) \cong \mathrm{Ext}^2_{\mathrm{Mod}(S)^r}(\mathbb{1}, M)$ *?*

4. Proof of the main result

4.1. Construction of $H^1_r(K,-)$.

Lemma 4.1.1. *Let T be a crystalline $\mathbb{Z}_p$-representation of G_K. The inclusion $H^1_{\mathrm{fin}}(K,T) \subseteq H^1(K,T)$ induces, for every integer $n \geq 1$, an injection*

$$H^1_{\mathrm{fin}}(K,T)/p^n H^1_{\mathrm{fin}}(K,T) \hookrightarrow H^1(K,T/p^nT). \tag{4.1.1.1}$$

Proof. The map $H^1_{\mathrm{fin}}(K,T)/p^n H^1_{\mathrm{fin}}(K,T) \to H^1(K,T/p^nT)$ factors through

$$H^1_{\mathrm{fin}}(K,T)/p^n H^1_{\mathrm{fin}}(K,T) \to H^1(K,T)/p^n H^1(K,T) \to H^1(K,T/p^nT).$$

The second morphism is injective since it is part of the long exact sequence associated to $0 \to T \xrightarrow{p^n} T \to T/p^nT \to 0$.

To show that the first one is injective we have to prove that $H^1_{\mathrm{fin}}(K,T) \cap p^n H^1(K,T) \subset p^n H^1_{\mathrm{fin}}(K,T)$. Let h be a class of a cocycle in $H^1_{\mathrm{fin}}(K,T) \cap p^n H^1(K,T)$. By definition the image of h in $H^1(K, T \otimes_{\mathbb{Z}_p} \mathrm{B}_{\mathrm{cris}})$ is trivial and since $H^1(K, T \otimes_{\mathbb{Z}_p} \mathrm{B}_{\mathrm{cris}})$ is a $\mathbb{Q}_p$-vector space the same is true for the image of h/p^n. This exactly means that h belongs to $p^n H^1_{\mathrm{fin}}(K,T)$. □

Definition 4.1.2. For every integers $n \geq 1$, $i \in \{0,1,2\}$ and $0 \leq r \leq p-2$ we define functors

$$H^i_r(K,-)\colon \mathrm{Rep}_{\mathbb{Z}_p}(G_{K_0})^{[0,r]}_{\mathrm{crys}} \to \mathrm{Mod}(\mathbb{Z}_p), \quad H^i_r(K,-)\colon \mathrm{Rep}_{\mathbb{Z}/p^n\mathbb{Z}}(G_{K_0})^{[0,r]}_{\mathrm{crys}} \to \mathrm{Mod}(\mathbb{Z}/p^n\mathbb{Z}),$$

by putting, for every T in $\mathrm{Rep}_{\mathbb{Z}_p}(G_{K_0})^{[0,r]}_{\mathrm{crys}}$ and $\overline{T}$ in $\mathrm{Rep}_{\mathbb{Z}/p^n\mathbb{Z}}(G_{K_0})^{[0,r]}_{\mathrm{crys}}$,

$$H^i_r(K,T) := H^i(\mathrm{Fil}^r C^\bullet(M(D_{\mathrm{cris}}(T)))), \quad H^i_r(K,\overline{T}) := H^i(\mathrm{Fil}^r C^\bullet(M(D^\infty_{\mathrm{cris}}(\overline{T})))),$$

where $M(-)$, $D_{\mathrm{cris}}(-)$ and $D^\infty_{\mathrm{cris}}(-)$ are the functors defined in 3.1.3 and 3.2.1, and $\mathrm{Mod}(\mathbb{Z}_p)$ denotes the category of $\mathbb{Z}_p$-modules.

Remark 4.1.3. For every $r_{\max} \leq r \leq p-2$ we have $H^i_r(K,T) = H^i_{r_{\max}}(K,T)$ and $H^i_r(K,\overline{T}) = H^i_{r_{\max}}(K,\overline{T})$. Indeed by Remark 3.1.4, we have

$$\begin{aligned}
\mathrm{Fil}^r M(D^{[0,r]}_{\mathrm{cris}}(T)) &= \mathrm{Fil}^r(D^{[0,r]}_{\mathrm{cris}}(T) \otimes_W S) \\
&= \sum_{j=0}^{r} \mathrm{Fil}^j(D^{[0,r]}_{\mathrm{cris}}(T)) \otimes_W \mathrm{Fil}^{r-j} S = \\
&= \sum_{j=0}^{r} \mathrm{Fil}^{j+r_{\max}-r}(D^{[0,r_{\max}]}_{\mathrm{cris}}(T)) \otimes_W \mathrm{Fil}^{r-j} S = \\
&= \sum_{j=r_{\max}-r}^{r_{\max}} \mathrm{Fil}^j(D^{[0,r_{\max}]}_{\mathrm{cris}}(T)) \otimes_W \mathrm{Fil}^{r_{\max}-j} S = \\
&= \sum_{j=0}^{r_{\max}} \mathrm{Fil}^j(D^{[0,r_{\max}]}_{\mathrm{cris}}(T)) \otimes_W \mathrm{Fil}^{r_{\max}-j} S = \\
&= \mathrm{Fil}^{r_{\max}} M(D^{[0,r_{\max}]}_{\mathrm{cris}}(T)),
\end{aligned}$$

and so $H^1_r(K,T) = H^i(\mathrm{Fil}^r C^\bullet(M(D^{[0,r]}_{\mathrm{cris}}(T)))) = H^i(\mathrm{Fil}^{r_{\max}} M(D^{[0,r_{\max}]}_{\mathrm{cris}}(T))) = H^1_{r_{\max}}(K,T)$. The case of $\overline{T}$ is the same.

Proposition 4.1.4. *Let T be in $\mathrm{Rep}_{\mathbb{Z}_p}(G_{K_0})^{[0,r]}_{\mathrm{crys}}$, with $0 \le r \le p-2$. The short exact sequence*

$$0 \to T \xrightarrow{p^n\cdot} T \to T/p^nT \to 0$$

induces two long exact sequences of abelian groups connected as in the commutative diagram below.

(4.1.4.1)
$$\begin{array}{ccccc}
\dots & & & & \dots \\
\downarrow{\scriptstyle \delta_r^{(1)}} & & & & \downarrow{\scriptstyle \delta^{(1)}} \\
H^1_r(K,T) & \xrightarrow[\cong]{t_{\mathrm{st}}} & H^1_{\mathrm{fin}}(K,T) & \hookrightarrow & H^1(K,T) \\
\downarrow{\scriptstyle p^n\cdot} & & \downarrow{\scriptstyle p^n\cdot} & & \downarrow{\scriptstyle p^n\cdot} \\
H^1_r(K,T) & \xrightarrow[\cong]{t_{\mathrm{st}}} & H^1_{\mathrm{fin}}(K,T) & \hookrightarrow & H^1(K,T) \\
\downarrow{\scriptstyle \pi_r^{(1)}} & & & & \downarrow{\scriptstyle \pi^{(1)}} \\
H^1_r(K,T/p^nT) & & \xrightarrow{t_{\mathrm{st}}^{\infty}} & & H^1(K,T/p^nT) \\
\downarrow{\scriptstyle \delta_r^{(2)}} & & & & \downarrow{\scriptstyle \delta^{(2)}} \\
H^2_r(K,T) & & & & H^2(K,T) \\
\downarrow{\scriptstyle p^n\cdot} & & & & \downarrow{\scriptstyle p^n\cdot} \\
H^2_r(K,T) & & & & H^2(K,T) \\
\downarrow & & & & \downarrow \\
H^2_r(K,T/p^nT) & & & & H^2(K,T/p^nT) \\
\downarrow & & & & \downarrow \\
0 & & & & 0
\end{array}$$

Proof. The right vertical sequence is the long exact sequence of continous Galois cohomology. Set $\overline{T} = T/p^nT$, $D = D_{\mathrm{crys}}(T)$, $\overline{D} = D^{\infty}_{\mathrm{cris}}(\overline{T})$, $M = M(D_{\mathrm{crys}}(T))$ and $\overline{M} = M(D^{\infty}_{\mathrm{cris}}(\overline{T}))$. We have a short exact sequence $0 \to D \overset{p^n}{\to} D \to \overline{D} \to 0$, cf. (3.1.3.1), and by applying $M(-)$ another short exact sequence $0 \to M \overset{p^n}{\to} M \to \overline{M} \to 0$, cf. 3.2.1. Therefore, by using Lemma 3.3.2, we have a short exact sequence of complexes

(4.1.4.2)
$$0 \to \mathrm{Fil}^r C^\bullet(M) \overset{p^n}{\to} \mathrm{Fil}^r C^\bullet(M) \to \mathrm{Fil}^r C^\bullet(\overline{M}) \to 0.$$

The left vertical sequence in (4.1.4.1) is the long exact sequence of cohomology associated to the short exact sequence (4.1.4.2).

To construct $t_{\mathrm{st}}^{\infty}\colon H^1_r(K,\overline{T}) \to H^1(K,T/p^nT)$ we proceed as follow.

$$\begin{aligned}
H^1_r(K,\overline{T}) &= H^1(\mathrm{Fil}^r C^{\bullet}(\overline{M})) \\
&\cong \mathrm{Ext}^1_{\mathrm{ModFI}(S/p^nS)^r}(\mathbb{1},\overline{M}) && \text{(by Prop. 3.3.4)} \\
&\to \mathrm{Ext}^1_{\mathrm{Rep}_{\mathbb{Z}/p^n\mathbb{Z}}(G_K)^{[0,r]}_{\mathrm{crys}}}(\mathbb{Z}/p^n\mathbb{Z}, T^{\infty}_{\mathrm{st}}(\overline{M})) && \text{(by exactness of } T^{\infty}_{\mathrm{st}}, \\
& && \text{cf. 2.3.7)} \\
&= \mathrm{Ext}^1_{\mathrm{Rep}_{\mathbb{Z}/p^n\mathbb{Z}}(G_K)^{[0,r]}_{\mathrm{crys}}}(\mathbb{Z}/p^n\mathbb{Z}, \overline{T}_{|G_K}) && \text{(by Prop. 3.2.2)} \\
&\subseteq \mathrm{Ext}^1_{\mathrm{Rep}_{\mathbb{Z}/p^n\mathbb{Z}}(G_K)}(\mathbb{Z}/p^n\mathbb{Z}, \overline{T}_{|G_K}) = H^1(K,T/p^nT)
\end{aligned}$$

To construct the morphism t_{st} we proceed similarly.

$$\begin{aligned}
H^1_r(K,T) &= H^1(\mathrm{Fil}^r C^{\bullet}(M)) \\
&= \mathrm{Ext}^1_{\mathrm{Mod}(S)^r}(\mathbb{1},M) && \text{(by Prop. 3.3.4)} \\
&\xrightarrow{\sim} \mathrm{Ext}^1_{\mathrm{Rep}_{\mathbb{Z}_p}(G_K)^{[0,r]}_{\mathrm{crys}}}(\mathbb{Z}_p, T_{\mathrm{st}}(M)) && \text{(by Cor. 2.3.5)} \\
&= \mathrm{Ext}^1_{\mathrm{Rep}_{\mathbb{Z}_p}(G_K)^{[0,r]}_{\mathrm{crys}}}(\mathbb{Z}_p, T_{|G_K}) && \text{(by Prop. 3.2.2)} \\
&= H^1_{\mathrm{fin}}(K,T) && \text{(by definition)} \\
&\subseteq \mathrm{Ext}^1_{\mathrm{Rep}_{\mathbb{Z}_p}(G_K)}(\mathbb{Z}_p, T_{|G_K}) = H^1(K,T) && \text{(by definition)}
\end{aligned}$$

We get a morphism of $\mathbb{Z}$-modules $t_{\mathrm{st}}\colon H^1_r(K,T) \to H^1(K,T)$ with image $H^1_{\mathrm{fin}}(K,T)$.

Finally the commutativity for the upper left square in the diagram follows from the additivity of T_{st}, for the upper right square is evident and for the lower square it follows from the the commutativity of the diagram (2.3.7.1). □

Corollary 4.1.5. *The map $H^1_{\mathrm{fin}}(K,T)/p^nH^1_{\mathrm{fin}}(K,T) \hookrightarrow H^1(K,T/p^nT)$, cf. (4.1.1.1), factors canonically through $t^{\infty}_{\mathrm{st}}\colon H^1_r(K,T/p^nT) \to H^1(K,T/p^nT)$. We denote by*

$$\varepsilon\colon H^1_{\mathrm{fin}}(K,T)/p^nH^1_{\mathrm{fin}}(K,T) \hookrightarrow H^1_r(K,T/p^nT)$$

the monomorphism obtained.

Proof. The morphism ε is induced by the composition $\pi_r^{(1)} \circ t_{\mathrm{st}}^{-1}$. The factorisation through t^{∞}_{st} follows by the commutativity of the diagram (4.1.4.1). □

We now state and prove the main result of this article.

Theorem 4.1.6. *Let $p \geq 3$ be a prime integer and we fix an integer r with $0 \leq r \leq p-2$. We consider a crystalline $\mathbb{Z}_p$-representations of G_{K_0}, T, with Hodge-Tate weights in $[0,r]$ and assume $e(r_{\max}-1) \leq p-1$. Then for every integer $n \geq 1$, $H^1_{\mathrm{fin}}(K,T)/p^nH^1_{\mathrm{fin}}(K,T)$ is isomorphic to $H^1_r(K,T/p^nT)$ via ε.*

Proof. Let T be as in Theorem 4.1.6. To show that $\varepsilon\colon H^1_{\mathrm{fin}}(K,T)/p^nH^1_{\mathrm{fin}}(K,T) \to H^1_r(K,T/p^nT)$ is an isomorphism we have to prove that $\pi_r^{(1)}$ in the diagram (4.1.4.1) is surjective. It is clear that if $H^2_r(K,T)=0$ (or more generally if $H^2_r(K,T)$ had not p-torsion) then $\pi_r^{(1)}$ would be surjective. We will prove below that $H^2_r(K,T)=0$ if $e=1$ or $r\le 1$ (cf. Remark 4.1.11) but in general this does not seem true. What we do instead is:

(1) we prove that if $H^2_r(K,T/p^nT)=0$ then $\pi_r^{(1)}$ is surjective, see Proposition 4.1.7 ;
(2) we prove $H^2_r(K,T/p^nT)=0$ under the hypothesis $e(r_{\max}(T)-1)\le p-1$, see Proposition 4.1.10.

Proposition 4.1.7. *Let T be in $\mathrm{Rep}_{\mathbb{Z}_p}(G_{K_0})^{[0,r]}_{\mathrm{crys}}$ with $r\le p-2$. If $H^2_r(K,T/p^nT)=0$ then*

$$\pi_r^{(1)}\colon H^1_r(K,T)\to H^1_r(K,T/p^nT)$$

is surjective.

Proof. Set $M=M(D_{\mathrm{cris}}(T))$ and $\overline{M}=M/p^nM$. It is enough to show that the reduction mod p^n map $Z^1(\mathrm{Fil}^rC^\bullet(M))\to Z^1(\mathrm{Fil}^rC^\bullet(\overline{M}))$ on 1-cocycles is surjective. Let $\overline{a}\in\overline{M}$ and $\overline{\gamma}\in\mathrm{Fil}^{r-1}\overline{M}\otimes\Omega^1$ be such that

$$\nabla_{\overline{M}}(\overline{a})=\Psi_{\overline{M}}(\overline{\gamma}),$$

where $\Psi_{\overline{M}}(\overline{\gamma})=(\varphi^{\overline{M}}_{r-1}\otimes\varphi_1)(\overline{\gamma})-\overline{\gamma}$. Let $a_0\in M$, (resp. $\gamma_0\in\mathrm{Fil}^{r-1}M\otimes\Omega^1$) be any lifting of a (resp. γ). We have $\nabla_M(a_0)-\Psi_M(\gamma_0)=p^n\xi_1$, for some $\xi_1\in M\otimes\Omega^1$. Denote by $\overline{\xi}_1$ the class of ξ_1 in $\overline{M}\otimes\Omega^1$. Since $H^2_r(K,T/p^nT)=0$, there exist $\overline{z_1}\in\overline{M}$ and $\overline{\omega}_1\in\mathrm{Fil}^{r-1}\overline{M}\otimes\Omega^1$, such that

$$\nabla_{\overline{M}}(\overline{z}_1)-\Psi_{\overline{M}}(\overline{\omega}_1)=-\overline{\xi}_1.$$

Take any lifting $z_1\in M$ (resp. $\omega_1\in\mathrm{Fil}^{r-1}M\otimes\Omega^1$) of $\overline{z}_1$ (resp. $\overline{\omega}_1$) and set $a_1=a_0+p^nz_1$ and $\gamma_1=\gamma_0+p^n\omega_1$. We have $\nabla_M(z_1)-\Psi_M(\omega_1)=-\xi_1+p^n\xi_2$ for some $\xi_2\in M\otimes\Omega^1$ and

$$\nabla_M(a_1)-\Psi_M(\gamma_1)=p^n\xi_1-p^n\xi_1+p^{2n}\xi_2=p^{2n}\xi_2.$$

Since M and $\mathrm{Fil}^{r-1}M\otimes\Omega^1$ are p-adically complete and separated, it is clear that we can finish by induction. ☐

4.1.8. Let us introduce some notations on the ring S/pS. We have $E(u)\equiv u^e \mod p$. For any integers $j\ge 0$ and $0\le\delta\le e-1$ we denote by $u^{\langle ej+\delta\rangle}$ the e-partial divided power of u (it satisfies $j!u^{\langle ej+\delta\rangle}=u^{ej+\delta}$). We set $\Omega^1_{S/pS}=S/pS\otimes_S\Omega^1_S=S/pSdu$. The canonical differential $d\colon S\to\Omega^1_S$ composed with the projection $\Omega^1_S\to\Omega^1_{S/pS}$ factors through a derivation $S/pS\to\Omega^1_{S/pS}$ that we denote still by d. For any $f\in S/pS$ we will write $df=\frac{d}{du}(f)\otimes du$.

Lemma 4.1.9. *Let $f:=\alpha u^{\langle ej+\delta\rangle}$ be in S/pS, with $j\ge 0$, $0\le\delta\le e-1$ and $\alpha\in k^*$. Then f is integrable in S/pS, i.e. there is $g\in S/pS$ such that $\frac{d}{du}(g)=f$, if and only if:*

- *for $\delta\ne e-1$, p does not divide $ej+\delta+1$;*
- *for $\delta=e-1$, p does not divide e.*

Proof. It is obvious that the conditions are sufficient. Indeed if $\delta\ne e-1$ and p does not divide $ej+\delta+1$, a primitive of f is $\frac{\alpha}{ej+\delta+1}u^{\langle ej+\delta+1\rangle}$; when $\delta=e-1$ and $p\nmid e$, a primitive is $\frac{\alpha}{e}u^{\langle e(j+1)\rangle}$.

The conditions of the lemma are also necessary because if they are not satisfied, we have $\frac{d}{du}(u^{\langle ej+\delta+1\rangle}) = 0$. $\square$

Proposition 4.1.10. *Let T be in $\mathrm{Rep}_{\mathbb{Z}_p}(G_{K_0})^{[0,r]}_{\mathrm{crys}}$, with $0 \le r \le p-2$. For every $n \ge 1$, the $\mathbb{Z}/p^n\mathbb{Z}$-module $H^2_r(K, T/p^nT)$ is of finite type and if $e(r_{\max}(T) - 1) \le p-1$, then $H^2_r(K, T/p^nT) = 0$.*

Proof. Thanks to Remark 4.1.3 we may assume $r = r_{\max}(T)$. The $n = 1$ case implies the case $n \ge 2$ by induction on n and *dévissage* on the exact sequence

$$0 \to p^nT/p^{n+1}T \to T/p^{n+1}T \to T/p^nT \to 0.$$

Let us compute $H^2_r(K, T/pT)$. Set $\overline{T} = T/pT$, $D = D_{\mathrm{cris}}(T)$ and $\overline{D} = D^{\infty}_{\mathrm{cris}}(\overline{T}) = D/pD$, cf. 3.1.3. By construction the smallest jump in the filtration $\mathrm{Fil}^i D$ is $r_{\min}(D) = 0$ and the biggest jump is $r_{\max}(D) = r - r_{\min}(T)$, where $r_{\min}(T)$ is the smallest Hodge-Tate weight of T. Put $\overline{M} := M(D)/pM(D) \in \mathrm{ModFI}(S/pS)^r$ and $\Psi := \Psi_{\overline{M}} := (\varphi^{\overline{M}}_{r-1} \otimes \varphi_1) - \mathrm{Id}$. The idea is to show first that

$$C := \mathrm{Coker}(\Psi \colon \mathrm{Fil}^{r-1}\overline{M} \otimes_S \Omega^1 \to \overline{M} \otimes_S \Omega^1)$$

is a finite dimensional $\mathbb{F}_p$-vector space, which implies of course that $H^2_r(K, T/pT)$ is also finite. Then, we show that, under the hypothesis $e(r-1) \le p-1$, every element in this cokernel C is in the image of $\nabla_{\overline{M}}$ (see the diagram below), which means $H^2_r(K, T/pT) = 0$.

$$\begin{array}{ccccc}
\mathrm{Fil}^r\overline{M} & \longrightarrow & \mathrm{Fil}^{r-1}\overline{M} \otimes_S \Omega^1 & & \\
\downarrow{\scriptstyle \varphi_r - \mathrm{Id}} & & \downarrow{\scriptstyle \Psi = (\varphi_{r-1}\otimes\varphi_1) - \mathrm{Id}} & & \\
\overline{M} & \xrightarrow{\nabla_{\overline{M}}} & \overline{M} \otimes_S \Omega^1 & & \\
\downarrow & & \downarrow & & \\
\overline{M}/(\varphi_r - \mathrm{Id})(\mathrm{Fil}^r\overline{M}) & \xrightarrow{\overline{\nabla}_{\overline{M}}} & C & \twoheadrightarrow & H^2_r(K, T/pT)
\end{array}$$

In the particular case $r \le 1$, the cokernel C is zero. Indeed, in this case, $\mathrm{Fil}^{r-1}\overline{M} = \overline{M}$ so that the operator $\varphi_{r-1} \otimes \varphi_1 \colon \overline{M} \otimes \Omega^1 \to \overline{M} \otimes \Omega^1$ can be iterated. The divided Frobenius $\varphi_1 \colon \Omega^1 \to \Omega^1$ is nilpotent:

$$\varphi_1^m(du) = u^{p^m-1}du = \left[\frac{p^m-1}{e}\right]! \frac{u^{p^m-1}}{\left[\frac{p^m-1}{e}\right]!} du\,;$$

therefore $\varphi_{r-1} \otimes \varphi_1$ is also nilpotent, Ψ is invertible and C is zero.

For the rest of the proof we suppose $r \ge 2$. Let us prove

$$\dim_{\mathbb{F}_p} C \le e(r-1)\dim_{\mathbb{F}_p}(T/pT). \qquad (4.1.10.1)$$

We write

$$\overline{M} \otimes_S \Omega^1 = D \otimes_W S/pS \otimes_S Sdu.$$

Any γ in $\overline{M} \otimes_S \Omega^1$ is a finite sum of elements of the form $v \otimes u^{\langle ej+\delta\rangle} \otimes du$, where $v \in D$, $j \ge 0$ and $0 \le \delta \le e-1$. We want to find conditions such that γ belongs to the image of

Ψ. It is enough to treat the case $\gamma = v \otimes u^{\langle ej+\delta\rangle} \otimes du$, and $v \neq 0$. Let s be the weight such that $v \in \mathrm{Fil}^s D \backslash \mathrm{Fil}^{s+1} D$; then γ belongs to $\mathrm{Fil}^{r-1}\overline{M} \otimes_S \Omega^1$ if and only if $j \geq r-1-s$.

We compute

$$\begin{aligned}(\varphi_{r-1} \otimes \varphi_1)(\gamma) &= \varphi_s(v) \otimes \varphi_{r-1-s}(u^{\langle ej+\delta\rangle}) \otimes u^{p-1}du \\ &= \varphi_s(v) \otimes 1 \otimes \varphi_{r-1-s}(\tfrac{u^{ej+\delta}}{j!})u^{p-1}du \\ &= \varphi_s(v) \otimes 1 \otimes \frac{u^{p(ej+\delta)}}{j!p^{r-1-s}} u^{p-1}du \\ &= \varphi_s(v) \otimes u^{\langle p(ej+\delta)\rangle} \otimes \frac{\left[\frac{p(ej+\delta)}{e}\right]!}{j!p^{r-1-s}} u^{p-1}du.\end{aligned}$$

Since $\left[\frac{p(ej+\delta)}{e}\right] \geq pj$, we have

$$\mathrm{v}_p\left(\frac{\left[\frac{p(ej+\delta)}{e}\right]!}{j!p^{r-1-s}}\right) \geq \mathrm{v}_p\left(\frac{pj!}{j!p^{r-1-s}}\right) = \frac{1}{p-1}(pj - \sigma_{pj} - j + \sigma_j) - (r-1-s),$$

where, for every m, σ_m denotes the sum of p-adic digits of m. Clearly, we have $\sigma_{pj} = \sigma_j$ and so

$$\mathrm{v}_p\left(\frac{\left[\frac{p(ej+\delta)}{e}\right]!}{j!p^{r-1-s}}\right) \geq j - (r-1-s).$$

By hypothesis $s \geq r_{\min}(D) = 0$; thus if $j \geq r$ then $\varphi_{r-1} \otimes \varphi_1(\gamma) = 0$, and $\Psi(-\gamma) = \gamma$. If $j = r-1$ then $(\varphi_{r-1} \otimes \varphi_1)(\gamma)$ belongs to $\mathrm{Fil}^{r-1}\overline{M} \otimes_S \Omega^1$ and the same kind of computation shows that $(\varphi_{r-1} \otimes \varphi_1)^2(\gamma) = 0$. Thus $\Psi(-\gamma - (\varphi_{r-1} \otimes \varphi_1)(\gamma)) = \gamma$. In conclusion the element $\gamma = v \otimes u^{\langle ej+\delta\rangle} \otimes du$ belongs to the image of Ψ for every $j \geq r-1$ and the inequality (4.1.10.1) follows.

To prove $H^2_r(K, T/pT) = 0$ it is enough to show that every $\gamma = v \otimes u^{\langle ej+\delta\rangle} \otimes du$ in $\overline{M} \otimes \Omega^1$, with $v \in D$, $j \leq (r-2)$ and $0 \leq \delta \leq e-1$, is in the image of $\nabla_{\overline{M}}$. Recall that every $v \in D$ is a horizontal section (by definition of $\nabla_{\overline{M}}$) and e is coprime with p (because of $r \geq 2$ and $e(r-1) \leq p-1$). By Lemma 4.1.9 we may assume moreover that $0 \leq \delta \leq e-2$ and $ej + \delta + 1 = pm$ for some integer $m \geq 1$. We have

$$pm = ej + \delta + 1 \leq e(r-2) + \delta + 1 \leq (p-1) - e + \delta + 1 \leq p-2,$$

which shows that there are no such elements. □

Remark 4.1.11. It is clear by the proof of Proposition 4.1.10 that if $e = 1$ or $r \leq 1$, then $H^2_r(K, T) = 0$. Indeed, if $e = 1$, every element in S is integrable, and so ∇_M is surjective. If $r \leq 1$, then $\mathrm{Fil}^{r-1}M = M$, which implies that $\Psi_M := \varphi_{r-1} \otimes \varphi_1 - \mathrm{Id}$ is an isomorphism: the series $\sum_{i=0}^{+\infty}(\varphi_{r-1} \otimes \varphi_1)^i$ converges to an inverse of Ψ_M. This was the idea of the proof in [GIP07]. We do not know if under the hypothesis of Proposition 4.1.10, we have $H^2_r(K, T) = 0$ or not.

End of the proof of Theorem 4.1.6. □

Corollary 4.1.12. *Let $p \geq 3$ and let T_1 and T_2 be two crystalline $\mathbb{Z}_p$-representations of G_{K_0} with Hodge-Tate weights in $[0, r] \subseteq [0, p-2]$ and assume $e(r-1) \leq p-1$. Then for every morphism (resp. isomorphism) $\iota\colon T_1/p^nT_1 \to T_2/p^nT_2$ of $\mathbb{Z}/p^n\mathbb{Z}[G_{K_0}]$-modules, there exists a morphism (resp. isomorphism) $\tilde{\iota}$ of $\mathbb{Z}/p^n\mathbb{Z}$-modules making the following diagram commutative.*

(4.1.12.1)
$$\begin{array}{ccc} H^1_{\mathrm{fin}}(K,T_1)/p^nH^1_{\mathrm{fin}}(K,T_1) & \hookrightarrow & H^1(K,T_1/p^nT_1) \\ \downarrow \tilde{\iota} & & \downarrow H^1(K,\iota) \\ H^1_{\mathrm{fin}}(K,T_2)/p^nH^1_{\mathrm{fin}}(K,T_2) & \hookrightarrow & H^1(K,T_2/p^nT_2) \end{array}$$

Proof. By the constructions above (cf. (4.1.4.1) and Corollary 4.1.5) we have the following commutative diagram

$$\begin{array}{ccccc} H^1_{\mathrm{fin}}(K,T_1)/p^nH^1_{\mathrm{fin}}(K,T_1) & \xrightarrow[\cong]{\varepsilon_1} & H^1_r(K,T_1/p^nT_1) & \xrightarrow{t^{\infty}_{\mathrm{st}}} & H^1(K,T_1/p^nT_1) \\ & & \downarrow H^1_r(K,\iota) & & \downarrow H^1(K,\iota) \\ H^1_{\mathrm{fin}}(K,T_2)/p^nH^1_{\mathrm{fin}}(K,T_2) & \xrightarrow[\cong]{\varepsilon_2} & H^1_r(K,T_2/p^nT_2) & \xrightarrow{t^{\infty}_{\mathrm{st}}} & H^1(K,T_2/p^nT_2) \end{array}$$

where ε_1 and ε_2 are isomorphisms by Theorem 4.1.6. Set $\tilde{\iota} = \varepsilon_2^{-1} \circ H^1_r(K,\iota) \circ \varepsilon_1$. □

4.2. **Some complements.** Let us finish this section with a variant of Corollary 4.1.12 and some examples.

Corollary 4.2.1. *Let T_1 and T_2 be two crystalline $\mathbb{Z}_p$-representations of G_{K_0} with Hodge-Tate weights in $[0, r] \subseteq [0, p-2]$ and assume $er \leq p-2$. Then for every morphism (resp. isomorphism) $\iota\colon T_1/p^nT_1 \to T_2/p^nT_2$ of $\mathbb{Z}/p^n\mathbb{Z}[G_K]$-modules, there exists a morphism (resp. isomorphism) $\tilde{\iota}$ of $\mathbb{Z}/p^n\mathbb{Z}$-modules making the following diagram commutative.*

(4.2.1.1)
$$\begin{array}{ccc} H^1_{\mathrm{fin}}(K,T_1)/p^nH^1_{\mathrm{fin}}(K,T_1) & \hookrightarrow & H^1(K,T_1/p^nT_1) \\ \downarrow \tilde{\iota} & & \downarrow H^1(K,\iota) \\ H^1_{\mathrm{fin}}(K,T_2)/p^nH^1_{\mathrm{fin}}(K,T_2) & \hookrightarrow & H^1(K,T_2/p^nT_2) \end{array}$$

Remark 4.2.2.

(1) The hypothesis $er \leq p-2$ is more restrictive than that of Corollary 4.1.12 (it implies $(e,p) = 1$ even for $r = 1$), but Corollary 4.2.1 gives functoriality for G_K-morphismes.

(2) The hypothesis $er \leq p-2$ is necessary. Indeed consider the following counterexample for $e = p-1$ and $r = 1$. Take $K = \mathbb{Q}_p(\mu_p(\overline{\mathbb{Q}}_p))$, $T_1 = \mathbb{Z}_p(1)$ and $T_2 = \mathbb{Z}_p$. Clearly T_1 and T_2 are congruent modulo p but $H^1_{\mathrm{fin}}(K,T_1) \otimes_{\mathbb{Z}_p} \mathbb{F}_p$ and $H^1_{\mathrm{fin}}(K,T_2) \otimes_{\mathbb{Z}_p} \mathbb{F}_p$ do not even have the same dimension over $\mathbb{F}_p$.

Proof. The proof is similar but simpler than that of Corollary 4.1.12 so we only sketch it. For $i = 1, 2$ set $D_i = D_{\mathrm{cris}}(T_i)$ and $M_i = M(D_i)$; we have $T_i/p^nT_i = T^{\infty}_{\mathrm{st}}(M_i/p^nM_i)$. Under the hypothesis $er \leq p-2$, the functor T^{∞}_{st} is fully faithful (cf. [Car06, Théorème 1.0.4]) therefore the morphism ι is induced by a unique morphism $\iota'\colon M_1/p^nM_1 \to M_2/p^nM_2$ in $\mathrm{ModFI}(S/p^nS)^r$ and we have a commutative diagram

$$\begin{array}{ccccc} H^1_{\mathrm{fin}}(K,T_1)/p^nH^1_{\mathrm{fin}}(K,T_1) & \xrightarrow[\cong]{\varepsilon_1} & H^1(\mathrm{Fil}^rC^\bullet(M_1/p^nM_1)) & \xrightarrow{t^\infty_{\mathrm{st}}} & H^1(K,T_1/p^nT_1) \\ & & \downarrow{\scriptstyle H^1(\mathrm{Fil}^rC^\bullet(\iota'))} & & \downarrow{\scriptstyle H^1(K,\iota)} \\ H^1_{\mathrm{fin}}(K,T_2)/p^nH^1_{\mathrm{fin}}(K,T_2) & \xrightarrow[\cong]{\varepsilon_2} & H^1(\mathrm{Fil}^rC^\bullet(M_2/p^nM_2)) & \xrightarrow{t^\infty_{\mathrm{st}}} & H^1(K,T_2/p^nT_2) \end{array}$$

where ε_1 and ε_2 are isomorphisms by Theorem 4.1.6. Set $\tilde{\iota} = \varepsilon_2^{-1} \circ H^1(\mathrm{Fil}^rC^\bullet(\iota')) \circ \varepsilon_1$. □

For an abelian group A we denote by $A[n]$ the n-torsion subgroup of A.

Proposition 4.2.3. *Let $2 \le r \le p-2$ be an integer and assume $e \ge p$, then:*

(1) *The groups $H^2_r(K,\mathbb{Z}_p(r))[p^n]$ and $H^2_r(K,\mathbb{Z}/p^n\mathbb{Z}(r))$ are non-zero.*
(2) *The monomorphism $\varepsilon\colon H^1_{\mathrm{fin}}(K,\mathbb{Z}_p(r))/p^nH^1_{\mathrm{fin}}(K,\mathbb{Z}_p(r)) \hookrightarrow H^1_r(K,T/p^nT)$ defined in Corollary* 4.1.5 *is not surjective.*

Proof. The statement (2) follows from (1) by using (4.1.4.1). Let us prove (1): it is enough to treat the case $n=1$. Set $D = D_{\mathrm{cris}}(\mathbb{Z}_p(r))$. We have $D = D^*_{\mathrm{cris}}(\mathbb{Z}_p(-r)(r)) = \mathrm{Hom}_{\mathbb{Z}_p[G_{K_0}]}(\mathbb{Z}_p, A_{\mathrm{cris}}) = W$, $\mathrm{Fil}^0D = D$, $\mathrm{Fil}^1D = 0$, and $\varphi^D_0 = \sigma$. Hence $M(D) = D \otimes_W S = S$, $\mathrm{Fil}^rM(D) = \mathrm{Fil}^rS$ as S-modules, and $M(D) = \mathbb{1}(r)$, cf. 2.3.6. We have $H^1_r(K,\mathbb{Z}_p(r)) = H^1(\mathrm{Fil}^rC^\bullet(\mathbb{1}(r)))$ and $H^1_r(K,\mathbb{F}_p(r)) = H^1(\mathrm{Fil}^rC^\bullet(\overline{\mathbb{1}(r)}))$, where $\overline{\mathbb{1}(r)} = \mathbb{1}(r)/p^n\mathbb{1}(r)$. Denote by ξ the class of $u^{p-1}\otimes_S du$ in $H^2(\mathrm{Fil}^rC^\bullet(\mathbb{1}(r)))$. We have $d(u^p) = p(u^{p-1}\otimes_S du)$, so $p\xi = 0$. We claim that the class of $u^{p-1}\otimes du$ in $H^2(\mathrm{Fil}^rC^\bullet(\overline{\mathbb{1}(r)}))$ is not zero, which implies at the same time $\xi \ne 0$, $H^2_r(K,\mathbb{F}_p(r)) \ne 0$ and $H^2_r(K,\mathbb{Z}_p(r))[p] \ne 0$. Let us prove this claim by contradiction: let a be in $\overline{\mathbb{1}(r)} = S/pS$ and $g\otimes du$ be in $\mathrm{Fil}^{r-1}\overline{\mathbb{1}(r)}\otimes\Omega^1 = \mathrm{Fil}^{r-1}(S/pS)\otimes_S\Omega^1$, such that

$$\tfrac{d}{du}(a)\otimes du - (\varphi_{r-1}\otimes\varphi_1)(g\otimes du) + g\otimes du = u^{p-1}\otimes du,$$

or equivalently

$$\tfrac{d}{du}(a) - u^{p-1}\varphi_{r-1}(g) + g = u^{p-1}.$$

Since $r-1 \ge 1$ and $e \ge p$, by writing $g = \sum_{j\ge r-1} g_ju^{\langle ej+\delta\rangle}$ (with $g_j \in k$ and $0 \le \delta \le e-1$), and by taking reduction modulo $\mathrm{Fil}^p(S/pS)$, we get that there exists $\alpha \in S/(pS+\mathrm{Fil}^pS) = k[u]/u^p$, such that

$$\tfrac{d}{du}(\alpha) = u^{p-1}$$

which is impossible. □

Remark 4.2.4. Let $2 \le r \le p-2$ be an integer, assume $e \ge p$ and that the residue field of K is finite. We have

$$H^1_{\mathrm{fin}}(K,\mathbb{Z}_p(r))/pH^1_{\mathrm{fin}}(K,\mathbb{Z}_p(r)) \overset{\ncong\,\varepsilon}{\hookrightarrow} H^1_r(K,\mathbb{F}_p(r)) \xrightarrow{t^\infty_{\mathrm{st}}} H^1(K,\mathbb{F}_p(r)),$$

and by [BK90, Example 3.9, pg. 359] $H^1_{\mathrm{fin}}(K,\mathbb{Z}_p(r)) = H^1(K,\mathbb{Z}_p(r))$; in particular t^∞_{st} is surjective but not injective and $H^1_{\mathrm{fin}}(K,\mathbb{Z}_p(r))/pH^1_{\mathrm{fin}}(K,\mathbb{Z}_p(r))$ can be identified to $t^\infty_{\mathrm{st}}(H^1_r(K,\mathbb{F}_p(r)))$. We suspect that this phenomenon happens more generally.

Proposition 4.2.5. *Let T be a crystalline $\mathbb{Z}_p$-representation of G_{K_0} with Hodge-Tate weights in $[0,r]$, $0 \le r \le p-2$. We have an isomorphism*

$$t_{\mathrm{st}}^{\infty}\colon H^0_r(K, T/p^nT) \xrightarrow{\sim} (T/p^nT)^{G_K}.$$

Proof. Set $D = D_{\mathrm{cris}}(T)$ and $M = M(D)$. A direct computation shows

$$\begin{aligned} H^0_r(K, T/p^nT) = \mathrm{Hom}_{'\mathrm{Mod}(S)^r}\big(\mathbb{1}/p^n\mathbb{1}, M/p^nM\big) &= \mathrm{Fil}^r((S/p^nS)^{\nabla=0} \otimes_{W_n} D/p^nD)^{\varphi_r=\mathrm{Id}} \\ &\supseteq \mathrm{Fil}^r(D/p^nD)^{\varphi_r=\mathrm{Id}} \cong (T/p^nT)^{G_{K_0}}. \end{aligned}$$

The last inclusion is strict in general. Indeed we have the exact sequence

$$0 \to H^0_r(K,T) \xrightarrow{p^n\cdot} H^0_r(K,T) \to H^0_r(K, T/p^nT) \xrightarrow{\delta} H^1_r(K,T) \xrightarrow{p^n\cdot} H^1_r(K,T) \to \dots$$

induced by the short exact sequence $0 \to M \xrightarrow{p^n} M \to M/p^nM \to 0$. By the theorem of Liu (2.3.4) T_{st} is an equivalence of categories and therefore we have

$$H^0_r(K,T) = \mathrm{Hom}_{'\mathrm{Mod}(S)^r}\big(\mathbb{1}, M\big) = \mathrm{Hom}_{\mathrm{Mod}(S)^r}\big(\mathbb{1}, M\big) \xrightarrow{\sim} T^{G_K}$$

and $H^1_r(M) \cong H^1_{\mathrm{fin}}(K,T)$, cf. (4.1.4.1). By definition $H^1_{\mathrm{fin}}(K,T)$ contains all the torsion of $H^1(K,T)$, so the kernel of $H^1_r(K,T) \xrightarrow{p^n\cdot} H^1_r(K,T)$ is equal to the p^n-torsion subgroup $H^1(K,T)[p^n]$ of $H^1(K,T)$. As in the proof of (4.1.4.1), the exact functors T_{st} and T^{∞}_{st} induce the commutative diagram

$$\begin{array}{ccccccccc} 0 \to & H^0_r(M) & \xrightarrow{p^n\cdot} & H^0_r(M) & \to & H^0_r(M/p^nM) & \xrightarrow{\delta} & H^1_r(K,T)[p^n] & \to 0 \\ & \cong\downarrow t_{\mathrm{st}} & & \cong\downarrow t_{\mathrm{st}} & & \downarrow t^{\infty}_{\mathrm{st}} & & \cong\downarrow t_{\mathrm{st}} & \\ 0 \to & T^{G_K} & \xrightarrow{p^n\cdot} & T^{G_K} & \to & (T/p^nT)^{G_K} & \xrightarrow{\delta} & H^1(K,T)[p^n] & \to 0 \end{array}$$

where the rows are exact. This proves the claim. □

Appendix A. The characteristic two case

In all this section we assume $p = 2$.

Theorem A.1. *Let $K/\mathbb{Q}_2$ be a finite extension. Let T_1 and T_2 be two crystalline $\mathbb{Z}_2$-representations of G_K with Hodge-Tate weights equal to 0. Let $\iota\colon T_1/2^nT_1 \to T_2/2^nT_2$ be a morphism (resp. isomorphism) of $\mathbb{Z}/2^n\mathbb{Z}[G_K]$-modules. Then there exists a morphism (resp. isomorphism) $\tilde{\iota}$ of $\mathbb{Z}/2^n\mathbb{Z}$-modules making the following diagram commutative.*

$$\begin{array}{ccc} H^1_{\mathrm{fin}}(K,T_1)/2^nH^1_{\mathrm{fin}}(K,T_1) & \hookrightarrow & H^1(K, T_1/2^nT_1) \\ \downarrow \tilde{\iota} & & \downarrow H^1(K,\iota) \\ H^1_{\mathrm{fin}}(K,T_2)/2^nH^1_{\mathrm{fin}}(K,T_2) & \hookrightarrow & H^1(K, T_2/2^nT_2) \end{array}$$

Remark A.2. Actually from the proof it follows that Theorem A.1 works for any complete discrete valuation field K of mixed characteristic $(0,2)$ whose residue field k has cohomological dimension smaller than one.

Lemma A.3. *Let T be crystalline with Hodge-Tate weights* 0*. We have* $H^1_{\text{fin}}(K,T) = H^1(k,T)$*.*

Proof. Since T is crystalline with Hodge-Tate weights 0 the inertia subgroup I_K of G_K acts trivially on T and T can be considered a G_k-module.

Let T' be an extension representing a class x in $H^1(K,T)$. We have $x \in H^1_{\text{fin}}(K,T)$ if and only if T' is crystalline with Hodge-Tate weights 0, if and only if the inertia I_K acts trivially on T'. Thus $H^1_{\text{fin}}(K,T) = \ker(H^1(K,T) \xrightarrow{\text{Res}} H^1(\widehat{K}^{\text{nr}},T))$. By the Inflation-Restriction short exact sequence [Ser68, VII §6 Prop.4], we have $H^1_{\text{fin}}(K,T) = H^1(G_k,T)$. □

Proof of Theorem A.1. Apply Lemma (A.3) to the long exact sequence associated to

$$0 \to T \xrightarrow{\cdot 2^n} T \to T/2^nT \to 0.$$

Since k is finite $H^2(k,T) = 0$ by [Ser68, XIII, §1, Prop. 2] therefore $H^1_{\text{fin}}(K,T)/2^nH^1_{\text{fin}}(K,T) = H^1(k,T/2^nT)$ and the statement follows.

□

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