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Tomoyuki Abe and Adriano Marmora

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PRODUCT FORMULA FOR p-ADIC EPSILON FACTORS

TOMOYUKI ABE[1] AND ADRIANO MARMORA[2]

[1] *Kavli Institute for the Physics and Mathematics of the Universe (WPI), The University of Tokyo, 5-1-5 Kashiwanoha, Kashiwa, Chiba, 277-8583, Japan* (tomoyuki.abe@ipmu.jp)

[2] *Institut de Recherche Mathématique Avancée, UMR 7501, Université de Strasbourg et CNRS, 7 rue René Descartes, 67000 Strasbourg, France* (marmora@math.unistra.fr)

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Abstract Let X be a smooth proper curve over a finite field of characteristic p. We prove a product formula for p-adic epsilon factors of arithmetic $\mathscr{D}$-modules on X. In particular we deduce the analogous formula for overconvergent F-isocrystals, which was conjectured previously. The p-adic product formula is a counterpart in rigid cohomology of the Deligne–Laumon formula for epsilon factors in ℓ-adic étale cohomology (for $\ell \neq p$). One of the main tools in the proof of this p-adic formula is a theorem of regular stationary phase for arithmetic $\mathscr{D}$-modules that we prove by microlocal techniques.

Keywords: rigid cohomology; p-adic differential equations; arithmetic D-modules; p-adic Fourier transform; epsilon factors

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Contents

Introduction

Inspired by the Langlands program, Deligne suggested that the constant appearing in the functional equation of the L-function of an ℓ-adic sheaf, on a smooth proper curve over a finite field of characteristic $p \neq \ell$, should factor as product of local contributions (later called *epsilon factors*) at each closed point of the curve. He conjectured a *product formula* and showed some particular cases of it, cf. [33]. This formula was proven by Laumon in the outstanding paper [50].

The goal of this article is to prove a product formula for p-adic epsilon factors of arithmetic $\mathscr{D}$-modules on a curve. This formula generalizes the conjecture formulated in [54] for epsilon factors of overconvergent F-isocrystals, and it is an analog in rigid cohomology of the Deligne–Laumon formula.

Let us give some notation. In this introduction we simplify the exposition by assuming more hypotheses than necessary, and we refer to the article for the general statements. Let k be a finite field of characteristic p, and let $q = p^f$ be its cardinality. Let X be a smooth, proper and geometrically connected curve over k.

We are interested in rigid cohomology [12] on X, which is a good p-adic theory in the sense that it is a Weil cohomology. The coefficients for this theory are the overconvergent F-isocrystals: they play the role of the smooth sheaves in ℓ-adic cohomology, or vector bundles with (flat) connection in complex analytic geometry. These coefficients are also known in the literature as p-adic differential equations. As their ℓ-adic and complex analogs, the overconvergent F-isocrystals form a category which is not stable under push-forward in general. Berthelot [11], inspired by algebraic analysis, proposed a framework to remedy this problem by introducing arithmetic F-$\mathscr{D}$-modules (shortly F-$\mathscr{D}^\dagger$-modules) and in particular the subcategory of holonomic modules, see [16] for a survey. Thanks to works of many people (*e.g.* [22, 30], ...), we have a satisfactory theory, at least in the curve case. We note that another approach to p-adic cohomologies has been initiated by Mebkhout and Narvaez-Macarro [58], and is giving interesting developments, for example see [8]. Although it would certainly be interesting to transpose our calculations into this theory, we place ourselves exclusively in the context of Berthelot's arithmetic $\mathscr{D}$-modules throughout this paper. Nevertheless, we point out that Christol–Mebkhout's results in the local theory of p-adic differential equations are indispensable both explicitly and implicitly in this article. The local theory of arithmetic $\mathscr{D}$-modules has been developed by Crew (cf. [30, 31]) and we will use it extensively in our work.

To state the p-adic product formula let us review the definitions of local and global epsilon factors for holonomic $\mathscr{D}^\dagger$-modules. The Poincaré duality was established for overconvergent F-isocrystals in the works of Berthelot [14], Crew [28] and Kedlaya [46], and for the theory of $\mathscr{D}^\dagger$-modules by the first author [4] based on the results of Virrion [70]. This gives a functional equation for the L-function (Caro [22], Etesse–Le Stum [38]) of an arithmetic $\mathscr{D}$-module $\mathscr{M}$, see section 7.2. The constant appearing in this functional equation is $\varepsilon(\mathscr{M}) := \prod_{r\in\mathbb{Z}} \det(-F; H^r f_+\mathscr{M})^{(-1)^{r+1}}$, where $f\colon X \to \mathrm{Spec}(k)$ is the structural morphism, and it is called the *global* epsilon factor of $\mathscr{M}$.

The *local* epsilon factor of $\mathscr{M}$ at a closed point x of X is defined up to the choice of a meromorphic differential form $\omega \neq 0$ on X. To define it, we restrict $\mathscr{M}$ to the complete *trait* S_x of X at x. To define the local factors $\varepsilon(\mathscr{M}|_{S_x}, \omega)$, we consider a localizing triangle, cf. 3.1.9.1; hence, by linearity, it remains to define the epsilon factors for punctual modules and for free differential modules on the Robba ring with Frobenius structure. The former case is explicit; the latter was done in [54] via the Weil–Deligne representation attached to free differential modules by the p-adic monodromy theorem.

The product formula (Theorem 7.2.5) states that for any holonomic F-$\mathscr{D}^\dagger$-module $\mathscr{M}$ on X, we have

$$\varepsilon(\mathscr{M}) = q^{r(\mathscr{M})(1-g)} \prod_{x\in|X|} \varepsilon(\mathscr{M}|_{S_x}, \omega), \tag{PF}$$

where g is the genus of X, $r(\mathscr{M})$ denotes the (opposite of) the generic rank of $\mathscr{M}$, $|X|$ is the set of closed points of X, and $\omega \neq 0$ is a meromorphic differential form on X. This formula can be seen as a multiplicative generalization of Grothendieck–Ogg–Shafarevich formulas for rigid cohomology on a curve, cf. [41, 5.3.2], [26, 5.0], 2.3.1.1 or 4.1.2.1.

The proof of (PF) starts by following the track of Laumon: a geometric argument (see [50, proof of 3.3.2]) reduces to prove the fundamental case where $X = \mathbb{P}^1_k$ and $\mathscr{M}$ is an F-isocrystal overconvergent along a closed set S of rational points of X (by refining the argument we can even take $S = \{0, \infty\}$, cf. [43, p. 121]). By saying that $\mathscr{M}$ is an F-isocrystal, we mean that it is an arithmetic $\mathscr{D}$-module corresponding to an F-isocrystal via the specialization map, see the convention section 0.0.7. In order to conclude, we need four components: (i) a canonical extension functor $\mathscr{M} \mapsto \mathscr{M}^{\mathrm{can}}$ from the category of holonomic F-$\mathscr{D}^{\dagger}$-modules on the formal disk to that of holonomic F-$\mathscr{D}^{\dagger}$-modules on the projective line, overconvergent at ∞; (ii) the proof of (PF) for $\mathscr{D}^{\dagger}$-modules in the essential image of this functor; (iii) the 'principle' of stationary phase (for modules whose differential slopes at infinity are less than 1); (iv) an exact sequence in the style 'nearby-vanishing cycles' for certain kinds of $\mathscr{D}^{\dagger}$-modules.

The first component is provided by the work of Crew [31], extending the canonical extension of Matsuda for overconvergent F-isocrystals. The second is technical but not difficult to achieve. The third is the deepest among these four, and a large part of this paper is devoted to it. This 'principle' can roughly be described by saying that it provides a description of the behaviour at infinity of the Fourier–Huyghe transform of $\mathscr{M}$, in terms of local contributions at closed points s in $\mathbb{A}^1_k$ where $\mathscr{M}$ is singular (*i.e.* the characteristic cycle of $\mathscr{M}$ does contain a vertical component at s, cf. paragraph 1.3.8). These 'local contributions' are called local Fourier transforms (LFT) of $\mathscr{M}$, and one of the key points of our work is to provide a good construction of them. Here, we differentiate from the work of Laumon, who used vanishing cycles to construct the local Fourier transform of an ℓ-adic sheaf.

A definition of local Fourier transform has been given by Crew [31, 8.3] following the classical path: take the canonical extension of a holonomic F-$\mathscr{D}^{\dagger}$-module at zero, then apply the Fourier–Huyghe transform, and finally restrict around ∞. However, we need more information on the internal structure of LFT, and therefore, we redefine it. Our approach is based on microlocalization inspired by the classical works of Malgrange [53] and Sabbah [63]. Yet, there are many more technical difficulties in our case because we need to deal with differential operators of infinite order. We note that the definition is still not completely local in the sense that it uses the canonical extension and the Frobenius structure is constructed by global methods. Once we have established some fundamental properties of LFT, the proof of the regular stationary phase is analogous to that of Sabbah [63] in the classical case (see also [52] for its generalizations).

The fourth component is proved using an exact sequence of Crew [31], Noot-Huyghe's results on Fourier transform, and the properties of cohomological operations proven in [4].

The end of the proof of (PF) is classical and it follows again Laumon, although there are still some differences from the ℓ-adic case that we have carefully pointed out in section 7.5. In particular, in this subsection, we detail the proof of a determinant formula for the p-adic epsilon factor. This p-adic formula gives a differential interpretation of the local epsilon factors and promises to have new applications. Indeed, in section 5.2 we give an explicit description of the Frobenius acting on the Fourier–Huyghe transform. This might provide explicit information on the p-adic epsilon factors, and moreover have arithmetic spin-offs. For example, in the case of a Kummer isocrystal, by carrying out this calculation and applying the product formula we can re-prove the Gross–Koblitz formula. This and related questions will be addressed in a future paper.

Concerning ℓ-adic theory, we point out that Abbes and Saito [1] have recently given an interesting new local description of LFT as well as an alternative proof of Laumon's determinant formula for ℓ-adic representations satisfying a certain ramification condition.

Finally, regarding Langlands correspondence for p-adic coefficients, we mention the preprint [5], where the p-adic product formula is used to show the equivalence between the (conjectural) *p-adic* Langlands correspondence for GL_n over function fields and Deligne's hope for *petits camarades cristallins* for curves, cf. [34, Conjecture (1.2.10-vi)] and also [27, Conjecture 4.13]. These conjectures have been proven recently in [6], in which the p-adic product formula plays an essential role.

After this introduction, this article is divided into seven sections. Here, we briefly describe their content; more information can be found in the text at the beginning of each section and subsection.

The aim of section 1 is to define the characteristic cycles of holonomic $\mathscr{D}$-modules on curves over the field k (which is supposed here only of characteristic $p > 0$), and prove some relations with the microlocalizations. For this, we prove a level stability theorem using microlocal techniques of [3]. The section starts with a short survey of microdifferential operators of [3].

Section 2 begins the study of local Fourier transforms for holonomic $\mathscr{D}$-modules. This section is the technical core of the paper. We start in section 2.1 by a review of Crew's theory of arithmetic $\mathscr{D}$-modules on a formal disk; then we study in section 2.2 the relations between microlocalization and analytification of $\mathscr{D}^\dagger$-modules. This gives several applications: namely the equality between Garnier's and Christol–Mebkhout's definitions of irregularity (section 2.3). We finish the section by giving an alternative definition of local Fourier transform (except for the Frobenius structure) in section 2.4. We will see in section 4 that this LFT coincides with that of Crew and we will complete the definition in section 5 by endowing it with the Frobenius structure.

Section 3 reviews the cohomological operations on arithmetic $\mathscr{D}$-modules. In particular, in section 3.1 we recall the results of [4] which are used in this paper, and in section 3.2 we review the global Fourier transform of Noot-Huyghe.

Section 4 is devoted to the regular stationary phase. In section 4.1 we establish some numerical results analogous to those of Laumon for perverse ℓ-adic sheaves. In section 4.2 we prove the stationary phase for regular holonomic modules on the projective line.

It is in section 5 that we finally implement the Frobenius in the theory. In section 5.1 we endow the local Fourier transform with the Frobenius induced by that of the global Fourier transform via the stationary phase isomorphism. In section 5.2 we explicitly describe the Frobenius on the naive Fourier transform.

Section 6 provides a key exact sequence for the proof of the product formula. This sequence should be seen as an analog of the exact sequence of vanishing cycles appearing in Laumon's proof of the ℓ-adic product formula. The section begins with a result on commutation of the Frobenius in section 6.1 and we then prove the exactness of the sequence in section 6.2.

Finally, in the last section, we state and prove the p-adic product formula. We begin in section 7.1 with the definition of local factors of holonomic modules; then, in section 7.2 we recall the definition of the L-function attached to a holonomic module and define the global epsilon factor. We state the product formula and we show that it is in fact equivalent to the product formula for overconvergent F-isocrystals conjectured in [54]. The section continues with the proof of the product formula: some preliminary particular cases in section 7.4, and the general case, as well as the determinant formula for local epsilon factors, in section 7.5.

Conventions and notation

0.0.1. Unless otherwise stated, the filtration of a filtered ring (resp. module) is assumed to be increasing. Let $(A, F_i A)_{i\in\mathbb{Z}}$ be a filtered ring or module. For $i \in \mathbb{Z}$, we will often denote $F_i A$ by A_i. Recall that the filtered ring A is said to be a *noetherian filtered ring* if its associated Rees ring $\bigoplus_{i\in\mathbb{Z}} F_i A$ is noetherian.

0.0.2. Let A be a topological ring, and let M be a finitely generated A-module. We consider the product topology on $A^{\oplus n}$ for any positive integer n. Let $\phi\colon A^{\oplus n} \to M$ be a surjection, and we denote by $\mathscr{T}_\phi$ the quotient topology on M induced by ϕ. Then $\mathscr{T}_\phi$ does not depend on the choice of ϕ up to equivalence of topologies. We call this topology the *A-module topology on M*.

0.0.3. Let K be a field, and $\sigma\colon K \xrightarrow{\sim} K$ be an automorphism. A σ-K-vector space is a K-vector space V equipped with a σ-semi-linear endomorphism $\varphi\colon V \to V$ such that the induced homomorphism $K \otimes_{\sigma,K} V \to V$ is an isomorphism.

0.0.4. Let R be a complete discrete valuation ring of mixed characteristic $(0, p)$, k be its residue field, and K be its field of fractions. We denote a uniformizer of R by ϖ. For any integer $i \geqslant 0$, we put $R_i := R/\varpi^{i+1}R$. The residue field k is not assumed to be perfect in general; we assume k to be perfect from the middle of section 2, and in the last section (section 7), we assume moreover k to be finite. We denote by $|\cdot|$ the p-adic norm on R or K normalized as $|p| = p^{-1}$.

In principle, we use Roman fonts (*e.g.* X) for schemes and script fonts (*e.g.* $\mathscr{X}$) for formal schemes. For a smooth formal scheme $\mathscr{X}$ over $\mathrm{Spf}(R)$, we denote by X_i the reduction $\mathscr{X} \otimes_R R_i$ over $\mathrm{Spec}(R_i)$. We denote X_0 by X unless otherwise stated. In this paper, *curve* (resp. *formal curve*) means dimension one smooth separated connected scheme (resp. formal scheme) of finite type over its basis.

When X (resp. $\mathscr{X}$) is an affine scheme (resp. formal scheme), we sometimes denote $\Gamma(X, \mathcal{O}_X)$ (resp. $\Gamma(\mathscr{X}, \mathcal{O}_{\mathscr{X}})$) simply by $\mathcal{O}_X$ (resp. $\mathcal{O}_{\mathscr{X}}$) if this is unlikely to cause any confusion.

0.0.5. Let $\mathscr{X}$ be a smooth formal scheme over $\mathrm{Spf}(R)$ of dimension d. A system of global coordinates on $\mathscr{X}$ is a subset $\{x_1, \ldots, x_d\}$ of $\Gamma(\mathscr{X}, \mathcal{O}_{\mathscr{X}})$ such that the morphism $\mathscr{X} \to \widehat{\mathbb{A}}^d_R$ defined by these functions is étale. A system of local coordinates is a system of global coordinates on an open subscheme $\mathscr{U}$ of $\mathscr{X}$.

Let $s \in \mathscr{X}$ be a closed point. A system of local parameters at s is a subset $\{y_1, \ldots, y_d\}$ of $\Gamma(\mathscr{U}, \mathcal{O}_{\mathscr{U}})$, for some open neighbourhood $\mathscr{U}$ of s, such that its image in $\mathcal{O}_{X,s}$ forms a system of regular local parameters in the sense of [EGA 0_{IV}, 17.1.6]. When $d = 1$, we say 'a (local) coordinate' instead of saying 'a system of (local) coordinates', and the same for 'a local parameter'.

0.0.6. We freely use the language of arithmetic $\mathscr{D}$-modules. For details see [13, 15, 16]. In particular, we use the notation $\mathscr{D}^{(m)}_X$, $\widehat{\mathscr{D}}^{(m)}_{\mathscr{X}}$, $\mathscr{D}^{\dagger}_{\mathscr{X}}$. An index $\mathbb{Q}$ means tensor with $\mathbb{Q}$.

0.0.7. Let X be a scheme of finite type over k, and Z be a closed subscheme of X. We put $U := X \setminus Z$. We denote by $(F\text{-})\mathrm{Isoc}(U, X/K)$ the category of convergent $(F\text{-})$isocrystal on U over K overconvergent along Z. If X is proper, we say, for sake of brevity, *overconvergent $(F\text{-})$isocrystal on U over K*, instead of convergent $(F\text{-})$isocrystal on U over K overconvergent along Z, and we denote the category by $(F\text{-})\mathrm{Isoc}^{\dagger}(U/K)$.

Now, let $\mathscr{X}$ be a smooth formal scheme, and Z be a divisor of its special fibre. Let $\mathscr{U} := \mathscr{X} \setminus Z$, X and U be the special fibres of $\mathscr{X}$ and $\mathscr{U}$ respectively. In this paper, we denote $\mathscr{D}^{\dagger}_{\mathscr{X},\mathbb{Q}}({}^{\dagger}Z)$ by $\mathscr{D}^{\dagger}_{\mathscr{X},\mathbb{Q}}(Z)$ for short. In the same way, we denote $\mathcal{O}_{\mathscr{X},\mathbb{Q}}({}^{\dagger}Z)$ by $\mathcal{O}_{\mathscr{X},\mathbb{Q}}(Z)$. Let $\mathscr{M}$ be a coherent $(F\text{-})\mathscr{D}^{\dagger}_{\mathscr{X},\mathbb{Q}}(Z)$-module such that $\mathscr{M}|_{\mathscr{U}}$ is coherent as an $\mathcal{O}_{\mathscr{U},\mathbb{Q}}$-module. Then we know that $\mathscr{M}$ is a coherent $\mathcal{O}_{\mathscr{X},\mathbb{Q}}(Z)$-module by [17]. Let $\mathcal{C}$ be the full subcategory of the category of coherent $(F\text{-})\mathscr{D}^{\dagger}_{\mathscr{X},\mathbb{Q}}(Z)$-modules consisting of such $\mathscr{M}$. By [13, 4.4.12] and [15, 4.6.7] the specialization functors sp_* and sp^* induce an equivalence between $\mathcal{C}$ and the category $(F\text{-})\mathrm{Isoc}^{\dagger}(U, X/K)$. We will say that $\mathscr{M}$ is a convergent $(F\text{-})$isocrystal on $\mathscr{U}$ overconvergent along Z by abuse of language.

0.0.8. The shift of a complex $\mathscr{C}$ will be denoted always by brackets $\mathscr{C}[d]$. When parenthesis appear, like $\mathscr{C}(i)[d]$, it means that all the terms of the complex are Tate twisted i times; cf. 3.1.3 for a definition of Tate twist.

1. Stability theorem for characteristic cycles on curves

The definition of stable holonomic module 1.3.8 and the stability theorem 1.5.1 are the goals of this section. Theorem 1.5.1 is needed to prove the product formula for holonomic $\mathscr{D}^{\dagger}$-modules with Frobenius structure on a curve over a finite field. Nevertheless, in this section, we tried to state the theorems in the more possible generality: in particular, we *do not* require k to be perfect, neither we assume the existence of a Frobenius structure on $\mathscr{D}^{\dagger}$-modules. Even if we put Frobenius structures, we do not know if the proof of the stability theorem could be simplified.

1.1. Review of microdifferential operators

We review the definitions and properties of the arithmetic microdifferential sheaves on curves, which are going to be used extensively in this paper. For the general definitions in higher dimensional settings and more details, see [3].

1.1.1. Let $\mathscr{X}$ be a formal curve over R. We denote its special fibre by X. Let T^*X be the cotangent bundle of X and $\pi\colon T^*X \to X$ be the canonical projection. We put $\mathring{T}^*X := T^*X \setminus s(X)$ where $s\colon X \to T^*X$ denotes the zero section. Let $m \geqslant 0$ be an integer and $\mathscr{M}$ be a coherent $\widehat{\mathscr{D}}^{(m)}_{\mathscr{X},\mathbb{Q}}$-module. One of the basic ideas of microlocalization is to 'localize' $\mathscr{M}$ over T^*X to make possible a more detailed analysis on $\mathscr{M}$. For this, we define step by step the sheaves of rings $\mathscr{E}^{(m)}_{X_i}$, $\widehat{\mathscr{E}}^{(m)}_{\mathscr{X}}$, $\widehat{\mathscr{E}}^{(m)}_{\mathscr{X},\mathbb{Q}}$ on the cotangent bundle.

Let i be a non-negative integer. Let us define $\mathscr{E}^{(m)}_{X_i}$ first. For the detail of this construction, see [3, 2.2, Remark 2.16]. There are mainly two types of rings of sections of $\mathscr{E}^{(m)}_{X_i}$. Let U be an open subset of T^*X. If $U \cap s(X)$ is non empty, we get $\Gamma(U, \mathscr{E}^{(m)}_{X_i}) \cong \Gamma(\pi(U \cap s(X)), \mathscr{D}^{(m)}_{X_i})$. Suppose that the intersection is empty. Let $U' := \pi^{-1}(\pi(U)) \cap$

$\mathring{T}^*X$. Then the ring of sections of $\mathscr{E}^{(m)}_{X_i}$ on U is equal to that on U', and the ring of these sections is the 'microlocalization' of $\Gamma(\pi(U), \mathscr{D}^{(m)}_{X_i})$. Let us describe locally the sections $\Gamma(U', \mathscr{E}^{(m)}_{X_i})$. Shrink $\mathscr{X}$ so that it possesses a global coordinate denoted by x. We denote the corresponding differential operator by ∂. There exists an integer N such that $\partial^{\langle Np^m\rangle_{(m)}}$ is in the centre of $\mathscr{D}^{(m)}_{X_i}$. Let S be the multiplicative system generated by $\partial^{\langle Np^m\rangle_{(m)}}$ in $\mathscr{D}^{(m)}_{X_i}$. The positive filtration on $\Gamma(\pi(U'), \mathscr{D}^{(m)}_{X_i})$, given by the order of differential operators, induces a ring filtration, necessarily indexed by $\mathbb{Z}$, on the localization $S^{-1}\Gamma(\pi(U'), \mathscr{D}^{(m)}_{X_i})$. Thus the elements in $S^{-1}\Gamma(\pi(U'), \mathscr{D}^{(m)}_{X_i})$ can have negative order, but they are 'finite' in the sense that only finitely many negative powers of $\partial^{\langle Np^m\rangle_{(m)}}$ can appear in (the total symbol of) each of them. We define

$$\Gamma(U', \mathscr{E}^{(m)}_{X_i}) := \big(S^{-1}\Gamma(\pi(U'), \mathscr{D}^{(m)}_{X_i})\big)^{\wedge},$$

as the completion of this filtered ring with respect to negative order (see [3, 1.1.5] for our conventions on the completion of a filtered ring).

Taking an inverse limit over i, we define $\widehat{\mathscr{E}}^{(m)}_{\mathscr{X}}$. The sections of $\widehat{\mathscr{E}}^{(m)}_{\mathscr{X}}$ can be described concretely as follows. Let U be an open subset of $\mathring{T}^*X$, and assume $V := \pi(U)$ to be affine. Let $\mathscr{V}$ be the open formal subscheme sitting over V. Suppose moreover that $\mathscr{V}$ possesses a global coordinate x, and we denote the corresponding differential operator by ∂. For an integer $k \geqslant 0$, take the minimal integer i such that $k \leqslant ip^m$, and let $l := ip^m - k$. By the construction of $\widehat{\mathscr{E}}^{(m)}_{\mathscr{X}}$, the operator $\partial^{\langle ip^m\rangle_{(m)}}$ in $\widehat{\mathscr{D}}^{(m)}_{\mathscr{X}}$ considered as a section of $\widehat{\mathscr{E}}^{(m)}_{\mathscr{X}}$ is invertible, and the inverse is denoted by $\partial^{\langle -ip^m\rangle_{(m)}}$. Then we put $\partial^{\langle -k\rangle_{(m)}} := \partial^{\langle l\rangle_{(m)}} \cdot \partial^{\langle -ip^m\rangle_{(m)}}$. By using [3, Example 2.13], we get

$$\Gamma(U, \widehat{\mathscr{E}}^{(m)}_{\mathscr{X}}) \cong \left\{ \sum_{k\in\mathbb{Z}} a_k \partial^{\langle k\rangle_{(m)}} \;\middle|\; a_k \in \mathcal{O}_{\mathscr{V}},\ \lim_{k\to+\infty} a_k = 0 \right\}.$$

Finally, by tensoring with $\mathbb{Q}$, we define $\widehat{\mathscr{E}}^{(m)}_{\mathscr{X},\mathbb{Q}}$. One of the most important properties of $\widehat{\mathscr{E}}^{(m)}_{\mathscr{X},\mathbb{Q}}$ is that we get an equality for any coherent $\widehat{\mathscr{D}}^{(m)}_{\mathscr{X},\mathbb{Q}}$-module $\mathscr{M}$ (cf. [3, Proposition 2.15])

$$\mathrm{Char}^{(m)}(\mathscr{M}) = \mathrm{Supp}(\widehat{\mathscr{E}}^{(m)}_{\mathscr{X},\mathbb{Q}} \otimes_{\pi^{-1}\widehat{\mathscr{D}}^{(m)}_{\mathscr{X},\mathbb{Q}}} \pi^{-1}\mathscr{M}),$$

where $\mathrm{Char}^{(m)}$ denotes the characteristic variety of level m (cf. [16, 5.2.5]). The module $\widehat{\mathscr{E}}^{(m)}_{\mathscr{X},\mathbb{Q}} \otimes_{\pi^{-1}\widehat{\mathscr{D}}^{(m)}_{\mathscr{X},\mathbb{Q}}} \pi^{-1}\mathscr{M}$ is called the (naive) microlocalization of $\mathscr{M}$ of level m. The ring $\widehat{\mathscr{E}}^{(m)}_{\mathscr{X},\mathbb{Q}}$ is called the (naive) microdifferential operators of level m.

1.1.2. In the last paragraph, we fixed the level m to construct the ring of microdifferential operators. However, to deal with microlocalizations of $\mathscr{D}^{\dagger}_{\mathscr{X},\mathbb{Q}}$-modules, we need to change levels and see the asymptotic behaviour. The problem is that there are no reasonable transition homomorphism $\widehat{\mathscr{E}}^{(m)}_{\mathscr{X},\mathbb{Q}} \to \widehat{\mathscr{E}}^{(m')}_{\mathscr{X},\mathbb{Q}}$ for non-negative integers $m' > m$. To remedy this, we need to take an 'intersection'. Let $\mathscr{E}^{(m)}_{\mathscr{X}} := \bigcup_{n\in\mathbb{Z}} (\widehat{\mathscr{E}}^{(m)}_{\mathscr{X}})_n$, where

$(\widehat{\mathscr{E}}^{(m)}_{\mathscr{X}})_n$ denotes the sub-$\pi^{-1}\mathcal{O}_{\mathscr{X}}$-module of $\widehat{\mathscr{E}}^{(m)}_{\mathscr{X}}$ consisting of microdifferential operators of order less than or equal to n, and put $\mathscr{E}^{(m)}_{\mathscr{X},\mathbb{Q}} := \mathscr{E}^{(m)}_{\mathscr{X}} \otimes \mathbb{Q}$. Then there exists a canonical homomorphism of $\pi^{-1}\mathcal{O}_{\mathscr{X},\mathbb{Q}}$-algebras $\psi_{m,m'}\colon \mathscr{E}^{(m')}_{\mathscr{X},\mathbb{Q}} \to \mathscr{E}^{(m)}_{\mathscr{X},\mathbb{Q}}$ sending $\partial^{\langle 1\rangle_{(m')}}$ to $\partial^{\langle 1\rangle_{(m)}}$. We define $\mathscr{E}^{(m,m')}_{\mathscr{X}} := \psi^{-1}_{m,m'}(\mathscr{E}^{(m)}_{\mathscr{X}}) \cap \mathscr{E}^{(m')}_{\mathscr{X}}$. By definition, $\widehat{\mathscr{E}}^{(m,m')}_{\mathscr{X}}$ is the p-adic completion of $\mathscr{E}^{(m,m')}_{\mathscr{X}}$, and $\widehat{\mathscr{E}}^{(m,m')}_{\mathscr{X},\mathbb{Q}}$ is $\widehat{\mathscr{E}}^{(m,m')}_{\mathscr{X}} \otimes \mathbb{Q}$. When $\mathscr{X}$ possesses a global coordinate x, we may write

$$\Gamma(\mathring{T}^*X, \widehat{\mathscr{E}}^{(m,m')}_{\mathscr{X},\mathbb{Q}}) \cong \left\{ \sum_{k\in\mathbb{Z}} a_k\partial^k \;\middle|\; a_k \in \mathcal{O}_{\mathscr{X},\mathbb{Q}}, \sum_{k<0} a_k\partial^k \in \widehat{\mathscr{E}}^{(m')}_{\mathscr{X},\mathbb{Q}}, \sum_{k\geqslant 0} a_k\partial^k \in \widehat{\mathscr{E}}^{(m)}_{\mathscr{X},\mathbb{Q}} \right\}.$$

We note that the last condition $\sum_{k\geqslant 0} a_k\partial^k \in \widehat{\mathscr{E}}^{(m)}_{\mathscr{X},\mathbb{Q}}$ is equivalent to $\sum_{k\geqslant 0} a_k\partial^k \in \widehat{\mathscr{D}}^{(m)}_{\mathscr{X},\mathbb{Q}}$. Moreover, by construction of $\Gamma(\mathring{T}^*X, \widehat{\mathscr{E}}^{(m,m')}_{\mathscr{X},\mathbb{Q}})$, the expansion $\sum_{k\in\mathbb{Z}} a_k\partial^k$ satisfying the above conditions is unique, once the coordinate x is fixed (and so is ∂). For an integer k, we put $(\widehat{\mathscr{E}}^{(m,m')}_{\mathscr{X},\mathbb{Q}})_k := \widehat{\mathscr{E}}^{(m,m')}_{\mathscr{X},\mathbb{Q}} \cap (\widehat{\mathscr{E}}^{(m)}_{\mathscr{X},\mathbb{Q}})_k$. We have canonical homomorphisms $\widehat{\mathscr{E}}^{(m-1,m')}_{\mathscr{X}} \to \widehat{\mathscr{E}}^{(m,m')}_{\mathscr{X}}$ and $\widehat{\mathscr{E}}^{(m,m'+1)}_{\mathscr{X}} \to \widehat{\mathscr{E}}^{(m,m')}_{\mathscr{X}}$ (cf. [3, Lemma 4.7]). We call these sheaves the intermediate rings of microdifferential operators.

We also recall the following definitions,

$$\mathscr{E}^{(m,\dagger)}_{\mathscr{X},\mathbb{Q}} := \varprojlim_{m'\to+\infty} \widehat{\mathscr{E}}^{(m,m')}_{\mathscr{X},\mathbb{Q}}, \qquad \mathscr{E}^{\dagger}_{\mathscr{X},\mathbb{Q}} := \varinjlim_{m\to+\infty} \mathscr{E}^{(m,\dagger)}_{\mathscr{X},\mathbb{Q}},$$

where the transition maps are induced by the canonical homomorphisms above, cf. [3, 4.11] for more details.

1.1.3. Now, let us explain the relation between the supports of the microlocalizations of a $\widehat{\mathscr{D}}^{(m)}_{\mathscr{X},\mathbb{Q}}$-module with respect to intermediate rings and the characteristic variety. Let $\mathscr{X}$ be a formal curve as in the last paragraph, and let $\mathscr{M}$ be a coherent $\widehat{\mathscr{D}}^{(m)}_{\mathscr{X},\mathbb{Q}}$-module. One might expect that, for integers $m'' \geqslant m' \geqslant m$,

$$\mathrm{Char}^{(m')}(\widehat{\mathscr{D}}^{(m')}_{\mathscr{X},\mathbb{Q}} \otimes_{\widehat{\mathscr{D}}^{(m)}_{\mathscr{X},\mathbb{Q}}} \mathscr{M}) = \mathrm{Supp}(\widehat{\mathscr{E}}^{(m',m'')}_{\mathscr{X},\mathbb{Q}} \otimes_{\pi^{-1}\widehat{\mathscr{D}}^{(m)}_{\mathscr{X},\mathbb{Q}}} \pi^{-1}\mathscr{M}). \tag{1.1.3.1}$$

This does not hold in general as we can see by the counter-example [3, 7.1]. However, the statement holds for m' large enough. The following is one of main results of [3].

Theorem ([3, Theorem 7.2])**.** *There exists an integer N such that 1.1.3.1 holds for any $m'' \geqslant m' \geqslant N$.*

1.2. Setup and preliminaries

1.2.1. In this paragraph, we introduce some situations and notation. In this paper, especially in the first two sections, we often consider the following setting, which is called *Situation (L)*.

> Let $\mathscr{X}$ be an affine formal curve over R. Recall the convention 0.0.4, especially X_i and X. Suppose that there exists a global coordinate x in $\Gamma(\mathscr{X}, \mathcal{O}_{\mathscr{X}})$ and fix it. We denote the corresponding differential operator by ∂.

If moreover we assume the following, we say we are in *Situation (Ls)*.

> Let s be a closed point in $\mathscr{X}$, and we fix it. We suppose that there exists a local parameter at s denoted by y_s on $\mathscr{X}$.

We use the following notation.

Notation. Let $\mathscr{X}$ be a formal curve over R, and $\mathscr{U}$ be an open affine formal subscheme of $\mathscr{X}$. We denote $\mathscr{U} \otimes R_i$ by U_i as usual. Let $m' \geqslant m$ be non-negative integers.

1. We put $E^{\dagger}_{\mathscr{U},\mathbb{Q}} := \Gamma(\mathring{T}^*U, \mathscr{E}^{\dagger}_{\mathscr{X},\mathbb{Q}})$, $E^{(m,\dagger)}_{\mathscr{U},\mathbb{Q}} := \Gamma(\mathring{T}^*U, \mathscr{E}^{(m,\dagger)}_{\mathscr{X},\mathbb{Q}})$, $\widehat{E}^{(m,m')}_{\mathscr{U},\mathbb{Q}} := \Gamma(\mathring{T}^*U, \widehat{\mathscr{E}}^{(m,m')}_{\mathscr{X},\mathbb{Q}})$, $\widehat{E}^{(m,m')}_{\mathscr{U}} := \Gamma(\mathring{T}^*U, \widehat{\mathscr{E}}^{(m,m')}_{\mathscr{X}})$, $(\widehat{E}^{(m,m')}_{\mathscr{U}})_k := \Gamma(\mathring{T}^*U, (\widehat{\mathscr{E}}^{(m,m')}_{\mathscr{X}})_k)$, $E^{(m,m')}_{U_i} := \Gamma(\mathring{T}^*U, \mathscr{E}^{(m,m')}_{X_i})$.
2. Let E be one of $E^{\dagger}_{\mathscr{U},\mathbb{Q}}$, $E^{(m,\dagger)}_{\mathscr{U},\mathbb{Q}}$, $\widehat{E}^{(m,m')}_{\mathscr{U},\mathbb{Q}}$. For a coherent $\widehat{\mathscr{D}}^{(m)}_{\mathscr{X},\mathbb{Q}}$-module $\mathscr{M}$, we denote by $E \otimes_{\widehat{\mathscr{D}}^{(m)}} \mathscr{M}$ or $E \otimes \mathscr{M}$ the E-module $E \otimes_{\Gamma(\mathscr{U},\widehat{\mathscr{D}}^{(m)}_{\mathscr{X},\mathbb{Q}})} \Gamma(\mathscr{U}, \mathscr{M})$.
3. Let x be a closed point in $\mathscr{X}$. Take a point ξ_x in $\pi^{-1}(x)$ which is not in the zero section. Let $\mathscr{E}$ be one of sheaves of rings $\widehat{\mathscr{E}}^{(m,m')}_{\mathscr{X}}$, $\widehat{\mathscr{E}}^{(m,m')}_{\mathscr{X},\mathbb{Q}}$, $\mathscr{E}^{(m,\dagger)}_{\mathscr{X},\mathbb{Q}}$ or $\mathscr{E}^{\dagger}_{\mathscr{X},\mathbb{Q}}$. By the construction of $\mathscr{E}$, the fibre $\mathscr{E}_{\xi_x}$ does not depend on the choice of ξ_x and we denote it by $\mathscr{E}_x$. Let a pair $(\mathscr{E}', \mathscr{E})$ be one of the four pairs $(\widehat{\mathscr{E}}^{(m,m')}_{x}, \widehat{\mathscr{E}}^{(m,m')}_{\mathscr{X}})$, $(\widehat{\mathscr{E}}^{(m,m')}_{x,\mathbb{Q}}, \widehat{\mathscr{E}}^{(m,m')}_{\mathscr{X},\mathbb{Q}})$, $(\mathscr{E}^{(m,\dagger)}_{x,\mathbb{Q}}, \mathscr{E}^{(m,\dagger)}_{\mathscr{X},\mathbb{Q}})$, $(\mathscr{E}^{\dagger}_{x,\mathbb{Q}}, \mathscr{E}^{\dagger}_{\mathscr{X},\mathbb{Q}})$. For a coherent $\widehat{\mathscr{D}}^{(m)}_{\mathscr{X}}$ or $\widehat{\mathscr{D}}^{(m)}_{\mathscr{X},\mathbb{Q}}$-module $\mathscr{M}$ we write $\mathscr{E}' \otimes_{\widehat{\mathscr{D}}^{(m)}} \mathscr{M}$ for $(\mathscr{E} \otimes_{\pi^{-1}\widehat{\mathscr{D}}^{(m)}_{\mathscr{X}}} \pi^{-1}\mathscr{M})_{\xi_x}$. This does not depend either on the choice of ξ_x. We warn the reader that even if $\mathscr{E}$ is complete with respect to some topology we do not take the completion when we take the fibre at ξ_x.

Moreover, assume that we are in Situation (L).

4. We denote by $R_{\mathscr{X}}\{\partial\}^{(m,m')}$ the subring of $\widehat{E}^{(m,m')}_{\mathscr{X}}$ whose elements are 'horizontal with respect to x'. More precisely, we define
$$R_{\mathscr{X}}\{\partial\}^{(m,m')} := \left\{ P = \sum_{n\in\mathbb{Z}} a_n \partial^n \in \widehat{E}^{(m,m')}_{\mathscr{X}} \;\middle|\; P\partial^k = \partial^k P \text{ for any } k \geqslant 0 \right\}.$$
We put $K_{\mathscr{X}}\{\partial\}^{(m,m')} := R_{\mathscr{X}}\{\partial\}^{(m,m')} \otimes \mathbb{Q}$ and $R_{X_i}\{\partial\}^{(m,m')} := R_{\mathscr{X}}\{\partial\}^{(m,m')}/\varpi^{i+1}$. We note that by the hypothesis of Situation (L), the coordinate x is fixed, and so is ∂.

Remark. (i) Let $\mathscr{E}$ be a sheaf of rings on a topological space. Then an $\mathscr{E}$-module $\mathscr{M}$ is said to be *globally finitely presented* if there exist integers $a, b \geqslant 0$ and an exact sequence $\mathscr{E}^a \to \mathscr{E}^b \to \mathscr{M} \to 0$ on the topological space. By [3, Corollary 5.3], when $\mathscr{X}$ is affine, there exists an equivalence of categories between the category of globally finitely presented $\widehat{\mathscr{E}}^{(m,m')}_{\mathscr{X}}$-modules (resp. $\widehat{\mathscr{E}}^{(m,m')}_{\mathscr{X},\mathbb{Q}}$-modules) on $\mathring{T}^*X$ and that of $\widehat{E}^{(m,m')}_{\mathscr{X}}$-modules (resp. $\widehat{E}^{(m,m')}_{\mathscr{X},\mathbb{Q}}$-modules). We remind here that if there exists a coherent $\widehat{\mathscr{D}}^{(m)}_{\mathscr{X}}$-module $\mathscr{N}$ such that $\mathscr{M} \cong \widehat{\mathscr{E}}^{(m,m')}_{\mathscr{X}} \otimes \mathscr{N}$, then $\mathscr{M}$ is globally finitely presented, and the same for $\widehat{\mathscr{E}}^{(m,m')}_{\mathscr{X},\mathbb{Q}}$-modules.

(ii) Let $\mathring{\pi}\colon \mathring{T}^*X \to X$ be the natural projection. A sheaf $\mathscr{E}$ on $\mathring{T}^*X$ is called *conic* if the natural morphism $\mathring{\pi}^{-1}\mathring{\pi}_*\mathscr{E} \to \mathscr{E}$ is an isomorphism. In particular, the support $\mathrm{Supp}(\mathscr{E})$ of such a sheaf is a conic subset, *i.e.* if $z \in \mathrm{Supp}(\mathscr{E}) \subset \mathring{T}^*X$, then $\mathring{\pi}^{-1}(\mathring{\pi}(z)) \subset \mathrm{Supp}(\mathscr{E})$.

By construction the sheaves $\mathscr{E}$ of microdifferentials operators are conic (where $\mathscr{E}$ stands for any of the sheaves $\widehat{\mathscr{E}}^{(m,m')}_{\mathscr{X},\mathbb{Q}}$, $\mathscr{E}^{(m,\dagger)}_{\mathscr{X},\mathbb{Q}}$, *etc.*, introduced before). Since this is clearly a local property, it follows that any coherent $\mathscr{E}$-module is also conic.

(iii) Every element Q in the ring $R_{\mathscr{X}}\{\partial\}^{(m,m')}$ can be written uniquely as $Q = \sum_{l<0} a_l \partial^{\langle l\rangle_{(m')}} + \sum_{k\geqslant 0} a_k \partial^{\langle k\rangle_{(m)}}$, with a_k in a finite étale extension $R(\mathscr{X})$ of R (cf. 1.3.5, 1.3.6), which coincides with R if and only if $\mathscr{X}$ is geometrically connected. The rings $K_{\mathscr{X}}\{\partial\}^{(m,m')}$ and $R_{X_i}\{\partial\}^{(m,m')}$ are subrings of $\widehat{E}^{(m,m')}_{\mathscr{X},\mathbb{Q}}$ and $E^{(m,m')}_{X_i}$ respectively. We remind that they *do* depend on the choice of coordinate x, and in particular, we are not able to globalize the construction.

1.2.2. Let $\mathscr{X}$ be an affine formal curve over R and let E be one of the rings $\widehat{E}^{(m,m')}_{\mathscr{X}}$, $\widehat{E}^{(m,m')}_{\mathscr{X},\mathbb{Q}}$ or $K_{\mathscr{X}}\{\partial\}^{(m,m')}$. We finish this subsection by introducing some useful topologies on E and on finitely generated E-modules. Let us start with $\widehat{E}^{(m,m')}_{\mathscr{X}}$.

For integers $k, l \geqslant 0$, we define a sub-R-module of $\widehat{E}^{(m,m')}_{\mathscr{X}}$ by

$$U_{k,l} := (\widehat{E}^{(m,m')}_{\mathscr{X}})_{-k} + \varpi^l \widehat{E}^{(m,m')}_{\mathscr{X}}.$$

We endow $\widehat{E}^{(m,m')}_{\mathscr{X}}$ with the topology denoted by $\mathscr{T}_0$ where the base of neighbourhoods of zero is given by the system $\{U_{k,l}\}_{k,l\geqslant 0}$. With this topology, $\widehat{E}^{(m,m')}_{\mathscr{X}}$ is a complete topological ring. Let us see that it is complete with respect to this topology. Let $\{P_i\}$ be a Cauchy sequence in $\widehat{E}^{(m,m')}_{\mathscr{X}}$. Then we may write $P_i = Q_i + R_i$ such that: for any integers k and l, there exists an integer N such that $Q_n - Q_N \in (\widehat{E}^{(m,m')}_{\mathscr{X}})_{-k}$ and $R_n - R_N \in \varpi^l \widehat{E}^{(m,m')}_{\mathscr{X}}$ for any $n \geqslant N$. By the definition of the topology and the construction of the ring $\widehat{E}^{(m,m')}_{\mathscr{X}}$, the limits $\lim_{i\to\infty} Q_i$ and $\lim_{i\to\infty} R_i$ exist, and they are denoted by Q and R respectively. Then we see that $\lim_{i\to\infty} P_i = Q + R$ by definition.

Now, let us define topologies $\mathscr{T}$ and $\mathscr{T}_n$ for any integer $n \geqslant 0$ on $\widehat{E}^{(m,m')}_{\mathscr{X},\mathbb{Q}}$. Let $n \geqslant 0$ be an integer. For integers $k, l \geqslant 0$, we can consider $\varpi^{-n} U_{k,l}$ as a sub-R-module of $\widehat{E}^{(m,m')}_{\mathscr{X},\mathbb{Q}}$, and we denote by $\mathscr{T}_n$ the topology on $\widehat{E}^{(m,m')}_{\mathscr{X},\mathbb{Q}}$ generated by the open basis $\{\varpi^{-n} U_{k,l}\}_{k,l\geqslant 0}$. This topology makes it a locally convex topological space, and moreover a Fréchet space by [66, Théorème 3.12]. The identity map $(\widehat{E}^{(m,m')}_{\mathscr{X},\mathbb{Q}}, \mathscr{T}_n) \to (\widehat{E}^{(m,m')}_{\mathscr{X},\mathbb{Q}}, \mathscr{T}_{n+1})$ is continuous by construction. By taking the inductive limit (of locally convex spaces), we define a topology, denoted by $\mathscr{T}$, on $\widehat{E}^{(m,m')}_{\mathscr{X},\mathbb{Q}}$. It makes $(\widehat{E}^{(m,m')}_{\mathscr{X},\mathbb{Q}}, \mathscr{T})$ an LF-space in the sense of [28, 3.1]. The separateness can be seen from the fact that the convex subset $(\widehat{E}^{(m,m')}_{\mathscr{X},\mathbb{Q}})_{-k} + \varpi^l \widehat{E}^{(m,m')}_{\mathscr{X}}$ in $\widehat{E}^{(m,m')}_{\mathscr{X},\mathbb{Q}}$ is open in the $\mathscr{T}_n$-topology for any n and thus in the $\mathscr{T}$-topology.

Let M be a finitely generate $\widehat{E}^{(m,m')}_{\mathscr{X},\mathbb{Q}}$-module. We denote the $(\widehat{E}^{(m,m')}_{\mathscr{X},\mathbb{Q}}, \mathscr{T}_n)$-module topology on M (cf. 0.0.2) by $\mathscr{T}'_n$. Let us prove that the topology $\mathscr{T}'_n$ is separated. Let $\varphi\colon (\widehat{E}^{(m,m')}_{\mathscr{X},\mathbb{Q}})^{\oplus a} \to M$ be a surjective homomorphism, and put $M' := \varphi((\widehat{E}^{(m,m')}_{\mathscr{X}})^{\oplus a})$. Consider the quotient topology on M' using the topology $\mathscr{T}_0$ on $\widehat{E}^{(m,m')}_{\mathscr{X}}$. It suffices to show that M' is separated. Indeed, take $\alpha, \alpha' \in M$ such that $\alpha \neq \alpha'$. There exists an integer $i \geqslant n$ such that $\varpi^i\alpha, \varpi^i\alpha' \in M'$. If M' is separated, there exists $U_{k,l}$, $U_{k',l'}$ such that $(\varpi^i\alpha + \varphi(U_{k,l})) \cap (\varpi^i\alpha' + \varphi(U_{k',l'})) = \emptyset$. Since $\varpi^{-i}U_{k,l} \supset \varpi^{-n}U_{k,l}$, we get the claim. Let us show that M' is separated. Since $\widehat{E}^{(m,m')}_{\mathscr{X}}$ is a noetherian complete p-adic ring (cf. [3, Proposition 4.12]), M' is also p-adically complete, and in particular, p-adically separated. Thus, it suffices to show that $M' \otimes R_i$ is separated for any $i \geqslant 0$ using the quotient topology $\mathscr{Q}$ from M'. Consider the topology defined by the filtration by order on $E^{(m,m')}_{X_i}$. The topology $\mathscr{Q}$ coincides with the quotient topology via $(E^{(m,m')}_{X_i})^{\oplus n} \to M' \otimes R_i$ induced by φ. Since $E^{(m,m')}_{X_i}$ is a noetherian complete filtered ring by [3, Proposition 4.9], we get that $M' \otimes R_i$ is separated, and thus the topology $\mathscr{T}'_n$ on M is separated.

This shows that, $\mathrm{Ker}(\varphi)$ is a closed sub-$(\widehat{E}^{(m,m')}_{\mathscr{X},\mathbb{Q}}, \mathscr{T}_n)$-module. Thus the topological vector space $(M, \mathscr{T}'_n)$ is a Fréchet space. Of course, the identity map $(M, \mathscr{T}'_n) \to (M, \mathscr{T}'_{n+1})$ is continuous. We define the inductive limit topology (of locally convex spaces) $\mathscr{T}'$ on M, which is called the *natural topology* on M. If the natural topology is separated, then $(M, \mathscr{T}')$ is an LF-space. When it is separated, the open mapping theorem [28, 3.4] implies that $(M, \mathscr{T}')$ coincides with the $(\widehat{E}^{(m,m')}_{\mathscr{X},\mathbb{Q}}, \mathscr{T})$-module topology.

In the same way, we define topologies $\mathscr{S}$ and $\mathscr{S}_n$ on $K_{\mathscr{X}}\{\partial\}^{(m,m')}$ and on finitely generated $K_{\mathscr{X}}\{\partial\}^{(m,m')}$-modules when we are in Situation (L) of 1.2.1.

1.2.3 Lemma. *Suppose we are in Situation* (L) *of 1.2.1. Let M be a finitely generated $\widehat{E}^{(m,m')}_{\mathscr{X},\mathbb{Q}}$-module. We assume that it is also finite as $K_{\mathscr{X}}\{\partial\}^{(m,m')}$-module. Then the natural topology as $\widehat{E}^{(m,m')}_{\mathscr{X},\mathbb{Q}}$-module and the natural topology as $K_{\mathscr{X}}\{\partial\}^{(m,m')}$-module are equivalent. In particular, if moreover M is a free $K_{\mathscr{X}}\{\partial\}^{(m,m')}$-module, then the topologies are separated, and M becomes an LF-space.*

Proof. Let us see the equivalence. Let $\phi\colon \big(K_{\mathscr{X}}\{\partial\}^{(m,m')}\big)^{\oplus a} \to M$ be a surjection. This surjection induces the surjection $(\widehat{E}^{(m,m')}_{\mathscr{X},\mathbb{Q}})^{\oplus a} \to M$, and the quotient topology $(M, \mathscr{T}'_n)$ is defined. Let $(M, \mathscr{S}'_n)$ be the Fréchet topology defined using the surjection ϕ and the $(K_{\mathscr{X}}\{\partial\}^{(m,m')}, \mathscr{S}_n)$-module structure, as done above in 1.2.2 for $\mathscr{T}'_n$. Since $(M, \mathscr{T}'_n)$ is a topological $(K_{\mathscr{X}}\{\partial\}^{(m,m')}, \mathscr{S}_n)$-module by the definition, the homomorphism ϕ defines a continuous surjective homomorphism of topological modules $(K_{\mathscr{X}}\{\partial\}^{(m,m')}, \mathscr{S}_n)^{\oplus a} \to (M, \mathscr{T}'_n)$. By the open mapping theorem of Fréchet spaces, we see that this homomorphism is strict, which implies that $\mathscr{T}'_n$ and $\mathscr{S}'_n$ are equivalent. The first claim follows by taking the inductive limit over n. When M is free as a $K_{\mathscr{X}}\{\partial\}^{(m,m')}$-module, then it is obvious that it is separated. □

1.3. Relations between microlocalizations at different levels

In this subsection, we investigate the behaviour of microlocalizations when we raise levels. In general, this is very difficult. However, once we know that the supports of the microlocalizations are stable (cf. 1.3.8), the behaviour is very simple at least in the case of a curve.

1.3.1 Lemma. *Suppose we are in Situation* (Ls) *of 1.2.1. Let $m' \geqslant m$ be non-negative integers, and I be a left ideal of $\widehat{E}^{(m,m')}_{\mathscr{X}}$; we put $M := \widehat{E}^{(m,m')}_{\mathscr{X}}/I$. Let $\mathscr{M}$ be the $\widehat{\mathscr{E}}^{(m,m')}_{\mathscr{X}}$-module associated with M (cf. Remark 1.2.1*(i)*) on $\mathring{T}^*X$, and assume*

$$\mathrm{Supp}(\mathscr{M}) = \pi^{-1}(s) \cap \mathring{T}^*X.$$

Then for any integer k, there exists a positive integer N, $R \in \widehat{E}^{(m,m')}_{\mathscr{X}}$ and $S \in (\widehat{E}^{(m,m')}_{\mathscr{X}})_k$ such that

$$y_s^N - \varpi R - S \in I.$$

Proof. Since $\widehat{E}^{(m,m')}_{\mathscr{X}}$ is a noetherian ring (cf. [3, Proposition 4.12]), there exist n operators $P_i \in \widehat{E}^{(m,m')}_{\mathscr{X}}$ for $i = 1, \ldots, n$ such that I is generated by $\{P_i\}_{1 \leqslant i \leqslant n}$. Let $\mathscr{U} := \mathscr{X} \setminus \{s\}$, and U be its special fibre. Then by the assumption on the support, there exists $Q_i \in \widehat{E}^{(m,m')}_{\mathscr{U}}$ for each $1 \leqslant i \leqslant n$, such that

$$\sum_{1 \leqslant i \leqslant n} Q_i \cdot P_i = 1. \tag{1.3.1.1}$$

For $P \in \widehat{E}^{(m,m')}_{\mathscr{U}}$, we denote by $\overline{P}$ the image of P in $E^{(m,m')}_{U}$, and define $\varpi\text{-ord}(P) := \mathrm{ord}(\overline{P})$. We put $\mu := \max_i\{\varpi\text{-ord}(P_i), 0\}$.

For any $f \in \mathcal{O}_{\mathscr{U}}$, there exists an integer n such that $\overline{y_s}^n \overline{f} \in \mathcal{O}_X$ where the overlines denote the images in $\mathcal{O}_U$ or $\mathcal{O}_X$, thus $y_s^n f \in \mathcal{O}_{\mathscr{X}} + \varpi \mathcal{O}_{\mathscr{U}}$. This shows that there exists an integer N such that for any $i = 1, \ldots, n$, we can write

$$y_s^N Q_i = Q'_i + \varpi R_i + S_i,$$

where $Q'_i \in \widehat{E}^{(m,m')}_{\mathscr{X}}$, $R_i \in \widehat{E}^{(m,m')}_{\mathscr{U}}$, and $S_i \in (\widehat{E}^{(m,m')}_{\mathscr{U}})_{k-\mu}$. Then by 1.3.1.1, there exist $R' \in \widehat{E}^{(m,m')}_{\mathscr{U}}$ and $S' \in (\widehat{E}^{(m,m')}_{\mathscr{U}})_k$ such that

$$y_s^N = \sum Q'_i \cdot P_i + \varpi R' + S'.$$

Let us show that for any integer k'

$$\big(\varpi \widehat{E}^{(m,m')}_{\mathscr{U}} + (\widehat{E}^{(m,m')}_{\mathscr{U}})_{k'}\big) \cap \widehat{E}^{(m,m')}_{\mathscr{X}} = \varpi \widehat{E}^{(m,m')}_{\mathscr{X}} + (\widehat{E}^{(m,m')}_{\mathscr{X}})_{k'}. \tag{1.3.1.2}$$

It is evident that the right hand side is included in the left one, let us prove the opposite inclusion. Take elements $P \in \varpi \widehat{E}^{(m,m')}_{\mathscr{U}}$ and $Q \in (\widehat{E}^{(m,m')}_{\mathscr{U}})_{k'}$ such that $P + Q \in \widehat{E}^{(m,m')}_{\mathscr{X}}$. Write $P = \sum_{n \in \mathbb{Z}} a_n \partial^n$ with $a_n \in \mathcal{O}_{\mathscr{U},\mathbb{Q}}$, and put $P_{>k'} := \sum_{n > k'} a_n \partial^n$ and $P_{\leqslant k'} :=$

$\sum_{n\leqslant k'} a_n\partial^n$. Then we get $P_{>k'}\in \widehat{E}^{(m)}_{\mathscr{X}}$ and in particular contained in $\widehat{E}^{(m)}_{\mathscr{X}}\cap\varpi\widehat{E}^{(m)}_{\mathscr{U}}$, which is $\varpi\widehat{E}^{(m)}_{\mathscr{X}}$ thanks to uniqueness of the expansion $\sum_{n\in\mathbb{Z}} a_n\partial^n$ considered in 1.1.2. By assumption we have $P_{\leqslant k'}+Q\in(\widehat{E}^{(m,m')}_{\mathscr{X}})_{k'}$, which implies the equality 1.3.1.2.

Since $y_s^N-\sum Q'_i\cdot P_i\in\widehat{E}^{(m,m')}_{\mathscr{X}}$, we get, by using 1.3.1.2, that

$$\varpi R'+S'\in\varpi\widehat{E}^{(m,m')}_{\mathscr{X}}+(\widehat{E}^{(m,m')}_{\mathscr{X}})_k.$$

Thus the lemma follows. □

1.3.2 Lemma. *Suppose we are in Situation* (Ls) *of 1.2.1. Let $m'\geqslant m$ be non-negative integers, and $\mathscr{M}$ be a globally finitely presented $\widehat{\mathscr{E}}^{(m,m')}_{\mathscr{X},\mathbb{Q}}$-module such that*

$$\mathrm{Supp}(\mathscr{M})\cap\mathring{T}^*X=\pi^{-1}(s)\cap\mathring{T}^*X.$$

We denote $\Gamma(\mathring{T}^*X,\mathscr{M})$ by M. Then we have the following.

(i) *The module M is finite over $K_{\mathscr{X}}\{\partial\}^{(m,m')}$. Moreover, if it is monogenic as an $\widehat{E}^{(m,m')}_{\mathscr{X},\mathbb{Q}}$-module, there exists a p-torsion free $\widehat{E}^{(m,m')}_{\mathscr{X}}$-module M', such that $M'\otimes\mathbb{Q}\cong M$, and M' is finitely generated over $R_{\mathscr{X}}\{\partial\}^{(m,m')}$.*

(ii) *There exists a finite set of elements in M such that for any open affine neighbourhood W of s in X, $\Gamma(\mathring{T}^*W,\mathscr{M})$ is generated by these elements over $K_{\mathscr{X}}\{\partial\}^{(m,m')}$. If M is generated by $\alpha\in M$, then there exists an integer $N>0$ such that we can take this set to be $\{(x^i y_s^j)\,\alpha\}_{0\leqslant i,j<N}$.*

Proof. Let $\mathscr{N}$ be a coherent sub-$\widehat{\mathscr{E}}^{(m,m')}_{\mathscr{X},\mathbb{Q}}$-module of $\mathscr{M}$. Let $\mathscr{N}'$ be either $\mathscr{N}$ or $\mathscr{M}/\mathscr{N}$. Then by the additivity of supports we know that

$$\mathrm{Supp}(\mathscr{N}')\cap\mathring{T}^*X=\pi^{-1}(s)\cap\mathring{T}^*X \text{ or } \emptyset.$$

If it is $\emptyset$, then $\mathscr{N}'|_{\mathring{T}^*X}=0$, and in particular, $N':=\Gamma(\mathring{T}^*X,\mathscr{N}')=0$ is finite over $K_{\mathscr{X}}\{\partial\}^{(m,m')}$. By induction on the number of generators of M, we reduce the verification of both (i) and (ii) to the monogenic case.

From now on, we assume that M is a monogenic module. Fix a generator $\alpha\in M$. Let M' be the sub-$\widehat{E}^{(m,m')}_{\mathscr{X}}$-module of M generated by α. Let I be the kernel of the homomorphism $\widehat{E}^{(m,m')}_{\mathscr{X}}\to M'$ of left $\widehat{E}^{(m,m')}_{\mathscr{X}}$-modules sending one to α. We note that, by definition, $M'\otimes\mathbb{Q}\cong M$. Since M' is p-torsion free, we get that $\mathrm{Supp}(\widehat{\mathscr{E}}^{(m,m')}_{\mathscr{X},\mathbb{Q}}\otimes M)=\mathrm{Supp}(\widehat{\mathscr{E}}^{(m,m')}_{\mathscr{X}}\otimes M')$ where $\widehat{\mathscr{E}}^{(m,m')}_{\mathscr{X},\mathbb{Q}}\otimes M$ denotes the $\widehat{\mathscr{E}}^{(m,m')}_{\mathscr{X},\mathbb{Q}}$-module associated with M (which is equal to $\mathscr{M}$ by 1.2.1), and the same for $\widehat{\mathscr{E}}^{(m,m')}_{\mathscr{X}}\otimes M'$. Thus by Lemma 1.3.1 for $k=-1$, there exists a positive integer N' and $T\in\varpi\widehat{E}^{(m,m')}_{\mathscr{X}}+(\widehat{E}^{(m,m')}_{\mathscr{X}})_{-1}$ such that $y_s^{N'}\equiv T \bmod I$.

To conclude, it suffices to show that $M'':=\widehat{E}^{(m,m')}_{\mathscr{X}}/(y_x^{N'}-T)$ is generated over $R_{\mathscr{X}}\{\partial\}^{(m,m')}$ by $\mathfrak{S}:=\{x^i\,y_s^j\}_{0\leqslant i,j<N}$ where $N:=N'+\deg(s)$ since there is a surjection $M''\twoheadrightarrow M'$. Since M'' and $R_{\mathscr{X}}\{\partial\}^{(m,m')}$ are ϖ-adically complete and p-torsion free, the

conditions of [18, III.2.11, Proposition 14] are fulfiled, and thus, it suffices to see that M''/ϖ is generated by $\mathfrak{S}$ over $R_{\mathscr{X}}\{\partial\}^{(m,m')}/\varpi$. It remains to show that $E_X^{(m,m')}/(\bar{y_s}^{N'}-\bar{T})$ is generated over $R_X\{\partial\}^{(m,m')}$ by $\mathfrak{S}$, where $\bar{T}\in(E_X^{(m,m')})_{-1}$. Since $E_X^{(m,m')}/(\bar{y_s}^{N'}-\bar{T})$ and $R_X\{\partial\}^{(m,m')}$ are complete with respect to the filtrations by order, it is enough to prove the claim after taking gr by [18, III.2.9, Proposition 12]. Since the order of $\bar{T}$ is less than zero, this amounts to prove that the commutative algebra $\mathrm{gr}(E_X^{(m,m')})/(\bar{y_s}^{N'})$ is generated over $R_{X_0}\{\partial\}^{(m,m')}$ by $\mathfrak{S}$. This is a straightforward verification which is left to the reader. □

1.3.3 Lemma. *We assume that we are in Situation* (L)*.*

(i) *Let $m'>m$. Then we have a canonical isomorphism*

$$\widehat{E}_{\mathscr{X},\mathbb{Q}}^{(m+1,m')}\cong K_{\mathscr{X}}\{\partial\}^{(m+1,m')}\widehat{\otimes}_{K_{\mathscr{X}}\{\partial\}^{(m,m')}}\widehat{E}_{\mathscr{X},\mathbb{Q}}^{(m,m')}$$

of bi-($K_{\mathscr{X}}\{\partial\}^{(m+1,m')}$, $\widehat{E}_{\mathscr{X},\mathbb{Q}}^{(m,m')}$)-modules. Here the complete tensor product is taken with respect to the p-adic topology.

(ii) *Let $m'\geqslant m$. Then we have a canonical isomorphism*

$$\widehat{E}_{\mathscr{X}}^{(m,m')}\cong R_{\mathscr{X}}\{\partial\}^{(m,m')}\widehat{\otimes}^f_{R_{\mathscr{X}}\{\partial\}^{(m,m'+1)}}\widehat{E}_{\mathscr{X}}^{(m,m'+1)}$$

of bi-($R_{\mathscr{X}}\{\partial\}^{(m,m')}$, $\widehat{E}_{\mathscr{X}}^{(m,m'+1)}$)-modules. Here the complete tensor product $\widehat{\otimes}^f$ is taken with respect to the filtration by order.

(iii) *Let $m'\geqslant m$. For any $i\geqslant 0$, we have a canonical isomorphism*

$$E_{X_i}^{(m,m')}\cong R_{X_i}\{\partial\}^{(m,m')}\widehat{\otimes}^f_{R_{X_i}\{\partial\}^{(m,m'+1)}}E_{X_i}^{(m,m'+1)}$$

of bi-($R_{X_i}\{\partial\}^{(m,m')}$, $E_{X_i}^{(m,m'+1)}$)-modules. Here the complete tensor product is taken with respect to the filtration by order.

Proof. Let us see (i). There exists a canonical homomorphism

$$\varphi\colon \widehat{E}_{\mathscr{X},\mathbb{Q}}^{(m,m')}\to K_{\mathscr{X}}\{\partial\}^{(m+1,m')}\widehat{\otimes}_{K_{\mathscr{X}}\{\partial\}^{(m,m')}}\widehat{E}_{\mathscr{X},\mathbb{Q}}^{(m,m')},$$

sending P to $1\otimes P$. For short, we denote $\Gamma(\mathscr{X},\mathscr{E}_{\mathscr{X}}^{(m+1,m')})$ by E, which is considered to be a subring of $\widehat{E}_{\mathscr{X},\mathbb{Q}}^{(m,m')}$ using the canonical inclusion. We know that $\widehat{E}\otimes\mathbb{Q}\cong\widehat{E}_{\mathscr{X},\mathbb{Q}}^{(m+1,m')}$ where ^ denotes the p-adic completion. The image $\varphi(E)$ is contained in the image of $R_{\mathscr{X}}\{\partial\}^{(m+1,m')}\widehat{\otimes}_{R_{\mathscr{X}}\{\partial\}^{(m,m')}}\widehat{E}_{\mathscr{X}}^{(m,m')}$. Indeed, let $P\in E$. We denote $\Gamma(\mathscr{X},\mathscr{D}_{\mathscr{X}}^{(m+1)})$ by $D_{\mathscr{X}}^{(m+1)}$. Then we may write $P=P_{\geqslant 0}+P_{<0}$ where $P_{\geqslant 0}\in D_{\mathscr{X}}^{(m+1)}$ and $P_{<0}\in E_{-1}\subset\widehat{E}_{\mathscr{X}}^{(m,m')}$. For $P_{\geqslant 0}\in D_{\mathscr{X}}^{(m+1)}$, we can write $P_{\geqslant 0}=\sum_{i\geqslant 0}\partial^{\langle i\rangle_{(m+1)}}a_i$ where $a_i\in\mathcal{O}_{\mathscr{X}}$. Since this is a finite sum, $1\otimes P_{\geqslant 0}$ is the image of $\sum_{i\geqslant 0}\partial^{\langle i\rangle_{(m+1)}}\otimes a_i$, and the claim follows.

This implies that the homomorphism φ induces the canonical homomorphism

$$\widehat{\varphi}\colon \widehat{E}_{\mathscr{X},\mathbb{Q}}^{(m+1,m')}\to K_{\mathscr{X}}\{\partial\}^{(m+1,m')}\widehat{\otimes}_{K_{\mathscr{X}}\{\partial\}^{(m,m')}}\widehat{E}_{\mathscr{X},\mathbb{Q}}^{(m,m')}.$$

On the other hand, we have the canonical homomorphism

$$\widehat{\psi}\colon K_{\mathscr{X}}\{\partial\}^{(m+1,m')}\widehat{\otimes}_{K_{\mathscr{X}}\{\partial\}^{(m,m')}}\widehat{E}^{(m,m')}_{\mathscr{X},\mathbb{Q}}\to\widehat{E}^{(m+1,m')}_{\mathscr{X},\mathbb{Q}}.$$

To conclude the proof, it suffices to show that $\widehat{\psi}\circ\widehat{\varphi}=\mathrm{id}$, $\widehat{\varphi}\circ\widehat{\psi}=\mathrm{id}$. Since ψ and φ are continuous, to see the former equality, it suffices to verify the identity on E, which is obvious. Let $A:=K_{\mathscr{X}}\{\partial\}^{(m+1,m')}\cap E\subset\widehat{E}^{(m+1,m')}_{\mathscr{X}}$. To check the latter equality, it suffices to see after restricting to $\mathrm{Im}(A\otimes_{\mathbb{Z}}\widehat{E}^{(m,m')}_{\mathscr{X}}\to K_{\mathscr{X}}\{\partial\}^{(m+1,m')}\widehat{\otimes}\widehat{E}^{(m,m')}_{\mathscr{X},\mathbb{Q}})$. Since this is straightforward, we leave the argument to the reader.

Now let us prove (ii). We have a canonical homomorphism

$$\widehat{\psi}\colon R_{\mathscr{X}}\{\partial\}^{(m,m')}\widehat{\otimes}^{f}_{R_{\mathscr{X}}\{\partial\}^{(m,m'+1)}}\widehat{E}^{(m,m'+1)}_{\mathscr{X}}\to\widehat{E}^{(m,m')}_{\mathscr{X}}.$$

On the other hand, we also have the homomorphism

$$\iota\colon \widehat{E}^{(m,m'+1)}_{\mathscr{X}}\to R_{\mathscr{X}}\{\partial\}^{(m,m')}\otimes_{R_{\mathscr{X}}\{\partial\}^{(m,m'+1)}}\widehat{E}^{(m,m'+1)}_{\mathscr{X}}$$

sending P to $1\otimes P$. Since $\widehat{\psi}\circ\iota$ is the canonical inclusion, we get that ι is injective. The target $B:=R_{\mathscr{X}}\{\partial\}^{(m,m')}\otimes_{R_{\mathscr{X}}\{\partial\}^{(m,m'+1)}}\widehat{E}^{(m,m'+1)}_{\mathscr{X}}$ of ι is filtered by the tensor product filtration [51, p. 57] denoted by $\{B_n\}$. Let $n<0$, and take $S:=\sum_i P_i\otimes Q_i$ in B_n. Then there exists $f\in\mathcal{O}_{\mathscr{X}}$ such that

$$\sum_i P_i\otimes Q_i\equiv\partial^{\langle n\rangle_{(m')}}\otimes f \bmod B_{n-1}.$$

Suppose $S\notin B_{n-1}$. Then $f\neq 0$. There exists an integer N such that $p^N\partial^{\langle n\rangle_{(m')}}\in R_{\mathscr{X}}\{\partial\}^{(m,m'+1)}$. Thus for $N'\geqslant N$, we get $p^{N'}S\equiv 1\otimes(p^{N'}\partial^{\langle n\rangle_{(m')}})\cdot f \bmod B_{n-1}$. If $1\otimes(p^{N'}\partial^{\langle n\rangle_{(m')}})\cdot f\in B_{n-1}$, we would get $\widehat{\psi}\circ\iota((p^{N'}\partial^{\langle n\rangle_{(m')}})\cdot f)\in(\widehat{E}^{(m,m')}_{\mathscr{X}})_{n-1}$, which is impossible. Thus, we get $1\otimes(p^{N'}\partial^{\langle n\rangle_{(m')}})\cdot f\notin B_{n-1}$. This shows that $p^{N'}S\notin B_{n-1}$ for any large enough N'. Thus, $\mathrm{gr}(R_{\mathscr{X}}\{\partial\}^{(m,m')}\otimes_{R_{\mathscr{X}}\{\partial\}^{(m,m'+1)}}\widehat{\mathscr{E}}^{(m,m'+1)}_{\mathscr{X}})$ is p-torsion free. In particular, the canonical homomorphism

$$i\colon R_{\mathscr{X}}\{\partial\}^{(m,m')}\widehat{\otimes}^{f}_{R_{\mathscr{X}}\{\partial\}^{(m,m'+1)}}\widehat{\mathscr{E}}^{(m,m'+1)}_{\mathscr{X}}\to(R_{\mathscr{X}}\{\partial\}^{(m,m')}\widehat{\otimes}^{f}_{R_{\mathscr{X}}\{\partial\}^{(m,m'+1)}}\widehat{\mathscr{E}}^{(m,m'+1)}_{\mathscr{X}})\otimes\mathbb{Q}$$

is injective.

Now, let $E^{[m,m']}_{\mathscr{X}}:=\rho^{-1}_{m,m'}(\widehat{E}^{(m,m')}_{\mathscr{X}})$ where $\rho_{m,m'}\colon\widehat{E}^{(m,m'+1)}_{\mathscr{X},\mathbb{Q}}\to\widehat{E}^{(m,m')}_{\mathscr{X},\mathbb{Q}}$ is the canonical inclusion (cf. [3, 5.4]). Consider the following diagram.

$$\begin{tikzcd} E^{[m,m']}_{\mathscr{X}} \arrow[r, "j"] \arrow[dr, dotted] & (R_{\mathscr{X}}\{\partial\}^{(m,m')}\widehat{\otimes}^{f}_{R_{\mathscr{X}}\{\partial\}^{(m,m'+1)}}\widehat{E}^{(m,m'+1)}_{\mathscr{X}})\otimes\mathbb{Q} \\ & R_{\mathscr{X}}\{\partial\}^{(m,m')}\widehat{\otimes}^{f}_{R_{\mathscr{X}}\{\partial\}^{(m,m'+1)}}\widehat{E}^{(m,m'+1)}_{\mathscr{X}}. \arrow[u, "i"'] \end{tikzcd}$$

Let us construct the dotted arrow making the diagram commutative. It suffices to see that $\mathrm{Im}(j)\subset\mathrm{Im}(i)$. Let $P:=\sum_{k\in\mathbb{Z}}\partial^k a_k\in E^{[m,m']}_{\mathscr{X}}$ where $a_k\in\mathcal{O}_{\mathscr{X},\mathbb{Q}}$. Since there exists an integer N such that $j(p^N \cdot P) \in \mathrm{Im}(i)$, it suffices to show that $j(\partial^k a_k) \in \mathrm{Im}(i)$ for any integer k, which is easy. Thus j induces the canonical homomorphism

$$E^{[m,m']}_{\mathscr{X}} \to R_{\mathscr{X}}\{\partial\}^{(m,m')}\widehat{\otimes}^f_{R_{\mathscr{X}}\{\partial\}^{(m,m'+1)}}\widehat{E}^{(m,m'+1)}_{\mathscr{X}}$$

of filtered rings since i is injective. By taking the completion with respect to the filtration by order, we get the canonical homomorphism

$$\widehat{\varphi}\colon \widehat{E}^{(m,m')}_{\mathscr{X}} \to R_{\mathscr{X}}\{\partial\}^{(m,m')}\widehat{\otimes}^f_{R_{\mathscr{X}}\{\partial\}^{(m,m'+1)}}\widehat{E}^{(m,m'+1)}_{\mathscr{X}}.$$

We see easily that $\widehat{\varphi}\circ\widehat{\psi} = \mathrm{id}$, $\widehat{\psi}\circ\widehat{\varphi} = \mathrm{id}$ as in the proof of (i), which concludes the proof.

Let us see (iii). The above argument shows that B/B_n is p-torsion free for any $n \in \mathbb{Z}$. Since B/B_n is p-torsion free and the inverse system $\{B/B_n\}_n$ satisfies the Mittag-Leffler condition, we get $(\varprojlim_n B/B_n)\otimes R_i \cong \varprojlim_n (B/B_n \otimes R_i)$ for any i. Let $\widehat{B}$ be the completion with respect to the filtration by order. We get

$$\begin{aligned}\widehat{B}\otimes R_i &\cong (\varprojlim_n B/B_n)\otimes R_i \cong \varprojlim_n (B/B_n\otimes R_i)\\ &\cong \varprojlim_n((B\otimes R_i)/\mathrm{Im}(B_n)) \cong R_{X_i}\{\partial\}^{(m,m')}\widehat{\otimes}^f_{R_{X_i}\{\partial\}^{(m,m'+1)}}E^{(m,m'+1)}_{X_i}.\end{aligned}$$

By using (ii), the claim follows. $\square$

1.3.4 Lemma. *Let $\mathscr{X}$ be a smooth formal curve over R, and $\pi : T^*X \to X$ as usual. Then the algebra $\widehat{\mathscr{E}}^{(m,m')}_{\mathscr{X},\mathbb{Q}}$ is flat over $\pi^{-1}\widehat{\mathscr{D}}^{(m)}_{\mathscr{X},\mathbb{Q}}$.*

Proof. By [3, Corollary 2.9] and [13, 3.5.3], we know that $\widehat{\mathscr{E}}^{(m')}_{\mathscr{X},\mathbb{Q}}$ is flat over $\pi^{-1}\widehat{\mathscr{D}}^{(m)}_{\mathscr{X},\mathbb{Q}}$. It suffices to show that $\mathrm{Tor}_1^{\pi^{-1}\widehat{\mathscr{D}}^{(m)}_{\mathscr{X},\mathbb{Q}}}(\widehat{\mathscr{E}}^{(m,m')}_{\mathscr{X},\mathbb{Q}}, \bullet) = 0$. This amounts to prove

$$\mathrm{Tor}_2^{\pi^{-1}\widehat{\mathscr{D}}^{(m)}_{\mathscr{X},\mathbb{Q}}}(\widehat{\mathscr{E}}^{(m')}_{\mathscr{X},\mathbb{Q}}/\widehat{\mathscr{E}}^{(m,m')}_{\mathscr{X},\mathbb{Q}}, \bullet) = 0$$

by the flatness of $\widehat{\mathscr{E}}^{(m')}_{\mathscr{X},\mathbb{Q}}$. However, by [3, Corollary 5.10 and Lemma 7.8], this is equivalent to showing that

$$\mathrm{Tor}_2^{\pi^{-1}\widehat{\mathscr{D}}^{(m)}_{\mathscr{X},\mathbb{Q}}}(\pi^{-1}\widehat{\mathscr{D}}^{(m')}_{\mathscr{X},\mathbb{Q}}/\widehat{\mathscr{D}}^{(m)}_{\mathscr{X},\mathbb{Q}}, \bullet) = 0,$$

which follows from the flatness of $\widehat{\mathscr{D}}^{(m')}_{\mathscr{X},\mathbb{Q}}$ over $\widehat{\mathscr{D}}^{(m)}_{\mathscr{X},\mathbb{Q}}$. $\square$

Remark. We do not know if $\widehat{\mathscr{E}}^{(m,m')}_{\mathscr{X},\mathbb{Q}}$ is flat over $\widehat{\mathscr{D}}^{(m)}_{\mathscr{X},\mathbb{Q}}$ when the dimension of $\mathscr{X}$ is greater than one and $m' > m$.

1.3.5. Let $\mathscr{X}$ be a connected smooth formal scheme over R. Let η be the generic point of $\mathscr{X}$, and denote by $R(\mathscr{X})$ the integral closure of R in the field $\mathcal{O}_{\mathscr{X},\eta}$. The ring $R(\mathscr{X})$ is a discrete valuation ring as well since it is finite over R and connected. Thus by [37, IV, 17.7.7], $R(\mathscr{X})$ is étale over R. Moreover $R(\mathscr{X})/\varpi$ is the separable closure of k in $\mathcal{O}_{X,\eta}$, and $\mathscr{X}$ is geometrically connected over $R(\mathscr{X})$ by [37, II, 4.5.15]. Put $K(\mathscr{X}) := R(\mathscr{X}) \otimes \mathbb{Q}$. This is an unramified field extension of K. The field $K(\mathscr{X})$ is called the *field of constants of* $\mathscr{X}$. The reason for this naming comes from the next lemma.

Lemma. *Suppose $\mathscr{X}$ is affine and possesses a system of global coordinates $\{x_1, \dots, x_d\}$. We denote the corresponding differential operators by $\{\partial_1, \dots, \partial_d\}$. Then we get*

$$K(\mathscr{X}) = \bigl\{ f \in \mathcal{O}_{\mathscr{X},\mathbb{Q}} \mid \partial_i(f) = 0 \text{ for } 1 \leqslant i \leqslant d \bigr\}.$$

Proof. For this, we may assume that $R = R(\mathscr{X})$. There exists a finite étale extension R' of R such that R' is a discrete valuation ring and $\mathscr{X}' := \mathscr{X} \otimes_R R'$ has a R'-rational point. Note that $\mathscr{X}'$ is also connected and $K(\mathscr{X}') = R' \otimes \mathbb{Q}$. By Galois descent, it suffices to show the lemma for $\mathscr{X}'$. In this case the lemma follows from [37, IV, 17.5.3]. □

1.3.6. Let L be a finite field extension of K. Let I be a connected interval in $\mathbb{R}_{\geqslant 0}$. We denote by $\mathcal{A}_{x,L}(I)$ the ring of analytic functions $\bigl\{\sum_{i\in\mathbb{Z}} \alpha_i x^i \mid \alpha_i \in L, \lim_{i\to\pm\infty} |\alpha_i|\rho^i = 0 \text{ for any } \rho \in I\bigr\}$. For short, we often denote this by $\mathcal{A}(I)$. For real numbers $0 < a \leqslant b < \infty$, we put

$$\mathcal{A}_{x,L}(\{a,b]) := \left\{ \sum_{i\in\mathbb{Z}} \alpha_i x^i \;\middle|\; \alpha_i \in L,\ \sup_{i\in\mathbb{Z}}\bigl\{|\alpha_i| a^i\bigr\} < \infty,\ \lim_{i\to+\infty} |\alpha_i|\, b^i = 0 \right\}.$$

Similarly, we define $\mathcal{A}_{x,L}([a,b\})$. For a series $f(x) = \sum_{i\in\mathbb{Z}} \alpha_i x^i$ in $\mathcal{A}_{x,L}(\{a,b])$ and a real number $c \in [a,b]$, we put $|f(x)|_c := \sup_{i\in\mathbb{Z}}\bigl\{|\alpha_i|\, c^i\bigr\} \in \mathbb{R}$.

Let $\omega_m := p^{-1/p^m(p-1)} < 1$, and $\omega := \omega_0$. Note that if $m' \geqslant m$, we get $\omega < \omega/\omega_{m'} \leqslant \omega/\omega_m \leqslant 1$. Then by the definition of $K\{\partial\}^{(m,m')}$ we get the following explicit description.

Lemma. *Suppose we are in Situation* (L) *of 1.2.1. Let L be the field of constants of $\mathscr{X}$. Then for any non-negative integers $m' \geqslant m$, we have an isomorphism*

$$K_{\mathscr{X}}\{\partial\}^{(m,m')} \xrightarrow{\sim} \mathcal{A}_{x,L}(\{\omega/\omega_{m'}, \omega/\omega_m])$$

sending ∂ to x.

1.3.7. We follow the notation of 1.3.6. Let $f := \sum_{n\in\mathbb{Z}} \alpha_n x^n \in \mathcal{A}_{x,L}(]a,b])$ and set $I := \bigl[-\log_p(b), -\log_p(a)\bigl[$, where $\log_p$ is the logarithm to base p. As in [47, Definition 8.2.1] we define the function $\nu(f)_\rho \colon I \to \mathbb{R}$, $\rho \mapsto -\log_p(|f|_{p^{-\rho}}) = \inf_{n\in\mathbb{Z}}\bigl\{v_p(\alpha_n) + n\rho\bigr\}$, where v_p denotes the valuation of L normalized by $v_p(p) = 1$. If f belongs to $\mathcal{A}_{x,L}(\{a,b])$, $\nu(f)$ is defined also at $-\log_p(a)$. The following lemma generalize [47, Proposition 8.3.2] to the ring[1] $\mathcal{A}_{x,L}(\{a,b])$.

Lemma. *Assume $a \leqslant b \in |\overline{K}^\times|$. For every f in $\mathcal{A}_{x,L}(\{a,b])$, we have the following:*

(i) *The number of slopes of the graph of the function $\nu(f)_\rho$ is finite.*

[1] In [47] $\mathcal{A}_{x,L}(\{a,b])$ is denoted by $L[\![a/x, x/b\rangle_0$ or $L\langle b^{-1}/t, t/a^{-1}]\!]_0$ with $t = x^{-1}$.

(ii) *(Weierstrass preparation) There exists $P \in L[x]$ and $g \in \mathcal{A}_{x,L}(\{a,b])^*$ such that $f = Pg$.*

Moreover, the ring $\mathcal{A}_{x,L}(\{a,b])$ is a principal ideal domain and, if $a = b$, it is a field.

Proof. Let us show (i). Since $a, b \in |\overline{K}|$ and $\nu(f)_\rho$ does not change under extensions of L, we may assume $a, b \in |L|$. Take $\gamma \in L$ such that $|\gamma| = a$. Changing the coordinate from x to γx, we get $\nu(f(\gamma x))_\rho = \nu(f(x))_{a+\rho}$ and $f(\gamma x) \in \mathcal{A}_{x,L}(\{1, b/a])$, so we may assume $a = 1$. It is enough to prove that the number of slopes of $\nu(f)_\rho$ is finite in a left neighbourhood of $-\log_p(a) = 0$. The valuation of L is discrete, so there exists an integer n_0 such that $v_p(\alpha_{n_0}) = \inf_{n\in\mathbb{Z}}\{v_p(\alpha_n)\} = \nu(f)_0$. On the other hand, take $c \in]1, b[$, and denote $\rho_c := -\log_p(c) < 0$. Take an integer N such that $v_p(\alpha_N) + N\rho_c = \nu(f)_{\rho_c}$. Then we have

$$v_p(\alpha_N) \geqslant v_p(\alpha_{n_0}) \geqslant v_p(\alpha_N) + \rho_c \cdot (N - n_0),$$

where the first (resp. second) inequality holds by the definition of n_0 (resp. N). This implies that $n_0 \leqslant N$, which concludes the proof of (i).

Thanks to (i), the Newton polygon of f (see [47, 8.2.2] for the definition) also has a finite number of slopes by [47, Remark 2.1.7 and proof of Proposition 8.2.3]. Therefore, the proof of (ii) goes exactly as in Proposition 8.3.2 of [47] by using Proposition 8.3.1. The rest of the proof is now classical: by Weierstrass preparation, every ideal of $\mathcal{A}_{x,L}(\{a,b])$ is spanned by polynomials in $L[x]$, and hence principal. Finally, if $a = b$, it is enough to show that every non zero polynomial in $L[x]$ is invertible in $\mathcal{A}_{x,L}(\{a,a])$. By a finite extension of L, we can reduce to checking this for degree one polynomials, which is straightforward recalling that $\mathcal{A}_{x,L}(\{a,a])$ is a Banach space with respect to $|\cdot|_a$, cf. [47, Proposition 8.2.5]. □

Remark. The above lemma (i) and (ii) are stated in [47, Chapter 8, Exercise (4)], but we point out that they are not true for general $a, b \in \mathbb{R}$. Indeed, take a such that $\log_p(a)$ is a Liouville number in $\mathbb{R}$. Then, we can take a series $f(x) = \sum_{n\in\mathbb{Z}} \alpha_n x^n \in \mathcal{A}_{x,\mathbb{Q}_p}(\{a,b])$ such that the infimum $-\log_p(|f(x)|_a) := \inf_{n\in\mathbb{Z}}\{v_p(\alpha_n) - n\log_p(a)\}$ cannot be attained by any n (*e.g.* $\log_p(a) := \sum_{n=0}^{+\infty} 10^{-n!}$, $b \geqslant a$, and $f(x) := \sum_{n<0} \alpha_n x^n$, with $\alpha_n = p^{\lceil n \log_p(a)\rceil}$ for $n < 0$). By construction, the graph of $\nu(f)_\rho$ has infinitely many slopes approaching $-\log_p(a)$ on the left.

For similar reasons it is also necessary that the valuation of L be discrete, as stated in [47, Remark 8.2.4].

Corollary. *Suppose we are in Situation* (L)*. For any non-negative integers $m' \geqslant m$, the commutative ring $K_{\mathscr{X}}\{\partial\}^{(m,m')}$ is a principal ideal domain. Moreover, $K_{\mathscr{X}}\{\partial\}^{(m)}$ is a field.*

Proof. Let L be the field of constants of $\mathscr{X}$. Apply Lemmas 1.3.6 and 1.3.7. □

By this corollary, we get that any finitely generated $K_{\mathscr{X}}\{\partial\}^{(m,m')}$-module with a connection is a free $K_{\mathscr{X}}\{\partial\}^{(m,m')}$-module by [24, Corollaire 4.3].

1.3.8. Let $\mathscr{X}$ be a formal curve, and $\mathscr{M}$ be a coherent $\widehat{\mathscr{D}}^{(m)}_{\mathscr{X},\mathbb{Q}}$-module. We say that $\mathscr{M}$ is *holonomic* if the dimension of $\mathrm{Char}^{(m)}(\mathscr{M})$ is one or if $\mathscr{M}=0$. For an integer $m'\geqslant m$, let $\mathscr{M}^{(m')}:=\widehat{\mathscr{D}}^{(m')}_{\mathscr{X},\mathbb{Q}}\otimes\mathscr{M}$. We say $\mathscr{M}$ is *stable* if for any $m''\geqslant m'\geqslant m$, we have

$$\mathrm{Supp}(\widehat{\mathscr{E}}^{(m',m'')}_{\mathscr{X},\mathbb{Q}}\otimes_{\pi^{-1}\widehat{\mathscr{D}}^{(m)}_{\mathscr{X},\mathbb{Q}}}\pi^{-1}\mathscr{M})=\mathrm{Char}^{(m)}(\mathscr{M}).$$

In particular, we have $\mathrm{Char}^{(m')}(\mathscr{M}^{(m')})=\mathrm{Char}^{(m)}(\mathscr{M})$. By Theorem 1.1.3, any coherent $\widehat{\mathscr{D}}^{(m)}_{\mathscr{X},\mathbb{Q}}$-module is stable after raising the level sufficiently. We say that a point $s\in X$ is a *singular point of* $\mathscr{M}$ if $\pi^{-1}(s)\subset\mathrm{Char}^{(m)}(\mathscr{M})$. From now on, to avoid too heavy notation, we sometimes denote $\mathrm{Char}^{(m)}$ just by Char. For a coherent $\mathscr{D}^{\dagger}_{\mathscr{X},\mathbb{Q}}$-module $\mathscr{M}$, there exists a stable coherent $\widehat{\mathscr{D}}^{(m)}_{\mathscr{X},\mathbb{Q}}$-module $\mathscr{M}'$ for some m such that $\mathscr{D}^{\dagger}_{\mathscr{X},\mathbb{Q}}\otimes\mathscr{M}'\cong\mathscr{M}$. We define $\mathrm{Char}(\mathscr{M}):=\mathrm{Char}^{(m)}(\mathscr{M}')$, and we say that $\mathscr{M}$ *is holonomic* if $\mathscr{M}'$ is. We define the set of singular points of $\mathscr{M}$ as that of $\mathscr{M}'$. Note that when $\mathscr{M}$ is a coherent F-$\mathscr{D}^{\dagger}_{\mathscr{X},\mathbb{Q}}$-module (cf. 3.1.1), the definition of holonomicity is equivalent to that of Berthelot as written in [3, Corollary 7.5]. For the later use, we remind here the following lemma.

1.3.9 Lemma. *Suppose we are in Situation* (L) *of 1.2.1. Let $\mathscr{M}$ be a monogenic stable holonomic $\widehat{\mathscr{D}}^{(m)}_{\mathscr{X},\mathbb{Q}}$-module, and $\alpha\in\Gamma(\mathscr{X},\mathscr{M})$ be a generator. Let S be the set of singular points of $\mathscr{M}$. Take $s\in S$, and let y_s be a local parameter of $\mathcal{O}_{\mathscr{X},s}$. Then for any integers $m''\geqslant m'\geqslant m$, there exists an integer N such that $\{x^i y_s^j\,\alpha\}_{0\leqslant i,j<N}$ generate $(\widehat{\mathscr{E}}^{(m',m'')}_{\mathscr{X},\mathbb{Q}}\otimes\mathscr{M})_s$ over $K_{\mathscr{X}}\{\partial\}^{(m',m'')}$.*

Proof. We may shrink $\mathscr{X}$ so that we are in Situation (Ls) and $S=\{s\}$. Then this is just a direct consequence of Lemma 1.3.2(ii). □

1.3.10. In this paragraph, we consider Situation (L) of 1.2.1 and we follow the notation fixed there. Let $\mathscr{M}$ be a stable holonomic $\widehat{\mathscr{D}}^{(m)}_{\mathscr{X},\mathbb{Q}}$-module and s be a closed point of $\mathscr{X}$ such that $\mathrm{Char}(\mathscr{M})\supset\pi^{-1}(s)$. We consider then $\widehat{\mathscr{E}}^{(m,m')}_{s,\mathbb{Q}}$-module $\widehat{\mathscr{E}}^{(m,m')}_{s,\mathbb{Q}}\otimes_{\widehat{\mathscr{D}}^{(m)}}\mathscr{M}$ (cf. Notation 1.2.1.3). It can be seen as a $K_{\mathscr{X}}\{\partial\}^{(m,m')}$-module. When we are especially interested in this $K_{\mathscr{X}}\{\partial\}^{(m,m')}$-module structure, we denote this module by $\widehat{\mathscr{E}}^{(m,m')}_{s,\mathbb{Q}}(\mathscr{M})$. *We caution here that this definition is only for this section, and from* 2.4.2.1, *we use the same notation for a slightly different object.* In the same way, for a $\widehat{\mathscr{D}}^{(m)}_{\mathscr{X}}$-module $\mathscr{M}'$, we put $\widehat{\mathscr{E}}^{(m,m')}_{s}(\mathscr{M}'):=\widehat{\mathscr{E}}^{(m,m')}_{s}\otimes_{\widehat{\mathscr{D}}^{(m)}}\mathscr{M}'$ and the same for $\mathscr{D}^{(m)}_{X}$-modules *etc.*

By the condition on the characteristic variety and Lemma 1.3.9, we get that $\widehat{\mathscr{E}}^{(m,m')}_{s,\mathbb{Q}}(\mathscr{M})$ is finitely generated over $K_{\mathscr{X}}\{\partial\}^{(m,m')}$. Let L be the field of constants of $\mathscr{X}$. We have the isomorphism of Lemma 1.3.6

$$\mathcal{A}^{(m,m')}_{x',L}:=\mathcal{A}_{x',L}(\{\omega/\omega_{m'},\omega/\omega_m])\xrightarrow{\sim}K_{\mathscr{X}}\{\partial\}^{(m,m')}$$

sending x' to ∂. We consider $\widehat{\mathscr{E}}^{(m,m')}_{s,\mathbb{Q}}(\mathscr{M})$ as a finitely generated $\mathcal{A}^{(m,m')}_{x',L}$-module using this isomorphism, and equip it with the following connection: for $\alpha\in\widehat{\mathscr{E}}^{(m,m')}_{s,\mathbb{Q}}(\mathscr{M})$, we put

$$\nabla(\alpha):=(-x\alpha)\otimes\mathrm{d}x'.$$

Lemma. *The $\mathcal{A}^{(m,m')}_{x',L}$-module $\widehat{\mathscr{E}}^{(m,m')}_{s,\mathbb{Q}}(\mathscr{M})$ is finite free and ∇ is a connection. We denote its rank by* $\mathrm{rk}(\widehat{\mathscr{E}}^{(m,m')}_{s,\mathbb{Q}}(\mathscr{M}))$.

Proof. Let us check that ∇ defines a connection. The additivity is evident and Leibniz rule follows from the relation $\partial\cdot(-x)+1=(-x)\cdot\partial$ in the ring of microdifferentials. The $\mathcal{A}^{(m,m')}_{x',L}$-module $\widehat{\mathscr{E}}^{(m,m')}_{s,\mathbb{Q}}(\mathscr{M})$ is finitely generated by Lemma 1.3.9, and it is endowed with a connection, hence it is torsion free by [28, 6.1]. Since the ring $\mathcal{A}^{(m,m')}_{x',L}\cong K_{\mathscr{X}}\{\partial\}^{(m,m')}$ is principal by Lemma 1.3.7, it follows that $\widehat{\mathscr{E}}^{(m,m')}_{s,\mathbb{Q}}(\mathscr{M})$ is finite free. □

1.3.11 Proposition. *Suppose we are in Situation* (L) *of 1.2.1. Let $\mathscr{M}$ be a stable holonomic $\widehat{\mathscr{D}}^{(m)}_{\mathscr{X},\mathbb{Q}}$-module. Let S be the set of singular points of $\mathscr{M}$.*

(i) *For an integer $m'>m$ and $s\in S$, we get an isomorphism*

$$\widehat{\mathscr{E}}^{(m+1,m')}_{s,\mathbb{Q}}\otimes_{\widehat{\mathscr{E}}^{(m,m')}_{s,\mathbb{Q}}}\widehat{\mathscr{E}}^{(m,m')}_{s,\mathbb{Q}}(\mathscr{M})\cong K_{\mathscr{X}}\{\partial\}^{(m+1,m')}\otimes_{K_{\mathscr{X}}\{\partial\}^{(m,m')}}\widehat{\mathscr{E}}^{(m,m')}_{s,\mathbb{Q}}(\mathscr{M}).$$

In particular, $\mathrm{rk}(\widehat{\mathscr{E}}^{(m,m')}_{s,\mathbb{Q}}(\mathscr{M}))=\mathrm{rk}(\widehat{\mathscr{E}}^{(m+1,m')}_{s,\mathbb{Q}}(\mathscr{M}))$.

(ii) *For $m'\geqslant m$ and $s\in S$, we get an isomorphism*

$$\widehat{\mathscr{E}}^{(m,m')}_{s,\mathbb{Q}}\otimes_{\widehat{\mathscr{E}}^{(m,m'+1)}_{s,\mathbb{Q}}}\widehat{\mathscr{E}}^{(m,m'+1)}_{s,\mathbb{Q}}(\mathscr{M})\cong K_{\mathscr{X}}\{\partial\}^{(m,m')}\otimes_{K_{\mathscr{X}}\{\partial\}^{(m,m'+1)}}\widehat{\mathscr{E}}^{(m,m'+1)}_{s,\mathbb{Q}}(\mathscr{M}).$$

In particular, $\mathrm{rk}(\widehat{\mathscr{E}}^{(m,m'+1)}_{s,\mathbb{Q}}(\mathscr{M}))=\mathrm{rk}(\widehat{\mathscr{E}}^{(m,m')}_{s,\mathbb{Q}}(\mathscr{M}))$.

Proof. Let us see (i). Let $\mathscr{U}$ be an open affine neighbourhood of s such that $S\cap\mathscr{U}=\{s\}$. We put $M':=\Gamma(\mathring{T}^*U,\widehat{\mathscr{E}}^{(m,m')}_{\mathscr{X},\mathbb{Q}}\otimes\mathscr{M})$, where U denotes the special fibre of $\mathscr{U}$. Since tensor products commute with direct limits, it suffices to show that

$$\widehat{E}^{(m+1,m')}_{\mathscr{U},\mathbb{Q}}\otimes_{\widehat{E}^{(m,m')}_{\mathscr{U},\mathbb{Q}}}M'\cong K_{\mathscr{X}}\{\partial\}^{(m+1,m')}\otimes_{K_{\mathscr{X}}\{\partial\}^{(m,m')}}M'.$$

By Lemma 1.3.3(i), we get

$$\begin{aligned}\widehat{E}^{(m+1,m')}_{\mathscr{U},\mathbb{Q}}\otimes_{\widehat{E}^{(m,m')}_{\mathscr{U},\mathbb{Q}}}M'&\cong\widehat{E}^{(m+1,m')}_{\mathscr{U},\mathbb{Q}}\widehat{\otimes}_{\widehat{E}^{(m,m')}_{\mathscr{U},\mathbb{Q}}}M'\\&\cong(K_{\mathscr{X}}\{\partial\}^{(m+1,m')}\widehat{\otimes}_{K_{\mathscr{X}}\{\partial\}^{(m,m')}}\widehat{E}^{(m,m')}_{\mathscr{U},\mathbb{Q}})\widehat{\otimes}_{\widehat{E}^{(m,m')}_{\mathscr{U},\mathbb{Q}}}M'\\&\cong K_{\mathscr{X}}\{\partial\}^{(m+1,m')}\widehat{\otimes}_{K_{\mathscr{X}}\{\partial\}^{(m,m')}}M'\\&\cong K_{\mathscr{X}}\{\partial\}^{(m+1,m')}\otimes_{K_{\mathscr{X}}\{\partial\}^{(m,m')}}M'.\end{aligned}$$

The last isomorphism follows from the fact that $\widehat{\mathscr{E}}^{(m,m')}_{s,\mathbb{Q}}(\mathscr{M})$ is finite over $K_{\mathscr{X}}\{\partial\}^{(m,m')}$. Since moreover $\widehat{\mathscr{E}}^{(m,m')}_{s,\mathbb{Q}}(\mathscr{M})$ is free over $K_{\mathscr{X}}\{\partial\}^{(m,m')}$, the claim for the rank follows from the preceding isomorphism.

Let us prove (ii). Since we know that $\widehat{\mathscr{E}}^{(m,m')}_{s,\mathbb{Q}}$ is flat over $\widehat{\mathscr{E}}^{(m,m'+1)}_{s,\mathbb{Q}}$ by [3, Theorem 5.12], we may suppose that $\mathscr{M}$ is a monogenic module using an extension argument. Let $\mathscr{U}$ be an open affine neighbourhood of s such that $S\cap\mathscr{U}=\{s\}$, and U be its special fibre.

As in the proof of (i), it suffices to show the claim over $\mathscr{U}$. By using Lemma 1.3.2, there exists a p-torsion free $\widehat{E}_{\mathscr{U}}^{(m,m'+1)}$-module M' such that $\Gamma(\mathring{T}^*\mathscr{U}, \widehat{\mathscr{E}}_{\mathscr{X},\mathbb{Q}}^{(m,m'+1)} \otimes \mathscr{M}) \cong M' \otimes \mathbb{Q}$ and which is finitely generated as an $R\{\partial\}^{(m,m'+1)}$-module. Now, we get

$$\widehat{E}_{\mathscr{U}}^{(m,m')} \otimes_{\widehat{E}_{\mathscr{U}}^{(m,m'+1)}} M' \cong \varprojlim_i E_{U_i}^{(m,m')} \otimes_{E_{U_i}^{(m,m'+1)}} M'_i$$
$$\cong \varprojlim E_{U_i}^{(m,m')} \widehat{\otimes}^f_{E_{U_i}^{(m,m'+1)}} M'_i. \tag{1.3.11.1}$$

Indeed, the first isomorphism holds since $\widehat{E}_{\mathscr{U}}^{(m,m')} \otimes M'$ is p-adically complete. To check the second isomorphism, take a good filtration on M'_i. The tensor filtration is good by [51, Chapter I, Lemma 6.15]. Thus $E_{U_i}^{(m,m')} \otimes_{E_{U_i}^{(m,m'+1)}} M'_i$ is complete by [51, Chapter II, Theorem 10] since $E_{U_i}^{(m,m')}$ is a noetherian filtered complete ring by [3, Proposition 4.9]. Now by Lemma 1.3.3(iii), we get

$$E_{U_i}^{(m,m')} \widehat{\otimes}^f_{E_{U_i}^{(m,m'+1)}} M'_i \cong R_{X_i}\{\partial\}^{(m,m')} \widehat{\otimes}^f_{R_{X_i}\{\partial\}^{(m,m'+1)}} M_i$$

by the same calculation as in the proof of (i). (For careful readers, we note here that the same statement of [19, 2.1.7/6,7] holds for filtered rings by exactly the same arguments. The detail is left to the reader.) Thus by the same calculation as 1.3.11.1, we get

$$\widehat{E}_{\mathscr{U}}^{(m,m')} \otimes_{\widehat{E}_{\mathscr{U}}^{(m,m'+1)}} M' \cong R_{\mathscr{X}}\{\partial\}^{(m,m')} \otimes_{R_{\mathscr{X}}\{\partial\}^{(m,m'+1)}} M'.$$

By tensoring with $\mathbb{Q}$, we get what we wanted. □

1.4. Characteristic cycles and microlocalizations

We will see how we can compute the multiplicities of holonomic modules from its microlocalizations. In general, it is very difficult to calculate the characteristic cycles in terms of intermediate microlocalizations. However, the construction of the rings of *naive* microdifferential operators are simple and formal, and we can calculate the multiplicities easier.

1.4.1. For a graded ring $(A, F_i A)_{i\in\mathbb{Z}}$ and a finite graded A-module $(M, F_i M)_{i\in\mathbb{Z}}$, the graded length of M is the length of M in the category of graded A-modules, and we denote it by $\mathrm{g.lg}_A(M)$. When $\mathrm{g.lg}_A(M) = 1$, we say that M is *gr-simple*. We say that A is *gr-Artinian* if $\mathrm{g.lg}_A(A) < \infty$.

Let A be a positively graded commutative ring, and M be a finite graded A-module. Let $\mathfrak{p} \in \mathrm{Proj}(A)$, and

$$S_{\mathfrak{p}} := \{1\} \cup \big\{ f \in A \setminus A_0 \mid f \text{ is a homogeneous element which is not contained in } \mathfrak{p} \big\}.$$

We denote by $A_{\widetilde{\mathfrak{p}}}$ the localization $S_{\mathfrak{p}}^{-1}A$, and $S_{\mathfrak{p}}^{-1}M$ by $M_{\widetilde{\mathfrak{p}}}$. We note that since $S_{\mathfrak{p}}$ consists of homogeneous elements, these are respectively a graded ring and a graded module. Let $X := \mathrm{Spec}(A_0)$, $V := \mathrm{Spec}(A)$, $P := \mathrm{Proj}(A)$. The schemes V and P are schemes over X, and there exists a canonical section $s\colon X \to V$. We put $\mathring{V} := V \setminus s(X)$. Let us denote by $q\colon \mathring{V} \to P$ the canonical surjection defined in [37, II, 8.3]. Now we get the following.

Lemma. *Let* $\widetilde{M}(*) := \bigoplus_{n\in\mathbb{Z}} \widetilde{M}(n)$ *be the quasi-coherent* $\mathcal{O}_P$*-module associated with* M*. Let* $\mathfrak{p}$ *be a generic point of* $\mathrm{Supp}(\widetilde{M}(*)) \subset P$*, and* $\mathfrak{q}$ *be a generic point of the fibre* $q^{-1}(\mathfrak{p})$*. Then we get*

$$\mathrm{g.lg}_{A_{\widetilde{\mathfrak{p}}}}(M_{\widetilde{\mathfrak{p}}}) = \mathrm{lg}_{A_{\mathfrak{q}}}(M_{\mathfrak{q}}).$$

Proof. By [37, II, 8.3.6], we get that the fibre of q at $\mathfrak{p}$ is $f\colon \mathrm{Spec}(A_{\widetilde{\mathfrak{p}}}) \to \mathrm{Spec}(A_{(\mathfrak{p})})$, and $A_{\widetilde{\mathfrak{p}}}$ is (non-canonically) isomorphic to $A_{(\mathfrak{p})}[t, t^{-1}]$. This implies that $\mathfrak{q}A_{\widetilde{\mathfrak{p}}} = \mathfrak{p}A_{\widetilde{\mathfrak{p}}}$. Since $f_*((M_{\widetilde{\mathfrak{p}}})^{\sim}) \cong \widetilde{M}(*)$, the support of the sheaf $(M_{\widetilde{\mathfrak{p}}})^{\sim}$ in $\mathrm{Spec}(A_{\widetilde{\mathfrak{p}}})$ is contained in $V(\mathfrak{p})$, and there exists an integer n such that $\mathfrak{p}^n M_{\widetilde{\mathfrak{p}}} = 0$. Thus $M_{\widetilde{\mathfrak{p}}}$ is a graded $A_{\widetilde{\mathfrak{p}}}/A_{\widetilde{\mathfrak{p}}}\mathfrak{p}^n$-module. Let N be a graded $A_{\widetilde{\mathfrak{p}}}/A_{\widetilde{\mathfrak{p}}}\mathfrak{p}^n$-module such that $N \neq 0$. Then $N_{\mathfrak{q}} \neq 0$ by the definition of $A_{\widetilde{\mathfrak{p}}}$. This shows that given a chain $0 \subsetneq N_1 \subsetneq \cdots \subsetneq N_l = M_{\widetilde{\mathfrak{p}}}$ of graded sub-$A_{\widetilde{\mathfrak{p}}}$-modules, we can attach a chain $0 \subsetneq (N_1)_{\mathfrak{q}} \subsetneq \cdots \subsetneq (N_l)_{\mathfrak{q}} = M_{\mathfrak{q}}$ of sub-$A_{\mathfrak{q}}$-modules. Thus we see that $\mathrm{g.lg}_{A_{\widetilde{\mathfrak{p}}}}(M_{\widetilde{\mathfrak{p}}}) \leqslant \mathrm{lg}_{A_{\mathfrak{q}}}(M_{\mathfrak{q}})$. To see the opposite inequality, it suffices to show that given a $A_{\widetilde{\mathfrak{p}}}/\mathfrak{p}^n$-module N such that $\mathrm{g.lg}_{A_{\widetilde{\mathfrak{p}}}}(N) = 1$ then $\mathrm{lg}_{A_{\mathfrak{q}}}(N_{\mathfrak{q}}) = 1$. Since $A_{\widetilde{\mathfrak{p}}}/\mathfrak{p}$ is the only gr-simple $A_{\widetilde{\mathfrak{p}}}/\mathfrak{p}^n$-module, it suffices to show that $\mathrm{lg}(A_{\mathfrak{q}} \otimes_{A_{\widetilde{\mathfrak{p}}}} (A_{\widetilde{\mathfrak{p}}}/\mathfrak{p})) = 1$, which follows by the fact that $\mathfrak{q}A_{\widetilde{\mathfrak{p}}} = \mathfrak{p}A_{\widetilde{\mathfrak{p}}}$. $\square$

1.4.2. Let A be a noetherian filtered ring, and M be a finite A-module. Let F be a good filtration on M. Then consider $\mathrm{g.lg}_{\mathrm{gr}(A)}(\mathrm{gr}(M))$. Exactly as in the classical way (e.g. [20, A.III.3.23]), we are able to show that this does not depend on the choice of good filtrations. Recall that we say an increasingly filtered ring $(A, F_iA)_{i\in\mathbb{Z}}$ is *Zariskian* if it is noetherian filtered and $F_{-1}A$ is contained in the Jacobson radical $J(F_0A)$ of F_0A.

Example. The field $A = k[\![t^{-1}]\!][t]$ of formal Laurent series in t^{-1} is Zariskian, filtered by the filtration given by the degree in t. We have $\mathrm{gr}(A) = k[t^{-1}, t]$, which is gr-Artinian, of graded length one, and $\mathrm{lg}_A(A) = \mathrm{g.lg}_{\mathrm{gr}(A)}(\mathrm{gr}(A)) = 1$.

Lemma. *Let* A *be a Zariskian filtered ring such that* $\mathrm{gr}(A)$ *is a gr-Artinian ring. Suppose moreover that* $\mathrm{lg}_A(A)$ *is finite and*

$$\mathrm{lg}_A(A) = \mathrm{g.lg}_{\mathrm{gr}(A)}(\mathrm{gr}(A)).$$

Then for any good filtered A*-module* $(M, M_i)_{i\in\mathbb{Z}}$*, we get*

$$\mathrm{lg}_A(M) = \mathrm{g.lg}_{\mathrm{gr}(A)}(\mathrm{gr}(M)).$$

Proof. First, let $(M, M_i)_{i\in\mathbb{Z}}$ be a good filtered A-module and take a sequence of sub-A-modules

$$M \supsetneq M^{(1)} \supsetneq \cdots \supsetneq M^{(l)} = 0.$$

We equip $M^{(k)}$ with the induced filtration. These filtrations are good by [51, Chapter II, 2.1.2] since A is a Zariskian filtered ring. Since the filtration on $M^{(k)}/M^{(k+1)}$ is good and A is a Zariskian filtered ring, the filtration is separated, and $\mathrm{gr}(M^{(k)}/M^{(k+1)}) \neq 0$. Thus we get the strictly decreasing sequence

$$\mathrm{gr}(M) \supsetneq \mathrm{gr}(M^{(1)}) \supsetneq \cdots \supsetneq \mathrm{gr}(M^{(l)}) = 0.$$

This shows that $\lg_A(M) \leqslant \mathrm{g.lg}_{\mathrm{gr}(A)}(\mathrm{gr}(M))$. It suffices to show that if (N, N_i) is a good filtered A-module such that N is a simple A-module, then $\mathrm{gr}(N)$ is a gr-simple $\mathrm{gr}(A)$-module.

Let

$$A =: I^{(0)} \supsetneq I^{(1)} \supsetneq \cdots \supsetneq I^{(l)} = 0$$

be a composition series. The hypothesis and the above observation imply that

$$\mathrm{gr}(A) \supsetneq \mathrm{gr}(I^{(1)}) \supsetneq \cdots \supsetneq \mathrm{gr}(I^{(l)}) = 0$$

is also a composition series in the category of graded modules. Since any simple graded A-module is appearing in the series, we get that for any simple A-module N, there exists a good filtration on N such that the $\mathrm{gr}(A)$-module $\mathrm{gr}(N)$ is gr-simple. Indeed, there exists $0 \leqslant k < l$ such that $N \cong I^{(k)}/I^{(k+1)}$. We put the good filtration induced by that of $I^{(k)}/I^{(k+1)}$ on N. Then since $\mathrm{gr}(I^{(k)}/I^{(k+1)})$ is a gr-simple $\mathrm{gr}(A)$-module, we get the claim. Since for any good filtrations F and G on N, we know that $\mathrm{gr}_F(N)$ and $\mathrm{gr}_G(N)$ have the same gr-length, we get that $\mathrm{gr}(N)$ is gr-simple for any good filtration. This concludes the proof of the lemma. $\square$

1.4.3. Let us consider Situation (L) of 1.2.1. Recall the notation of 1.3.10. Let $\mathscr{M}$ be a holonomic $\widehat{\mathscr{D}}^{(m)}_{\mathscr{X},\mathbb{Q}}$-module (not necessarily stable). Let $\mathrm{Cycl}^{(m)}(\mathscr{M}) = \sum_{s\in S} m_s \cdot [\pi^{-1}(s)] + r\cdot[X]$ be the characteristic cycle of $\mathscr{M}$ (cf. [2, 2.1.17]). We note that, by construction, $\mathrm{Cycl}^{(m)}$ is additive with respect to short exact sequences of $\widehat{\mathscr{D}}^{(m)}_{\mathscr{X},\mathbb{Q}}$-modules. The integer m_s is called the *vertical multiplicity* of $\mathscr{M}$ at s and the integer r is called the *generic rank* (or *horizontal multiplicity*) of $\mathscr{M}$.

Proposition. *We get*

$$p^m \cdot \mathrm{rk}_{K_{\mathscr{X}}\{\partial\}^{(m)}}(\widehat{\mathscr{E}}^{(m)}_{s,\mathbb{Q}}(\mathscr{M})) = \deg_L(s)\cdot m_s,$$

where $L := K(\mathscr{X})$ *and* $\deg_L(s) := \deg(s)\cdot[L:K]^{-1}$.

Proof. We may assume that $\mathscr{M}$ is a monogenic module by an extension argument using the flatness of $\widehat{\mathscr{E}}^{(m)}_{\mathscr{X},\mathbb{Q}}$ over $\pi^{-1}\widehat{\mathscr{D}}^{(m)}_{\mathscr{X},\mathbb{Q}}$, cf. Lemma 1.3.4. Let $\mathscr{M}'$ be a monogenic $\widehat{\mathscr{D}}^{(m)}_{\mathscr{X}}$-module without p-torsion such that $\mathscr{M}'\otimes\mathbb{Q} \cong \mathscr{M}$. Since $\widehat{\mathscr{E}}^{(m)}_{\mathscr{X}}$ is flat over $\widehat{\mathscr{D}}^{(m)}_{\mathscr{X}}$ by [3, Proposition 2.8(ii)], we note that $\widehat{\mathscr{E}}^{(m)}_s(\mathscr{M}')$ is also p-torsion free. Let $L\{\partial^{\langle p^m\rangle_{(m)}}\}^{(0)}$ be the subring of $K_{\mathscr{X}}\{\partial\}^{(m)}$ topologically generated by $\partial^{\langle \pm p^m\rangle_{(m)}}$ over L. Then we get

$$p^m \cdot \mathrm{rk}_{K_{\mathscr{X}}\{\partial\}^{(m)}}(\widehat{\mathscr{E}}^{(m)}_{s,\mathbb{Q}}(\mathscr{M})) = \mathrm{rk}_{L\{\partial^{\langle p^m\rangle_{(m)}}\}^{(0)}}(\widehat{\mathscr{E}}^{(m)}_{s,\mathbb{Q}}(\mathscr{M})).$$

We know that $\widehat{\mathscr{E}}^{(m)}_s(\mathscr{M}')$ is finite over $R_{\mathscr{X}}\{\partial\}^{(m)}$ by Lemma 1.3.2, and in particular, finite over $R_L\{\partial^{\langle p^m\rangle_{(m)}}\}^{(0)} := L\{\partial^{\langle p^m\rangle_{(m)}}\}^{(0)} \cap R_{\mathscr{X}}\{\partial\}^{(m)}$. Since $R_L\{\partial^{\langle p^m\rangle_{(m)}}\}^{(0)}$ is a discrete valuation ring whose uniformizer is ϖ and $\widehat{\mathscr{E}}^{(m)}_s(\mathscr{M}')$ is p-torsion free, we get that $\widehat{\mathscr{E}}^{(m)}_s(\mathscr{M}')$ is free over $R_L\{\partial^{\langle p^m\rangle_{(m)}}\}^{(0)}$. Thus, we get

$$\mathrm{rk}_{L\{\partial^{\langle p^m\rangle_{(m)}}\}^{(0)}}(\widehat{\mathscr{E}}^{(m)}_{s,\mathbb{Q}}(\mathscr{M})) = \mathrm{rk}_{R_L\{\partial^{\langle p^m\rangle_{(m)}}\}^{(0)}}(\widehat{\mathscr{E}}^{(m)}_{s}(\mathscr{M}'))$$
$$= \deg(L/K)^{-1}\cdot \mathrm{rk}_{k[\![\partial^{\langle -p^m\rangle_{(m)}}]\!][\partial^{\langle p^m\rangle_{(m)}}]}(\widehat{\mathscr{E}}^{(m)}_{s}(\mathscr{M}/\varpi)).$$

Here $k[\![\partial^{\langle -p^m\rangle_{(m)}}]\!][\partial^{\langle p^m\rangle_{(m)}}]$ is considered as a subring of $R_L\{\partial^{\langle p^m\rangle_{(m)}}\}^{(0)}/\varpi$. It is the ring given in Example 1.4.2, with $t := \partial^{\langle p^m\rangle_{(m)}}$, so it satisfies the hypotheses of Lemma 1.4.2. Now, we put $\mathscr{N} := \mathscr{M}'/\varpi$. Take a good filtration $F_\bullet$ of $\mathscr{N}$. Let ξ_s be the generic point of the fibre $\pi^{-1}(s)$. Then $(\mathrm{gr}^F(\mathscr{N}))_{\xi_s}$ is an Artinian $(\mathcal{O}_{T^{(m)*}X})_{\xi_s}$-module. Note that $\mathscr{E}^{(m)}_s(\mathscr{N})$ possesses a natural filtration induced by the good filtration of $\mathscr{N}$ (cf. [3, 1.6] or [49, A.3.2.4]). Thus, we get

$$m_s = \mathrm{lg}_{(\mathcal{O}_{T^{(m)*}X})_{\xi_s}}(\mathrm{gr}^F(\mathscr{N})_{\xi_s}) = \mathrm{g.lg}_{\mathrm{gr}(\mathscr{E}^{(m)}_{X,\xi_s})}\big(\mathrm{gr}(\mathscr{E}^{(m)}_s(\mathscr{N}))\big)$$

by using Lemma 1.4.1. Since the injection $\mathcal{O}_{X,s}[\xi^{\langle \pm p^m\rangle_{(m)}}] \hookrightarrow \mathrm{gr}(\mathscr{E}^{(m)}_{X,\xi_s})$ induces the isomorphism $\mathcal{O}_{X,s}[\xi^{\langle \pm p^m\rangle_{(m)}}] \xrightarrow{\sim} \big(\mathrm{gr}(\mathscr{E}^{(m)}_{X,\xi_s})\big)^{\mathrm{red}}$ where $\xi^{\langle \pm p^m\rangle_{(m)}}$ denotes the class of $\partial^{\langle \pm p^m\rangle_{(m)}}$ in $\mathrm{gr}(\mathscr{E}^{(m)}_{X,\xi_s})$, we get

$$\mathrm{g.lg}_{\mathrm{gr}(\mathscr{E}^{(m)}_{X,\xi_s})}\big(\mathrm{gr}(\mathscr{E}^{(m)}_s(\mathscr{N}))\big) = \mathrm{g.lg}_{\mathcal{O}_{X,s}[\partial^{\langle \pm p^m\rangle_{(m)}}]}\big(\mathrm{gr}(\mathscr{E}^{(m)}_s(\mathscr{N}))\big)$$
$$= (\deg(s))^{-1}\cdot \mathrm{g.lg}_{k[\partial^{\langle \pm p^m\rangle_{(m)}}]}\big(\mathrm{gr}(\mathscr{E}^{(m)}_s(\mathscr{N}))\big).$$

By using Lemma 1.4.2, we get the proposition. □

1.5. Stability theorem

We summarize what we have got, and obtain the following characteristic cycle version of Theorem 1.1.3, which is one of the main theorems of this paper. Recall the notation of paragraph 1.3.10.

1.5.1 Theorem (Stability Theorem for Curves). *Let $\mathscr{X}$ be an affine formal curve over R in Situation* (L) *of 1.2.1, and $\mathscr{M}$ be a stable holonomic $\widehat{\mathscr{D}}^{(m)}_{\mathscr{X},\mathbb{Q}}$-module. Let S be the set of singular points of $\mathscr{M}$, and suppose that $\mathscr{M}|_{\mathscr{X}\setminus S}$ is a convergent isocrystal. Let r be the generic rank of $\mathscr{M}$.*

(i) *For any $m'' \geqslant m' \geqslant m$, we get*

$$\mathrm{rk}_{K_{\mathscr{X}}\{\partial\}^{(m)}}\big(\widehat{\mathscr{E}}^{(m)}_{s,\mathbb{Q}}(\mathscr{M})\big) = \mathrm{rk}_{K_{\mathscr{X}}\{\partial\}^{(m',m'')}}\big(\widehat{\mathscr{E}}^{(m',m'')}_{s,\mathbb{Q}}(\mathscr{M})\big) = \mathrm{rk}_{K_{\mathscr{X}}\{\partial\}^{(m')}}\big(\widehat{\mathscr{E}}^{(m')}_{s,\mathbb{Q}}(\mathscr{M})\big).$$

This number is denoted by r_s.

(ii) *For any $m' \geqslant m$, we get*

$$\mathrm{Cycl}^{(m')}(\widehat{\mathscr{D}}^{(m')}_{\mathscr{X},\mathbb{Q}} \otimes_{\widehat{\mathscr{D}}^{(m)}_{\mathscr{X},\mathbb{Q}}} \mathscr{M}) = r\cdot[X] + p^{m'}[K(\mathscr{X}):K]\cdot \sum_{s\in S}\deg(s)^{-1}\cdot r_s\cdot[\pi^{-1}(s)].$$

Proof. Since $\mathscr{M}$ is stable, we get for any $m'' \geqslant m' \geqslant m$,

$$\mathrm{rk}(\widehat{\mathscr{E}}^{(m'')}_{s,\mathbb{Q}}(\mathscr{M})) = \mathrm{rk}(\widehat{\mathscr{E}}^{(m',m'')}_{s,\mathbb{Q}}(\mathscr{M})) = \mathrm{rk}(\widehat{\mathscr{E}}^{(m')}_{s,\mathbb{Q}}(\mathscr{M}))$$

by Proposition 1.3.11. Thus (i) follows.

Let us see (ii). For this, the vertical multiplicities are the only problem. By Proposition 1.4.3, we get for any $m' \geqslant m$

$$p^{m'} \cdot \mathrm{rk}_{K_{\mathscr{X}}\{\partial\}^{(m')}}(\widehat{\mathscr{E}}^{(m')}_{s,\mathbb{Q}}(\mathscr{M})) = \deg(s) \cdot [K(\mathscr{X}) : K]^{-1} \cdot m_s(\mathscr{M}^{(m')}).$$

Thus combining with (i), we get (ii). □

1.5.2. Thanks to Theorem 1.5.1 we can now define the characteristic cycles of holonomic $\mathscr{D}^{\dagger}_{\mathscr{X},\mathbb{Q}}$-modules and prove Corollary 1.5.3. In fact, these have already been defined and proven for the category of holonomic F-$\mathscr{D}^{\dagger}_{\mathscr{X},\mathbb{Q}}$-modules, cf. [16, 5.4.1 and 5.4.3(ii)]. See also 3.1.1 for the definition of Frobenius structure. In [16] Berthelot used his Frobenius descent theorem, and we are able to generalize his definition by using the stability theorem as follows:

Definition. Let $\mathscr{X}$ be a formal curve, and let $\mathscr{M}$ be a holonomic $\mathscr{D}^{\dagger}_{\mathscr{X},\mathbb{Q}}$-module. Take a stable coherent $\widehat{\mathscr{D}}^{(m)}_{\mathscr{X},\mathbb{Q}}$-module $\mathscr{M}'$ such that $\mathscr{D}^{\dagger}_{\mathscr{X},\mathbb{Q}} \otimes \mathscr{M}' \cong \mathscr{M}$. Let S be the set of singular points of $\mathscr{M}$. We define

$$\mathrm{Cycl}(\mathscr{M}) = r \cdot [X] + [K(\mathscr{X}) : K] \cdot \sum_{s \in S} \deg(s)^{-1} \cdot r_s \cdot [\pi^{-1}(s)],$$

where r is the generic rank of $\mathscr{M}'$, and $r_s := \mathrm{rk}\big(\widehat{\mathscr{E}}^{(m)}_{s,\mathbb{Q}}(\mathscr{M}')\big) \in \mathbb{N}$, which do not depend on the choice of $\mathscr{M}'$ by Theorem 1.5.1.

Remark. (i) When $\mathscr{M}$ possesses a Frobenius structure, the characteristic cycle here coincides with that of Berthelot [16, 5.4.2] (or [2, 2.3.13]).

(ii) By Theorem 1.5.1(ii) and 1.4.3, Cycl is additive with respect to short exact sequences of holonomic $\mathscr{D}^{\dagger}_{\mathscr{X},\mathbb{Q}}$-modules.

(iii) The characteristic cycle has integral coefficients. To prove this we may first assume that $\mathscr{X}$ is geometrically connected, indeed $\mathrm{Cycl}(\mathscr{M})$ does not change if we consider $\mathscr{X}$ as a smooth formal scheme over $\mathrm{Spf}(R(\mathscr{X}))$. Secondly we note that if we base change $\mathscr{X}$ by a finite extension, the multiplicities r_s do not change by construction. Thus, we may assume that S is rational over K and the integrality of $\mathrm{Cycl}(\mathscr{M})$ is evident.

1.5.3 Corollary. *Let $\mathscr{X}$ be a formal curve over R. The category of holonomic $\mathscr{D}^{\dagger}_{\mathscr{X},\mathbb{Q}}$-modules is both noetherian and artinian.*

Proof. The argument of the proof is the same as that of [16, 5.4.3 (ii)]. We recall it for the convenience of the reader. We can prove the ascending chain condition as follows: let $(\mathscr{M}_n \subseteq \mathscr{M})_{n \in \mathbb{N}}$ be an ascending filtration by holonomic sub-modules. We may assume $\mathscr{M} \neq 0$. The support of $\mathrm{Cycl}(\mathscr{M})$ has dimension one because $\mathscr{M}$ is holonomic. By additivity of Cycl (cf. Remark 1.5.2(ii)), we have, for all n,

$$\mathrm{Cycl}(\mathscr{M}) = \mathrm{Cycl}(\mathscr{M}_0) + \sum_{i=1}^{n} \mathrm{Cycl}(\mathscr{M}_i/\mathscr{M}_{i-1}) + \mathrm{Cycl}(\mathscr{M}/\mathscr{M}_n).$$

Since T^*X is a noetherian space and the coefficients (r and r_s) appearing in $\mathrm{Cycl}(\mathscr{M})$ belong to $\mathbb{N}$, we get, for n big enough, $\mathrm{Cycl}(\mathscr{M}_n/\mathscr{M}_{n-1}) = 0$; therefore $\mathscr{M}_n = \mathscr{M}_{n-1}$. We can prove the descending chain condition in a similar way. □

2. Local Fourier transform

The aim of this section is to define the local Fourier transform. We note that the definition itself is not difficult anymore thanks to works of Huyghe and Matsuda: we can take the canonical extension, the geometric Fourier transform, and take the differential module around ∞ as presented in [31, 8.3]. However, with this definition, we are not able to prove the stationary phase formula in the way used in the complex case. In this section, we instead define the local Fourier transform using microlocalizations following the classical techniques, and prove some basic properties.

2.1. Local theory of arithmetic $\mathscr{D}$-modules

Since the main goal of this paper (Theorem 7.2.5) is to prove a theorem connecting local and global invariants, it is indispensable to work in local situations. In the ℓ-adic case, this was the theory of étale sheaves on *traits*, in other words Galois representations of local fields. In our setting, the theory of arithmetic $\mathscr{D}$-modules on a formal disk by Crew [30, 31], which can be seen as a generalization of the theory of solvable p-adic differential equations, should be the corresponding theory. We briefly review the theory in this subsection.

2.1.1. Let R, k, K be as usual (cf. 0.0.4). We recall that ϖ denotes a uniformizer of R. The field k will be assumed perfect in all section 2, with exception of 2.1.1–2.1.3 and 2.2. Denote by $W(k)$ a Cohen ring with residue field k (ring of Witt vectors if k is perfect). For any commutative local ring A, we will denote by $\mathfrak{m}_A$ (or $\mathfrak{m}$) its maximal ideal. If A is an I-adic ring, for an ideal I of A, we will denote it by (A, I) when we want to specify the ideal of definition explicitly.

Let $(A, \mathfrak{m})$ be a two-dimensional formally smooth local noetherian R-algebra complete with respect to the $\mathfrak{m}$-adic topology, such that $p \in \mathfrak{m}$, whose residue field k_A is finite over k. In this situation, A is complete with respect to the p-adic topology by [37, 0_{I}, 7.2.4]. Let R_A be the normalization of R inside A, and $K_A := \mathrm{Fr}(R_A)$. Note that R_A is a discrete valuation ring. Now we get the following.

Lemma. *The R-algebra A is isomorphic to $R_A[[t]]$.*

Proof. By [37, 0_{IV}, 19.6.5], we get $A/\varpi A \cong k_A[[t]]$. Moreover, the ring $R_A[[t]]$ is a complete noetherian local ring, formally smooth over R, such that its reduction over k is isomorphic to $A/\varpi A$. Thus by [37, 0_{IV}, 19.7.2], we get the lemma. □

The situation we have in mind is the following: the R-algebra A is the completion $\widehat{\mathcal{O}}_{\mathscr{X},x}$ of the local ring $\mathcal{O}_{\mathscr{X},x}$ of a formal curve $\mathscr{X}$ at a closed point x, with respect to the filtration by the powers of its maximal ideal, denoted by $\mathfrak{m}_{\mathscr{X},x}$. In this case we put $k_x := k_{\widehat{\mathcal{O}}_{\mathscr{X},x}}$, $R_x := R_{\widehat{\mathcal{O}}_{\mathscr{X},x}}$, and denotes by $K_x := \mathrm{Fr}(R_x)$ the field of fractions of R_x.

We simply denote $\mathrm{Spf}(A, \varpi A)$ by $\mathrm{Spf}(A)$. The formal scheme $\mathscr{S} := \mathrm{Spf}(A)$ is called a *formal disk* and it consists of two points: an open point $\eta_{\mathscr{S}}$ and a closed point s. We put $\widetilde{\mathscr{S}} := \mathrm{Spf}(A, \mathfrak{m})$, which consists of only one point. Note that Crew in [31] used the notation $\mathrm{Spf}(A)$ for $\mathrm{Spf}(A, \mathfrak{m})$. The reason why we introduced $\mathscr{S}$ and $\widetilde{\mathscr{S}}$ will be clarified in Remark 2.1.4.

2.1.2. In [30, 31], Crew checked that on $\widetilde{\mathscr{S}}$, the theory of arithmetic $\mathscr{D}$-modules can be constructed in the same manner. He constructed the ring $\mathscr{D}^{\dagger}_{\widetilde{\mathscr{S}},\mathbb{Q}}$ (resp. $\mathscr{D}^{\dagger}_{\widetilde{\mathscr{S}},\mathbb{Q}}(0)$) of Berthelot differential operators (resp. overconvergent at $t = 0$), cf. [31, (3.1.5)]. He also constructed analytic variants $\mathscr{D}^{\mathrm{an}}_{\widetilde{\mathscr{S}},\mathbb{Q}}$ and $\mathscr{D}^{\mathrm{an}}_{\widetilde{\mathscr{S}},\mathbb{Q}}(0)$ of these rings by 'analytifying' $\mathscr{D}^{\dagger}_{\mathscr{X},\mathbb{Q}}$ and $\mathscr{D}^{\dagger}_{\mathscr{X},\mathbb{Q}}(0)$, where $\mathscr{X} := \mathrm{Spf}(R\{t\})$. Here analytification roughly means to tensor with $\mathcal{A}_{u,K}([0,1[) \subset K[\![t]\!]$ and take the completion with respect to a suitable topology. Let us briefly review the constructions of $\mathscr{D}^{\mathrm{an}}_{\widetilde{\mathscr{S}},\mathbb{Q}}$. Choose an isomorphism $\widehat{\mathcal{O}}_{\mathscr{X},x} \cong R_x[\![t]\!]$ by Lemma 2.1.1. For a positive integer r and a non-negative integer i, we define $\mathcal{O}_{r,i}$ to be $\widehat{\mathcal{O}}_{X_i,x}[T]/(pT - t^r)$ as defined in [31, 4.1]. When r is divisible by p^{m+1}, $\mathcal{O}_{r,i}$ possesses a $\mathscr{D}^{(m)}_{X_i}$-module structure, cf. [31, Lemma 3.1.1]. We define a *ring*

$$(\widehat{\mathscr{D}}^{(m)}_{x,\mathbb{Q}})^{\mathrm{an}} := \varprojlim_{n}\big((\varprojlim_{i} \mathcal{O}_{np^{m+1},i} \otimes_{\mathcal{O}_{X_i}} \mathscr{D}^{(m)}_{X_i}) \otimes \mathbb{Q}\big).$$

Although it is not defined explicitly in [31], this ring is used to define $\mathscr{D}^{\mathrm{an}}_{x,\mathbb{Q}}$ as inductive limit of $(\widehat{\mathscr{D}}^{(m)}_{x,\mathbb{Q}})^{\mathrm{an}}$ over m. By the remark below this ring depends only on $\widetilde{\mathscr{S}}$ and not on the parameter t used to define it, so we denote it by $\mathscr{D}^{\mathrm{an}}_{\widetilde{\mathscr{S}},\mathbb{Q}}$. The construction of $\mathscr{D}^{\mathrm{an}}_{\widetilde{\mathscr{S}},\mathbb{Q}}(0)$ is analogous, by using $\mathscr{D}^{(m)}_{X_i}(0)$ in place of $\mathscr{D}^{(m)}_{X_i}$.

Remark. In the construction of the rings $\mathscr{D}^{\mathrm{an}}_{\widetilde{\mathscr{S}},\mathbb{Q}}$ and $\mathscr{D}^{\mathrm{an}}_{\widetilde{\mathscr{S}},\mathbb{Q}}(0)$, one uses the parameter t. Thus, *a priori* the construction depends on this choice. However, if we use another t' such that its image in $k[\![t]\!]$ is a uniformizer to construct the rings, the resulting rings are canonically isomorphic to those constructed using the uniformizer t. Indeed, using the notation of [31, 4.1], let $\mathcal{O}_r(t)$ be the ring $\mathcal{O}_r$ using the uniformizer t. Let t' be another uniformizer. Then there exists a canonical isomorphism $\mathcal{O}_{r,\mathbb{Q}} := \mathcal{O}_r(t) \otimes \mathbb{Q} \cong \mathcal{O}_r(t') \otimes \mathbb{Q}$. Moreover, in $\mathcal{O}_{r,\mathbb{Q}}$, there exists an inclusion $\mathcal{O}_r(t) \subset p^{-r}\mathcal{O}_r(t')$ and $\mathcal{O}_r(t') \subset p^{-r}\mathcal{O}_r(t)$. Thus the claim follows from the definition of $\mathscr{D}^{\mathrm{an}}$ and $\mathscr{D}^{\mathrm{an}}(0)$.

Moreover, Crew generalizes these constructions to define analytification functors, cf. [31, 4.1],

$$(-)^{\mathrm{an}} \colon \mathrm{Coh}(\mathscr{D}^{\dagger}_{\widetilde{\mathscr{S}},\mathbb{Q}}) \to \mathrm{Mod}(\mathscr{D}^{\mathrm{an}}_{\widetilde{\mathscr{S}},\mathbb{Q}}), \quad (-)^{\mathrm{an}} \colon \mathrm{Coh}(\mathscr{D}^{\dagger}_{\widetilde{\mathscr{S}},\mathbb{Q}}(0)) \to \mathrm{Mod}(\mathscr{D}^{\mathrm{an}}_{\widetilde{\mathscr{S}},\mathbb{Q}}(0)),$$

where $\mathrm{Coh}(-)$ denotes the category of coherent modules and $\mathrm{Mod}(-)$ denotes the category of modules. They send $\mathscr{D}^{\dagger}_{\widetilde{\mathscr{S}},\mathbb{Q}}$ to $\mathscr{D}^{\mathrm{an}}_{\widetilde{\mathscr{S}},\mathbb{Q}}$ and $\mathscr{D}^{\dagger}_{\widetilde{\mathscr{S}},\mathbb{Q}}(0)$ to $\mathscr{D}^{\mathrm{an}}_{\widetilde{\mathscr{S}},\mathbb{Q}}(0)$ respectively. We have injective morphisms $\mathscr{D}^{\dagger}_{\widetilde{\mathscr{S}},\mathbb{Q}} \hookrightarrow \mathscr{D}^{\mathrm{an}}_{\widetilde{\mathscr{S}},\mathbb{Q}}$ and $\mathscr{D}^{\dagger}_{\widetilde{\mathscr{S}},\mathbb{Q}}(0) \hookrightarrow \mathscr{D}^{\mathrm{an}}_{\widetilde{\mathscr{S}},\mathbb{Q}}(0)$, which are flat (both left and right); the analytification functors are exact; and we have $\mathscr{M}^{\mathrm{an}} \cong$

$\mathscr{D}^{\mathrm{an}}_{\widetilde{\mathscr{S}},\mathbb{Q}} \otimes_{\mathscr{D}^{\dagger}_{\widetilde{\mathscr{S}},\mathbb{Q}}} \mathscr{M}$ (resp. $\mathscr{M}^{\mathrm{an}} \cong \mathscr{D}^{\mathrm{an}}_{\widetilde{\mathscr{S}},\mathbb{Q}}(0) \otimes_{\mathscr{D}^{\dagger}_{\widetilde{\mathscr{S}},\mathbb{Q}}(0)} \mathscr{M}$) for any coherent $\mathscr{D}^{\dagger}_{\widetilde{\mathscr{S}},\mathbb{Q}}$-module (resp. $\mathscr{D}^{\dagger}_{\widetilde{\mathscr{S}},\mathbb{Q}}(0)$-module) $\mathscr{M}$, cf. [31, Theorems 4.1.1 and 4.1.2].

We define the sheaves $\mathscr{D}^{\dagger}_{\mathscr{S},\mathbb{Q}}$, $\mathscr{D}^{\mathrm{an}}_{\mathscr{S},\mathbb{Q}}$ by

$$\Gamma(\mathscr{S}, \mathscr{D}^{\dagger}_{\mathscr{S},\mathbb{Q}}) := \mathscr{D}^{\dagger}_{\widetilde{\mathscr{S}},\mathbb{Q}}, \quad \Gamma(\eta_{\mathscr{S}}, \mathscr{D}^{\dagger}_{\mathscr{S},\mathbb{Q}}) := \mathscr{D}^{\dagger}_{\widetilde{\mathscr{S}},\mathbb{Q}}(0),$$
$$\Gamma(\mathscr{S}, \mathscr{D}^{\mathrm{an}}_{\mathscr{S},\mathbb{Q}}) := \mathscr{D}^{\mathrm{an}}_{\widetilde{\mathscr{S}},\mathbb{Q}}, \quad \Gamma(\eta_{\mathscr{S}}, \mathscr{D}^{\mathrm{an}}_{\mathscr{S},\mathbb{Q}}) := \mathscr{D}^{\mathrm{an}}_{\widetilde{\mathscr{S}},\mathbb{Q}}(0).$$

By the remark above, these sheaves are well-defined. For any $\mathscr{D}^{\dagger}_{\widetilde{\mathscr{S}},\mathbb{Q}}$-module $\mathscr{M}$, we can define a sheaf $\mathscr{M}^{\triangle}$ on $\mathscr{S}$ by

$$\Gamma(\mathscr{S}, \mathscr{M}^{\triangle}) := \mathscr{M}, \quad \Gamma(\eta_{\mathscr{S}}, \mathscr{M}^{\triangle}) := \mathscr{D}^{\dagger}_{\widetilde{\mathscr{S}},\mathbb{Q}}(0) \otimes_{\mathscr{D}^{\dagger}_{\widetilde{\mathscr{S}},\mathbb{Q}}} \mathscr{M};$$

and similarly we can associate a sheaf on $\mathscr{S}$ to any $\mathscr{D}^{\mathrm{an}}_{\widetilde{\mathscr{S}},\mathbb{Q}}$-module, by replacing $\mathscr{D}^{\dagger}_{\widetilde{\mathscr{S}},\mathbb{Q}}(0)$ with $\mathscr{D}^{\mathrm{an}}_{\widetilde{\mathscr{S}},\mathbb{Q}}(0)$ in the definition above. By construction, the fibres of any sheaf of $\mathscr{D}^{\dagger}_{\mathscr{S},\mathbb{Q}}$-modules (resp. $\mathscr{D}^{\mathrm{an}}_{\mathscr{S},\mathbb{Q}}$-modules) $\mathscr{M}$ are given by $\mathscr{M}_s = \Gamma(\mathscr{S}, \mathscr{M})$ and $\mathscr{M}_\eta = \Gamma(\eta, \mathscr{M})$. Hence the functors $\Gamma(\mathscr{S}, -)$ and $(-)^{\triangle}$ are equivalence of categories, quasi-inverse each other, between the category of coherent sheaves of $\mathscr{D}^{\dagger}_{\mathscr{S},\mathbb{Q}}$-modules (resp. $\mathscr{D}^{\mathrm{an}}_{\mathscr{S},\mathbb{Q}}$-modules) and coherent $\mathscr{D}^{\dagger}_{\widetilde{\mathscr{S}},\mathbb{Q}}$-modules (resp. $\mathscr{D}^{\mathrm{an}}_{\widetilde{\mathscr{S}},\mathbb{Q}}$-modules). We will often abusively use the same symbol $\mathscr{M}$ to denote a sheaf of coherent $\mathscr{D}^{\dagger}_{\mathscr{S},\mathbb{Q}}$-modules (resp. $\mathscr{D}^{\mathrm{an}}_{\mathscr{S},\mathbb{Q}}$-modules) and its global sections on $\mathscr{S}$.

2.1.3. Put $\mathcal{O}^{\mathrm{an}}_{u,K} := \mathcal{A}_{u,K}([0,1[)$, define the bounded Robba ring (in fact a field) by $\mathcal{R}^b_{u,K} := \bigcup_{r<1} \mathcal{A}_{u,K}([r,1\})$ and the Robba ring by $\mathcal{R}_{u,K} := \bigcup_{r<1} \mathcal{A}_{u,K}([r,1[)$, cf. 1.3.6.

By Remark 2.1.2, the rings $\mathcal{A}_{t,K_A}([0,1[)$, $\mathcal{R}_{t,K_A}$ and $\mathcal{R}^b_{t,K_A}$ depend on the choice of the coordinate t of $\mathscr{S}$ only up to a canonical isomorphism: indeed they are the sub-rings of order zero differential operators in $\mathscr{D}^{\mathrm{an}}_{\widetilde{\mathscr{S}},\mathbb{Q}}$, $\mathscr{D}^{\mathrm{an}}_{\widetilde{\mathscr{S}},\mathbb{Q}}(0)$ and $\mathscr{D}^{\dagger}_{\widetilde{\mathscr{S}},\mathbb{Q}}(0)$ respectively. We denote these rings by $\mathcal{O}^{\mathrm{an}}_{\mathscr{S}}$, $\mathcal{R}_{\mathscr{S}}$ and $\mathcal{R}^b_{\mathscr{S}}$. We often omit the subscripts and write simply $\mathcal{O}^{\mathrm{an}}$, $\mathcal{R}$ and $\mathcal{R}^b$.

Let us fix our conventions on the definition of differential modules. In this paper, we will adopt the definition of Kedlaya [47, 8.4.3]. Namely, we define a differential $\mathcal{A}_K([r,1[)$-module to be a locally free sheaf of finite rank on the rigid analytic annulus $\mathcal{C}([r,1[)$ over K with a connection. In other words, it is a collection $\{M_{r'}\}_{r<r'<1}$ where $M_{r'}$ is a finite differential $\mathcal{A}_K([r,r'])$-module equipped with isomorphisms $\mathcal{A}([r,r_1]) \otimes M_{r_2} \cong M_{r_1}$, for $r_1 < r_2$, of differential modules, compatible with each other in the obvious sense.

Let us define the category $\mathcal{C}$ of differential $\mathcal{R}$-modules. An object consists of a differential $\mathcal{A}([r,1[)$-modules for some $0 < r < 1$. Let M be a differential $\mathcal{A}([r,1[)$-module, and N be a differential $\mathcal{A}\big([r',1\big[)$-module where $0 < r, r' < 1$. Then we define the set of homomorphisms by

$$\mathrm{Hom}_{\mathcal{C}}(M,N) := \varinjlim_{s\to 1^-} \mathrm{Hom}_{\partial}\big(\mathcal{A}([s,1[) \otimes M, \mathcal{A}([s,1[) \otimes N\big),$$

where $s \geqslant \max\{r, r'\}$, and $\mathbf{Hom}_\partial$ denotes the homomorphism of differential modules.

For a differential $\mathcal{A}([r,1[)$-module M, we denote by $\Gamma(M)$ its global sections. The Fréchet algebra $\mathcal{A}([r,1[) = \varprojlim_{r<r'<1} \mathcal{A}([r,r'])$ is Fréchet–Stein in the sense of [65, Section 3], and any differential $\mathcal{A}([r,1[)$-module M is in particular a coherent sheaf for $\mathcal{A}([r,1[)$, under the terminology of [65]. Therefore by [65, Corollary 3.1], the natural map $\Gamma(M) \otimes_{\mathcal{A}([r,1[)} \mathcal{A}([r,r']) \to M_{r'}$ is an isomorphism for any r', and the functor of global sections is an equivalence on its essential image [65, Corollary 3.3].

Let M be a differential $\mathcal{R}$-module which is defined by a differential $\mathcal{A}([r,1[)$-module also denoted by M. We define

$$\Gamma(M) := \varinjlim_{s \to 1^-} \Gamma\big(\mathcal{A}([s,1[) \otimes M\big).$$

We note that this is an $\mathcal{R}$-module with a connection.

Definition. (i) We say that M is a *free differential $\mathcal{R}$-module* (*resp. $\mathcal{A}([r,1[)$-module*) if the module of global sections is a finite free $\mathcal{R}$-module (resp. $\mathcal{A}([r,1[)$-module). This is equivalent to saying that $\Gamma(M)$ is finitely presented by [28, 4.8, 6.1].

(ii) We denote[2] by $\mathrm{Hol}'(\eta_{\mathscr{S}})$ (resp. $\mathrm{Hol}(\eta_{\mathscr{S}})$) the category of differential (resp. free differential) modules on $\mathcal{R}_{\mathscr{S}}$.

Lemma. *Let M be a* free *differential $\mathcal{R}$-module, and N be a differential $\mathcal{R}$-module. Suppose there exists a homomorphism $\varphi\colon \Gamma(M) \to \Gamma(N)$ which is compatible with the connections. Then there exists a homomorphism of differential modules $M \to N$ inducing φ.*

Proof. Indeed, there exists $0 < r < 1$ and a free differential $\mathcal{A}([r,1[)$-module M' which induces M. Take a finite basis $\{e_i\}_{i\in I}$ of $\Gamma(M')$. Then there exists $r < r' < 1$ and a differential $\mathcal{A}([r',1[)$-module N' inducing N such that $\varphi(e_i) \in \Gamma(N')$ for any $i \in I$. By [65, Corollary 3.3] this defines a homomorphism $\mathcal{A}([r',1[) \otimes M' \to N'$ inducing φ. Taking the inductive limit, we get what we want. □

Remark. For any differential ideal J of $\mathcal{A}([r,1[)$ we can define a differential $\mathcal{A}([r,1[)$-module by $\widetilde{J} := \{J \otimes_{\mathcal{A}([r,1[)} \mathcal{A}([r,r'])\}_{r<r'<1}$. However $\Gamma(\widetilde{J})$ is not equal to J in general. For example take a differential ideal J of $\mathcal{A}([r,1[)$ of infinite type (cf. [25, Example 3.2]). Since $\mathcal{A}([r,r'])$ is differentially simple (cf. [25, Remark 3.1]), the ideal $J \otimes_{\mathcal{A}([r,1[)} \mathcal{A}([r,r'])$ is equal to $\mathcal{A}([r,r'])$, and the differential module $\widetilde{J}$ is free of rank one.

2.1.4. Starting from here, with the exception of 2.2, we assume that k is perfect. Let $\sigma\colon \mathscr{S} \to \mathscr{S}$ be a lifting of the h-th absolute Frobenius morphism. An F-$\mathscr{D}^\dagger_{\widetilde{\mathscr{S}},\mathbb{Q}}$-module (resp. F-$\mathscr{D}^{\mathrm{an}}_{\mathscr{S},\mathbb{Q}}$-module) is a $\mathscr{D}^\dagger_{\widetilde{\mathscr{S}},\mathbb{Q}}$-module (resp. $\mathscr{D}^{\mathrm{an}}_{\mathscr{S},\mathbb{Q}}$-module) M endowed with an isomorphism $M \xrightarrow{\sim} \sigma^* M$. For F-$\mathscr{D}^\dagger_{\widetilde{\mathscr{S}},\mathbb{Q}}$-modules the definition of holonomicity is the same as that for formal smooth schemes, cf. [16, 5.3.5], [31, 3.4].

[2] The choice of notation Hol is motivated by Lemma 2.1.4.

An F-$\mathscr{D}^{\mathrm{an}}_{\widetilde{\mathscr{S}},\mathbb{Q}}$-module (resp. F-$\mathscr{D}^{\mathrm{an}}_{\widetilde{\mathscr{S}},\mathbb{Q}}(0)$-module) M is said *holonomic* if there exists a holonomic F-$\mathscr{D}^{\dagger}_{\widetilde{\mathscr{S}},\mathbb{Q}}$-module $\mathscr{N}$ such that $\mathscr{N}^{\mathrm{an}} \cong M$, cf. [31, 5.2]. We denote by F-$\mathrm{Hol}(\mathscr{S})$ (resp. F-$\mathrm{Hol}(\eta_{\mathscr{S}})$, resp. F-$\mathrm{Hol}'(\eta_{\mathscr{S}})$) the category of holonomic F-$\mathscr{D}^{\mathrm{an}}_{\widetilde{\mathscr{S}}}$-modules (resp. holonomic F-$\mathscr{D}^{\mathrm{an}}_{\widetilde{\mathscr{S}}}(0)$-modules, resp. the category of differential module on $\mathcal{R}_{\mathscr{S}}$ with Frobenius structure).

Lemma. *The category of free differential $\mathcal{R}_{\mathscr{S}}$-modules with Frobenius structure is equivalent to the category of holonomic F-$\mathscr{D}^{\mathrm{an}}_{\widetilde{\mathscr{S}},\mathbb{Q}}(0)$-modules.*

Proof. By [68, 4.2.1], we get that, given a free differential $\mathcal{R}$-module M with Frobenius structure, there exists a free differential $\mathcal{R}^b$-module M' with Frobenius structure such that $\mathcal{R} \otimes M' \cong M$. By [31, Proposition 4.1.1 and Corollary 4.1.1], we have $M \cong (M')^{\mathrm{an}}$ which is an F-$\mathscr{D}^{\mathrm{an}}(0)$-module, holonomic by definition. This gives a functor from the category of free differential $\mathcal{R}_{\mathscr{S}}$-modules with Frobenius structure to F-$\mathrm{Hol}(\eta_{\mathscr{S}})$ and, by construction, this is clearly an equivalence. □

The scalar extension defines a functor denoted by $(\cdot)^{\mathrm{an}}$ from the category of holonomic F-$\mathscr{D}^{\dagger}_{\mathscr{S},\mathbb{Q}}$-modules to F-$\mathrm{Hol}(\mathscr{S})$.

Let $\mathscr{S} := \mathrm{Spf}(A)$ and $\mathscr{S}' := \mathrm{Spf}(B)$ be formal disks. If we are given a finite étale morphism $\tau\colon \mathscr{S} \to \mathscr{S}'$, this induces a functor $\tau_*\colon F\text{-}\mathrm{Hol}(\mathscr{S}) \to F\text{-}\mathrm{Hol}(\mathscr{S}')$, and also the pull-back τ^*. In the same way, if we are given a finite étale morphism of generic points $\tau'\colon \eta_{\mathscr{S}} \to \eta_{\mathscr{S}'}$, this defines a functor $\tau'_*\colon F\text{-}\mathrm{Hol}(\eta_{\mathscr{S}}) \to F\text{-}\mathrm{Hol}(\eta_{\mathscr{S}'})$ and the pull-back τ'^*, and the same for $\mathrm{Hol}'(\eta_{\mathscr{S}})$ etc.

Remark. We note here that $\widetilde{\mathscr{S}}$ consists of a single point. There is no problem as long as we only consider finite étale morphisms of $\widetilde{\mathscr{S}}$ like τ above, but in this paper, we need to use push-forwards and pull-backs in the situation where only morphisms on $\eta_{\mathscr{S}}$ like τ' are defined. Under this situation, adding the generic point $\eta_{\mathscr{S}}$ by considering $\mathscr{S}$ instead of $\widetilde{\mathscr{S}}$ makes descriptions much simpler.

2.1.5. Let us note here some definitions for F-$\mathscr{D}^{\dagger}$-modules on a formal disk, that are used in rest of this section; for more details cf. [31, 3.4] or 3.1.9.

For any $\mathscr{D}^{\dagger}_{\widetilde{\mathscr{S}},\mathbb{Q}}$-module $\mathscr{M}$ we put $j^+\mathscr{M} := \mathscr{D}^{\dagger}_{\widetilde{\mathscr{S}},\mathbb{Q}}(0) \otimes_{\mathscr{D}^{\dagger}_{\widetilde{\mathscr{S}},\mathbb{Q}}} \mathscr{M}$; conversely for any $\mathscr{D}^{\dagger}_{\widetilde{\mathscr{S}},\mathbb{Q}}(0)$-module $\mathscr{N}$ we define $j_+\mathscr{N}$ the $\mathscr{D}^{\dagger}_{\widetilde{\mathscr{S}},\mathbb{Q}}$-module obtained from $\mathscr{N}$ by restriction of scalars via $\mathscr{D}^{\dagger}_{\widetilde{\mathscr{S}},\mathbb{Q}} \hookrightarrow \mathscr{D}^{\dagger}_{\widetilde{\mathscr{S}},\mathbb{Q}}(0)$. Similar definitions hold for $\mathscr{D}^{\mathrm{an}}_{\widetilde{\mathscr{S}},\mathbb{Q}}$-modules and $\mathscr{D}^{\mathrm{an}}_{\widetilde{\mathscr{S}},\mathbb{Q}}(0)$-modules. The F-$\mathscr{D}^{\dagger}_{\widetilde{\mathscr{S}},\mathbb{Q}}$-module δ is by definition the holonomic F-$\mathscr{D}^{\dagger}_{\widetilde{\mathscr{S}},\mathbb{Q}}$-module

$$\delta := \mathscr{D}^{\dagger}_{\widetilde{\mathscr{S}},\mathbb{Q}} / \mathscr{D}^{\dagger}_{\widetilde{\mathscr{S}},\mathbb{Q}} \cdot t = \mathcal{R}^b / \mathcal{O}_{\mathscr{S},\mathbb{Q}} = \mathcal{R}/\mathcal{O}^{\mathrm{an}}.$$

By construction δ is holonomic and $\delta = \delta^{\mathrm{an}}$. We say that a $\mathscr{D}^{\dagger}_{\widetilde{\mathscr{S}},\mathbb{Q}}$-module $\mathscr{M}$ is *punctual* (or *punctual type*) if there exists a finite dimensional K-vector space such that $\mathscr{M} \cong i_+V := \delta \otimes_K V$. By construction a punctual F-$\mathscr{D}^{\dagger}_{\widetilde{\mathscr{S}},\mathbb{Q}}$-module is automatically holonomic.

We say that a holonomic $\mathscr{D}^{\dagger}_{\widetilde{\mathscr{S}},\mathbb{Q}}$-module $\mathscr{M}$ is of *connection type* if the canonical homomorphism $\mathscr{M} \to j_+ j^+ \mathscr{M}$ is an isomorphism. We have similar definitions of punctual and connection type for $\mathscr{D}^{\mathrm{an}}_{\widetilde{\mathscr{S}},\mathbb{Q}}$-modules.

2.1.6. Let us recall more notation and definitions from [31], see [31, §1] for details. The bounded Robba ring $\mathcal{R}^b = \mathcal{R}^b_{u,K}$ is a discrete valuation field with respect to the 1-Gauss norm. Let us denote by $\mathcal{O}_{\mathcal{R}^b}$ its integer ring and by $\mathcal{K} = k(\!(\bar{u})\!)$ its residue field, where $\bar{u}$ is the class of u. Choose a separable closure $\mathcal{K}^{\mathrm{sep}}$. We set $G_{\mathcal{K}} := \mathrm{Gal}(\mathcal{K}^{\mathrm{sep}}/\mathcal{K})$, and let $I_{\mathcal{K}}$ be the inertia subgroup. Since $\mathcal{O}_{\mathcal{R}^b}$ is henselian, given a finite separable extension $\mathcal{L}$ of $\mathcal{K}$, there exists a unique finite unramified extension $\mathcal{R}^b(\mathcal{L})$ of $\mathcal{R}^b$ whose residue field (of its integer ring) is $\mathcal{L}$. Put $\mathcal{R}(\mathcal{L}) := \mathcal{R} \otimes_{\mathcal{R}^b} \mathcal{R}^b(\mathcal{L})$. Let h be a positive integer, and we put $q := p^h$. We fix a lifting of h-th Frobenius σ of $\mathcal{K}$ on $\mathcal{O}_{\mathcal{R}^b}$, which induces the Frobenius homomorphism on $\mathcal{R}^b$ and $\mathcal{R}$, also denoted by σ. This extends canonically to $\mathcal{R}(\mathcal{L})$. Now, we put $\mathcal{B}_0 := \varinjlim_{\mathcal{L}} \mathcal{R}(\mathcal{L})$ where $\mathcal{L}$ runs through finite separable extensions of $\mathcal{K}$ inside $\mathcal{K}^{\mathrm{sep}}$. Then this ring is naturally equipped with a $G_{\mathcal{K}}$-action, and a Frobenius homomorphism σ. We formally add 'log' to get the ring of hyperfunctions: we define $\mathcal{B} := \mathcal{B}_0[\log(u)]$. The action of $G_{\mathcal{K}}$ extends canonically to $\mathcal{B}$, so does σ. We also have the monodromy operator, which is the derivation by $\log(u)$. See [31, §1.4]. We put[3] $\mathcal{O}^{\mathrm{an}}_{K^{\mathrm{ur}}} := \mathcal{O}^{\mathrm{an}} \otimes_K K^{\mathrm{ur}}$, where K^{ur} denotes the maximal unramified field extension of K and $\mathcal{O}^{\mathrm{an}}$ is $\mathcal{A}_{u,K}([0,1[)$, cf. 2.1.3. Crew defined an $\mathcal{O}^{\mathrm{an}}_{K^{\mathrm{ur}}}$-module by $\mathcal{C} := \mathcal{B}/\mathcal{O}^{\mathrm{an}}_{K^{\mathrm{ur}}}$, cf. [31, (6.1.1)]. The action of $G_{\mathcal{K}}$, the endomorphism σ, and the nilpotent operator N, induce, by quotient, analogous structures on $\mathcal{C}$. We denote by $\mathsf{can}\colon \mathcal{B} \to \mathcal{C}$ the canonical projection. By definition, the derivation $N\colon \mathcal{B} \to \mathcal{B}$ factors through can, and we get $\mathsf{var}\colon \mathcal{C} \to \mathcal{B}$. These homomorphisms satisfy the relations $N = \mathsf{can} \circ \mathsf{var}$ and $N = \mathsf{var} \circ \mathsf{can}$, cf. [31, §6.1].

2.1.7. Let $\mathrm{Del}_{K^{\mathrm{ur}}}(G_{\mathcal{K}})$ denote the category of Deligne modules[4]: *i.e.* finite dimensional K^{ur}-vector spaces, endowed with a semi-linear action of $G_{\mathcal{K}}$ (which acts on the constants K^{ur} via its residual action), a Frobenius isomorphism φ, and a monodromy operator N, satisfying $N\varphi = q\,\varphi N$ where $q = p^h$ in 2.1.6. See [54, Section 3.1] for more details.

In the following, for simplicity, we denote $\mathscr{D}^{\mathrm{an}}_{\mathscr{S},\mathbb{Q}}$ by $\mathscr{D}^{\mathrm{an}}$. Crew classifies holonomic F-$\mathscr{D}^{\mathrm{an}}$-modules in terms of linear data (cf. [31, 6.1]). To do this, let M be a holonomic F-$\mathscr{D}^{\mathrm{an}}$-module. He defined in [31],

$$\mathbb{V}(M) := \mathrm{Hom}_{\mathscr{D}^{\mathrm{an}}}(M, \mathcal{B}), \qquad \mathbb{W}(M) := \mathrm{Hom}_{\mathscr{D}^{\mathrm{an}}}(M, \mathcal{C}).$$

These are Deligne modules, and define (contravariant) functors $\mathbb{V}, \mathbb{W}\colon F\text{-}\mathrm{Hol}(\mathscr{D}^{\mathrm{an}}) \to \mathrm{Del}_{K^{\mathrm{ur}}}(G_{\mathcal{K}})$. There are the canonical homomorphism $\mathbb{V}(M) \to \mathbb{W}(M)$ induced by can, and the variation homomorphism $\mathbb{W}(M) \to \mathbb{V}(M)$ induced by var. These satisfy many compatibilities, they are endowed with an extra structure (the 'Galois variation', which we do not recall here). These kind of objects are called *solution data*, they form an artinian category, denoted Sol_K, and we have an exact functor $\mathbb{S}\colon F\text{-}\mathrm{Hol}(\mathscr{S}) \to \mathrm{Sol}_K$, $\mathbb{S}(M) =$

[3] In [31, 6.1], he defined $\mathcal{O}^{\mathrm{an}}_{K^{\mathrm{ur}}}$ to be $\mathcal{O}^{\mathrm{an}} \widehat{\otimes}_K K^{\mathrm{ur}}$, but this should be a typo.

[4] This terminology was first introduced by Fontaine in [39, Section 1]. These are also called $(\varphi, N, G_{\mathcal{K}})$-modules, and this terminology is used more widely, especially in p-adic Hodge theory. However, in our context, we believe that 'Deligne module' is more suitable.

$(\mathbb{V}(M), \mathbb{W}, \text{can}, \text{var}, \dots)$; for the details see [31]. The main point is that we can retrieve the original module M from these linear data. Indeed Crew constructs a quasi-inverse functor $\mathbb{M}^{\text{an}}\colon \text{Sol}_K \to F\text{-Hol}(\mathscr{S})$ factorizing through the category of holonomic F-$\mathscr{D}^{\dagger}_{\mathscr{S},\mathbb{Q}}$-modules, cf. [31, 7.1].

We can characterize some properties of M in terms of these linear data. For example M is connection type if and only if the canonical map $\mathbb{V}(M) \to \mathbb{W}(M)$ is an isomorphism, cf. [31, Corollary 6.1.2]. The most important property for us is the existence of the following exact sequence of Deligne modules, cf. [31, Corollary 6.1.1, Lemma 6.1.2].

$$0 \to \text{Hom}_{\mathscr{D}^{\text{an}}}(M, \mathcal{O}^{\text{an}}_{K^{\text{ur}}}) \to \mathbb{V}(M) \to \mathbb{W}(M) \to \text{Ext}^1_{\mathscr{D}^{\text{an}}}(M, \mathcal{O}^{\text{an}}_{K^{\text{ur}}}) \to 0, \quad (2.1.7.1)$$

$$\forall i \geqslant 2, \quad \text{Ext}^i_{\mathscr{D}^{\text{an}}}(M, \mathcal{O}^{\text{an}}_{K^{\text{ur}}}) = \text{Ext}^i_{\mathscr{D}^{\text{an}}}(M, \mathcal{B}) = \text{Ext}^i_{\mathscr{D}^{\text{an}}}(M, \mathcal{C}) = 0. \quad (2.1.7.2)$$

Let us mention another property which will be useful later.

Lemma. *Let M be a holonomic F-$\mathscr{D}^{\text{an}}_{\mathscr{S},\mathbb{Q}}$-module. If $\mathbb{W}(M) = 0$ then M is free as differential $\mathcal{O}^{\text{an}}_{\mathscr{S}}$-module, cf. 2.1.3.*

Proof. By 2.1.7.1 it follows that $\mathbb{V}(M)$ is geometrically constant, which means that we have an isomorphism $\mathbb{V}(M)^{G_K} \otimes_K K^{\text{ur}} \cong \mathbb{V}(M)$ of Deligne modules. By the construction of the functor $\mathbb{M}^{\text{an}}$ it follows immediately that natural evaluation map $M \to \text{Hom}_{K^{\text{ur}}}(\mathbb{V}(M), \mathcal{O}^{\text{an}}_{K^{\text{ur}}})^{G_K}$ is an isomorphism of $\mathscr{D}^{\text{an}}_{\mathscr{S},\mathbb{Q}}$-modules (here $\text{Hom}_{K^{\text{ur}}}$ denotes homomorphisms of K^{ur}-vector spaces). We have isomorphisms of $\mathscr{D}^{\text{an}}_{\mathscr{S},\mathbb{Q}}$-modules:

$$\begin{aligned}\text{Hom}_{K^{\text{ur}}}(\mathbb{V}(M), \mathcal{O}^{\text{an}}_{K^{\text{ur}}})^{G_K} &\cong \text{Hom}_{K^{\text{ur}}}(\mathbb{V}(M)^{G_K} \otimes_K K^{\text{ur}}, \mathcal{O}^{\text{an}}_{K^{\text{ur}}})^{G_K} \cong \text{Hom}_K(\mathbb{V}(M)^{G_K}, \mathcal{O}^{\text{an}}_{K^{\text{ur}}})^{G_K} \\ &\cong \text{Hom}_K(\mathbb{V}(M)^{G_K}, \mathcal{O}^{\text{an}}_{\mathscr{S}}) \cong (\mathcal{O}^{\text{an}}_{\mathscr{S}})^{\oplus \text{rk}(M)},\end{aligned}$$

which concludes the proof. $\square$

2.1.8. Let $\mathscr{X}$ be a formal curve over R, and let $\mathscr{M}$ be a holonomic $\mathscr{D}^{\dagger}_{\mathscr{X},\mathbb{Q}}$-module. Let $x \in \mathscr{X}$ be a closed point. We denote by $\mathscr{S}_x := \text{Spf}(\widehat{\mathcal{O}}_{\mathscr{X},x})$ where $\widehat{\mathcal{O}}_{\mathscr{X},x}$ is the completion of $\mathcal{O}_{\mathscr{X},x}$ for the $\mathfrak{m}_{\mathscr{X},x}$-topology. Let $\mathscr{M}$ be a coherent $\mathscr{D}^{\dagger}_{\mathscr{X},\mathbb{Q}}$-module on $\mathscr{X}$. Take an open affine neighbourhood $\mathscr{U}$ of x, and we denote by $\mathscr{D}^{\dagger}_{\mathscr{S}_x,\mathbb{Q}} \otimes \mathscr{M}$ the coherent $\mathscr{D}^{\dagger}_{\mathscr{S}_x,\mathbb{Q}}$-module on $\mathscr{S}_x$

$$\left(\mathscr{D}^{\dagger}_{\widetilde{\mathscr{S}}_x,\mathbb{Q}} \otimes_{\Gamma(\mathscr{U},\mathscr{D}^{\dagger}_{\mathscr{X},\mathbb{Q}})} \Gamma(\mathscr{U},\mathscr{M})\right)^{\triangle},$$

cf. 2.1.2. This does not depend on the choice of $\mathscr{U}$, and we will also denote by $\mathscr{D}^{\dagger}_{\mathscr{S}_x,\mathbb{Q}} \otimes \mathscr{M}$ its global sections on $\mathscr{S}_x$. For a holonomic F-$\mathscr{D}^{\dagger}_{\mathscr{X},\mathbb{Q}}$-module $\mathscr{M}$ (cf. 3.1.1), we put

$$\mathscr{M}|_{S_x} := (\mathscr{D}^{\dagger}_{\mathscr{S}_x,\mathbb{Q}} \otimes \mathscr{M})^{\text{an}}, \quad \mathscr{M}|_{\eta_x} := (\mathscr{D}^{\dagger}_{\mathscr{S}_x,\mathbb{Q}}(0) \otimes \mathscr{M})^{\text{an}},$$

which are defined in $F\text{-Hol}(\mathscr{S}_x)$ and $F\text{-Hol}(\eta_{\mathscr{S}_x})$ respectively. We note that they do not depend on the choice of a local parameter of $\mathscr{X}$ at x (cf. Remark 2.1.2). For example we have $\mathcal{O}_{\mathscr{X},\mathbb{Q}}|_{S_x} = \mathcal{O}^{\text{an}}_{\mathscr{S}_x}$ and $\mathcal{O}_{\mathscr{X},\mathbb{Q}}|_{\eta_x} = \mathcal{R}_{\mathscr{S}_x}$. The following lemma combined with [31, Theorem 4.1.1] shows that the functors $|_{S_x}$ and $|_{\eta_x}$ are exact.

Lemma. *The functor $\mathscr{D}^\dagger_{\mathscr{S}_x,\mathbb{Q}}\otimes(-)$ from the category of coherent $\mathscr{D}^\dagger_{\mathscr{X},\mathbb{Q}}$-modules to that of coherent $\mathscr{D}^\dagger_{\mathscr{S}_x,\mathbb{Q}}$-modules is exact.*

Proof. Let S_i be $\widetilde{\mathscr{S}}_x\otimes R_i$. In this proof, we denote $\Gamma(\mathscr{U}\otimes R_i,\mathscr{D}^{(m)}_{X_i})$ by $\mathscr{D}^{(m)}_{U_i}$. It suffices to show that the canonical homomorphism $\mathscr{D}^{(m)}_{U_i}\to\mathscr{D}^{(m)}_{S_i}$ is flat. For this, it suffices to show that $\mathrm{gr}(\mathscr{D}^{(m)}_{U_i})\to\mathrm{gr}(\mathscr{D}^{(m)}_{S_i})$ is flat where the gr is taken with respect to the filtration by order. This follows from the flatness of $\mathcal{O}_{X_i}\to(\mathcal{O}_{X_i,x})^\wedge$ where the completion is taken with respect to the $\mathfrak{m}_x$-topology. $\square$

2.1.9. Let us recall the canonical extension, which is one of key tools in this paper. For the detailed argument, one can refer to [31, Section 8]. Let $\widehat{\mathbb{P}}:=\widehat{\mathbb{P}}^1_R$ and $\mathscr{S}_0:=\mathrm{Spf}(\widehat{\mathcal{O}}_{\widehat{\mathbb{P}},0})$. Then there exists a functor

$$F\text{-Hol}(\mathscr{S}_0)\to F\text{-Hol}(\mathscr{D}^\dagger_{\widehat{\mathbb{P}},\mathbb{Q}}(\infty));\quad M\mapsto M^{\mathrm{can}}$$

where $F\text{-Hol}(\mathscr{D}^\dagger_{\widehat{\mathbb{P}},\mathbb{Q}}(\infty))$ denotes the category of holonomic F-$\mathscr{D}^\dagger_{\widehat{\mathbb{P}},\mathbb{Q}}(\infty)$-modules. By construction, this functor is fully faithful, exact, and it commutes with tensor products and duals. Moreover, we have the following properties (cf. [31, Theorem 8.2.1] and paragraph after its proof):

1. $M^{\mathrm{can}}|_{\widehat{\mathbb{P}}\setminus\{0\}}$ is a 'special' convergent isocrystal;
2. $M^{\mathrm{can}}|_{S_0}\cong M$;
3. M^{can} regular at ∞ (for the definition of regularity cf. 2.3.1).

This M^{can} is called the *canonical extension of M*. By these properties, we remind that when M is a free differential module, M^{can} coincides with the canonical extension of Matsuda in [56, 7.3], and in this case, it sends unit-root objects to unit-root objects.

2.1.10. In 2.1.4 we have characterized holonomic F-$\mathscr{D}^{\mathrm{an}}_{\widetilde{\mathscr{S}},\mathbb{Q}}(0)$-modules. Let us conclude this subsection with a lemma characterizing holonomic F-$\mathscr{D}^{\mathrm{an}}$-modules.

Lemma. *Let M be an F-$\mathscr{D}^{\mathrm{an}}_{\widetilde{\mathscr{S}},\mathbb{Q}}$-module. Assume that $M(0):=\mathscr{D}^{\mathrm{an}}_{\widetilde{\mathscr{S}},\mathbb{Q}}(0)\otimes M$ is a free differential $\mathcal{R}$-module, and the kernel and cokernel of the canonical homomorphism $\alpha\colon M\to M(0)$ are punctual $\mathscr{D}^\dagger_{\widetilde{\mathscr{S}},\mathbb{Q}}$-modules. Then M is a holonomic F-$\mathscr{D}^{\mathrm{an}}_{\widetilde{\mathscr{S}},\mathbb{Q}}$-module. In particular, if there exists a holonomic F-$\mathscr{D}^\dagger_{\widetilde{\mathscr{S}},\mathbb{Q}}$-module $\mathscr{M}$ such that $\mathscr{M}^{\mathrm{an}}\cong M$ as $\mathscr{D}^{\mathrm{an}}_{\widetilde{\mathscr{S}},\mathbb{Q}}$-modules* without Frobenius structures*, then M is a holonomic F-$\mathscr{D}^{\mathrm{an}}_{\widetilde{\mathscr{S}},\mathbb{Q}}$-module.*

Proof. We denote by $\mathcal{C}$ the full subcategory of the category of $\mathscr{D}^{\mathrm{an}}_{\widetilde{\mathscr{S}},\mathbb{Q}}$-modules with Frobenius structure consisting of objects considered in the statement of the lemma. We define functors $\mathbb{V}$ and $\mathbb{W}$ in the same way as [31, 6.1] or especially [31, (6.1.9)], (cf. also 2.1.7). We first claim that $\mathbb{V}$ and $\mathbb{W}$ are exact functors. To see this, consider the following exact sequences

$$0\to N_1\to M\to M'\to 0,$$

$$0 \to M' \to M(0) \to N_2 \to 0, \tag{2.1.10.1}$$

where N_1 and N_2 are the kernel and cokernel of α respectively, and thus punctual $\mathscr{D}^\dagger_{\widetilde{\mathscr{S}},\mathbb{Q}}$-modules by assumption. By hypothesis and Lemma 2.1.4, we know that $M(0)$ and N_2 are holonomic F-$\mathscr{D}^{\mathrm{an}}_{\widetilde{\mathscr{S}},\mathbb{Q}}$-modules, and thus by considering the canonical extensions (cf. 2.1.9), we get that M' is a holonomic F-$\mathscr{D}^{\mathrm{an}}_{\mathscr{S},\mathbb{Q}}$-module as well. Thus, we get $\mathrm{Ext}^i(M', \mathcal{B}) = \mathrm{Ext}^i(M', \mathcal{C}) = 0$ for $i > 0$. Considering the long exact sequence induced by the first short exact sequence above, we get $\mathrm{Ext}^i(M, \mathcal{B}) = \mathrm{Ext}^i(M, \mathcal{C}) = 0$ for $i > 0$, cf. [31, Lemma 6.1.2].

Repeating the same construction of [31, (6.1.10)] and arguments of [31, Theorem 6.1.1], we get an exact functor $\mathbb{S}' = (\mathbb{V}, \mathbb{W}, \dots)$ from $\mathcal{C}$ to the category of solution data. It suffices to construct a canonical isomorphism $M \to \mathbb{M}^{\mathrm{an}}(\mathbb{S}'(M))$. This can be shown in the same way as [31, Theorem 7.1.1].

For any holonomic F-$\mathscr{D}^\dagger_{\widetilde{\mathscr{S}},\mathbb{Q}}$-module $\mathscr{M}$, we have $\mathscr{M}^{\mathrm{an}} \cong \mathscr{D}^{\mathrm{an}}_{\mathscr{S},\mathbb{Q}} \otimes_{\mathscr{D}^\dagger_{\widetilde{\mathscr{S}},\mathbb{Q}}} \mathscr{M}$ (cf. 2.1.2), $\mathscr{M}^{\mathrm{an}}(0) \cong (j_+ j^+ \mathscr{M})^{\mathrm{an}}$ (cf. [31, Lemma 4.1.4]), and the analytification of a punctual module is punctual (cf. [31, Proposition 5.1.1]). Thus, the last claim of the lemma follows from the first because the analytification functor is exact (cf. 2.1.2). □

2.2. Analytification

In section 2.4, we will define a local Fourier transform. The local Fourier transform should be local not only in the scheme theoretic sense, but also 'rigid analytically'. In this subsection, we will prove a crucial tool (cf. Proposition 2.2.3) which is indispensable to prove such properties.

2.2.1. In this subsection, we do not assume k to be perfect. Let $\mathscr{X}$ be a formal curve, and take a closed point x in $\mathscr{X}$. Choose an isomorphism $\widehat{\mathcal{O}}_{\mathscr{X},x} \cong R_x[\![t]\!]$, where $R_x := R_{\widehat{\mathcal{O}}_{\mathscr{X},x}}$, cf. 2.1.1. We will define a ring $(\widehat{\mathscr{E}}_x^{(m,m')})^{\mathrm{an}}$ in the following way. Let us use the notation of 2.1.2. To construct the analytification of $\widehat{\mathscr{E}}^{(m)}_{\mathscr{X},\mathbb{Q}}$, we follow exactly the same way as in [3]; there are several steps. For the first step, we take the microlocalization of $\mathcal{O}_{np^{m+1},i} \otimes_{\mathcal{O}_{X_i}} \mathscr{D}^{(m)}_{X_i}$ with respect to the filtration by order (cf. [3, 2.1]) and denote it by $\mathscr{E}^{(m)}_{np^{m+1},x,i}$. Second step, take the inverse limit over i, namely $\varprojlim_i \mathscr{E}^{(m)}_{np^{m+1},x,i}$ and denote it by $\widehat{\mathscr{E}}^{(m)}_{np^{m+1},x}$. We put $\widehat{\mathscr{E}}^{(m)}_{np^{m+1},x,\mathbb{Q}} := \widehat{\mathscr{E}}^{(m)}_{np^{m+1},x} \otimes \mathbb{Q}$, and we take the inverse limit over n to define $(\widehat{\mathscr{E}}^{(m)}_{x,\mathbb{Q}})^{\mathrm{an}}$.

Now, for an integer $m' \geqslant m$, we want to define the analytification of $\widehat{\mathscr{E}}^{(m,m')}_{\mathscr{X},\mathbb{Q}}$. Also for this, we follow the same way as [3]. Put $c := p^{m'+1}$. Let a be either m or m'. Then we define $\mathscr{E}^{(a)}_{nc,x}$ to be the subring of $\widehat{\mathscr{E}}^{(a)}_{nc,x}$ consisting of the finite order operators. Then we may prove in the same way as [3] that there exists a canonical homomorphism

$$\psi_{m,m'} \colon \mathscr{E}^{(m')}_{nc,x} \otimes \mathbb{Q} \to \mathscr{E}^{(m)}_{nc,x} \otimes \mathbb{Q}.$$

We define $\widehat{\mathscr{E}}^{(m,m')}_{nc,x}$ to be the p-adic completion of $\psi^{-1}_{m,m'}(\mathscr{E}^{(m)}_{nc,x}) \cap \mathscr{E}^{(m')}_{nc,x}$. This ring is noetherian by the same argument as [3, Proposition 4.12]. We define $\widehat{\mathscr{E}}^{(m,m')}_{nc,x,\mathbb{Q}} := \widehat{\mathscr{E}}^{(m,m')}_{nc,x} \otimes \mathbb{Q}$.

Finally we define

$$(\widehat{\mathscr{E}}^{(m,m')}_{x,\mathbb{Q}})^{\mathrm{an}} := \varprojlim_n \widehat{\mathscr{E}}^{(m,m')}_{nc,x,\mathbb{Q}}.$$

Obviously, there exists the canonical inclusion $\widehat{\mathscr{E}}^{(m,m')}_{x,\mathbb{Q}} \to (\widehat{\mathscr{E}}^{(m,m')}_{x,\mathbb{Q}})^{\mathrm{an}}$. In the same way as for $\mathscr{E}^{\dagger}_{\mathscr{X},\mathbb{Q}}$, we define

$$(\mathscr{E}^{(m,\dagger)}_{x,\mathbb{Q}})^{\mathrm{an}} := \varprojlim_k (\widehat{\mathscr{E}}^{(m,m+k)}_{x,\mathbb{Q}})^{\mathrm{an}}, \quad \mathscr{E}^{\mathrm{an}}_{x,\mathbb{Q}} := \varinjlim_m (\mathscr{E}^{(m,\dagger)}_{x,\mathbb{Q}})^{\mathrm{an}}.$$

We point out that the rings $(\widehat{\mathscr{E}}^{(m,m')}_{x,\mathbb{Q}})^{\mathrm{an}}$, $(\mathscr{E}^{(m,\dagger)}_{x,\mathbb{Q}})^{\mathrm{an}}$, $\mathscr{E}^{\mathrm{an}}_{x,\mathbb{Q}}$ does not depend on the choice of the isomorphism $\widehat{\mathcal{O}}_{\mathscr{X},x} \cong R_x[\![t]\!]$ (cf. Remark 2.1.1). As an example, we have the following explicit description whose verification is left to the reader. We recall that $|\cdot|_r$ denotes the r-Gauss norm on $\mathcal{O}^{\mathrm{an}} = \mathcal{A}_{K_x,t}([0,1[)$ for $0<r<1$ (cf. 1.3.6). We have

$$(\widehat{\mathscr{E}}^{(m,m')}_{x,\mathbb{Q}})^{\mathrm{an}} = \left\{ \sum_{k<0} a_k \partial^{\langle k \rangle_{(m')}} + \sum_{k \geqslant 0} b_k \partial^{\langle k \rangle_{(m)}} \;\middle|\; \begin{array}{l} a_k, b_k \in \mathcal{O}^{\mathrm{an}}, \text{ and for any } 0<r<1, \text{ there} \\ \text{exists } C_r>0 \text{ such that } |a_k|_r < C_r \text{ for any } k \\ \text{and } \lim_{k\to\infty} |b_k|_r = 0. \end{array} \right\}.$$

At last, let us fix some notation. Let $\mathscr{M}$ be a coherent $\widehat{\mathscr{D}}^{(m)}_{\mathscr{X},\mathbb{Q}}$-module. Let $m'' \geqslant m' \geqslant m$, and $\mathscr{E}$ be one of $(\widehat{\mathscr{D}}^{(m')}_{x,\mathbb{Q}})^{\mathrm{an}} = (\widehat{\mathscr{D}}^{(m')}_{\mathscr{S},\mathbb{Q}})^{\mathrm{an}}$, $(\widehat{\mathscr{E}}^{(m',m'')}_{x,\mathbb{Q}})^{\mathrm{an}}$, $(\mathscr{E}^{(m',\dagger)}_{x,\mathbb{Q}})^{\mathrm{an}}$, $\mathscr{E}^{\mathrm{an}}_{x,\mathbb{Q}}$. We denote $\mathscr{E} \otimes_{(\widehat{\mathscr{D}}^{(m)}_{\mathscr{X},\mathbb{Q}})_x} \mathscr{M}_x$ by $\mathscr{E} \otimes_{\widehat{\mathscr{D}}^{(m)}} \mathscr{M}$ or $\mathscr{E} \otimes \mathscr{M}$. This notation goes together with Notation 1.2.1.

2.2.2. We put topologies $\mathscr{T}_{n'}$ for $n' \geqslant 0$ and $\mathscr{T}$ on $\widehat{\mathscr{E}}^{(m,m')}_{nc,x,\mathbb{Q}}$ in exactly the same way as in 1.2.2. Precisely, we put $U_{k,l} := (\widehat{\mathscr{E}}^{(m,m')}_{nc,x})_{-k} + \varpi^l \widehat{\mathscr{E}}^{(m,m')}_{nc,x}$, and we define a topology $\mathscr{T}_0$ on $\widehat{\mathscr{E}}^{(m,m')}_{nc,x}$ as the topology generated by $\{U_{k,l}\}$ as a base of neighbourhoods of zero. The topology $\mathscr{T}_{n'}$ on $\widehat{\mathscr{E}}^{(m,m')}_{nc,x,\mathbb{Q}}$ is the locally convex topology generated by $\{\varpi^{-n'} U_{k,l}\}$ as a base of neighbourhoods of zero, and $\mathscr{T}$ is the inductive limit topology. We get that $\widehat{\mathscr{E}}^{(m,m')}_{\mathscr{X},\mathbb{Q}} \cap \varpi^{-n'} \widehat{\mathscr{E}}^{(m,m')}_{nc,x}$ is dense in $(\varpi^{-n'} \widehat{\mathscr{E}}^{(m,m')}_{nc,x,\mathbb{Q}}, \mathscr{T}_{n'})$ where the intersection is taken in $\widehat{\mathscr{E}}^{(m,m')}_{nc,x,\mathbb{Q}}$. Indeed putting $\mathcal{O}_r := \varprojlim_i \mathcal{O}_{r,i}$, the intersection $\mathcal{O}_{\mathscr{X},\mathbb{Q}} \cap \mathcal{O}_{nc}$ is dense in $\mathcal{O}_{nc}$. In the same way as 1.2.2, for any finitely generated $\widehat{\mathscr{E}}^{(m,m')}_{nc,x,\mathbb{Q}}$-module, the $(\widehat{\mathscr{E}}^{(m,m')}_{nc,x,\mathbb{Q}}, \mathscr{T}_{n'})$-module topology is separated.

2.2.3 Proposition. *Suppose we are in Situation* (Ls) *of 1.2.1. Moreover, we assume $x = y_s$. Let $m' \geqslant m$ be non-negative integers, and $\mathscr{M}$ be a holonomic $\widehat{\mathscr{D}}^{(m)}_{\mathscr{X},\mathbb{Q}}$-module (not necessarily stable). We assume that*

$$\mathrm{Supp}(\widehat{\mathscr{E}}^{(m,m'+1)}_{\mathscr{X},\mathbb{Q}} \otimes_{\widehat{\mathscr{D}}^{(m)}_{\mathscr{X},\mathbb{Q}}} \mathscr{M}) \cap \mathring{T}^*X = \pi^{-1}(s) \cap \mathring{T}^*X.$$

Then the canonical homomorphism

$$\widehat{E}^{(m,m')}_{\mathscr{X},\mathbb{Q}} \otimes_{\widehat{\mathscr{D}}^{(m)}} \mathscr{M} \to (\widehat{\mathscr{E}}^{(m,m')}_{s,\mathbb{Q}})^{\mathrm{an}} \otimes_{\widehat{\mathscr{D}}^{(m)}} \mathscr{M} \tag{2.2.3.1}$$

is an isomorphism.

Proof. Suppose that $\mathbf{Supp}(\widehat{\mathscr{E}}^{(m,m')}_{\mathscr{X},\mathbb{Q}} \otimes_{\widehat{\mathscr{D}}^{(m)}_{\mathscr{X},\mathbb{Q}}} \mathscr{M}) \cap \mathring{T}^*X$ is empty. In this case, the source of the homomorphism 2.2.3.1 is zero. Thanks to the following isomorphism

$$(\widehat{\mathscr{E}}^{(m,m')}_{s,\mathbb{Q}})^{\mathrm{an}} \otimes_{\widehat{\mathscr{D}}^{(m)}} \mathscr{M} \cong (\widehat{\mathscr{E}}^{(m,m')}_{s,\mathbb{Q}})^{\mathrm{an}} \otimes_{\widehat{E}^{(m,m')}_{\mathscr{X},\mathbb{Q}}} \widehat{E}^{(m,m')}_{\mathscr{X},\mathbb{Q}} \otimes_{\widehat{\mathscr{D}}^{(m)}} \mathscr{M},$$

the target of the homomorphism is also zero, and we get the proposition. Thus we may assume that $\mathbf{Supp}(\widehat{\mathscr{E}}^{(m,m')}_{\mathscr{X},\mathbb{Q}} \otimes_{\widehat{\mathscr{D}}^{(m)}_{\mathscr{X},\mathbb{Q}}} \mathscr{M}) \cap \mathring{T}^*X = \pi^{-1}(s)$.

First, we will see that the source of the homomorphism has an $\widehat{\mathscr{E}}^{(m,m')}_{r,s,\mathbb{Q}}$-module structure for some integer r. For this, let us start by remarking that $\mathscr{M}$ is monogenic. Indeed, since $\mathscr{X}$ is affine, the m-th relative Frobenius homomorphism can be lifted. Let us denote this lifting by $\mathscr{X}'$ and $F\colon \mathscr{X} \to \mathscr{X}'$. Let the $\widehat{\mathscr{D}}^{(0)}_{\mathscr{X}',\mathbb{Q}}$-module $\mathscr{N}$ be the Frobenius descent of $\mathscr{M}$ (cf. [15, 4.1.3]). Then by [40, Proposition 5.3.1], there exists a surjection $\widehat{\mathscr{D}}^{(0)}_{\mathscr{X}',\mathbb{Q}} \to \mathscr{N}$. Apply F^* to both sides. By composing with the canonical surjection $\widehat{\mathscr{D}}^{(m)}_{\mathscr{X},\mathbb{Q}} \to F^*\widehat{\mathscr{D}}^{(0)}_{\mathscr{X}',\mathbb{Q}}$, we get a surjective morphism $\widehat{\mathscr{D}}^{(m)}_{\mathscr{X},\mathbb{Q}} \to \mathscr{M}$.

Denote by $\varphi\colon \widehat{\mathscr{D}}^{(m)}_{\mathscr{X},\mathbb{Q}}/I \cong \mathscr{M}$ the induced isomorphism, and put $I' := (\widehat{E}^{(m,m'+1)}_{\mathscr{X},\mathbb{Q}} \cdot I) \cap \widehat{E}^{(m,m'+1)}_{\mathscr{X}}$. By Lemma 1.3.1, there exist $Q \in \widehat{E}^{(m,m'+1)}_{\mathscr{X}} \subset \widehat{E}^{(m,m')}_{\mathscr{X}}$, $R \in (\widehat{E}^{(m,m'+1)}_{\mathscr{X}})_{-p^{m'+1}}$, and a positive integer d such that $x^d - \varpi Q - R \in I'$. Since the order of R is less than $-p^{m'+1}$, there exists $R' \in \widehat{E}^{(m,m')}_{\mathscr{X}}$ such that $R = \varpi R'$, and we get $x^d - \varpi S \in \widehat{E}^{(m,m')}_{\mathscr{X}} \cdot I'$ where $S = Q + R' \in \widehat{E}^{(m,m')}_{\mathscr{X}}$. By increasing the integer d, we may assume that d is divisible by $p^{m'+1}$. For any element P in $\widehat{D}^{(m')}_{\mathscr{X}} := \Gamma(\mathscr{X}, \widehat{\mathscr{D}}^{(m')}_{\mathscr{X}})$, we get $x^d \cdot P \in P \cdot x^d + p\widehat{D}^{(m')}_{\mathscr{X}}$. Thus for any operator $D \in \widehat{E}^{(m,m')}_{\mathscr{X}}$ we also get $x^d \cdot D \in D \cdot x^d + p\widehat{E}^{(m,m')}_{\mathscr{X}}$. This implies that for any integer $n > 0$, there exists $S_n \in \widehat{E}^{(m,m')}_{\mathscr{X}}$ such that

$$x^{nd} - \varpi^n S_n \in \widehat{E}^{(m,m')}_{\mathscr{X}} \cdot I'. \tag{2.2.3.2}$$

Let e be the absolute ramification index of R, and take $r \geqslant (e+1)d$, which is divisible by $p^{m'+1}$. Let $\alpha \in \Gamma(\mathscr{X}, \mathscr{M})$. We claim the following.

Claim. *For any sequence $\{P_i\}_{i\geqslant 0}$ in $\widehat{E}^{(m,m')}_{\mathscr{X},\mathbb{Q}} \cap \widehat{\mathscr{E}}^{(m,m')}_{r,s}$ which converges to zero seen as a sequence in $(\widehat{\mathscr{E}}^{(m,m')}_{r,s}, \mathscr{T}_0)$, the sequence $\{P_i \cdot (1 \otimes \alpha)\}$ in $\widehat{E}^{(m,m')}_{\mathscr{X},\mathbb{Q}} \otimes \mathscr{M}$ converges to zero using the $(\widehat{E}^{(m,m')}_{\mathscr{X},\mathbb{Q}}, \mathscr{T}_0)$-module topology. In particular the sequence converges to zero using the natural topology which makes the module an LF-space by Lemmas 1.2.3 and 1.3.2.*

Let us admit this claim first, and see that there exists a canonical $\widehat{\mathscr{E}}^{(m,m')}_{r,s,\mathbb{Q}}$-module structure on $\widehat{E}^{(m,m')}_{\mathscr{X},\mathbb{Q}} \otimes \mathscr{M}$. For $P \in \widehat{\mathscr{E}}^{(m,m')}_{r,s}$, we may write $P = \sum_{i\geqslant 0} P_i$ with $P_i \in \widehat{\mathscr{E}}^{(m,m')}_{\mathscr{X},\mathbb{Q}}$

and the sequence $\{P_i\}$ converges to zero in $\widehat{\mathscr{E}}^{(m,m')}_{r,s}$. Then the claim says that $\{P_i\cdot(1\otimes\alpha)\}$ converges to zero in $\widehat{E}^{(m,m')}_{\mathscr{X},\mathbb{Q}}\otimes\mathscr{M}$. So we may define $P\cdot(1\otimes\alpha)$ by $\sum_{i\geqslant 0}P_i\cdot(1\otimes\alpha)$, and we get the action of $\widehat{\mathscr{E}}^{(m,m')}_{r,s}$ on $\widehat{E}^{(m,m')}_{\mathscr{X},\mathbb{Q}}\otimes\mathscr{M}$ since the latter space is separated.

Let us verify the claim. By 2.2.3.2 and the choice of r, we get

$$\left(\frac{x^r}{p}\right)^n\equiv\varpi^nT_n \bmod \widehat{E}^{(m,m')}_{\mathscr{X},\mathbb{Q}}\cdot I',$$

where $T_n\in\widehat{E}^{(m,m')}_{\mathscr{X}}$ (*e.g.* $T_n=S_{(e+1)n}\in\widehat{E}^{(m,m')}_{\mathscr{X}}$ when $r=(e+1)d$). We denote by $d_{i,n}$ the order of the image of T_n in $E^{(m,m')}_{X_i}$. Put $E_r:=\widehat{E}^{(m,m')}_{\mathscr{X},\mathbb{Q}}\cap\widehat{\mathscr{E}}^{(m,m')}_{r,s}$. Let Q be an element of $(E_r)_N+\varpi^{N'}E_r$ for some integers N and $N'\geqslant 0$. We may write

$$Q=\sum_{n\geqslant 0}Q_n\left(\frac{x^r}{p}\right)^n,$$

where $Q_n\in A:=\bigoplus_{i=0}^{r-1}R_{\mathscr{X}}\{\partial\}^{(m,m')}x^i$. Then $Q_n\in(A)_N+\varpi^{N'}A$ for any $n\geqslant 0$. Thus, for any $n\geqslant 0$, we get

$$Q_n\left(\frac{x^r}{p}\right)^n\in(\widehat{E}^{(m,m')}_{\mathscr{X}})_{M+N}+\varpi^{N'}(\widehat{E}^{(m,m')}_{\mathscr{X}})+\widehat{E}^{(m,m')}_{\mathscr{X},\mathbb{Q}}\cdot I',$$

where $M:=\max\{d_{i,N'-1}\mid i=1,\ldots,N'-1\}=d_{N'-1,N'-1}$. Summing up, there exists an increasing sequence of integers $\{M_k\}_{k\geqslant 0}$ such that if $Q\in(E_r)_N+\varpi^{N'}E_r$, then $Q\in(\widehat{E}^{(m,m')}_{\mathscr{X}})_{M_{N'}+N}+\varpi^{N'}(\widehat{E}^{(m,m')}_{\mathscr{X}})+\widehat{E}^{(m,m')}_{\mathscr{X},\mathbb{Q}}\cdot I'$. We can find a sequence of integers $\{N_k\}_{k\geqslant 0}$ such that the sequence $\{N_k+M_k\}_{k\geqslant 0}$ is strictly decreasing. Back to the claim, for any integer $k\geqslant 0$, there exists n_k such that $P_j\in(E_r)_{N_k}+\varpi^kE_r$ for any $j\geqslant n_k$. Then, we have $P_j\cdot\alpha\in\varphi((\widehat{E}^{(m,m')}_{\mathscr{X}})_{M_k+N_k}+\varpi^k(\widehat{E}^{(m,m')}_{\mathscr{X}}))$, and we get the claim.

There are two natural homomorphisms

$$\alpha\colon\widehat{E}^{(m,m')}_{\mathscr{X},\mathbb{Q}}\otimes_{\widehat{\mathscr{D}}^{(m)}}\mathscr{M}\to\widehat{\mathscr{E}}^{(m,m')}_{r,s,\mathbb{Q}}\otimes_{\widehat{\mathscr{D}}^{(m)}}\mathscr{M},\quad\beta\colon\widehat{\mathscr{E}}^{(m,m')}_{r,s,\mathbb{Q}}\otimes\mathscr{M}\to\widehat{E}^{(m,m')}_{\mathscr{X},\mathbb{Q}}\otimes\mathscr{M},$$

where α is induced by the inclusion $\widehat{E}^{(m,m')}_{\mathscr{X},\mathbb{Q}}\to\widehat{\mathscr{E}}^{(m,m')}_{r,s,\mathbb{Q}}$, and β is defined by extending linearly the canonical homomorphism $\mathscr{M}\to\widehat{E}^{(m,m')}_{\mathscr{X},\mathbb{Q}}\otimes\mathscr{M}$ using the $\widehat{\mathscr{E}}^{(m,m')}_{r,s,\mathbb{Q}}$-module structure on $\widehat{E}^{(m,m')}_{\mathscr{X},\mathbb{Q}}\otimes\mathscr{M}$ defined above. We can check easily that $\beta\circ\alpha=\mathrm{id}$. Thus α is injective. Let us see that α is surjective. It suffices to show that α is a homomorphism of $\widehat{\mathscr{E}}^{(m,m')}_{r,s,\mathbb{Q}}$-modules. Take an element $P\in\widehat{\mathscr{E}}^{(m,m')}_{r,s,\mathbb{Q}}$. It suffices to show that

$$\alpha(P\cdot e)=P\cdot\alpha(e)\tag{2.2.3.3}$$

for any $e\in\widehat{E}^{(m,m')}_{\mathscr{X},\mathbb{Q}}\otimes_{\widehat{\mathscr{D}}^{(m)}}\mathscr{M}$. By density (cf. 2.2.2), there exists an integer $n\geqslant 0$ such that P is contained in the closure of $\widehat{E}^{(m,m')}_{\mathscr{X},\mathbb{Q}}$ in $(\widehat{\mathscr{E}}^{(m,m')}_{r,s,\mathbb{Q}},\mathscr{T}_n)$. Consider the $(\widehat{\mathscr{E}}^{(m,m')}_{r,s,\mathbb{Q}},\mathscr{T}_n)$-module topologies on the both sides of α. Since the both sides of α are finitely generated over $\widehat{\mathscr{E}}^{(m,m')}_{r,s,\mathbb{Q}}$, they are Fréchet spaces by 2.2.2, and by the open mapping theorem, the topology on the source of α is equivalent to the $(\widehat{E}^{(m,m')}_{\mathscr{X},\mathbb{Q}},\mathscr{T}_n)$-module topology. Since α

is $\widehat{E}^{(m,m')}_{\mathscr{X},\mathbb{Q}}$-linear, it is continuous. Since the target of α is separated, we get that 2.2.3.3 holds by the continuity of α. We conclude that α is an isomorphism.

Let us finish the proof. It suffices to see that

$$(\widehat{\mathscr{E}}^{(m,m')}_{s,\mathbb{Q}})^{\mathrm{an}} \otimes \mathscr{M} \to \varprojlim_n \widehat{\mathscr{E}}^{(m,m')}_{nc,s,\mathbb{Q}} \otimes \mathscr{M} \tag{2.2.3.4}$$

is an isomorphism where $c := p^{m'+1}$. When $\mathscr{M}$ is a finite projective $\widehat{\mathscr{D}}^{(m)}_{\mathscr{X},\mathbb{Q}}$-module, the equality is obvious. By the same argument as Lemma 1.3.4, $\widehat{\mathscr{E}}^{(m,m')}_{nc,s,\mathbb{Q}}$ is flat over $\widehat{\mathscr{D}}^{(m)}_{\mathscr{X},\mathbb{Q}}$ for any positive integer i. Since $\widehat{\mathscr{D}}^{(m)}_{\mathscr{X},\mathbb{Q}}$ is of finite homological dimension, there is a finite projective resolution $\mathscr{P}_\bullet \to \mathscr{M}$ whose length is l by [15, 4.4.6] (if fact we can take l to be 2). We have the following diagram where the bottom sequence is exact.

$$\begin{tikzcd}
0 \arrow[r] & (\widehat{\mathscr{E}}^{(m,m')}_{s,\mathbb{Q}})^{\mathrm{an}} \otimes \mathscr{P}_l \arrow[r] \arrow[d] & \dots \arrow[r] & (\widehat{\mathscr{E}}^{(m,m')}_{s,\mathbb{Q}})^{\mathrm{an}} \otimes \mathscr{P}_0 \arrow[r] \arrow[d] & (\widehat{\mathscr{E}}^{(m,m')}_{s,\mathbb{Q}})^{\mathrm{an}} \otimes \mathscr{M} \arrow[r] \arrow[d, "(*)"] & 0 \\
0 \arrow[r] & \widehat{\mathscr{E}}^{(m,m')}_{nc,s,\mathbb{Q}} \otimes \mathscr{P}_l \arrow[r] & \dots \arrow[r] & \widehat{\mathscr{E}}^{(m,m')}_{nc,s,\mathbb{Q}} \otimes \mathscr{P}_0 \arrow[r] & \widehat{\mathscr{E}}^{(m,m')}_{nc,s,\mathbb{Q}} \otimes \mathscr{M} \arrow[r] & 0
\end{tikzcd}$$

Let us show that $R^1 \varprojlim_n (\widehat{\mathscr{E}}^{(m,m')}_{nc,s,\mathbb{Q}} \otimes \mathscr{P}_j) = 0$ for any j. This is equivalent to saying $R^1 \varprojlim_n \widehat{\mathscr{E}}^{(m,m')}_{nc,s,\mathbb{Q}} = 0$ since $\mathscr{P}_j$ is finite projective. Let $|\cdot|_n$ be the p-adic norm on $\widehat{\mathscr{E}}^{(m,m')}_{nc,s,\mathbb{Q}}$. Let E_n be the closure of $\widehat{\mathscr{E}}^{(m,m')}_{(n+1)c,s,\mathbb{Q}}$ in $\widehat{\mathscr{E}}^{(m,m')}_{nc,s,\mathbb{Q}}$ with respect to $|\cdot|_n$-norm. Then $R^1 \varprojlim_n E_n \cong R^1 \varprojlim_n \widehat{\mathscr{E}}^{(m,m')}_{nc,s,\mathbb{Q}}$ (cf. the proof of [3, Theorem 5.8]). Now, apply [37, 0_{III}, 13.2.4 (i)] to the system $\{E_n\}$ with respect to the p-adic norm, and the claim follows.

By applying $\varprojlim_n$ to the bottom sequence of the diagram above, we get that the sequence

$$\varprojlim_n \widehat{\mathscr{E}}^{(m,m')}_{nc,s,\mathbb{Q}} \otimes \mathscr{P}_1 \to \varprojlim_n \widehat{\mathscr{E}}^{(m,m')}_{nc,s,\mathbb{Q}} \otimes \mathscr{P}_0 \to \varprojlim_n \widehat{\mathscr{E}}^{(m,m')}_{nc,s,\mathbb{Q}} \otimes \mathscr{M} \to 0$$

is exact. Since 2.2.3.4 is an isomorphism for projective modules, by using the right exactness of tensor product, 2.2.3.4 is an isomorphism in general, and we conclude the proof of the proposition. Moreover, when we apply $\varprojlim_n$ to the bottom sequence of the above diagram, we get that the vertical homomorphisms are isomorphisms, and thus the top sequence is also exact. This implies the flatness of $(\widehat{\mathscr{E}}^{(m,m')}_{s,\mathbb{Q}})^{\mathrm{an}}$ over $\widehat{\mathscr{D}}^{(m)}_{\mathscr{X},\mathbb{Q}}$. $\square$

2.2.4 Corollary. *Under the hypothesis of Proposition 2.2.3, suppose moreover that $\mathscr{M}$ is stable. Then for $m' \geqslant m$ the canonical homomorphisms*

$$E^{(m',\dagger)}_{\mathscr{X},\mathbb{Q}} \otimes_{\widehat{\mathscr{D}}^{(m)}} \mathscr{M} \to (\mathscr{E}^{(m',\dagger)}_{s,\mathbb{Q}})^{\mathrm{an}} \otimes_{\widehat{\mathscr{D}}^{(m)}} \mathscr{M}, \quad E^{\dagger}_{\mathscr{X},\mathbb{Q}} \otimes_{\widehat{\mathscr{D}}^{(m)}} \mathscr{M} \to \mathscr{E}^{\mathrm{an}}_{s,\mathbb{Q}} \otimes_{\widehat{\mathscr{D}}^{(m)}} \mathscr{M}$$

are isomorphisms.

Proof. Clearly the first equality implies the second one. To prove the first one, it suffices to show that $\varprojlim_{m''} (\widehat{\mathscr{E}}^{(m',m'')}_{s,\mathbb{Q}})^{\mathrm{an}} \otimes \mathscr{M} \cong (\mathscr{E}^{(m',\dagger)}_{s,\mathbb{Q}})^{\mathrm{an}} \otimes \mathscr{M}$ since $E^{(m'',\dagger)}_{\mathscr{X},\mathbb{Q}}$ is a Fréchet–Stein algebra (cf. [3, Theorem 5.8]). For this we only need to repeat the argument of the last part of the proof of the previous proposition. Namely, we prove that $R^1 \varprojlim_{m''}(\widehat{\mathscr{E}}^{(m',m'')}_{s,\mathbb{Q}})^{\mathrm{an}} = 0$ using [37, 0_{III}, 13.2.4 (i)]. The detail is left to the reader. $\square$

2.2.5. Let $\mathscr{X}$ be a formal curve over R. Let $\mathscr{M}$ be a stable holonomic $\widehat{\mathscr{D}}^{(m)}_{\mathscr{X},\mathbb{Q}}$-module, and let s be a singular point of $\mathscr{M}$. Then by Proposition 2.2.3, for any integers $m'' \geqslant m' \geqslant m$, the module $\widehat{\mathscr{E}}^{(m',m'')}_{s,\mathbb{Q}} \otimes \mathscr{M}$ (cf. 1.3.10) possesses a canonical $(\widehat{\mathscr{D}}^{(m')}_{s,\mathbb{Q}})^{\mathrm{an}}$-module structure, and in particular, we get a canonical homomorphism

$$\psi\colon (\widehat{\mathscr{D}}^{(m')}_{s,\mathbb{Q}})^{\mathrm{an}} \otimes \mathscr{M} \to \widehat{\mathscr{E}}^{(m',m'')}_{s,\mathbb{Q}} \otimes \mathscr{M}.$$

Taking the inductive limit over m', we get a canonical homomorphism

$$\mathscr{M}|_{S_s} \to \mathscr{E}^{\dagger}_{s,\mathbb{Q}} \otimes \mathscr{M}.$$

By an abuse of language, we sometimes denote the image of $\alpha \in M_s$ where M_s is either $(\widehat{\mathscr{D}}^{(m')}_{s,\mathbb{Q}})^{\mathrm{an}} \otimes \mathscr{M}$ or $\mathscr{M}|_{S_s}$ by $1 \otimes \alpha$.

Let $\mathscr{U}$ be an open affine neighbourhood of s such that there exists a local parameter at s and s is the unique singularity of $\mathscr{M}$ in $\mathscr{U}$. Then by the proposition, we get $\widehat{E}^{(m',m'')}_{\mathscr{U},\mathbb{Q}} \otimes \mathscr{M} \cong \widehat{\mathscr{E}}^{(m',m'')}_{s,\mathbb{Q}} \otimes \mathscr{M}$. Now we define topologies as follows.

Definition. (i) We equip $\widehat{\mathscr{E}}^{(m',m'')}_{s,\mathbb{Q}} \otimes \mathscr{M}$ with the natural topology as an $\widehat{E}^{(m',m'')}_{\mathscr{U},\mathbb{Q}}$-module which makes it an LF-space by Lemma 1.2.3 and Lemma 1.3.2. Note that this topology does not depend on the choice of $\mathscr{U}$ by the same lemma.

(ii) We equip $(\widehat{\mathscr{D}}^{(m')}_{s,\mathbb{Q}})^{\mathrm{an}} \otimes \mathscr{M}$ with the projective limit topology of the projective system of Banach spaces $\left\{\mathcal{O}_{np^{m+1}}\widehat{\otimes}(\widehat{\mathscr{D}}^{(m')}_{\mathscr{U},\mathbb{Q}} \otimes \mathscr{M})\right\}_{n\geqslant 0}$. This makes $(\widehat{\mathscr{D}}^{(m')}_{s,\mathbb{Q}})^{\mathrm{an}} \otimes \mathscr{M}$ a Fréchet space.

Remark. (i) The topology on $\widehat{\mathscr{E}}^{(m',m'')}_{s,\mathbb{Q}} \otimes \mathscr{M}$ is also equivalent to the $(\widehat{E}^{(m',m'')}_{\mathscr{U},\mathbb{Q}}, \mathscr{T})$-module topology by Lemma 1.2.3.

(ii) The homomorphism ψ is continuous by the claim in the proof of Proposition 2.2.3. In particular, if a sequence $\{\alpha_i\}$ converges to α in $(\widehat{\mathscr{D}}^{(m')}_{s,\mathbb{Q}})^{\mathrm{an}} \otimes \mathscr{M}$, the sequence $\left\{1 \otimes \alpha_i\right\}$ converges to $1 \otimes \alpha$ in $\widehat{\mathscr{E}}^{(m',m'')}_{s,\mathbb{Q}} \otimes \mathscr{M}$.

2.2.6. The following corollary of Proposition 2.2.3 plays an important role when we prove fundamental properties of local Fourier transforms (cf. Lemmas 2.4.9 and 2.4.10).

Corollary. *Let $\mathscr{X}$ and $\mathscr{X}'$ be two formal curves, and take points $x \in \mathscr{X}$ and $x' \in \mathscr{X}'$. Assume that there exists an isomorphism $\iota\colon \mathscr{S}_x \xrightarrow{\sim} \mathscr{S}_{x'}$ of formal disks over R.*

(i) *Let $\mathscr{M}$ and $\mathscr{M}'$ be holonomic $\widehat{\mathscr{D}}^{(m)}_{\mathscr{X},\mathbb{Q}}$ and $\widehat{\mathscr{D}}^{(m)}_{\mathscr{X}',\mathbb{Q}}$-module respectively, and assume that*

$$\iota_*\big((\widehat{\mathscr{D}}^{(m)}_{x,\mathbb{Q}})^{\mathrm{an}} \otimes \mathscr{M}\big) \xrightarrow{\sim} (\widehat{\mathscr{D}}^{(m)}_{x',\mathbb{Q}})^{\mathrm{an}} \otimes \mathscr{M}'$$

as $(\widehat{\mathscr{D}}^{(m)}_{x',\mathbb{Q}})^{\mathrm{an}}$*-modules. Then there exists canonical isomorphisms*

$$\iota_*\big(\widehat{\mathscr{E}}^{(m,m')}_{x,\mathbb{Q}}\otimes\mathscr{M}\big)\cong\widehat{\mathscr{E}}^{(m,m')}_{x',\mathbb{Q}}\otimes\mathscr{M}',\quad \iota_*\big(\mathscr{E}^{(m,\dagger)}_{x,\mathbb{Q}}\otimes\mathscr{M}\big)\cong\mathscr{E}^{(m,\dagger)}_{x',\mathbb{Q}}\otimes\mathscr{M}'$$

for $m'\geqslant m$*.*

(ii) *Let* $\mathscr{M}$ *and* $\mathscr{M}'$ *be holonomic* $\mathscr{D}^{\dagger}_{\mathscr{X},\mathbb{Q}}$ *and* $\mathscr{D}^{\dagger}_{\mathscr{X}',\mathbb{Q}}$*-module respectively, and assume that*

$$\iota_*\mathscr{M}|_{S_x}\cong\mathscr{M}'|_{S_{x'}}$$

as $\mathscr{D}^{\mathrm{an}}_{x',\mathbb{Q}}$*-modules. Then there exists a canonical isomorphism* $\iota_*\big(\mathscr{E}^{\dagger}_{x,\mathbb{Q}}\otimes\mathscr{M}\big)\cong\mathscr{E}^{\dagger}_{x',\mathbb{Q}}\otimes\mathscr{M}'$*.*

Proof. First, let us prove (i). We get

$$\begin{aligned}\iota_*\widehat{\mathscr{E}}^{(m,m')}_{x,\mathbb{Q}}\otimes\mathscr{M}&\xrightarrow[(*)]{\sim}\iota_*(\widehat{\mathscr{E}}^{(m,m')}_{x,\mathbb{Q}})^{\mathrm{an}}\otimes\mathscr{M}\xleftarrow{\sim}\iota_*(\widehat{\mathscr{E}}^{(m,m')}_{x,\mathbb{Q}})^{\mathrm{an}}\otimes((\widehat{\mathscr{D}}^{(m)}_{\mathscr{X},\mathbb{Q}})^{\mathrm{an}}\otimes\mathscr{M})\\&\xrightarrow[\iota']{\sim}(\widehat{\mathscr{E}}^{(m,m')}_{x',\mathbb{Q}})^{\mathrm{an}}\otimes((\widehat{\mathscr{D}}^{(m)}_{\mathscr{X}',\mathbb{Q}})^{\mathrm{an}}\otimes\mathscr{M})\xrightarrow{\sim}(\widehat{\mathscr{E}}^{(m,m')}_{x',\mathbb{Q}})^{\mathrm{an}}\otimes\mathscr{M}\\&\xleftarrow[(*)]{\sim}\widehat{\mathscr{E}}^{(m,m')}_{x',\mathbb{Q}}\otimes\mathscr{M}'.\end{aligned}$$

Here we used Proposition 2.2.3 two times at $(*)$, and ι' denotes the isomorphism induced by ι. To show the equality for $\mathscr{E}^{(m,\dagger)}_{\mathscr{X},\mathbb{Q}}$, it suffices to use Proposition 2.2.3, Corollary 2.2.4, and the Fréchet–Stein property of $E^{(m,\dagger)}_{\mathscr{U},\mathbb{Q}}$ where $\mathscr{U}$ is an affine neighbourhood of x.

Now, let us prove (ii). Let $\mathscr{M}$ be a $\mathscr{D}^{\dagger}_{\mathscr{X},\mathbb{Q}}$-module. Let $\mathscr{M}^{(m)}$ be a coherent $\widehat{\mathscr{D}}^{(m)}_{\mathscr{X},\mathbb{Q}}$-module such that $\mathscr{D}^{\dagger}_{\mathscr{X},\mathbb{Q}}\otimes\mathscr{M}^{(m)}\cong\mathscr{M}$, and the same for $\mathscr{M}'^{(m)}$. Then since these are coherent, there exists N such that

$$\mathscr{S}_{x'}\otimes_{\mathscr{S}_x}(\widehat{\mathscr{D}}^{(N)}_{\mathscr{X},\mathbb{Q}})^{\mathrm{an}}\otimes\mathscr{M}^{(m)}\xrightarrow{\sim}(\widehat{\mathscr{D}}^{(N)}_{\mathscr{X}',\mathbb{Q}})^{\mathrm{an}}\otimes\mathscr{M}'^{(m)}.$$

Thus (ii) follows from (i). □

2.3. Equality between two definitions of irregularity

Another important corollary of Proposition 2.2.3 is a comparison result of multiplicities of characteristic cycles (irregularity of Garnier) and irregularity of Christol–Mebkhout, cf. Corollary 2.3.2.

2.3.1. Let us review the definitions first. We assume that k is perfect and that there exists a lifting of h-th absolute Frobenius $R\xrightarrow{\sim}R$. Let M be a solvable differential $\mathcal{R}_K$-module, cf. [47, 12.6.4] or [25, 8.7]. By a result of Christol and Mebkhout (cf. [26, 2.4-1] for free differential modules, or in general [47, 12.6.4]) there exists a canonical decomposition $M\cong\bigoplus_{\beta\geqslant 0}M_\beta$ where M_β is a differential $\mathcal{R}_K$-module purely of differential slope β. The irregularity of M is defined by $\mathrm{irr}(M):=\sum_{\beta\geqslant 0}\beta\cdot\mathrm{rk}(M_\beta)$. We say that M is *regular* if $\mathrm{irr}(M)=0$, or equivalently $M=M_0$.

Let $\mathscr{X}$ be a formal curve over R, and S be a closed subset of $\mathscr{X}$ such that its complement is dense in $\mathscr{X}$. Let $\mathscr{M}$ be a convergent isocrystal on $\mathscr{U}:=\mathscr{X}\setminus S$

overconvergent along S. For x in S, we define the irregularity of Christol–Mebkhout

$$\mathrm{irr}^{\mathrm{CM}}_x(\mathscr{M}) := \mathrm{irr}(\mathscr{M}|_{\eta_x});$$

when $\mathrm{irr}(\mathscr{M}|_{\eta_x}) = 0$ we say that $\mathscr{M}$ *is regular at* x, or that x *is a regular singularity for* $\mathscr{M}$. We also have the irregularity of Garnier [41, 5.1.2]. For simplicity, we assume moreover that $\mathscr{M}$ possesses a Frobenius structure. Let us denote by $j_+\mathscr{M}$ the underlying $\mathscr{D}^\dagger_{\mathscr{X},\mathbb{Q}}$-module of $\mathscr{M}$, which is *a priori* a coherent $\mathscr{D}^\dagger_{\mathscr{X},\mathbb{Q}}(S)$-module. This is a holonomic module, and Garnier defined[5] for x in S

$$\mathrm{irr}^{\mathrm{Gar}}_x(\mathscr{M}) := \chi\big(R\,\mathcal{H}om_{\mathscr{D}^\dagger_{\mathscr{X},\mathbb{Q}}}(j_+\mathscr{M}, \mathcal{O}_{\mathscr{X},\mathbb{Q}}|_{S_x})\big) - (m_{\xi_x=0}(j_+\mathscr{M}) - m_x(j_+\mathscr{M})),$$

where m_x (resp. $m_{\xi_x=0}$) denotes the vertical multiplicity at x (resp. the generic rank) of $j_+\mathscr{M}$, cf. 1.4.3 and Definition 1.5.2. Since $\mathscr{M}$ is an isocrystal we have $m_{\xi_x=0}(j_+\mathscr{M}) = \mathrm{rk}(\mathscr{M})$, cf. [41, 2.2.5, 2.2.6]. The finiteness of the index $\chi\big(R\,\mathcal{H}om_{\mathscr{D}^\dagger_{\mathscr{X},\mathbb{Q}}}(j_+\mathscr{M}, \mathcal{O}_{\mathscr{X},\mathbb{Q}}|_{S_x})\big)$ was not known at the time Garnier defined the irregularity; now, using some results of Crew, based on the local monodromy theorem, this finiteness is easy. Let us mention that the definition of irregularity of Garnier holds more in general for holonomic $\mathscr{D}^\dagger_{\mathscr{X},\mathbb{Q}}$-modules; here, since $\mathscr{M}$ is an overconvergent F-isocrystal, the index is indeed zero, as proven in the next corollary, cf. 2.3.2.1.

Let us recall the Grothendieck–Ogg–Shafarevich type formula (GOS-type formula) as stated[6] by Garnier in [41, 5.3.2]: for an overconvergent F-isocrystal $\mathscr{M}$ as above, we have

$$\chi_{\mathrm{rig}}(\mathscr{U}, \mathscr{M}) = \mathrm{rk}(\mathscr{M})\chi_{\mathrm{rig}}(\mathscr{U}) - \sum_{x\in S} \deg(x)\cdot \mathrm{irr}^{\mathrm{Gar}}_x(\mathscr{M}), \tag{2.3.1.1}$$

where $\chi_{\mathrm{rig}}(\mathscr{U}, \mathscr{M})$ (resp. $\chi_{\mathrm{rig}}(\mathscr{U})$) denotes the Euler characteristic for rigid cohomology of the isocrystal $\mathscr{M}$ (resp. $\mathcal{O}_{\mathscr{X},\mathbb{Q}}(S)$).

2.3.2. The following corollary has already been announced[7] by the second author using a local computation, which is different from our method here.

Corollary. *Let $\mathscr{M}$ be a convergent F-isocrystal on $\mathscr{X}\setminus S$ overconvergent along S. Then we have*

$$R\,\mathcal{H}om_{\mathscr{D}^\dagger_{\mathscr{X},\mathbb{Q}}}(j_+\mathscr{M}, \mathcal{O}_{\mathscr{X},\mathbb{Q}}|_{S_x}) = 0. \tag{2.3.2.1}$$

Moreover, we have

$$\mathrm{irr}^{\mathrm{Gar}}_x(\mathscr{M}) = \mathrm{irr}^{\mathrm{CM}}_x(\mathscr{M}). \tag{2.3.2.2}$$

Proof. First of all, let us show the equality 2.3.2.1. The ring $\mathcal{O}_{\mathscr{X},\mathbb{Q}}|_{S_x}$ has a canonical $\mathscr{D}^{\mathrm{an}}_{\mathscr{S}_x,\mathbb{Q}}$-module structure and identifies with $\mathcal{O}^{\mathrm{an}}_{\mathscr{S}_x}$, cf. 2.1.8; let us denote it simply by $\mathcal{O}^{\mathrm{an}}$.

[5] In [41], he defined only in the case where x is a k-rational point, but we do not think we need this assumption here. In [41], $\mathcal{O}_{\mathscr{X},\mathbb{Q}}|_{S_x}$ is denoted by $\mathrm{sp}_*\mathcal{O}_{]x[}$.

[6] Again in [41], Garnier needed to assume $\chi(R\,\mathcal{H}om_{\mathscr{D}^\dagger_{\mathscr{X},\mathbb{Q}}}(j_+\mathscr{M}, \mathcal{O}_{\mathscr{X},\mathbb{Q}}|_{S_x})) = 0$.

[7] See A. Marmora, *About p-adic Local Fourier Transform*, Poster two at Journées de Géométrie Arithmétique de Rennes, available at http://perso.univ-rennes1.fr/ahmed.abbes/Conference/posters.html.

Let $\mathcal{A} \to \mathcal{B}$ be a *flat* homomorphism of sheaves of rings on a topological space T. For $\mathcal{M} \in D^-(\mathcal{A})$ and $\mathcal{N} \in D^+(\mathcal{B})$, we have $R\mathrm{Hom}_{\mathcal{A}}(\mathcal{M}, \mathcal{N}) \cong R\mathrm{Hom}_{\mathcal{B}}(\mathcal{B} \otimes_{\mathcal{A}} \mathcal{M}, \mathcal{N})$. Using this, we have

$$\begin{aligned} R^n\,\mathcal{H}om_{\mathscr{D}^\dagger_{\mathscr{X},\mathbb{Q}}}(j_+\mathscr{M}, \mathcal{O}_{\mathscr{X},\mathbb{Q}}|_{S_x}) &\cong R^n\,\mathcal{H}om_{\mathscr{D}^\dagger_{\mathscr{S}_x,\mathbb{Q}}}(\mathscr{D}^\dagger_{\mathscr{S}_x,\mathbb{Q}} \otimes j_+\mathscr{M}, \mathcal{O}^{\mathrm{an}}) \\ &\cong R^n\mathrm{Hom}_{\mathscr{D}^\dagger_{\mathscr{S}_x,\mathbb{Q}}}(\mathscr{D}^\dagger_{\mathscr{S}_x,\mathbb{Q}} \otimes j_+\mathscr{M}, \mathcal{O}^{\mathrm{an}}) \\ &\cong R^n\mathrm{Hom}_{\mathscr{D}^{\mathrm{an}}_{\mathscr{S}_x,\mathbb{Q}}}(\mathscr{D}^{\mathrm{an}}_{\mathscr{S}_x,\mathbb{Q}} \otimes j_+\mathscr{M}, \mathcal{O}^{\mathrm{an}}) \end{aligned}$$

where the first and the last isomorphism follow by Lemma 2.1.8 and [31, Theorem 4.1.1], and the second is obtained by taking global sections, cf. 2.1.2 and also 2.1.8.

Since $\mathscr{D}^{\mathrm{an}}_{\mathscr{S}_x,\mathbb{Q}} \otimes j_+\mathscr{M} = (j_+\mathscr{M})|_{S_x}$ is clearly of connection type, the canonical map

$$\mathrm{can}\colon \mathbb{V}((j_+\mathscr{M})|_{S_x}) \to \mathbb{W}((j_+\mathscr{M})|_{S_x})$$

is an isomorphism, cf. [31, Corollary 6.1.2]. We have $R^n\mathrm{Hom}_{\mathscr{D}^{\mathrm{an}}_{\mathscr{S}_x,\mathbb{Q}}}((j_+\mathscr{M})|_{S_x}, \mathcal{O}^{\mathrm{an}}_{K^{\mathrm{ur}}}) = 0$ by 2.1.7.1 and 2.1.7.2, from which it follows $R^n\mathrm{Hom}_{\mathscr{D}^{\mathrm{an}}_{\mathscr{S}_x,\mathbb{Q}}}((j_+\mathscr{M})|_{S_x}, \mathcal{O}^{\mathrm{an}}) = 0$. The first claim is proven.

Now, let us start the proof of the equality of the irregularities. The irregularity $\mathrm{irr}^{\mathrm{CM}}_x$ only depends on $\mathscr{M}|_{\eta_x}$ by definition. By Corollary 2.2.6 combined with Theorem 1.5.1, we get that $\mathrm{irr}^{\mathrm{Gar}}_x$ only depends on its analytification as well. This implies that we may assume $\mathscr{X} \cong \widehat{\mathbb{A}}^1$, $x = 0$, and that $\mathscr{M}$ is the canonical extension of $\mathscr{M}|_{\eta_x}$. Note that, thanks to 2.3.2.1, $\mathrm{irr}^{\mathrm{Gar}}_x$ satisfies GOS-type formula by [41, 5.3.2]. By [25, 1.2], we know that $\mathrm{irr}^{\mathrm{CM}}_x$ also satisfies GOS-type formula.

The equality 2.3.2.2 holds when $\mathscr{M}$ is regular singular at x. Indeed, by the definition of regularity we have $\mathrm{irr}^{\mathrm{CM}}_x(\mathscr{M}) = 0$, and it suffices to show that $\mathrm{irr}^{\mathrm{Gar}}_x(\mathscr{M}) = 0$. Now, by using the structure theorem of regular p-adic differential equation [25, 12.3] and the additivity of $\mathrm{irr}^{\mathrm{Gar}}_x$ (cf. [41, 5.1.3]), we may assume that $\mathscr{M}$ is of rank one. For this case we refer to [41, 5.3.1].

Finally, let us prove the general case. Set $\mathscr{U} := \widehat{\mathbb{P}} \backslash \{x, \infty\}$. By GOS-type formulas, we get

$$\begin{aligned} \chi_{\mathrm{rig}}(\mathscr{U}, \mathscr{M}) &= \mathrm{rk}(\mathscr{M}) \cdot \chi_{\mathrm{rig}}(\mathscr{U}) - \mathrm{irr}^{\mathrm{CM}}_x(\mathscr{M}) - \mathrm{irr}^{\mathrm{CM}}_\infty(\mathscr{M}), \\ \chi_{\mathrm{rig}}(\mathscr{U}, \mathscr{M}) &= \mathrm{rk}(\mathscr{M}) \cdot \chi_{\mathrm{rig}}(\mathscr{U}) - \mathrm{irr}^{\mathrm{Gar}}_x(\mathscr{M}) - \mathrm{irr}^{\mathrm{Gar}}_\infty(\mathscr{M}). \end{aligned}$$

By using the regular case we have proven, we get $\mathrm{irr}^{\mathrm{CM}}_\infty(\mathscr{M}) = \mathrm{irr}^{\mathrm{Gar}}_\infty(\mathscr{M}) = 0$ since $\mathscr{M}$ is regular at ∞. Thus comparing these two equalities, we get what we want. □

2.4. Definition of local Fourier transform

In this subsection, we define the local Fourier transform. We only define the so called $(0, \infty')$-local Fourier transforms. In section 6.2, we will define an analog of $(\infty, 0')$-local Fourier transform in very special cases, and we do not deal with (∞, ∞')-local Fourier transform in this paper.

2.4.1. Let us fix a situation under which we use the Fourier transforms. Let h be a positive integer, and put $q := p^h$. We assume that the residue field k of R is perfect.

We moreover assume that the absolute h-th Frobenius automorphism of k lifts to an automorphism $\sigma\colon R \xrightarrow{\sim} R$.

We assume that the field K contains a root π of the equation $X^{p-1}+p=0$ (so that it contains all of them). The Dwork exponential series $\theta_\pi(x)=\exp(\pi(x-x^p))$ in $K[\![x]\!]$ has a radius of convergence strictly greater than one and it converges for $x=1$ to a p-th root of unity $\zeta=\theta_\pi(1)$. The choice of π determines a non-trivial additive character $\psi_\pi\colon \mathbb{F}_p\to K^*$, by sending x to ζ^x. Conversely if $\psi\colon\mathbb{F}_p\to K^*$ is a non-trivial additive character, then $\psi(1)$ is a p-th root of unity in K and the polynomial $X^{p-1}+p$ splits completely in K. There exists a unique root π_ψ of $X^{p-1}+p$ such that $\pi_\psi\equiv\psi(1)-1$ modulo $(\psi(1)-1)^2$ and then we have $\theta_{\pi_\psi}(1)=\psi(1)$. For the details see [10, 1.3].

We denote by $\widehat{\mathbb{A}}^1_{R,x}:=\mathrm{Spf}(R\{x\})$ (resp. $\widehat{\mathbb{P}}^1_{R,x}$) the affine (resp. projective) line over R with the fixed coordinate x, and we denote by $\widehat{\mathbb{A}}^1_{R,x'}$ (resp. $\widehat{\mathbb{P}}^1_{R,x'}$) its dual line. To lighten the notation we often put $\widehat{\mathbb{A}}:=\widehat{\mathbb{A}}^1_{R,x}$, $\widehat{\mathbb{P}}:=\widehat{\mathbb{P}}^1_{R,x}$, $\widehat{\mathbb{A}}':=\widehat{\mathbb{A}}^1_{R,x'}$ and $\widehat{\mathbb{P}}':=\widehat{\mathbb{P}}^1_{R,x'}$. We denote by ∂ and ∂' (or ∂_x and $\partial_{x'}$ if we want to clarify the coordinates) the differential operators corresponding to x and x'. We denote by ∞ (resp. ∞') the point at infinity of $\widehat{\mathbb{P}}$ (resp. $\widehat{\mathbb{P}}'$). Let $\widehat{\mathbb{P}}'':=\widehat{\mathbb{P}}\times\widehat{\mathbb{P}}'$, and $Z:=(\{\infty\}\times\widehat{\mathbb{P}}')\cup(\widehat{\mathbb{P}}\times\{\infty'\})$.

To summarize notation once and for all, we use terminologies of the next section, and consider the following diagram of *couples*:

$$\begin{tikzcd} & (\widehat{\mathbb{P}}'', Z) \arrow[dl, "p"'] \arrow[dr, "p'"] & \\ (\widehat{\mathbb{P}}, \{\infty\}) & & (\widehat{\mathbb{P}}', \{\infty'\}), \end{tikzcd} \tag{2.4.1.1}$$

where p and p' are the projections. These morphisms are not used till the next section.

If we take $\mathscr{X}$ to be $\widehat{\mathbb{A}}$ (resp. $\widehat{\mathbb{A}}'$), we are in Situation (L) of 1.2.1 using the fixed global coordinate. We use freely the notation of 1.2.1, especially $K_{\widehat{\mathbb{A}}}\{\partial\}^{(m,m')}$.

For a smooth formal scheme $\mathscr{X}$ over R, we put $\mathscr{X}^{(1)}:=\mathscr{X}\otimes_{R,\sigma}R$. We denote by y (resp. y') the local coordinate of $\widehat{\mathbb{P}}^{(1)}$ (resp. $\widehat{\mathbb{P}}'^{(1)}$) induced by x (resp. x'). The relative Frobenius of $\mathbb{P}^1_k$ lifts to the morphism $F_{\widehat{\mathbb{P}}}\colon\widehat{\mathbb{P}}\to\widehat{\mathbb{P}}^{(1)}$ sending y to x^q. We have a similar morphism for $\widehat{\mathbb{P}}'$, and denote it by $F_{\widehat{\mathbb{P}}'}$.

Let us introduce some notation for a formal disk around a closed point of $\widehat{\mathbb{A}}$. We put $\mathscr{S}:=\mathrm{Spf}(R[\![u]\!])$ and $\mathscr{S}':=\mathrm{Spf}(R[\![u']\!])$. We denote $\eta_{\mathscr{S}}$ and $\eta_{\mathscr{S}'}$ by η and η' respectively. Let E be a finite unramified extension of K, and R_E the ring of integers of E. We put $\mathscr{S}_E:=\mathrm{Spf}(R_E[\![u]\!])$ and denote $\eta_{\mathscr{S}_E}$ by η_E. Let s be a closed point of $\widehat{\mathbb{A}}=\mathrm{Spf}(R\{x\})$, $\mathfrak{m}_s$ the maximal ideal of $k[x]$ corresponding to s. We denote by y_s the monic generator of $\mathfrak{m}_s$, and by $\widetilde{y}_s$ a lifting of y_s in $R\{x\}$, which is a local parameter of $\widehat{\mathbb{A}}$ at s, cf. 0.0.5; we denote also by $\widetilde{y}_s$ its image in $\widehat{\mathcal{O}}_{\widehat{\mathbb{A}},s}$, the completion of the local ring of $\widehat{\mathbb{A}}$ at s. We define $\tau_{\widetilde{y}_s}\colon\mathscr{S}_s:=\mathrm{Spf}(\widehat{\mathcal{O}}_{\widehat{\mathbb{A}},s})\to\mathscr{S}$ by sending u to $\widetilde{y}_s$. We note that if we take another lifting $\widetilde{y}'_s$, then there exists a canonical equivalence of functors $\tau^*_{\widetilde{y}_s}\cong\tau^*_{\widetilde{y}'_s}$ and $\tau_{\widetilde{y}_s*}\cong\tau_{\widetilde{y}'_s*}$, since $\widetilde{y}_s$ and $\widetilde{y}'_s$ are congruent modulo ϖ. We denote $\tau^*_{\widetilde{y}_s}$ and $\tau_{\widetilde{y}_s*}$ by τ^*_s and τ_{s*} respectively.

For $\mathfrak{s}\in\mathbb{A}^1_k(\bar{k})$, let s be the closed point of $\mathbb{A}^1_k$ defined by $\mathfrak{s}$. Let k_s be the residue field of s, R_s be the unique finite étale extension of R corresponding to k_s, namely $W(k_s)\otimes_{W(k)}R$, and K_s be its field of fractions. Let $\widetilde{s}$ be the closed point of $\widehat{\mathbb{A}}_s:=\widehat{\mathbb{A}}\otimes R_s$ defined by $\mathfrak{s}$, and $\widehat{\mathbb{P}}_s:=\widehat{\mathbb{P}}\otimes R_s$. The rational point $\mathfrak{s}$ corresponds to an element of k_s also denoted by

$\mathfrak{s}$, and take a lifting $\tilde{\mathfrak{s}}$ of $\mathfrak{s}$ in R_s. We have the homomorphism $\sigma_{\tilde{\mathfrak{s}}}\colon \mathscr{S}_{\tilde{\mathfrak{s}}} \to \mathscr{S}$ sending u to $x - \tilde{\mathfrak{s}}$. The functors $\sigma^*_{\tilde{\mathfrak{s}}}$ and $\sigma_{\tilde{\mathfrak{s}}*}$ do not depend on the choice of $\tilde{\mathfrak{s}}$ up to a canonical equivalence. We denote $\sigma^*_{\tilde{\mathfrak{s}}}$ and $\sigma_{\tilde{\mathfrak{s}}*}$ by $\sigma^*_{\mathfrak{s}}$ and $\sigma_{\mathfrak{s}*}$ respectively.

By the étaleness of R_s over R, there is a canonical inclusion $R_s \hookrightarrow \widehat{\mathcal{O}}_{\widehat{\mathbb{A}},s}$. We define $\tilde{\tau}_{\tilde{\mathfrak{s}}}\colon \mathscr{S}_s \to \mathscr{S}_{K_s}$ by sending u to $x - \tilde{\mathfrak{s}}$. Also in this case, $\tilde{\tau}^*_{\tilde{\mathfrak{s}}}$ and $\tilde{\tau}_{\tilde{\mathfrak{s}}*}$ do not depend on the choice of $\tilde{\mathfrak{s}}$ up to a canonical equivalence and we denote them by $\tilde{\tau}^*_{\mathfrak{s}}$ and $\tilde{\tau}_{\mathfrak{s}*}$ respectively. Let $\tau'\colon \eta_{\infty'} \to \eta'$ be the finite étale morphism induced by sending u' to $1/x'$. Summing up, we defined the following morphisms:

$$\begin{tikzcd}
\widehat{\mathbb{A}}^1_{R_s} & \mathscr{S}_{\tilde{\mathfrak{s}}} \arrow[dr, "\sigma_{\tilde{\mathfrak{s}}}"] & \mathscr{S}_{K_s} \arrow[d, "\iota"] \\
\widehat{\mathbb{A}}^1_{R} & \mathscr{S}_s \arrow[ur, "\tilde{\tau}_{\tilde{\mathfrak{s}}}"] \arrow[r, "\tau_s"'] & \mathscr{S},
\end{tikzcd} \qquad \tau'\colon \eta_{\infty'} \to \eta'.$$

Here ι is the base change morphism.

Now, we recall that $|\pi| = \omega$, and we get the following isomorphism

$$\tau\colon \mathcal{A}_{K,u'}([\omega_m, \omega_{m'}\}) \xrightarrow{\sim} \mathcal{A}_{K,x}(\{\omega/\omega_{m'}, \omega/\omega_m]) \xrightarrow{\sim} K_{\widehat{\mathbb{A}}}\{\partial\}^{(m,m')}, \qquad (2.4.1.2)$$

where the first isomorphism sends u' to πx^{-1}, the second is that of Lemma 1.3.6, thus $\tau(u') = \pi\partial^{-1}$.

2.4.2. Let $\mathscr{M}$ be a stable holonomic $\widehat{\mathscr{D}}^{(m)}_{\widehat{\mathbb{A}},\mathbb{Q}}$-module and let $S \subset \widehat{\mathbb{A}}$ be the set of its singular points, *i.e.* closed points s such that $\pi^{-1}(s) \subset \mathrm{Char}(\mathscr{M})$, cf. 1.3.8.

We follow the notation of 1.2.1. In particular, for $m'' \geqslant m' \geqslant m$ non-negative integers and $s \in S$, we consider the microlocalization

$$\widehat{\mathscr{E}}^{(m',m'')}_{s,\mathbb{Q}} \otimes \mathscr{M} := (\widehat{\mathscr{E}}^{(m',m'')}_{\mathscr{X},\mathbb{Q}} \otimes_{\pi^{-1}\widehat{\mathscr{D}}^{(m)}_{\mathscr{X}}} \pi^{-1}\mathscr{M})_{\xi_x}$$

(cf. Notation 1.2.1.3). It is a finite $K_{\widehat{\mathbb{A}}}\{\partial\}^{(m',m'')}$-module by Lemma 1.3.2, and using τ, this can be seen as a finite $\mathcal{A}_{K,u'}([\omega_{m'}, \omega_{m''}\})$-module. We put

$$\widehat{\mathscr{E}}^{(m',m'')}_{s,\mathbb{Q}}(\mathscr{M}) := \Big(\widehat{\mathscr{E}}^{(m',m'')}_{s,\mathbb{Q}} \otimes \mathscr{M}, \nabla\colon \alpha \mapsto (\pi^{-1}\partial^2 x)\cdot\alpha \otimes du'\Big), \qquad (2.4.2.1)$$

which is considered as a differential $\mathcal{A}_{K,u'}([\omega_{m'}, \omega_{m''}\})$-module. The fact that ∇ defines a connection follows formally from the relation $\partial x = x\partial + 1$ in $\widehat{\mathscr{E}}^{(m',m'')}_{s,\mathbb{Q}}$, and the verification is similar to that for formal microlocalization, cf. [52, around (1.1)]. We warn the reader that the same notation $\widehat{\mathscr{E}}^{(m',m'')}_{s,\mathbb{Q}}(\mathscr{M})$ has been used in 1.3.10, but from now on, we refer only to 2.4.2.1. The difference between the connections defined in 1.3.10 and 2.4.2.1 is due to the change of variable from zero to ∞ via the isomorphism $u' \mapsto \pi x^{-1}$.

For a fixed m', the projective system $\big\{\widehat{\mathscr{E}}^{(m',m'')}_{s,\mathbb{Q}}(\mathscr{M})\big\}_{m'' \geqslant m'}$ defines a differential module on $\mathcal{A}_{K,u'}([\omega_{m'}, 1[)$ by Proposition 1.3.11, which is denoted by $\mathscr{E}^{(m',\dagger)}_{s,\mathbb{Q}}(\mathscr{M})$. This defines a differential $\mathcal{R}_{u',K}$-module which does not depend on the choice of m' up to a canonical isomorphism by the same proposition.

Definition. (i) Let $\mathscr{M}$ be a holonomic $\mathscr{D}^{\dagger}_{\widehat{\mathbb{P}},\mathbb{Q}}(\infty)$-module. Let s in $\widehat{\mathbb{A}}$ be a singular point of $\mathscr{M}$. Take a stable coherent $\widehat{\mathscr{D}}^{(m)}_{\widehat{\mathbb{A}},\mathbb{Q}}$-module $\mathscr{M}^{(m)}$ such that $\mathscr{D}^{\dagger}_{\widehat{\mathbb{A}},\mathbb{Q}} \otimes \mathscr{M}^{(m)} \cong \mathscr{M}|_{\widehat{\mathbb{A}}}$. Then $\mathscr{M}^{(m)}$ defines the differential $\mathcal{R}_{u',K}$-module $\mathscr{E}^{(m',\dagger)}_{s,\mathbb{Q}}(\mathscr{M}^{(m)})$ which does not depend on the choice of m' up to a canonical isomorphism. We see easily that this does not depend on the choice of $\mathscr{M}^{(m)}$ as well in the category of differential $\mathcal{R}$-modules. We denote this differential $\mathcal{R}$-module by $\mathscr{F}^{(s,\infty')}_{\pi}(\mathscr{M})$, or $\mathscr{F}^{(s,\infty')}_{\pi,K}(\mathscr{M})$ if we want to indicate the base, and we call it *the local Fourier transform of $\mathscr{M}$ at s*. When s is not a singular point of $\mathscr{M}$, we put $\mathscr{F}^{(s,\infty')}_{\pi}(\mathscr{M}) = 0$. This defines a functor

$$\mathscr{F}^{(s,\infty')}_{\pi} \colon \mathrm{Hol}(\mathscr{D}^{\dagger}_{\widehat{\mathbb{P}},\mathbb{Q}}(\infty)) \to \mathrm{Hol}'(\eta')$$

for any closed point $s \in \widehat{\mathbb{A}}$. Here $\mathbf{Hol}$ denotes the category of holonomic modules, and $\mathrm{Hol}'(\eta')$ denotes the category of differential $\mathcal{R}_{\mathscr{S}'}$-modules (not necessarily free), cf. Definition 2.1.3. If no confusion can arise, we omit the subscript π.

(ii) Let M be a holonomic F-$\mathscr{D}^{\mathrm{an}}_{\mathscr{S},\mathbb{Q}}$-module, and let $\mathfrak{s} \in \mathbb{A}^1(\bar{k})$. Take the canonical extension M^{can} of M at $\mathfrak{s}$: this is an F-$\mathscr{D}^{\dagger}_{\widehat{\mathbb{A}}_s,\mathbb{Q}}$-module on $\widehat{\mathbb{P}}_s \setminus \{\widetilde{s}, \infty\}$ overconvergent along $\{\widetilde{s}, \infty\}$, we have $M^{\mathrm{can}}|_{S_{\widetilde{s}}} \cong \sigma^*_{\mathfrak{s}} M$, and ∞ is a regular singular point (cf. 2.1.9). We define the *local Fourier transform of M at $\mathfrak{s}$* to be $\mathscr{F}^{(\widetilde{s},\infty')}(M^{\mathrm{can}})$. We denote it by $\Phi^{(\mathfrak{s},\infty')}_{\pi}(M)$, or $\Phi^{(\mathfrak{s},\infty')}_{\pi,K}(M)$ if we want to indicate the base. This defines the functor

$$\Phi^{(\mathfrak{s},\infty')}_{\pi} \colon F\text{-}\mathrm{Hol}(\mathscr{S}) \to \mathrm{Hol}'(\eta'_{K_s}).$$

If no confusion can arise, we omit the subscript π.

We sometimes denote $m \in \Gamma(\mathscr{S}, M)$ by $m \in M$. As in 2.2.5, any $m \in M$ defines an element in $\Phi^{(\mathfrak{s},\infty')}(M)$. We denote this element by $\widehat{m}$.

2.4.3 Remark. (i) By using the stationary phase Theorem 4.2.2, we may prove that the local Fourier transforms are *free* differential $\mathcal{R}$-modules. Moreover, we can also prove that the local Fourier transform coincides with that defined by Crew (cf. Corollary 4.2.3). Using this, we will endow local Fourier transforms with a Frobenius structure later in section 5.

(ii) In Definition 2.4.2(ii) above, we can also construct $\Phi^{(0,\infty')}_{\pi}$ in a purely local way (*i.e.* without using canonical extensions). For a holonomic $\mathscr{D}^{\mathrm{an}}_{\mathscr{S},\mathbb{Q}}$-module M with Frobenius structure, we define the local Fourier transform to be $\mathscr{E}^{\mathrm{an}} \otimes_{\mathscr{D}^{\mathrm{an}}} M$, and put a connection in the same manner as in 2.4.2.1 above.

A problem of this construction is to see that this is a free differential $\mathcal{R}$-module. For this, we need to compare with Definition 2.4.2(ii), and this is why we did not adopt this definition. Namely, the module coincides with $\Gamma(\Phi^{(0,\infty')}(M))$ by Lemma 2.4.5 below. As written in (i), we will prove that $\Phi^{(0,\infty)}(M)$ is a free differential $\mathcal{R}$-module. This shows that $\mathscr{E}^{\mathrm{an}} \otimes M$ defines a free differential $\mathcal{R}$-module, which is what we wanted.

By this comparison, once we have the stationary phase formula, the functor $\mathscr{E}^{\mathrm{an}} \otimes_{\mathscr{D}^{\mathrm{an}}} (-)$ from the category of holonomic $\mathscr{D}^{\mathrm{an}}$-modules with Frobenius structure to the category of $\mathscr{E}^{\mathrm{an}}$-modules, is exact by Proposition 2.4.7 below. However, we do not know if $\mathscr{E}^{\mathrm{an}}$ is flat over $\mathscr{D}^{\mathrm{an}}$ or not.

2.4.4 Example. Here are some basic examples to illustrate Definition 2.4.2. More properties will be proven in the sections 4 and 5.

(i) Consider the holonomic F-$\mathscr{D}^{\mathrm{an}}_{\mathscr{S},\mathbb{Q}}$-module $\mathcal{O}^{\mathrm{an}}$ (with the trivial Frobenius structure). The point zero is not singular, hence by definition $\Phi^{(0,\infty')}(\mathcal{O}^{\mathrm{an}}) = 0$. *A posteriori* this can be also computed by Remark 2.4.3(ii): tensor the presentation $\mathscr{D}^{\mathrm{an}} \xrightarrow{\cdot\partial} \mathscr{D}^{\mathrm{an}} \to \mathcal{O}^{\mathrm{an}} \to 0$ with $\mathscr{E}^{\mathrm{an}}$.

(ii) Consider the F-$\mathscr{D}^{\mathrm{an}}_{\mathscr{S},\mathbb{Q}}$-module δ, cf. 2.1.5. The canonical extension $\delta_0 := \delta^{\mathrm{can}}$ at zero has the following presentation on $\widehat{\mathbb{A}}$:

$$\mathscr{D}^{\dagger}_{\widehat{\mathbb{A}},\mathbb{Q}} \xrightarrow{\cdot x} \mathscr{D}^{\dagger}_{\widehat{\mathbb{A}},\mathbb{Q}} \to \delta_0|_{\widehat{\mathbb{A}}} \to 0.$$

The $\mathcal{R}$-module $\Phi^{(0,\infty')}(\delta)$ is free of rank one (the easiest way to prove this is to use Corollary 4.1.3); the connection is trivial (it follows immediately by the definition). As we have mentioned, we endow it with Frobenius structure in section 5, and this structure is also trivial, which follows from the global Fourier transform and the stationary phase theorem, cf. 5.1.8.

(iii) Consider the connection type holonomic F-$\mathscr{D}^{\mathrm{an}}_{\mathscr{S},\mathbb{Q}}$-module $\mathcal{R}$. Again we can compute $\Phi^{(0,\infty')}(\mathcal{R})$ globally, or locally (with exception of Frobenius). To proceed locally, we can consider the presentation of $\mathcal{R}$

$$\mathscr{D}^{\mathrm{an}} \xrightarrow{\cdot\partial x} \mathscr{D}^{\mathrm{an}} \to \mathcal{R} \to 0,$$

as $\mathscr{D}^{\mathrm{an}}$-module (Crew's solution data functor $\mathbb{S}$ permits to show that this is a presentation, cf. 2.1.7); then tensoring with $\mathscr{E}^{\mathrm{an}}$ we reduce to the case (ii) above and so $\Phi^{(0,\infty')}(\mathcal{R}) = \mathcal{R}$, endowed with the trivial connection. To proceed globally we consider $\mathcal{R}^{\mathrm{can}} = \mathcal{O}_{\widehat{\mathbb{P}},\mathbb{Q}}(0,\infty)$ and use the global Fourier transform and the stationary phase. We obtain the trivial Frobenius structure on $\Phi^{(0,\infty')}(\mathcal{R})$. The details are left to the reader.

(iv) For any $m \geqslant 0$, the $\widehat{\mathscr{D}}^{(m)}_{\widehat{\mathbb{A}},\mathbb{Q}}$-module $\widehat{\mathscr{D}}^{(m)}_{\widehat{\mathbb{A}},\mathbb{Q}}$ is stable but not holonomic. For any closed point s of $\widehat{\mathbb{A}}$, and $m'' \geqslant m' \geqslant m$, we have $\widehat{\mathscr{E}}^{(m',m'')}_{s,\mathbb{Q}}(\widehat{\mathscr{D}}^{(m)}_{\widehat{\mathbb{A}},\mathbb{Q}}) = \widehat{\mathscr{E}}^{(m',m'')}_{s,\mathbb{Q}}$, which is not of finite type as $\mathcal{A}_{K,u'}([\omega_{m'},\omega_{m''}\})$-module.

2.4.5 Lemma. *Let $\mathscr{M}$ be a stable holonomic $\widehat{\mathscr{D}}^{(m)}_{\widehat{\mathbb{A}},\mathbb{Q}}$-module, and let s in $\widehat{\mathbb{A}}$ be a singular point of $\mathscr{M}$. We get an isomorphism $\Gamma(\mathscr{E}^{(m,\dagger)}_{s,\mathbb{Q}}(\mathscr{M})) \cong \mathscr{E}^{(m,\dagger)}_{s,\mathbb{Q}} \otimes \mathscr{M}$. In particular, for a holonomic $\mathscr{D}^{\dagger}_{\widehat{\mathbb{P}},\mathbb{Q}}(\infty)$-module $\mathscr{N}$ and its singularity $s \in \widehat{\mathbb{A}}$, we get $\Gamma(\mathscr{F}^{(s,\infty')}(\mathscr{N})) \cong \mathscr{E}^{\dagger}_{s,\mathbb{Q}} \otimes \mathscr{N}$.*

Proof. Using Proposition 2.2.3 and Corollary 2.2.4, there exists an affine open neighbourhood $\mathscr{U}$ of s such that for $m' \geqslant m$

$$\widehat{E}^{(m,m')}_{\mathscr{U},\mathbb{Q}} \otimes \mathscr{M} \cong \widehat{\mathscr{E}}^{(m,m')}_{s,\mathbb{Q}} \otimes_{\widehat{\mathscr{D}}^{(m)}} \mathscr{M}, \quad E^{(m,\dagger)}_{\mathscr{U},\mathbb{Q}} \otimes \mathscr{M} \cong \mathscr{E}^{(m,\dagger)}_{s,\mathbb{Q}} \otimes_{\widehat{\mathscr{D}}^{(m)}} \mathscr{M}.$$

Thus we get the lemma by the fact that $E^{(m,\dagger)}_{\mathscr{U},\mathbb{Q}}$ is a Fréchet–Stein algebra (cf. [3, Theorem 5.8]). □

2.4.6. Let E be an unramified finite extension of K. Then there exists a finite étale morphism $r\colon \mathscr{S}_E \to \mathscr{S}$ sending u to u. This defines a functor $\mathrm{Res}^E_K := r_*$. Then the following diagram of functors

$$\begin{tikzcd} F\text{-Hol}(\mathscr{S}_E) \arrow[r, "\mathrm{Res}^E_K"] \arrow[d, "\Phi^{(0,\infty')}_E"'] & F\text{-Hol}(\mathscr{S}_K) \arrow[d, "\Phi^{(0,\infty')}_K"] \\ F\text{-Hol}'(\eta'_E) \arrow[r, "\mathrm{Res}^E_K"'] & F\text{-Hol}'(\eta'_K) \end{tikzcd}$$

is commutative up to a canonical isomorphism. The verification is straightforward.

2.4.7 Proposition. *The functors $\mathscr{F}^{(s,\infty')}$ and $\Phi^{(\mathfrak{s},\infty')}$ are exact.*

Proof. For $\mathscr{F}^{(s,\infty')}$, it follows from Lemma 1.3.4. For $\Phi^{(\mathfrak{s},\infty')}$, use the fact that taking canonical extension is exact. □

2.4.8 Definition. For $\widetilde{s} \in R$, we define a free differential $\mathcal{R}_{u,K}$-module $L(\widetilde{s})$ in the following way. The underlying module is $\mathcal{R}_{u,K}$. The connection is defined as follows:

$$\nabla(1) = \widetilde{s} \cdot \pi u^{-2} \otimes du.$$

Let $\widetilde{s}'$ be an element of R whose class in k is equal to that of $\widetilde{s}'$. Then there exists a canonical isomorphism $L(\widetilde{s}) \xrightarrow{\sim} L(\widetilde{s}')$ sending one to $\exp(\pi(\widetilde{s}' - \widetilde{s})u^{-1})$. This shows that the differential module $L(\widetilde{s})$ only depends on the class s of $\widetilde{s}$ in k.

Now, take an element $\mathfrak{s} \in \mathbb{A}^1(\bar{k})$. Let $\widetilde{\mathfrak{s}} \in K_s$ be a lifting of $\mathfrak{s}$ in k_s. Then we get a differential $\mathcal{R}_{K_s}$-module $L(\widetilde{\mathfrak{s}})$. As proven above, this does not depend on the choice of liftings up to an isomorphism. We denote this abusively by $L(\mathfrak{s})$. This is called the *Dwork differential module.*

2.4.9. Using the notation of 2.4.1, we have two lemmas. The definition of F-$\mathscr{D}^\dagger_{\widehat{\mathbb{P}},\mathbb{Q}}(\infty)$-modules is recalled in 3.1.1.

Lemma. *Let $\mathscr{M}$ be a holonomic F-$\mathscr{D}^\dagger_{\widehat{\mathbb{P}},\mathbb{Q}}(\infty)$-module, and $\mathfrak{s} \in \mathbb{A}^1_k(\bar{k})$. Then there exists a canonical isomorphism*

$$\mathscr{F}^{(\widetilde{s},\infty')}_{K_s}(\mathscr{M} \otimes K_s) \xrightarrow{\sim} \Phi^{(\mathfrak{s},\infty')}_{K_s}(\widetilde{\tau}_{\mathfrak{s}*}\mathscr{M}|_{S_s})$$

in the category $\mathrm{Hol}'(\eta'_{K_s})$. *Here $\mathscr{M} \otimes K_s$ denotes the pull-back of $\mathscr{M}$ to $\widehat{\mathbb{A}}_s = \widehat{\mathbb{A}} \otimes_R R_s$.*

Proof. There exists a canonical isomorphism $\alpha\colon \mathscr{S}_{\tilde{s}} \xrightarrow{\sim} \mathscr{S}_s$. Using Corollary 2.2.6, it suffices to show that there exists a canonical isomorphism $\alpha_*(\mathscr{M}\otimes K_s)|_{S_{\tilde{s}}} \xrightarrow{\sim} \mathscr{M}|_{S_s}$. The verification is straightforward. $\square$

2.4.10 Lemma. *Let $\mathscr{M}$ be a holonomic F-$\mathscr{D}^\dagger_{\mathbb{P},\mathbb{Q}}(\infty)$-module, and $\mathfrak{s}\in\mathbb{A}^1_k(\bar{k})$. Then we have a canonical isomorphism*

$$\mathscr{F}^{(s,\infty')}(\mathscr{M}) \cong \mathrm{Res}^{K_s}_K(\Phi^{(0,\infty')}(\widetilde{\tau}_{\mathfrak{s}*}\mathscr{M}|_{S_s})\otimes_{\mathcal{R}_{K_s}} L(\mathfrak{s})).$$

Proof. When s is a k-rational point, the lemma follows by using Corollary 2.2.6. When s is not a rational point, we need to check that $\mathrm{Res}^{K_s}_K(\mathscr{F}^{(\tilde{s},\infty')}(\mathscr{M}\otimes K_s))\cong\mathscr{F}^{(s,\infty')}(\mathscr{M})$ by using the notation and result of Lemma 2.4.9 above. The verification is easy. $\square$

3. Complements to cohomological operations

In this section, we review some known results on six functors which are indispensable in this paper, and give complements to properties of geometric Fourier transforms defined by Noot-Huyghe. The proofs for the properties of geometric Fourier transforms are almost the same as that of [50], so we content ourselves by pointing out the differences.

3.1. Cohomological operations

3.1.1. In this section, we assume k to be perfect. Let $h>0$ be an integer, and we put $q:=p^h$ as usual. We assume that there exists an automorphism $\sigma\colon R\xrightarrow{\sim} R$ which is a lifting of the absolute h-th Frobenius on k. Let $\mathscr{X}$ be a smooth formal scheme over R. We define $\mathscr{X}'$ by the following Cartesian diagram.

$$\begin{tikzcd} \mathscr{X}' \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & \mathscr{X} \arrow[d] \\ \mathrm{Spf}(R) \arrow[r, "\sigma"'] & \mathrm{Spf}(R) \end{tikzcd}$$

For a $\mathscr{D}^\dagger_{\mathscr{X},\mathbb{Q}}$-module $\mathscr{M}$, we denote by $\mathscr{M}^\sigma$ the $\mathscr{D}^\dagger_{\mathscr{X}',\mathbb{Q}}$-module defined by changing base by σ. Note that even when there is no lifting of the relative Frobenius $F^{(h)}_{X/k}\colon X\to X'$ to a morphism of formal schemes $\mathscr{X}\to\mathscr{X}'$, we are able to define the pull-back functor F^* from the category of $\mathscr{D}^\dagger_{\mathscr{X}',\mathbb{Q}}$-modules to that of $\mathscr{D}^\dagger_{\mathscr{X},\mathbb{Q}}$-modules (cf. [15, Remarques 4.2.4]). Recall that an $F^{(h)}$-$\mathscr{D}^\dagger_{\mathscr{X},\mathbb{Q}}$-module is a pair of a $\mathscr{D}^\dagger_{\mathscr{X},\mathbb{Q}}$-module $\mathscr{M}$ and an isomorphism $\mathscr{M}\to F^{(h)*}\mathscr{M}^\sigma$. We often abbreviate $F^{(h)}$ by F if there is nothing to be confused. Let $D^b_{\mathrm{coh}}(\mathscr{D}^\dagger_{\mathscr{X},\mathbb{Q}})$ be the derived category of $\mathscr{D}^\dagger_{\mathscr{X},\mathbb{Q}}$-modules with bounded coherent cohomology. We define a complex of $F^{(h)}$-$\mathscr{D}^\dagger_{\mathscr{X},\mathbb{Q}}$-modules as a complex $\mathscr{M}^\bullet$ in $D^b_{\mathrm{coh}}(\mathscr{D}^\dagger_{\mathscr{X},\mathbb{Q}})$ endowed with an isomorphism $\Phi\colon\mathscr{M}^\bullet\to F^{(h)*}(\mathscr{M}^\bullet)^\sigma$ in $D^b_{\mathrm{coh}}(\mathscr{D}^\dagger_{\mathscr{X},\mathbb{Q}})$. We say that such a complex $(\mathscr{M}^\bullet,\Phi)$ is *holonomic* if its cohomology sheaves are holonomic $F^{(h)}$-$\mathscr{D}^\dagger_{\mathscr{X},\mathbb{Q}}$-modules, and we will denote by $F^{(h)}$-$D^b_{\mathrm{hol}}(\mathscr{D}^\dagger_{\mathscr{X},\mathbb{Q}})$ the category of holonomic $F^{(h)}$-$\mathscr{D}^\dagger_{\mathscr{X},\mathbb{Q}}$-complexes with bounded cohomology, cf. [16, 5.3.5]. Let Z be a divisor of the of X. We denote by $F^{(h)}\text{-}D^b_{\mathrm{hol}}(\mathscr{D}^\dagger_{\mathscr{X},\mathbb{Q}}(Z))$ the full subcategory of $F^{(h)}\text{-}D^b_{\mathrm{hol}}(\mathscr{D}^\dagger_{\mathscr{X},\mathbb{Q}})$ of complexes $\mathscr{M}$ such that the canonical homomorphism $\mathscr{M} \to \mathscr{D}^\dagger_{\mathscr{X},\mathbb{Q}}(Z) \otimes_{\mathscr{D}^\dagger_{\mathscr{X},\mathbb{Q}}} \mathscr{M}$ is an isomorphism, cf. [22, 2.1.2]. We call them holonomic (complex of) $F^{(h)}\text{-}\mathscr{D}^\dagger_{\mathscr{X},\mathbb{Q}}(Z)$-modules.

3.1.2. Let R' be a discrete valuation ring finite étale over R, and let $\mathscr{C}$ be an object of $F^{(h)}\text{-}D^b_{\mathrm{hol}}(\mathrm{Spf}(R'))$. Let $\Phi\colon \mathscr{C} \xrightarrow{\sim} F^{(h)*}\mathscr{C}$ be the Frobenius structure of $\mathscr{C}$. There exists a canonical σ-semi-linear homomorphism $\mathscr{C} \to F^{(h)*}\mathscr{C}^\sigma$ sending x to $1 \otimes x$. The composition $\mathscr{C} \to F^{(h)*}\mathscr{C}^\sigma \xrightarrow[\Phi^{-1}]{\sim} \mathscr{C}$, where the first homomorphism is the canonical homomorphism, makes $\mathscr{C}$ a complex of σ-K-vector spaces. We note that this homomorphism is in fact a quasi-isomorphism since σ is. This correspondence induces an equivalence between $F^{(h)}\text{-}D^b_{\mathrm{hol}}(\mathrm{Spf}(R))$ and the category of finite σ-K-complexes, and we identify them.

3.1.3. Let us fix notation for Dieudonné-Manin slopes and Tate twists. Recall that we denote by e the absolute ramification index of K. We denote by $K_\sigma\langle t\rangle$ the ring of non-commutative polynomials defined by the relation $t\alpha = \sigma(\alpha)t$, for every $\alpha \in K$. For any $\alpha \in K$ and integer $s > 1$, we put

$$K^{(\alpha;seh)} := K_\sigma\langle t\rangle / K_\sigma\langle t\rangle(t^s - \alpha),$$

endowed with the Frobenius action given by the multiplication on the left by t. It is a σ-K-module of rank s. When $\sigma(\varpi) = \varpi$, we normalize the Dieudonné-Manin slope so that it is purely of slope $\lambda := \frac{-v_K(\alpha)}{seh}$. For any smooth formal scheme $\mathscr{X}$ over R, we denote by $\mathcal{O}^{(\alpha;seh)}_{\mathscr{X},\mathbb{Q}}$ the pull-back of $K^{(\alpha;seh)}$ by the structural morphism of $\mathscr{X}$.

Let us define Tate twists. For any $\mathscr{D}^\dagger_{\mathscr{X},\mathbb{Q}}$-module $\mathscr{M}$ and integer n, we put

$$\mathscr{M}(n) := \mathscr{M} \otimes_{\mathcal{O}_{\mathscr{X},\mathbb{Q}}} \mathcal{O}^{(p^{-hn};eh)}_{\mathscr{X},\mathbb{Q}}.$$

This is called the *n-th Tate twist* of $\mathscr{M}$. Let us define the twist by a Dieudonné-Manin slope $\lambda \in \mathbb{Q}$. There is a unique way to write $\lambda = \frac{r}{seh}$, where r and s are coprime integers and $s > 0$ (if $\lambda = 0$ by convention we put $r = 0$ and $s = 1$). Let $\varpi \in \Lambda$ be a uniformizer; for any coherent $\mathscr{D}^\dagger_{\mathscr{X},\mathbb{Q}}$-module $\mathscr{M}$ and $\lambda \in \mathbb{Q}$, we put

$$\mathscr{M}^{(\lambda)} := \mathscr{M} \otimes_{\mathcal{O}_{\mathscr{X},\mathbb{Q}}} \mathcal{O}^{(\varpi^{-r};seh)}_{\mathscr{X},\mathbb{Q}},$$

which is usually called the *twist*[8] of $\mathscr{M}$ by the *slope* λ. When $\sigma(\varpi) = \varpi$, its Dieudonné-Manin slope is indeed shifted by $+\lambda$. The notation $\mathscr{M}^{(\lambda)}$ is slightly abusive because it depends on the choice of the uniformizer ϖ, whereas that for Tate twists is intrinsic. We give analogous definitions for overconvergent F-isocrystals[9] and free differentials modules with Frobenius structure over the the Robba ring.

[8] This is called *décalé* in French.

[9] For an overconvergent F-isocrystal, in [54], $M^{(\lambda)}$ was denoted by $M(\lambda)$. We modify here the notation to avoid any possible confusion with Tate twists.

3.1.4. To fix notation, let us review the theory of arithmetic $\mathscr{D}$-modules concerning this paper. For more thorough treatment of the formalism, see [6, 7].

We say that $(\mathscr{X}, Z)$ is a d-couple if $\mathscr{X}$ is a smooth formal scheme and Z is a divisor of its special fibre. Here, Z can be empty. Let $(\mathscr{Y}, W)$ be another d-couple. A morphism of d-couples $f\colon (\mathscr{X}, Z) \to (\mathscr{Y}, W)$ is a morphism $\widetilde{f}\colon \mathscr{X} \to \mathscr{Y}$ such that $\widetilde{f}(\mathscr{X} \setminus Z) \subset \mathscr{Y} \setminus W$ and $\widetilde{f}^{-1}(W)$ is a divisor. The morphism $\widetilde{f}$ is called the realization of f, and if it is unlikely to be confused, we often denote $\widetilde{f}$ by f. Let P be a property of morphisms. We say that the morphism of d-couples f satisfies the property P if $\widetilde{f}$ satisfies the property P.

Let $\mathscr{X}$ be a smooth formal scheme over R, X be its special fibre, and $\mathscr{X}_K$ be its Raynaud generic fibre. Then we have the specialization map $\mathrm{sp}\colon \mathscr{X}_K \to \mathscr{X}$ of topoi. Recall that, we say that a $\mathscr{D}^{\dagger}_{\mathscr{X},\mathbb{Q}}$-module $\mathscr{M}$ is a convergent $(F\text{-})$isocrystal if $\mathrm{sp}^*(\mathscr{M})$ is a convergent $(F\text{-})$isocrystal, and the same for overconvergent $(F\text{-})$isocrystals (cf. 0.0.7).

Let $(\mathscr{X}, Z)$ be a d-couple, and we denote by $\mathbb{D}_{\mathscr{X},Z}$ the dual functor with respect to $\mathscr{D}^{\dagger}_{\mathscr{X},\mathbb{Q}}(Z)$-modules. If it is unlikely to cause any confusion, we often denote this by $\mathbb{D}$. Let $f\colon (\mathscr{X}, Z) \to (\mathscr{Y}, W)$ be a morphism of d-couples. We have the extraordinary pull-back functor $\widetilde{f}^{!}$ from the category of coherent $\mathscr{D}^{\dagger}_{\mathscr{Y},\mathbb{Q}}(W)$-modules to that of $\mathscr{D}^{\dagger}_{\mathscr{X},\mathbb{Q}}(f^{-1}(W))$-modules (cf. [16, 4.3.3]). Let $\mathscr{M}$ be a bounded coherent $(F\text{-})\mathscr{D}^{\dagger}_{\mathscr{Y},\mathbb{Q}}(W)$-complex. When $\widetilde{f}^{!}(\mathscr{M})$ is coherent, we define $f^{!}(\mathscr{M})$ to be $\mathscr{D}^{\dagger}_{\mathscr{X},\mathbb{Q}}(Z) \otimes \widetilde{f}^{!}(\mathscr{M})$. Suppose in turn that $\widetilde{f}^{!} \circ \mathbb{D}_{\mathscr{Y},W}(\mathscr{M})$ is a bounded coherent $(F\text{-})\mathscr{D}^{\dagger}_{\mathscr{X},\mathbb{Q}}(f^{-1}(W))$-complex. In this case, we put

$$f^{+}\mathscr{M} := \big(\mathbb{D}_{\mathscr{X},Z} \circ f^{!} \circ \mathbb{D}_{\mathscr{Y},W}\big)(\mathscr{M}).$$

Now, suppose that the morphism of d-couples f is a proper morphism. We have the push-forward functor $\widetilde{f}_{+}$ from the category of coherent $\mathscr{D}^{\dagger}_{\mathscr{X},\mathbb{Q}}(f^{-1}(W))$-modules to that of coherent $\mathscr{D}^{\dagger}_{\mathscr{Y},\mathbb{Q}}(W)$-modules. Let $\mathscr{N}$ be a $(F\text{-})\mathscr{D}^{\dagger}_{\mathscr{X},\mathbb{Q}}(Z)$-module. Suppose $\mathscr{N}$ is coherent as a $\mathscr{D}^{\dagger}_{\mathscr{X},\mathbb{Q}}(f^{-1}(W))$-module. Then we denote by $j_{+}\mathscr{N}$ this coherent module. We define $f_{+}(\mathscr{N})$ to be the coherent $\mathscr{D}^{\dagger}_{\mathscr{Y},\mathbb{Q}}(W)$-module $\widetilde{f}_{+}(j_{+}(\mathscr{N}))$. Assume in turn that $\mathbb{D}_{\mathscr{X},Z}(\mathscr{N})$ is coherent as a $\mathscr{D}^{\dagger}_{\mathscr{X},\mathbb{Q}}(f^{-1}(W))$-module. We define

$$f_{!}\mathscr{N} := \big(\mathbb{D}_{\mathscr{Y},W} \circ f_{+} \circ \mathbb{D}_{\mathscr{X},Z}\big)(\mathscr{N}).$$

When we are given $\mathscr{M}$ and $\mathscr{N}$ in $\underrightarrow{LD}^{b}_{\mathbb{Q},\mathrm{qc}}(\widehat{\mathscr{D}}^{(\bullet)}_{\mathscr{X}}(Z))$ (cf. [16, 4.2.2, 4.2.3] for the notation when Z is empty, but the construction is the same), we denote the object $\mathscr{M} \otimes^{\mathbb{L}\dagger}_{\mathcal{O}_{\mathscr{X},\mathbb{Q}}(Z)} \mathscr{N}$ in $\underrightarrow{LD}^{b}_{\mathbb{Q},\mathrm{qc}}(\widehat{\mathscr{D}}^{(\bullet)}_{\mathscr{X}}(Z))$ by $\mathscr{M} \otimes \mathscr{N}$.

3.1.5. In this paragraph, let us summarize some properties of the cohomological functors defined in the previous paragraph which will be used later in this paper. We recall the convention 0.0.8 about shifts and Tate twists. Let $f\colon (\mathscr{X}, Z) \to (\mathscr{Y}, W)$ be a morphism of d-couples. We denote respectively by $d_{\mathscr{X}}$, $d_{\mathscr{Y}}$, d the dimension of $\mathscr{X}$, $\mathscr{Y}$, and $d_{\mathscr{X}} - d_{\mathscr{Y}}$. Then we get the following.

1. If f is smooth and of constant relative dimension, we get $f^{!} \cong f^{+}(d)[2d]$ (cf. [4, Theorem 5.5]). If f is a closed immersion of connected formal schemes and $\mathscr{M}$ is

an overconvergent F-isocrystal, we get $f^!(\mathscr{M}) \cong f^+(\mathscr{M})(d)[2d]$ (cf. [4, Theorem 5.6]).

2. If f is proper and $\widetilde{f}|_{\mathscr{X}\setminus Z}\colon \mathscr{X}\setminus Z \to \mathscr{Y}\setminus W$ is also proper, we get $f_! \xrightarrow{\sim} f_+$ (cf. [4, Theorem 5.4]).
3. Let $\mathscr{M}$ be an object of $D^b_{\mathrm{coh}}(\mathscr{D}^\dagger_{\mathscr{X},\mathbb{Q}}(Z))$ and $\mathscr{N}$ be an object of $D^b_{\mathrm{coh}}(\mathscr{D}^\dagger_{\mathscr{Y},\mathbb{Q}}(W))$. There exists a canonical isomorphism $f_+(\mathscr{M}\otimes f^!\mathscr{N}) \cong f_+\mathscr{M}\otimes\mathscr{N}$ (cf. [21, 2.1.4]. The compatibility with Frobenius pull-back can be seen easily from the definition of the homomorphism).

By using 1 above, we get $\mathbb{D}_{\mathscr{X},Z}(\mathcal{O}_{\mathscr{X},\mathbb{Q}}(Z)) \cong \mathcal{O}_{\mathscr{X},\mathbb{Q}}(Z)(-d_{\mathscr{X}})$. Now, we define[10] the *twisted tensor product* $\widetilde{\otimes}$ on $\mathscr{X}$ to be $\mathbb{D}_{\mathscr{X},Z}(\mathbb{D}_{\mathscr{X},Z}(-)\otimes\mathbb{D}_{\mathscr{X},Z}(-))$. We note that the definition is slightly different from that of [59]. The main reason we introduce this new tensor product is the following.

4. Assume that f is a morphism of connected formal schemes. There exists a canonical isomorphism $f^!((-)\otimes(-))[d] \cong f^!(-)\otimes f^!(-)$. Similarly, we also have $f^+((-)\widetilde{\otimes}(-))[-d] \cong f^+(-)\widetilde{\otimes} f^+(-)$ (cf. [4, 5.8]).

The following result enables us to compare these two tensor products in special cases.

5. Assume $\mathscr{X}$ is connected. If $\mathscr{M}$ be an overconvergent F-isocrystal, and $\mathscr{N}$ be a coherent F-$\mathscr{D}^\dagger_{\mathscr{X},\mathbb{Q}}(Z)$-module. Then we get $\mathscr{M}\widetilde{\otimes}\mathscr{N} \cong \mathscr{M}\otimes\mathscr{N}(d_{\mathscr{X}})$ (cf. [4, Proposition 5.8]).

3.1.6. Now, let us see the base change theorem. Consider the following Cartesian diagrams of d-couples.

$$\begin{tikzcd} (\mathscr{X}', Z') \arrow[r, "i'"] \arrow[d, "f'"'] \arrow[dr, phantom, "\square"] & (\mathscr{X}, Z) \arrow[d, "f"] \\ (\mathscr{Y}', W') \arrow[r, "i"'] & (\mathscr{Y}, W) \end{tikzcd}$$

Here, we say that the diagram of d-couples is *Cartesian* if it is Cartesian for the underlying smooth formal schemes, and $\mathscr{X}'\setminus Z' = (\mathscr{X}\setminus Z)\times_{(\mathscr{Y}'\setminus W')}(\mathscr{Y}\setminus W)$. Then we get $i^!\circ f_+ \cong f'_+\circ i'^!$ (cf. [4, Theorem 5.7]). This isomorphism is compatible with Frobenius structure by the same theorem. We call this the *base change isomorphism.*

Assume that f is proper. In this case, for a bounded coherent (F)-$\mathscr{D}^\dagger_{\mathscr{X},\mathbb{Q}}$-complex $\mathscr{M}$, we get $i^+\circ f_!(\mathscr{M}) \cong f'_!\circ i'^+(\mathscr{M})$ if the both sides are defined. This follows by using $\mathbb{D}\circ\mathbb{D} = \mathrm{id}$ [70, II, 3.5]. This is also called the base change isomorphism.

3.1.7. We also have the Künneth formula. Namely, let $f\colon \mathscr{X}\to\mathscr{X}'$ and $g\colon \mathscr{Y}\to\mathscr{Y}'$ be smooth morphisms between smooth formal schemes over a smooth formal scheme $\mathscr{T}$. Let D be a divisor of the special fibre of $\mathscr{T}$. We denote by $D_{\mathscr{X}^{(\prime)}}$ (resp. $D_{\mathscr{Y}^{(\prime)}}$, D') be the divisor of the special fibre of $\mathscr{X}^{(\prime)}$ (resp. $\mathscr{Y}^{(\prime)}$, $\mathscr{X}'\times_{\mathscr{Y}}\mathscr{Y}'$) which is the pull-back of D. Let $\mathscr{M}$ (resp. $\mathscr{N}$) be an element of $\underrightarrow{LD}^b_{\mathbb{Q},\mathrm{qc}}(\widehat{\mathscr{D}}^{(\bullet)}_{\mathscr{X}}(D_{\mathscr{X}}))$ (resp. $\underrightarrow{LD}^b_{\mathbb{Q},\mathrm{qc}}(\widehat{\mathscr{D}}^{(\bullet)}_{\mathscr{Y}}(D_{\mathscr{Y}}))$). Then

[10] In [7], the notation $\widetilde{\otimes}$ and $\otimes$ are used for slightly different functors.

we get

$$(f \times g)_+(\mathscr{M} \boxtimes^{\mathbb{L}}_{\mathcal{O}_{\mathscr{T}}(D)} \mathscr{N}) \cong (f_+\mathscr{M}) \boxtimes^{\mathbb{L}}_{\mathcal{O}_{\mathscr{T}}(D)} (g_+\mathscr{N})$$

in $\underrightarrow{LD}^b_{\mathbb{Q},\mathrm{qc}}(\widehat{\mathscr{D}}^{(\bullet)}_{\mathscr{X}'\times_{\mathscr{T}}\mathscr{Y}'}(D'))$. To check this, we apply the Künneth formula [4, Proposition 4.9] to the diagram

$$\begin{tikzcd} & \mathscr{X} \times_{\mathscr{T}} \mathscr{Y} \arrow[dl, "\mathrm{id}\times g"'] \arrow[d, "f\times g"] \arrow[dr, "f\times\mathrm{id}"] & \\ \mathscr{X} \times_{\mathscr{T}} \mathscr{Y}' \arrow[dr, "f\times\mathrm{id}"'] & & \mathscr{X}' \times_{\mathscr{T}} \mathscr{Y} \arrow[dl, "\mathrm{id}\times g"] \\ & \mathscr{X}' \times_{\mathscr{T}} \mathscr{Y}' & \end{tikzcd}$$

and $\mathscr{M} \boxtimes^{\mathbb{L}}_{\mathcal{O}_{\mathscr{T}}(D)} \mathcal{O}_{\mathscr{Y}'}(D_{\mathscr{Y}'})$ on $\mathscr{X} \times_{\mathscr{T}} \mathscr{Y}'$ and $\mathcal{O}_{\mathscr{X}'}(D_{\mathscr{X}'}) \boxtimes^{\mathbb{L}}_{\mathcal{O}_{\mathscr{T}}(D)} \mathscr{N}$ on $\mathscr{X}' \times_{\mathscr{T}} \mathscr{Y}$.

3.1.8. In this paragraph, let us see the relation between the rigid cohomology and the push-forward of arithmetic $\mathscr{D}$-modules. Let $\mathscr{X}$ be a *proper* smooth formal scheme of dimension d. Let Z be a divisor of the special fibre of $\mathscr{X}$, $\mathscr{U}$ be the complement, and U be its special fibre. We denote by $\mathsf{sp}\colon \mathscr{X}_K \to \mathscr{X}$ the specialization map where $\mathscr{X}_K$ denotes the Raynaud generic fibre of $\mathscr{X}$. Let $f\colon (\mathscr{X}, Z) \to \mathsf{Spf}(R)$ be the structural morphism. Let $\mathscr{M}$ be a coherent $\mathscr{D}^\dagger_{\mathscr{X},\mathbb{Q}}(Z)$-module which is overconvergent along Z. Suppose that it is coherent as a $\mathscr{D}^\dagger_{\mathscr{X},\mathbb{Q}}$-module. In 3.1.5, we noted that $\mathbb{D}_{\mathscr{X},Z}(\mathcal{O}_{\mathscr{X},\mathbb{Q}}(Z)) \cong \mathcal{O}_{\mathscr{X},\mathbb{Q}}(Z)(-d)$. For an isocrystal M, we denote by $M^\vee$ the dual as isocrystal. This isomorphism leads us to the following comparison of dual functors (cf. [4, Corollary 3.12]):

$$\mathrm{sp}^*(\mathbb{D}_{\mathscr{X},Z}(\mathscr{M})) \cong (\mathrm{sp}^*(\mathscr{M}))^\vee(-d). \tag{3.1.8.1}$$

We get the following relation with the rigid cohomology:

$$\mathscr{H}^i(f_+\mathscr{M}) \cong H^{d+i}_{\mathrm{rig}}(U, \mathrm{sp}^*\mathscr{M})(d), \tag{3.1.8.2}$$

where $d = \dim(U)$. For the details of the proof, see [4, 3.14]. To see the relation for cohomologies with compact support, we use the Poincaré duality of rigid cohomology to get

$$\mathscr{H}^i(f_!\mathscr{M}) \cong H^{d+i}_{\mathrm{rig},c}(U, \mathrm{sp}^*\mathscr{M})(d). \tag{3.1.8.3}$$

For the detailed account, one can refer to [4, 5.9].

When $\mathscr{X}$ is a curve, and for a holonomic $\mathscr{D}^\dagger_{\mathscr{X},\mathbb{Q}}(Z)$-module $\mathscr{M}$, we get that the following pairing

$$\mathscr{H}^i(f_+\mathscr{M}) \times \mathscr{H}^{-i}(f_!\,\mathbb{D}_{\mathscr{X},Z}(\mathscr{M})) \to K \tag{3.1.8.4}$$

is perfect. This can be seen from [4, 5.5].

3.1.9. For the later use, we review the cohomological functors i_+, j^+, $j_!$, $i^!$, and $\mathbb{D}$ in the theory of formal disks. Let $\mathscr{S}$ be the formal disk over K. For an object M in F-$\mathrm{Hol}(\eta)$, $j_+(M)$ is by definition the underlying F-$\mathscr{D}^{\mathrm{an}}_{\mathscr{S},\mathbb{Q}}$-module. For an object N in F-$\mathrm{Hol}(\mathscr{S})$, we denote $\mathscr{D}^{\mathrm{an}}_{\mathscr{S},\mathbb{Q}}(0) \otimes N$ in F-$\mathrm{Hol}(\eta)$ by j^+M. We denote by $i\colon \{0\} \hookrightarrow \mathscr{S}$ the closed immersion. The definitions of the functors i_+, $i^!$, $j^! \cong j^+$ are essentially the same as in the global case, and are used frequently in [31], so for the details see [31, 3.4, *etc.*].

Now, we denote by $\mathbb{D}_{\mathscr{S}}$ (resp. $\mathbb{D}_\eta$) the dual functor with respect to $\mathscr{D}^{\mathrm{an}}_{\mathscr{S},\mathbb{Q}}$ (resp. $\mathscr{D}^{\mathrm{an}}_{\mathscr{S},\mathbb{Q}}(0)$). These define functors from F-$\mathrm{Hol}(\mathscr{S})$ (resp. F-$\mathrm{Hol}(\eta)$) to itself by [31, 5.2]. For an object M in F-$\mathrm{Hol}(\eta)$, we define $j_!M := \mathbb{D}_{\mathscr{S}}\, j_+ \mathbb{D}_\eta(M)$, and for an object N in F-$\mathrm{Hol}(\mathscr{S})$, we put $i^+N := (i^!\,\mathbb{D}_{\mathscr{S}}N)^\vee$ where $^\vee$ denotes the dual of σ-K-vector spaces. By using facts in 3.1.5, we get isomorphisms $i_+\,\mathbb{D}_\eta \cong \mathbb{D}_{\mathscr{S}}\, i_+$ and $j^+\,\mathbb{D}_{\mathscr{S}} \cong \mathbb{D}_\eta\, j^+$. By using these isomorphisms, the localization triangle [31, 3.4.3] induces the following distinguished triangle:

$$j_!\, j^+ \to \mathrm{id} \to i_+\, i^+ \xrightarrow{+1} . \tag{3.1.9.1}$$

Definition. For a holonomic F-$\mathscr{D}^{\mathrm{an}}_{\mathscr{S},\mathbb{Q}}$-module M, we put $\Psi(M) := \mathbb{V}(\mathbb{D}_{\mathscr{S}}M)$ and $\Phi(M) := \mathbb{W}(\mathbb{D}_{\mathscr{S}}M)$, and call them the *nearby cycles* and the *vanishing cycles* respectively. These define functors

$$\Psi, \Phi\colon F\text{-}\mathrm{Hol}(\mathscr{D}^{\mathrm{an}}_{\mathscr{S},\mathbb{Q}}) \to \mathrm{Del}_{K^{\mathrm{ur}}}(G_{\mathcal{K}}).$$

We note here that when M is a free differential module on $\mathcal{R}$ with Frobenius structure, we get $\mathbb{D}_\eta(M) \cong M^\vee(-1)$, where $^\vee$ denotes the dual as a $\mathcal{R}$-module by 3.1.8.1. For example $\Psi(\mathcal{R}) \cong \mathbb{V}(\mathcal{R})(1) \cong K^{\mathrm{ur}}(1)$. On the other hand, if M is of punctual type such that $M = V \otimes_K \delta$ where V is a K-vector space with Frobenius structure, we get that $\Phi(M) \cong V_{K^{\mathrm{ur}}}$.

3.1.10 Lemma. *Let M be an object of F-$\mathrm{Hol}(\eta)$. Then we get the following exact sequence:*

$$0 \to M^{\partial=0}(1) \otimes_K \delta \to j_!M \to j_+M \to M/\partial M(1) \otimes_K \delta \to 0.$$

Proof. Since $j_+ j^+ j_!\, M \cong j_+M$, we get $\Psi(j_!\, M) \cong \Psi(j_+\, M)$ by [31, Proposition 6.1.1]. We also get $\Psi(j_!M) \xrightarrow{\sim} \Phi(j_!M)$. Thus, 2.1.7.1 induces the following exact sequence:

$$0 \to \mathrm{Hom}_{\mathscr{D}^{\mathrm{an}}}(\mathbb{D}(j_+M), \mathcal{O}^{\mathrm{an}}_{K^{\mathrm{ur}}}) \to \Phi(j_!\, M) \to \Phi(j_+\, M) \to \mathrm{Ext}^1_{\mathscr{D}^{\mathrm{an}}}(\mathbb{D}(j_+M), \mathcal{O}^{\mathrm{an}}_{K^{\mathrm{ur}}}) \to 0.$$

We get isomorphisms

$$R\mathrm{Hom}_{\mathscr{D}^{\mathrm{an}}}(\mathbb{D}_{\mathscr{S}}(j_+M), \mathcal{O}^{\mathrm{an}}) \cong R\mathrm{Hom}_{\mathscr{D}^{\mathrm{an}}}(\mathcal{O}^{\mathrm{an}}(-1), j_+M) \cong R\mathrm{Hom}_{\mathscr{D}^{\mathrm{an}}(0)}(\mathcal{R}, M)(1).$$

Here the first isomorphism follows from the fact that $\mathbb{D}_{\mathscr{S}}$ gives an anti-equivalence of categories combined with the isomorphism $\mathbb{D}_{\mathscr{S}}(\mathcal{O}^{\mathrm{an}}) \cong \mathcal{O}^{\mathrm{an}}(-1)$, and the second by adjunction. Thus the lemma follows. □

Remark. We note that the dimension of $M^{\partial=0}$ and $M/\partial M$ over K are the same by the index theorem of Christol–Mebkhout [25, 14.13].

3.2. Geometric Fourier transforms

3.2.1. We briefly review the definition of geometric Fourier transform due to Noot-Huyghe [59]. For simplicity, we only review under the situation of 2.4.1.

To define Fourier transforms, we need to define an integral kernel $\mathscr{L}_\pi$ of the transform. We define a convergent F-isocrystal on $\widehat{\mathbb{A}}^1$ overconvergent along ∞ denoted by $\mathscr{L}_\pi$ in the following way. Let t be the coordinate. As an $\mathcal{O}_{\widehat{\mathbb{P}}^1_{R,t},\mathbb{Q}}(\infty)$-module, it is $\mathcal{O}_{\widehat{\mathbb{P}}^1_{R,t},\mathbb{Q}}(\infty)$. We denote the element corresponding to one by e. We define its connection by

$$\nabla(e) = -\pi\, e \otimes \mathrm{d}t.$$

This module is equipped with Frobenius structure. The Frobenius structure $\Phi\colon F^*\mathscr{L}_\pi \xrightarrow{\sim} \mathscr{L}_\pi$ is defined by

$$\Phi(1 \otimes e) := \exp(\pi(t - t^q))e.$$

Now, let us consider the situation in 2.4.1. There exists the canonical coupling $\mu\colon \widehat{\mathbb{A}} \times \widehat{\mathbb{A}}' \to \widehat{\mathbb{A}}^1_{R,t}$ sending t to $x \otimes x'$. By the general theory of overconvergent F-isocrystals, the pull-back $\mu^*\mathscr{L}_\pi$ is a convergent F-isocrystal on $\widehat{\mathbb{A}} \times \widehat{\mathbb{A}}'$ overconvergent along Z on $\widehat{\mathbb{P}}''$. This is a coherent $\mathscr{D}^\dagger_{\widehat{\mathbb{P}}'',\mathbb{Q}}(Z)$-module, and its restriction to $\widehat{\mathbb{A}} \times \widehat{\mathbb{A}}'$ is nothing but $\mathscr{H}^{-1}(\mu^!\mathscr{L}_\pi)$. By abuse of language, we denote this $\mathscr{D}^\dagger_{\widehat{\mathbb{P}}'',\mathbb{Q}}(Z)$-module by $\mu^!\mathscr{L}_\pi[-1]$, or sometimes by $\mathscr{L}_{\pi,\mu}$. In the same way, there exists a unique coherent complex of $\mathscr{D}^\dagger_{\widehat{\mathbb{P}}'',\mathbb{Q}}(Z)$-modules whose restriction to $\widehat{\mathbb{A}} \times \widehat{\mathbb{A}}'$ is $\mathscr{H}^1(\mu^+\mathscr{L}_\pi)$. We also denote this by $\mu^+\mathscr{L}_\pi[1]$.

3.2.2. Now, let us recall the definition of the geometric Fourier transform. We continuously use the notation of 2.4.1. Recall the diagram 2.4.1.1. Let $\mathscr{M}$ be a coherent $\mathscr{D}^\dagger_{\widehat{\mathbb{P}},\mathbb{Q}}(\infty)$-module. Noot-Huyghe defined the geometric Fourier transform of $\mathscr{M}$ to be

$$\begin{aligned}\mathscr{F}_\pi(\mathscr{M}) &:= p'_+(\mathscr{L}_{\pi,\mu} \otimes^{\mathbb{L}}_{\mathcal{O}_{\widehat{\mathbb{P}}'',\mathbb{Q}}(\infty)} p^!\mathscr{M}[-2]) \\ &\bigl(= p'_+(\mu^!\mathscr{L}_\pi \otimes^{\mathbb{L}}_{\mathcal{O}_{\widehat{\mathbb{P}}'',\mathbb{Q}}(\infty)} p^!\mathscr{M}[-3])\bigr).\end{aligned} \tag{3.2.2.1}$$

She also proved that $p'_!\,(\mathscr{L}_{\pi,\mu} \otimes^{\mathbb{L}}_{\mathcal{O}_{\widehat{\mathbb{P}}'',\mathbb{Q}}(\infty)} p^!\mathscr{M})$ is well-defined, and also showed an analog of the result of Katz and Laumon [60, Theorem 3.2]. Namely, the canonical homomorphism

$$p'_!\,(\mathscr{L}_{\pi,\mu} \otimes^{\mathbb{L}}_{\mathcal{O}_{\widehat{\mathbb{P}}'',\mathbb{Q}}(\infty)} p^!\mathscr{M}[-2]) \to \mathscr{F}_\pi(\mathscr{M}) \tag{3.2.2.2}$$

is an isomorphism. Since Fourier transform is defined using three cohomological functors p'_+, $\otimes$, $p^!$, and there is a canonical Frobenius structure on $\mathscr{L}_{\pi,\mu}$, Fourier transform commutes with Frobenius pull-backs. In particular, if $\mathscr{M}$ is a coherent F-$\mathscr{D}^\dagger_{\mathscr{X},\mathbb{Q}}$-complex, there exists a canonical Frobenius structure on the complex $\mathscr{F}_\pi(\mathscr{M})$.

3.2.3 Lemma. *We have a canonical isomorphism*

$$\mu^+\mathscr{L}_\pi(1)[2] \cong \mu^!\mathscr{L}_\pi.$$

Proof. Note that $\mu|_{\widehat{\mathbb{A}}\times\widehat{\mathbb{A}}'\setminus\{(0,0')\}}$ is smooth. This shows that $\mu^!\mathscr{L}_\pi$ and $\mu^+\mathscr{L}_\pi(1)[2]$ are generically isomorphic by 3.1.5.1. The module $\mu^!\mathscr{L}_\pi[-1]$ is concentrated at degree zero, and it is finite over $\mathcal{O}_{\widehat{\mathbb{P}},\mathbb{Q}}(\infty)$. This implies that it is an overconvergent isocrystal along ∞. Since the dual of an isocrystal is also an isocrystal, $\mu^+\mathscr{L}_\pi$ is also an overconvergent isocrystal along ∞. Moreover, both sides are overconvergent F-isocrystals. Thus, the two modules in the statement are isomorphic by using [44] and [67, Theorem 4.1.1]. $\square$

3.2.4. Let s be a closed point of $\widehat{\mathbb{A}}$. Let k_s be the residue field and K_s be the corresponding unramified extension of K, R_s its valuation ring. Then there exists a closed immersion $i_s\colon \widehat{\mathbb{P}}'_{R_s} \hookrightarrow \widehat{\mathbb{P}}''_{R_s}$ sending x' to (s, x'). We define $\mathscr{L}(s\cdot x') := i_s^!(\mathscr{L}_{\pi,\mu})$. In the same way, given a closed point s' in $\widehat{\mathbb{A}}'$, we define $\mathscr{L}(x\cdot s')$ on $\widehat{\mathbb{P}}$. When s is a rational point, we can check that $\tau'_*(\mathscr{L}(s\cdot x')|_{\eta_{\infty'}}) \cong L'(s)$, where $L'(s)$ is the Dwork differential module 2.4.8 on η'.

3.2.5. Finally, let us review a fundamental property of Fourier transform shown by Noot-Huyghe. There exists an isomorphism of rings

$$\iota\colon \Gamma(\widehat{\mathbb{P}}', \mathscr{D}^\dagger_{\widehat{\mathbb{P}}',\mathbb{Q}}(\infty')) \xrightarrow{\sim} \Gamma(\widehat{\mathbb{P}}, \mathscr{D}^\dagger_{\widehat{\mathbb{P}},\mathbb{Q}}(\infty))$$

sending x' to $\pi^{-1}\partial$ and ∂' to $-\pi x$. It is also shown by Noot-Huyghe that coherent $\mathscr{D}^\dagger_{\widehat{\mathbb{P}},\mathbb{Q}}(\infty)$-modules corresponds to $\Gamma(\widehat{\mathbb{P}}, \mathscr{D}^\dagger_{\widehat{\mathbb{P}},\mathbb{Q}}(\infty))$-modules by taking global sections. Given a coherent $\Gamma(\widehat{\mathbb{P}}, \mathscr{D}^\dagger_{\widehat{\mathbb{P}},\mathbb{Q}}(\infty))$-module M, we denote by $\mathscr{F}_{\mathrm{naive},\pi}(M)$ the coherent $\Gamma(\widehat{\mathbb{P}}', \mathscr{D}^\dagger_{\widehat{\mathbb{P}}',\mathbb{Q}}(\infty'))$-module obtained from M via transport of structure by ι. For $m \in M$ we denote by $\widehat{m}$ the corresponding element of $\mathscr{F}_{\mathrm{naive},\pi}(M)$.

Let $\mathscr{M}$ be a coherent $\mathscr{D}^\dagger_{\widehat{\mathbb{P}},\mathbb{Q}}(\infty)$-module. We denote by $\mathscr{F}_{\mathrm{naive},\pi}(\mathscr{M})$ the coherent $\mathscr{D}^\dagger_{\widehat{\mathbb{P}}',\mathbb{Q}}$-module corresponding to $\mathscr{F}_{\mathrm{naive},\pi}(\Gamma(\widehat{\mathbb{P}}, \mathscr{M}))$. We call this the *naive Fourier transform* of $\mathscr{M}$. Then Noot-Huyghe shows in [59, 5.3.1] that there exists a canonical isomorphism

$$\mathscr{F}_\pi(\mathscr{M}) \xrightarrow{\sim} \mathscr{F}_{\mathrm{naive},\pi}(\mathscr{M})[-1]. \tag{3.2.5.1}$$

We often denote by $m \in \mathscr{M}$ to mean $m \in \Gamma(\widehat{\mathbb{P}}, \mathscr{M})$. For simplicity, we will often omit the index π in the notation $\mathscr{F}_{\mathrm{naive},\pi}(\mathscr{M})$ or $\mathscr{F}_\pi(\mathscr{M})$.

3.2.6. Standard properties of ℓ-adic geometric Fourier transform explained in [50, 1.2, 1.3] hold also for p-adic Fourier transform with suitable changes. Since the proof works well with few changes, we leave the reader to formulate and verify these properties. Although the most of these properties are not used in this paper, we still need a few analogous results. We will write the statements of these results with short comments of the proofs.

3.2.7 Proposition ([50, 1.2.2.2])**.** *Let V be a coherent F-$\mathscr{D}^\dagger_{\mathrm{Spf}(R),\mathbb{Q}}$-module (*i.e. *a finite dimensional σ–K-vector space). Then, we have a canonical isomorphism*

$$\mathscr{F}(q^!(V)) \cong i_{0'+}(V)(1),$$

where $q\colon (\widehat{\mathbb{P}}, \infty) \to \mathrm{Spf}(R)$ denotes the structural morphisms, and $i_{0'}\colon \mathrm{Spf}(R) \hookrightarrow \widehat{\mathbb{P}}'$ is the closed immersion defined by $0'$ in $\widehat{\mathbb{P}}'$.

Remark. When V is trivial, another calculation for this Fourier transform was carried out by Baldassarri and Berthelot in [9]. In their calculation, there is no Tate twist contrary to our calculation here. This is because the definitions of the Frobenius structures on the geometric Fourier transform are slightly different. For the precise argument, see [4, Remark 3.15(ii)].

3.2.8. Let $\mathscr{F}'$ be the dual geometric Fourier transform: the functor $\mathscr{F}'\colon D^b_{\mathrm{coh}}(\mathscr{D}^\dagger_{\widehat{\mathbb{P}}',\mathbb{Q}}) \to D^b_{\mathrm{coh}}(\mathscr{D}^\dagger_{\widehat{\mathbb{P}},\mathbb{Q}})$ defined in the same way as $\mathscr{F}$ except for reversing the role of p and p'. We get the following inversion formula.

Theorem ([50, 1.2.2.1])**.** *Let $\mathscr{M}$ be a coherent F-$\mathscr{D}^\dagger_{\widehat{\mathbb{P}},\mathbb{Q}}(\infty)$-module. Then, there exists a functorial isomorphism*

$$\mathscr{F}'\circ\mathscr{F}(\mathscr{M})[2] \cong \mathscr{M}(1).$$

Remark. We may also prove the theorem in more general cases: let $\mathscr{X}$ be a smooth formal scheme over $\mathrm{Spf}(R)$, and let $\mathscr{E}$ be a locally free sheaf of finite rank r on $\mathscr{X}$. Consider the projective bundle $p\colon \mathscr{P} := \widehat{\mathbb{P}}(\mathscr{E}\oplus\mathcal{O}_{\mathscr{X}}) \to \mathscr{X}$ to define the geometric Fourier transform (cf. [59, 3.2.1]). Then the theorem is reformulated as $\mathscr{F}'\circ\mathscr{F}(\mathscr{M})[4-2r] \cong \mathscr{M}(r)$. Let Z be the divisor at infinity of $\mathscr{P}$. For the proof, we need to show that there exists an isomorphism $\pi_+(\mathcal{O}_{\mathscr{P},\mathbb{Q}}(Z)) \cong \mathcal{O}_{\mathscr{X},\mathbb{Q}}[r](r)$. This can be seen from [61, Corollary 4.4] and [4, 3.14 or 3.15(i)].

3.2.9. Let $\alpha \in (q-1)^{-1}\mathbb{Z}$. We define a convergent F-isocrystal $\mathscr{K}_\alpha$ on $\widehat{\mathbb{P}}\setminus\{0,\infty\}$ overconvergent along $\{0,\infty\}$ in the following way. As an $\mathcal{O}_{\widehat{\mathbb{P}},\mathbb{Q}}(0,\infty)$-module, it is isomorphic to $\mathcal{O}_{\widehat{\mathbb{P}},\mathbb{Q}}(0,\infty)$. We denote the global section corresponding to 1 by e. We define its connection by

$$\nabla(e) = (\alpha x^{-1})\cdot e\otimes \mathrm{d}x.$$

The Frobenius structure $\Phi\colon F^*\mathscr{K}_\alpha \xrightarrow{\sim} \mathscr{K}_\alpha$ is defined by

$$\Phi(1\otimes e) := x^{\alpha(q-1)}\cdot e.$$

We often use the same notation $\mathscr{K}_\alpha$ for the underlying holonomic $\mathscr{D}^\dagger_{\widehat{\mathbb{P}},\mathbb{Q}}(\infty)$-module. This is called the *Kummer isocrystal*.

Proposition ([50, 1.4.3.2])**.** *Let $j\colon (\widehat{\mathbb{P}},\{0,\infty\}) \to (\widehat{\mathbb{P}},\{\infty\})$ be the canonical morphism of couples. Assume $\alpha\notin\mathbb{Z}$. Then we get that the canonical homomorphism*

$$j_! j^+\mathscr{K}_\alpha \to \mathscr{K}_\alpha$$

is an isomorphism. Moreover, let $G(\alpha,\pi)$ be the following K-vector space with Frobenius structure:

$$G(\alpha,\pi) := H^1_{\mathrm{rig}}(\mathbb{A}^1_k\setminus\{0\}, \mathscr{K}_\alpha\otimes\mathscr{L}_\pi).$$

Then, we have

$$\mathscr{F}_\pi(\mathscr{K}_\alpha)[1] \cong \mathscr{K}_{1-\alpha}\otimes G(\alpha,\pi)(1).$$

Proof. The first statement follows from Lemma 3.1.10. For the latter claim, the proof works essentially the same as in [50] by replacing m^* by $m^!$ and using the Künneth formula 3.1.7. The Tate twist appearing here comes from the isomorphism 3.1.8.2. $\square$

4. Stationary phase

In this section, we will prove the stationary phase formula when the differential slope at infinity is less than or equal to one. However, in this section, we do not consider Frobenius structures on the local Fourier transforms, so the stationary phase in this section is still temporary. This will be completed in the next section.

> Throughout this section, we continuously use the assumptions and notation of paragraph 2.4.1.

4.1. Geometric calculations

4.1.1. Following [50, 2.2.1], we will define several invariants which will be used throughout this section. Let M be a solvable differential module on the Robba ring, cf. [47, 12.6.4] or [25, 8.7]. We denote by $\mathrm{rk}(M)$ the rank, by $\mathrm{irr}(M)$ the irregularity (cf. 2.3.1), and by $\mathrm{pt}(M)$ the greatest differential slope of M as usual. By a result of Christol and Mebkhout (cf. [26, 2.4-1] or in general [47, 12.6.4]), we get the differential slope decomposition $M = \bigoplus M_\beta$ where M_β is purely of slope β. For any interval $I \subset [0, \infty[$, we put $M_I := \bigoplus_{\beta \in I} M_\beta \subset M$. Let $\mathscr{M}$ be a holonomic F-$\mathscr{D}^\dagger_{\widehat{\mathbb{P}},\mathbb{Q}}(\infty)$-module. For any closed point x in $\widehat{\mathbb{A}}$, we put

$$r(\mathscr{M}) := -\mathrm{rk}(\mathscr{M}|_{\eta_z}) \leqslant 0, \quad s_x(\mathscr{M}) := -\mathrm{irr}(\mathscr{M}|_{\eta_x}) \leqslant 0,$$
$$r_x(\mathscr{M}) := \dim_{K_x}(i_x^!\mathscr{M}), \quad a_x(\mathscr{M}) := r(\mathscr{M}) + s_x(\mathscr{M}) - r_x(\mathscr{M}),$$

where z can be taken to be any closed point in $\widehat{\mathbb{A}}$, and $i_x \colon \mathrm{Spf}(R_x) \hookrightarrow \widehat{\mathbb{A}}$ is the closed immersion for x. The following lemma compares these invariants to the generic rank and the vertical multiplicity, cf. 1.4.3.

4.1.2 Lemma. *We preserve the notation. Let*

$$\mathrm{Cycl}(\mathscr{M}|_{\widehat{\mathbb{A}}}) = r \cdot [\mathbb{A}^1] + \sum_{x \in |\mathbb{A}^1|} m_x \cdot [\pi^{-1}(x)].$$

Recall that $\pi : T^*\mathbb{A}^1 \to \mathbb{A}^1$ is the canonical projection and $|\mathbb{A}^1|$ is the set of closed points. Then we have $r(\mathscr{M}) = -r$ and, for any closed point $x \in \widehat{\mathbb{A}}$, $a_x(\mathscr{M}) = -m_x$.

Proof. This follows from Corollary 2.3.2. $\square$

Let C be a complex of $D^b_{\mathrm{coh}}(\mathscr{D}^\dagger_{\mathrm{Spf}(R),\mathbb{Q}}) \cong D^b_{\mathrm{fin}}(K\text{-mod})$, where the latter category is the derived category of complexes of K-vector spaces whose cohomology is finite dimensional. We put $\chi(C) := \sum_i (-1)^i \dim_K \mathscr{H}^i(C)$. Let S be a closed subset of $\widehat{\mathbb{P}}$, and $q\colon (\widehat{\mathbb{P}}, S) \to (\mathrm{Spf}(R), \emptyset)$ be the structural morphism of d-couples. For a holonomic F-$\mathscr{D}^\dagger_{\widehat{\mathbb{P}},\mathbb{Q}}(S)$-module $\mathscr{M}$, we put $\chi(\widehat{\mathbb{P}} \setminus S, \mathscr{M}) := \chi(q_+\mathscr{M})$.

By GOS-type formula for arithmetic $\mathscr{D}$-modules [2, 2.4.7] combining with the above lemma, we get the following variant of GOS-type formula, for a holonomic F-$\mathscr{D}^{\dagger}_{\widehat{\mathbb{P}},\mathbb{Q}}(\infty)$-module $\mathscr{M}$:

$$\chi(\widehat{\mathbb{A}}, \mathscr{M}) = r(\mathscr{M}) - \sum_{x \in |\widehat{\mathbb{A}}|} \deg(x) \cdot a_x(\mathscr{M}) + \mathrm{irr}(\mathscr{M}|_{\eta_\infty}). \tag{4.1.2.1}$$

This can be shown by the GOS-type formula 2.3.1.1 of Garnier as well by using the fact that $\chi_{\mathrm{rig}}(\mathscr{U}, -) = -\chi(\mathscr{U}, -)$ for $\mathscr{U} \subset \widehat{\mathbb{P}}$ (cf. 3.1.8.2) and *dévissage*.

4.1.3 Corollary. *Let $\mathscr{M}$ be a holonomic $\mathscr{D}^{\dagger}_{\widehat{\mathbb{P}},\mathbb{Q}}(\infty)$-module, and let $x \in |\widehat{\mathbb{A}}|$ be a singular point of $\mathscr{M}$. Then we get*

$$\mathrm{rk}(\mathscr{F}^{(x,\infty')}(\mathscr{M})) = -\deg(x) \cdot a_x(\mathscr{M}).$$

Proof. This follows from the definition of the local Fourier transform (cf. 2.4.2), using the stability theorem (cf. 1.5.1-i), combined with Proposition 1.4.3 and Lemma 4.1.2. ☐

4.1.4 Lemma. *Using the notation of Lemma 4.1.2, we get $r_x(\mathscr{M}) = r(\mathscr{M})$ if and only if $m_x = 0$.*

Proof. When $m_x = 0$, we know that $\mathscr{M}$ is a convergent isocrystal on an open neighbourhood of x, and the lemma follows easily. Let us check the 'only if' part. We know that $\dim_{K^{\mathrm{ur}}} \mathbb{V}(\mathscr{M}) = \mathrm{rk}(\mathscr{M})$ by [31, (6.1.11)]. By [30, 2.2], we get

$$i_x^! \mathscr{M} \cong R\mathrm{Hom}(\mathscr{M}|_{S_x}, \mathcal{O}^{\mathrm{an}})^* \,[1],$$

where * denotes the dual in the derived category $D^b_{\mathrm{fin}}(K\text{-mod})$. The exact sequence 2.1.7.1 implies $\dim_{K^{\mathrm{ur}}} \mathbb{W}(\mathscr{M}|_{S_x}) = 0$, and thus $\mathbb{W}(\mathscr{M}|_{S_x}) = 0$. By Lemma 2.1.7, $\mathscr{M}|_{S_x}$ is a free differential $\mathcal{O}^{\mathrm{an}}$-module, and in particular $m_x = 0$. ☐

4.1.5 Lemma. *Let M be a solvable free differential module on the Robba ring $\mathcal{R}$ over K. We further assume that M is purely of differential slope 1. Let $L(\mathfrak{s})$ be the Dwork differential module, for $\mathfrak{s} \in \mathbb{A}^1_k(\bar{k})$, cf. 2.4.8. We consider the tensor product $M \otimes_{\mathcal{R}} L(\mathfrak{s})$ as a differential $\mathcal{R}_{K_s}$-module. Then we get the following.*

(i) *For almost all $\mathfrak{s} \in \mathbb{A}^1_k(\bar{k})$, we get*

$$\mathrm{irr}_{K_s}(M \otimes_{\mathcal{R}} L(\mathfrak{s})) = \mathrm{rk}(M),$$

where irr_{K_s} denotes the irregularity as an $\mathcal{R}_{K_s}$-module.

(ii) *There exists an $\mathfrak{s} \neq 0$ in $\bar{k}$ such that the irregularity $\mathrm{irr}_{K_s}(M \otimes L(\mathfrak{s}))$ is less than $\mathrm{rk}(M)$.*

Proof. Let us prove (i). We use the induction on the rank of M over $\mathcal{R}_K$. Suppose there exists a geometric point $\mathfrak{s}$ such that

$$\mathrm{irr}(M \otimes L(\mathfrak{s})) < \mathrm{rk}(M), \quad \mathrm{pt}(M \otimes L(\mathfrak{s})) = 1.$$

These conditions show that $M \otimes L(\mathfrak{s})$ has at least two slopes including 1. Thus, there exists the canonical decomposition $M \otimes L(\mathfrak{s}) = M'_1 \oplus M'_{<1}$ where M'_1 is purely of slope one, and $M'_{<1}$ is purely of slope less than one, and these modules are non-zero. Thus, we get the decomposition

$$M \otimes_K K_s = M'_1 \otimes L(-\mathfrak{s}) \oplus M'_{<1} \otimes L(-\mathfrak{s}).$$

Since M is purely of slope one, both $M'_1 \otimes L(-\mathfrak{s})$ and $M'_{<1} \otimes L(-\mathfrak{s})$ are purely of slope one as well. Thus by the induction hypothesis, the lemma holds for these two modules. This implies that the lemma also holds for M.

If $\mathrm{pt}(M \otimes L(\mathfrak{s})) = 1$ for any $\mathfrak{s}$, we get the lemma by the above argument. Suppose $\mathrm{pt}(M \otimes L(\mathfrak{s})) < 1$ for some $\mathfrak{s}$. Then for any $\mathfrak{s}' \neq \mathfrak{s}$, we get $\mathrm{pt}(M \otimes L(\mathfrak{s}')) = 1$. If there exists $\mathfrak{s}' \neq \mathfrak{s}$ such that $\mathrm{irr}(M \otimes L(\mathfrak{s}')) < \mathrm{rk}(M)$, then we may use the above argument. Otherwise, the lemma is trivial.

Now, let us move to (ii). By using [57, 2.0-1], there exists a number a in $\overline{K}$ whose absolute value is one, and an integer h such that the irregularity of $M \otimes \exp(\pi a x^{-p^h})$ is less than $\mathrm{rk}(M)$. Here we are using the notation of [57]. We remind that in [57], there is an assumption on the spherically completeness of K. However, as mentioned in [57, 2.0-4], this hypothesis is used only to use a result of Robba, and when $p \neq 2$, the assumption was removed by Matsuda as written there. This result was extended also to the case $p = 2$ by Pulita [62, Theorem 4.6], and we no longer need to assume the spherically completeness here. Arguing as the proof of [41, 4.2.3(ii)] using [55, 1.5][11], there exists a number a' in $\overline{K}$ whose absolute value is one such that $\exp(\pi a x^{-p^h})$ is isomorphic to $\exp(\pi a' x^{-1})$ as differential $\mathcal{R}$-modules, and the latter is isomorphic to $L(\overline{a'})$ where the overline denotes the residue class. $\square$

4.1.6. Let $\mathscr{E}$ be a coherent F-$\mathscr{D}^{\dagger}_{\widehat{\mathbb{P}},\mathbb{Q}}(\infty)$-module. We denote by $\mathscr{E}' := \mathscr{H}^1(\mathscr{F}_\pi(\mathscr{E}))$ the geometric Fourier transform and by S (resp. S') the set of singular points of $\mathscr{E}$ (resp. $\mathscr{E}'$) in $\widehat{\mathbb{A}}$ (resp $\widehat{\mathbb{A}}'$). We have the following analog of [50, 2.3.1.1].

Proposition. (i) $r(\mathscr{E}') = \sum_{s \in S} \deg(s) \cdot a_s(\mathscr{E}) + \mathrm{rk}((\mathscr{E}|_{\eta_\infty})_{]1,\infty[}) - \mathrm{irr}((\mathscr{E}|_{\eta_\infty})_{]1,\infty[})$.

(i′) $r(\mathscr{E}) = \sum_{s' \in S'} \deg(s') \cdot a_{s'}(\mathscr{E}') + \mathrm{rk}((\mathscr{E}'|_{\eta_\infty})_{]1,\infty[}) - \mathrm{irr}((\mathscr{E}'|_{\eta_\infty})_{]1,\infty[})$.

(ii) *For* $s' \in \widehat{\mathbb{A}}' \setminus \{0'\}$*, we get*

$$r_{s'}(\mathscr{E}') = r(\mathscr{E}') + \mathrm{rk}((\mathscr{E}|_{\eta_\infty})_1) - \mathrm{irr}((\mathscr{E}|_{\eta_\infty})_1 \otimes \mathscr{L}(x \cdot s')|_{\eta_\infty}).$$

(ii′) *For* $s \in \widehat{\mathbb{A}} \setminus \{0\}$*, we get*

$$r_s(\mathscr{E}) = r(\mathscr{E}) + \mathrm{rk}((\mathscr{E}'|_{\eta_\infty})_1) - \mathrm{irr}((\mathscr{E}'|_{\eta_\infty})_1 \otimes \mathscr{L}(s \cdot x')|_{\eta_\infty}).$$

(iii) $r_{0'}(\mathscr{E}') = r(\mathscr{E}') + \mathrm{rk}((\mathscr{E}|_{\eta_\infty})_{[0,1[}) - \mathrm{irr}((\mathscr{E}|_{\eta_\infty})_{[0,1[})$.

(iii′) $r_0(\mathscr{E}) = r(\mathscr{E}) + \mathrm{rk}((\mathscr{E}'|_{\eta_{\infty'}})_{[0,1[}) - \mathrm{irr}((\mathscr{E}'|_{\eta_{\infty'}})_{[0,1[})$.

[11] In [55], $p \neq 2$ is assumed extensively. However, the proof of Lemma 1.5 works also for $p = 2$ without any change.

Proof. The idea of the proof is the same as that of [50]. We sketch the proof. By the base change Theorem 3.1.6, we get

$$r_{s'}(\mathscr{E}') = -\chi(\widehat{\mathbb{A}}, \mathscr{E} \otimes \mathscr{L}(x \cdot s')).$$

Let us prove (i). Since both sides of the equality are invariant under base extension, we may assume that S consists of k-rational points. We have $a_s(\mathscr{E}) = a_s(\mathscr{E} \otimes \mathscr{L}(x \cdot s'))$ for $s, s' \neq 0$ since $\mathscr{L}(x \cdot s')$ has no singular points in $\widehat{\mathbb{A}}$. Using 4.1.2.1, it remains to show

$$-r(\mathscr{E}) - \mathrm{irr}(\mathscr{E}|_{\eta_\infty} \otimes \mathscr{L}(x \cdot s')|_{\eta_\infty}) = \mathrm{rk}((\mathscr{E}|_{\eta_\infty})_{]1,\infty[}) - \mathrm{irr}((\mathscr{E}|_{\eta_\infty})_{]1,\infty[})$$

for almost all $s' \in \overline{k}$. This follows from Lemma 4.1.5(i). The claims (ii) and (iii) follow from (i) and 4.1.2.1, and (i)$'$, (ii)$'$, (iii)$'$ follow by the involutivity 3.2.8 of the geometric Fourier transform. □

4.1.7 Corollary. *Let $\mathscr{E}$ be a holonomic F-$\mathscr{D}^\dagger_{\widehat{\mathbb{P}},\mathbb{Q}}(\infty)$-module such that 0 is the only singularity, and it is regular at infinity (i.e. $\mathrm{irr}(\mathscr{E}|_{\eta_\infty}) = 0$). Then $\mathscr{E}'$, the first cohomology of the geometric Fourier transform, is not singular except for $0'$, we have $s_{0'}(\mathscr{E}') = 0$, $a_{0'}(\mathscr{E}') = r(\mathscr{E})$, and the differential slope at ∞' is strictly less than 1.*

Proof. By (ii) of Proposition 4.1.6 and the hypothesis that $\mathscr{E}$ is regular at ∞, we get for $s' \neq 0'$ in $\widehat{\mathbb{A}}'$,

$$r_{s'}(\mathscr{E}') = r(\mathscr{E}'),$$

which shows that $m_{s'} = 0$ for $s' \neq 0'$ by Lemma 4.1.4. Thus $\mathscr{E}'$ is not singular except for $0'$.

Now, by (ii$'$) of the proposition, we get for $s \neq 0, \infty$

$$\mathrm{rk}((\mathscr{E}'|_{\eta_\infty})_1) - \mathrm{irr}((\mathscr{E}'|_{\eta_\infty})_1 \otimes \mathscr{L}(s \cdot x')|_{\eta_\infty}) = 0. \tag{4.1.7.1}$$

Suppose $(\mathscr{E}'|_{\eta_\infty})_1 \neq 0$. Then by Lemma 4.1.5(ii), there exists $s \neq 0$ such that the irregularity of $(\mathscr{E}'|_{\eta_\infty})_1 \otimes \mathscr{L}(s \cdot x')|_{\eta_\infty}$ is less than $r((\mathscr{E}'|_{\eta_\infty})_1)$, which contradicts with 4.1.7.1. This shows that $(\mathscr{E}'|_{\eta_\infty})_1 = 0$. Now, we get

$$\begin{aligned} a_{0'}(\mathscr{E}') - s_{0'}(\mathscr{E}') &= r(\mathscr{E}') - r_{0'}(\mathscr{E}') \\ &= -\left(\mathrm{rk}((\mathscr{E}|_{\eta_\infty})_{[0,1[}) - \mathrm{irr}((\mathscr{E}|_{\eta_\infty})_{[0,1[})\right) \\ &= -\mathrm{rk}(\mathscr{E}|_{\eta_\infty}) = r(\mathscr{E}), \end{aligned} \tag{4.1.7.2}$$

where the second equality holds by (iii) of the proposition and the third by the assumption that $\mathscr{E}$ is regular at infinity. Combining with (i$'$), we get

$$r(\mathscr{E}) = r(\mathscr{E}) + s_{0'}(\mathscr{E}') + \mathrm{rk}((\mathscr{E}'|_{\eta_\infty})_{]1,\infty[}) - \mathrm{irr}((\mathscr{E}'|_{\eta_\infty})_{]1,\infty[}).$$

Thus,

$$s_{0'}(\mathscr{E}') = \mathrm{irr}((\mathscr{E}'|_{\eta_\infty})_{]1,\infty[}) - \mathrm{rk}((\mathscr{E}'|_{\eta_\infty})_{]1,\infty[}) \geqslant 0.$$

On the other hand, we have $s_{0'}(\mathscr{E}') \leqslant 0$ by definition. This shows that $s_{0'}(\mathscr{E}') = 0$, and $(\mathscr{E}'|_{\eta_\infty})_{]1,\infty[} = 0$. Thus $a_{0'}(\mathscr{E}') = r(\mathscr{E})$ by 4.1.7.2. □

4.2. Regular stationary phase formula

4.2.1. Let $\mathscr{M}$ be a holonomic $\mathscr{D}^{\dagger}_{\widehat{\mathbb{P}},\mathbb{Q}}(\infty)$-module. Let $s \in \widehat{\mathbb{A}}$ be a singular point of $\mathscr{M}$. Recall the notation of 2.4.1 and Definition 2.4.2. We have a canonical homomorphism

$$\mathscr{H}^1(\mathscr{F}_\pi(\mathscr{M}))|_{\eta_{\infty'}} \cong \mathscr{F}_{\mathrm{naive},\pi}(\mathscr{M})|_{\eta_{\infty'}} \to \tau'^*\big(\mathscr{E}^{\dagger}_{s,\mathbb{Q}} \otimes \mathscr{M}\big) \cong \tau'^*\Gamma(\mathscr{F}^{(s,\infty')}_{\pi}(\mathscr{M})),$$

where the second homomorphism sends $\alpha \otimes \widehat{m}$ ($\alpha \in \mathcal{R}_{\mathscr{S}'_\infty}$, and $m \in \Gamma(\widehat{\mathbb{P}}, \mathscr{M})$) to $\alpha(1 \otimes m)$. The first isomorphism is the isomorphism by Noot-Huyghe 3.2.5.1, and the second one is that of Lemma 2.4.5. We see easily that this homomorphism is compatible with the connections. By construction (cf. 2.1.8) the source is a free differential $\mathcal{R}$-module, thus we get a canonical homomorphism of differential $\mathcal{R}$-modules

$$\alpha_s \colon \mathscr{H}^1(\mathscr{F}_\pi(\mathscr{M}))|_{\eta_{\infty'}} \to \tau'^*\mathscr{F}^{(s,\infty')}_{\pi}(\mathscr{M})$$

by Lemma 2.1.3.

4.2.2 Theorem (Regular Stationary Phase)**.** *Let $\mathscr{M}$ be a holonomic F-$\mathscr{D}^{\dagger}_{\widehat{\mathbb{P}},\mathbb{Q}}(\infty)$-module whose* differential slopes at infinity are less than or equal to 1*. Let S be the set of singular points of $\mathscr{M}$ in $\widehat{\mathbb{A}}$. Then the canonical homomorphism*

$$(\alpha_s)_{s\in S} \colon \mathscr{H}^1(\mathscr{F}_\pi(\mathscr{M}))|_{\eta_{\infty'}} \to \bigoplus_{s\in S} \tau'^*\mathscr{F}^{(s,\infty')}_{\pi}(\mathscr{M}) \tag{4.2.2.1}$$

is an isomorphism.

Proof. First of all, we will reduce to the case where S consists of rational points. There exists an unramified Galois extension E of K such that S consists of k_E-rational points where k_E denotes the residue field of E as usual. Note that

$$\mathscr{F}_{\mathrm{naive},\pi}(\mathscr{M})|_{\eta_{\infty'}} \otimes_K E \cong \mathscr{F}_{\mathrm{naive},\pi}(\mathscr{M} \otimes_K E)|_{\eta_{\infty'}},$$
$$\mathscr{F}^{(s,\infty')}(\mathscr{M}) \otimes_K E \xrightarrow{\sim} \bigoplus_{s'\mapsto s} \mathscr{F}^{(s',\infty)}(\mathscr{M} \otimes_K E),$$

where the direct sum in the second isomorphism runs over the set of closed points of $\widehat{\mathbb{A}}^1_{R_E}$ which map to s. Indeed, the first isomorphism follows since the cohomological operators used in the definition of geometric Fourier transform are compatible with base change. The second isomorphism follows from Lemmas 2.4.10 and 2.4.9. Since the left hand sides of these two isomorphisms have the action of $G := \mathrm{Gal}(E/K)$, we define a G-action on the right hands sides by transport of structure. Note that the G-invariant parts are isomorphic to $\mathscr{F}_{\mathrm{naive},\pi}(\mathscr{M})|_{\eta_{\infty'}}$ and $\mathscr{F}^{(s,\infty')}(\mathscr{M})$ respectively. By definition, the following diagram is commutative

$$\begin{array}{ccc}
\mathscr{F}_{\mathrm{naive},\pi}(\mathscr{M})|_{\eta_{\infty'}} & \xrightarrow{\alpha_s} & \tau'^*\mathscr{F}^{(s,\infty')}(\mathscr{M}) \\
\downarrow & & \downarrow \\
\mathscr{F}_{\mathrm{naive},\pi}(\mathscr{M} \otimes_K E)|_{\eta_{\infty'}} & \xrightarrow[\bigoplus \alpha_{s'}]{} & \bigoplus_{s'\mapsto s} \tau'^*_E\mathscr{F}^{(s',\infty)}(\mathscr{M} \otimes_K E),
\end{array}$$

where τ'_E denotes τ' over E. Thus, if the theorem holds for $\mathscr{M} \otimes_K E$, then it also holds for $\mathscr{M}$ by taking G-invariant parts, and the claim follows. From now on, we assume that S consists of rational points.

Now, for a moment, let $\mathscr{M}$ be a holonomic $\mathscr{D}^{\dagger}_{\widehat{\mathbb{P}},\mathbb{Q}}(\infty)$-module which may not possess Frobenius structure. In this situation, let us check that α_s is surjective for any $s \in S$. By the exactness of Fourier transforms and local Fourier transforms (cf. Lemma 2.4.7 and 3.2.5.1), it suffices to show the claim when $\mathscr{M}$ is monogenic. Suppose it is generated by $m_0 \in \Gamma(\widehat{\mathbb{P}}, \mathscr{M})$. Let N be a differential $\mathcal{A}_{t'}([r, 1[)$-module, where $t' := 1/x'$, satisfying the following:

1. $\mathcal{R}_{\infty'} \otimes N \cong \mathscr{F}_{\mathrm{naive},\pi}(\mathscr{M}) \mid_{\eta_{\infty'}}$ where $\mathcal{R}_{\infty'} := \mathcal{R}_{\mathscr{S}_{\infty'}}$;
2. there exists an element $m'_0 \in N$ which is mapped to $1 \otimes \widehat{m}_0$ in $\mathscr{F}_{\mathrm{naive},\pi}(\mathscr{M})|_{\eta_{\infty'}}$ by the canonical injective homomorphism $N \to \mathscr{F}_{\mathrm{naive},\pi}(\mathscr{M})|_{\eta_{\infty'}}$.

Now, let $\mathscr{M}^{(m)}$ be a stable coherent $\widehat{\mathscr{D}}^{(m)}_{\widehat{\mathbb{P}},\mathbb{Q}}(\infty)$-module generated by a single element m''_0 in $\Gamma(\widehat{\mathbb{P}}, \mathscr{M}^{(m)})$ such that $\mathscr{D}^{\dagger}_{\widehat{\mathbb{P}},\mathbb{Q}}(\infty) \otimes \mathscr{M}^{(m)} \cong \mathscr{M}$ and $1 \otimes m''_0$ is sent to m_0 via this isomorphism. Then there exists an integer m' such that $m' \geqslant m$, $\omega_{m'} \geqslant r$ and α_s is induced by the homomorphism $\mathcal{A}([\omega_{m'}, 1[) \otimes N \to \tau'^{*}\mathscr{E}_s^{(m',\dagger)}(\mathscr{M}^{(m)})$ sending $1 \otimes m'_0$ to $1 \otimes (m''_0)^{\wedge}$. To see that this is surjective, it suffices to show that the canonical homomorphism

$$\theta \colon \mathcal{A}([\omega_{m'}, \omega_{m''}\}) \otimes N \to \tau'^{*}\widehat{\mathscr{E}}_s^{(m',m'')}(\mathscr{M})$$

is surjective for any $m'' \geqslant m'$. We have the local parameter $y_s = x - s$ around s. By the choice of N, $1 \otimes (y_s^n m_0)^{\wedge}$ is also contained in $\theta(N)$ for any positive integer n. Indeed, we have

$$1 \otimes (y_s^n m''_0)^{\wedge} = (-\pi^{-1}\partial_{x'} - s)^n \cdot \big(1 \otimes (m''_0)^{\wedge}\big) = \theta\big((\pi^{-1}t'^2\partial_{t'} - s)^n \cdot m'_0\big).$$

Thus, the surjectivity follows from Lemma 1.3.9.

Let M be an object of F-Hol$(\mathscr{S})$, and let M_0^{can} be the canonical extension of M at 0. Then the homomorphism

$$\mathscr{F}_{\mathrm{naive},\pi}(M_0^{\mathrm{can}})|_{\eta_{\infty'}} \to \tau'^{*}\Phi^{(0,\infty')}(M)$$

is surjective by the argument above. By Corollary 4.1.7, the greatest differential slope of $\mathscr{F}_{\mathrm{naive},\pi}(M_0^{\mathrm{can}})|_{\eta_{\infty'}}$ is strictly less than one. The surjectivity of the homomorphism implies the following:

> (*) The greatest differential slope of $\Phi^{(0,\infty')}(M)$ is strictly less than 1 for any object M in F-Hol$(\mathscr{S})$.

We get back to the situation where $\mathscr{M}$ possesses a Frobenius structure. For $s \in S$, we know that

$$\mathscr{F}^{(s,\infty')}(\mathscr{M}) \cong \Phi^{(0,\infty')}(\tau_{s*}\mathscr{M}|_{S_s}) \otimes L(s)$$

by Lemma 2.4.10. Thus we get the following:

> The greatest slope of $\mathscr{F}^{(s,\infty')}(\mathscr{M})$ is equal to one for $s \in S \setminus \{0\}$.

We put $M' := \mathscr{F}_{\mathrm{naive},\pi}(\mathscr{M})|_{\eta_{\infty'}}$. For a k-rational point s in $\widehat{\mathbb{A}}^1$, we define

$$M'^s := L(s) \otimes (L(-s) \otimes M')_{[0,1[} \subset M'.$$

By definition, M'^s is a direct factor of M', and in particular, there exists a canonical projection $p_s\colon M' \to M'^s$. Take two *distinct* points $s, t \in S$, and consider the canonical homomorphism

$$L(-s) \otimes M'^t \xrightarrow{\mathrm{id}\otimes\alpha_s} L(-s) \otimes \mathscr{F}^{(s,\infty')}(\mathscr{M}) \cong \Phi^{(0,\infty')}(\tau_{s*}\mathscr{M}|_{S_s}).$$

Since $L(-s) \otimes M'^t \cong L(-s+t) \otimes (L(-t) \otimes \mathscr{M}')_{[0,1[}$, we get that the source is purely of slope one. We know that the greatest slope of the target is strictly less than one by (*). This shows that the homomorphism is zero. Thus the homomorphism

$$M'^t \hookrightarrow M' \xrightarrow{\alpha_s} \tau'^* \mathscr{F}^{(s,\infty')}(\mathscr{M})$$

is 0 if $t \neq s$.

Let s be a point in S and let $S_s := S \setminus \{s\}$. By using the same argument, the canonical homomorphism $\sum_{t\in S_s} M'^t \hookrightarrow M \xrightarrow{p_s} M'^s$ is zero. This shows that $\sum_{t\in S_s} M'^t \cap M'^s = 0$ in M', and thus, the canonical homomorphism $\bigoplus_{s\in S} M'^s \to M'$ is injective.

Now, since $M' \to \mathscr{F}^{(s,\infty')}(\mathscr{M})$ is surjective, the homomorphism

$$L(-s) \otimes M' \to L(-s) \otimes \mathscr{F}^{(s,\infty')}(\mathscr{M}) \cong \Phi^{(0,\infty')}(\mathscr{M}|_{S_s})$$

is also surjective. Since the target has slope <1, the homomorphism

$$(L(-s) \otimes M')_{[0,1[} \to L(-s) \otimes \mathscr{F}^{(s,\infty')}(\mathscr{M})$$

is surjective, and we deduce that the composition

$$\beta_s\colon M'^s \hookrightarrow M' \xrightarrow{\alpha_s} \mathscr{F}^{(s,\infty')}(\mathscr{M}),$$

is surjective as well. Thus, we get that the canonical homomorphism

$$\bigoplus_{s\in S} M'^s \hookrightarrow M' \xrightarrow{(\alpha_s)} \bigoplus_{s\in S} \mathscr{F}^{(s,\infty')}(\mathscr{M}), \tag{4.2.2.2}$$

is also surjective and is equal to $\bigoplus_{s\in S} \beta_s$. This leads us to the following inequalities:

$$\mathrm{rk}(M') \geqslant \sum_{s\in S} \mathrm{rk}(M'^s) \geqslant \sum_{s\in S} \mathrm{rk}(\mathscr{F}^{(s,\infty')}(\mathscr{M})) = \mathrm{rk}(M'),$$

where the last equation holds by Corollary 4.1.3 combined with Proposition 4.1.6(i) using the assumption that the greatest slope of $\mathscr{M}$ at infinity is less than or equal to one. Since a surjection of differential $\mathcal{R}$-modules with the same ranks is an isomorphism, we get that 4.2.2.2 is an isomorphism. Since an injection of differential $\mathcal{R}$-modules with the same ranks is an isomorphism, we have $\bigoplus_{s\in S} M'^s = M'$, and combining these, we obtain the theorem. $\square$

4.2.3 Corollary. *Let M be a holonomic F-$\mathscr{D}^{\mathrm{an}}_{\mathscr{S},\mathbb{Q}}$-module. Then the local Fourier transform $\Phi^{(0,\infty)}(M)$ coincides with $\mathcal{F}^{0,\infty'}(M)$ of Crew* [31, (8.3.1)]. *In particular, the local Fourier transform is a free $\mathcal{R}$-module.*

Proof. Apply the stationary phase formula to the canonical extension of M. □

4.2.4 Corollary. *For any holonomic F-$\mathscr{D}^{\mathrm{an}}_{\mathscr{S},\mathbb{Q}}$-module M, the differential slope of $\Phi^{(0,\infty')}(M)$ is strictly less than 1. Moreover, when M is an $\mathcal{R}$-module, we have*

$$\operatorname{rk}(\Phi^{(0,\infty')}(j_+M)) = \operatorname{rk}(\Phi^{(0,\infty')}(j_!M)) = \operatorname{rk}(M) + \operatorname{irr}(M),$$
$$\operatorname{irr}(\Phi^{(0,\infty')}(j_+M)) = \operatorname{irr}(\Phi^{(0,\infty')}(j_!M)) = \operatorname{irr}(M),$$

using the cohomological functors of 3.1.9.

Proof. Let us see the first claim. Let $\mathscr{M}$ be the canonical extension of $\tau_0^*(M)$. By stationary phase formula, it suffices to see that the slope of $\mathscr{F}_{\mathrm{naive},\pi}(\mathscr{M})|_{\eta_{\infty'}}$ is strictly less than one. For this, apply Corollary 4.1.7.

Let us prove the latter claim. The first equalities hold by Lemma 3.1.10 and its remark. Let us calculate the rank and irregularity of $\Phi^{(0,\infty')}(j_+M)$. Let $\mathscr{M}$ be the canonical extension of $\tau_0^*(j_+M)$. For the rank, apply Corollary 4.1.3. It remains to calculate the irregularity. We put $\mathscr{M}' := \mathscr{F}_{\mathrm{naive},\pi}(\mathscr{M})$. By Proposition 4.1.6(i) and (iii′), we get

$$r(\mathscr{M}') = a_0(\mathscr{M}) = r(\mathscr{M}) - \operatorname{irr}(\mathscr{M}|_{\eta_0}) - r_0(\mathscr{M}),$$
$$r_0(\mathscr{M}) = r(\mathscr{M}) + \operatorname{rk}(\mathscr{M}'|_{\eta_{\infty'}}) - \operatorname{irr}(\mathscr{M}'|_{\eta_{\infty'}}).$$

Since $\operatorname{rk}(\mathscr{M}'|_{\eta_{\infty'}}) = -r(\mathscr{M}')$, we get $\operatorname{irr}(\mathscr{M}|_{\eta_0}) = \operatorname{irr}(\mathscr{M}'|_{\eta_{\infty'}})$. Thus the corollary follows. □

5. Frobenius structures

In this section, we endow the local Fourier transform with Frobenius structure. We define the Frobenius structure using that of geometric Fourier transform. In the first subsection, we show that, with this Frobenius structure, the stationary phase theorem is compatible with Frobenius structures. In the second subsection, we explicitly describe this Frobenius structure in terms of differential operators.

> Throughout this section, we continuously use the assumptions and notation of paragraph 2.4.1.

5.1. Frobenius structures on local Fourier transforms

5.1.1 Definition. For a coherent $\mathscr{D}^{\dagger}_{\widehat{\mathbb{P}},\mathbb{Q}}(\infty)$-module $\mathscr{M}$, recall that we have the canonical isomorphism $\mathscr{F}_{\mathrm{naive},\pi}(\mathscr{M}) \cong \mathscr{H}^1(\mathscr{F}_\pi(\mathscr{M}))$ (cf. 3.2.5.1). Since the geometric Fourier transform is defined by cohomological operators, $\mathscr{F}_\pi$ commutes with Frobenius pull-backs. By transporting structure, we have a canonical isomorphism

$$\epsilon_{\mathscr{M}} \colon F^*_{\widehat{\mathbb{P}}'}(\mathscr{F}_{\mathrm{naive},\pi}(\mathscr{M})^{\sigma}) \xrightarrow{\sim} \mathscr{F}_{\mathrm{naive},\pi}(F^*_{\widehat{\mathbb{P}}}\mathscr{M}^{\sigma}),$$

where $\mathscr{M}^{\sigma}$ is the pull-back by Frobenius morphism. When $\mathscr{M}$ is a coherent F-$\mathscr{D}^{\dagger}_{\widehat{\mathbb{P}},\mathbb{Q}}(\infty)$-module, we define a Frobenius structure on $\mathscr{F}_{\mathrm{naive},\pi}(\mathscr{M})$ using this isomorphism.

5.1.2 Lemma. *There exists an operator $\Upsilon_\pi \in \mathscr{D}^\dagger_{\widehat{\mathbb{P}}',\mathbb{Q}}(\infty')$ such that the following condition holds: let $\mathscr{M}$ be any coherent $\mathscr{D}^\dagger_{\widehat{\mathbb{P}}^{(1)},\mathbb{Q}}(\infty)$-module (which may not possess Frobenius structure). For any $m \in \Gamma(\widehat{\mathbb{P}}^{(1)}, \mathscr{M})$, we have*

$$\epsilon_{\mathscr{M}}(1 \otimes \widehat{m}) = \Upsilon_\pi \cdot \widehat{1 \otimes m}.$$

Moreover, the operator Υ_π is compatible with base changes. We omit π in the notation Υ_π if there is nothing to be confused. Note that Υ is not *unique in general.*

Proof. Let us take an operator Υ_π such that

$$\epsilon_{\mathscr{D}^\dagger_{\widehat{\mathbb{P}},\mathbb{Q}}}(1 \otimes \widehat{1}) = \Upsilon_\pi \cdot (\widehat{1 \otimes 1}).$$

The existence follows easily from the fact that the homomorphism

$$\mathscr{D}^\dagger_{\widehat{\mathbb{P}}',\mathbb{Q}}(\infty') \xrightarrow{\sim} \mathscr{F}_{\mathrm{naive},\pi}(\mathscr{D}^\dagger_{\widehat{\mathbb{P}},\mathbb{Q}}(\infty)) \to \mathscr{F}_{\mathrm{naive},\pi}(F^*\mathscr{D}^\dagger_{\widehat{\mathbb{P}}^{(1)},\mathbb{Q}}(\infty)),$$

where the second homomorphism is induced by the surjection $\mathscr{D}^\dagger_{\widehat{\mathbb{P}},\mathbb{Q}}(\infty) \to F^*\mathscr{D}^\dagger_{\widehat{\mathbb{P}}^{(1)},\mathbb{Q}}(\infty)$, is surjective; in other words, $\widehat{1 \otimes 1}$ is a generator of $\mathscr{F}_{\mathrm{naive},\pi}(F^*\mathscr{D}^\dagger_{\widehat{\mathbb{P}}^{(1)},\mathbb{Q}}(\infty))$ over $\mathscr{D}^\dagger_{\widehat{\mathbb{P}}',\mathbb{Q}}(\infty')$. For $m \in \mathscr{M}$, let us denote by $\rho\colon \mathscr{D}^\dagger_{\widehat{\mathbb{P}},\mathbb{Q}}(\infty) \to \mathscr{M}$ the homomorphism sending one to m. Then by the functoriality of geometric Fourier transform, we get the following commutative diagram:

$$\begin{tikzcd}
F^*(\mathscr{F}_{\mathrm{naive},\pi}(\mathscr{D}^\dagger_{\widehat{\mathbb{P}}^{(1)},\mathbb{Q}}(\infty))) \arrow[r, "F^*\mathscr{F}_{\mathrm{naive}}(\rho)"] \arrow[d, "\epsilon_{\mathscr{D}^\dagger}"'] & F^*(\mathscr{F}_{\mathrm{naive},\pi}(\mathscr{M})) \arrow[d, "\epsilon_{\mathscr{M}}"] \\
\mathscr{F}_{\mathrm{naive},\pi}(F^*\mathscr{D}^\dagger_{\widehat{\mathbb{P}}^{(1)},\mathbb{Q}}(\infty)) \arrow[r, "\mathscr{F}_{\mathrm{naive}}F^*(\rho)"'] & \mathscr{F}_{\mathrm{naive},\pi}(F^*\mathscr{M}).
\end{tikzcd}$$

Thus the lemma follows. □

5.1.3 Definition. Let M be a holonomic F-$\mathscr{D}^{\mathrm{an}}_{\mathscr{S},\mathbb{Q}}$-module. Let $\mathfrak{s} \in \mathbb{A}^1_k(\bar{k})$. We denote by $\mathscr{M}$ the canonical extension of $\sigma^*_{\mathfrak{s}}M$ at $\widetilde{\mathfrak{s}}$. Recall the notation of 2.4.1. Let

$$\widetilde{\alpha}_s\colon \tau'_*\mathscr{F}_{\mathrm{naive},\pi}(\mathscr{M})|_{\eta_\infty} \xrightarrow{\sim} \Phi^{(\mathfrak{s},\infty')}(M)$$

be the isomorphism given by the stationary phase theorem. Here we used abusively the notation τ' and ∞ on $\widehat{\mathbb{P}} \otimes K_s$. Since F^* and τ'_* commute, the isomorphisms $\widetilde{\alpha}_s$ and $\epsilon_{\mathscr{M}}$ induce an isomorphism $\epsilon_{M,\mathfrak{s}}$ as follows

$$\begin{aligned}
\epsilon_{M,\mathfrak{s}}\colon F^*\big(\Phi^{(\mathfrak{s},\infty')}(M)\big) &\xrightarrow[\sim]{F^*(\widetilde{\alpha}_s^{-1})} F^*\big(\tau'_*\mathscr{F}_{\mathrm{naive},\pi}(\mathscr{M})|_{\eta_\infty}\big) \\
&\xrightarrow[\sim]{} \tau'_*\big(F^*\mathscr{F}_{\mathrm{naive},\pi}(\mathscr{M})|_{\eta_\infty}\big) \\
&\xrightarrow[\sim]{\tau'_*(\epsilon_{\mathscr{M}})|_{\eta_\infty}} \tau'_*\big(\mathscr{F}_{\mathrm{naive},\pi}(F^*\mathscr{M})|_{\eta_\infty}\big) \xrightarrow[\sim]{F^*(\widetilde{\alpha}_s)} \Phi^{(\mathfrak{s},\infty')}(F^*M).
\end{aligned}$$

We define the Frobenius structure on $\Phi^{(0,\infty')}(M)$ by composing this isomorphism with the isomorphism of functoriality $\Phi^{(\mathfrak{s},\infty')}(F^*M) \xrightarrow{\sim} \Phi^{(\mathfrak{s},\infty')}(M)$ induced by the Frobenius structure of M.

5.1.4 Lemma. *Let $\mathscr{M}$ be a holonomic F-$\mathscr{D}^{\dagger}_{\widehat{\mathbb{P}},\mathbb{Q}}(\infty)$-module. Let s be a k-rational point of $\mathbb{A}^1_k$, and γ_s be the translation isomorphism of $\widehat{\mathbb{P}}$ sending s to zero. Then we have an isomorphism compatible with Frobenius structures*

$$\mathscr{F}_{\pi}(\gamma_s^!\mathscr{M}) \cong \mathscr{F}_{\pi}(\mathscr{M}) \otimes \mathscr{L}(-s\cdot x').$$

Proof. The proof is formally the same as that of [50, 1.2.3.2] using the Künneth formula 3.1.7, so we leave the details to the reader. □

5.1.5 Lemma. *Let M be a holonomic F-$\mathscr{D}^{\mathrm{an}}_{\mathscr{S},\mathbb{Q}}$-module, and s be a k-rational point of $\widehat{\mathbb{A}}^1_k$. Then we have the following isomorphism compatible with Frobenius structures:*

$$\Phi^{(s,\infty')}(M) \cong \Phi^{(0,\infty')}(M) \otimes L(s).$$

Proof. Let $\mathscr{M}_s$ denote the canonical extension of $\tau_s^*(M)$ at s. By the previous lemma, we know that $\mathscr{F}(\gamma_s^!\mathscr{M}_s) \otimes \mathscr{L}(s\cdot x') \cong \mathscr{F}(\mathscr{M}_s)$. By definition, the stationary phase isomorphism

$$\mathscr{F}(\gamma_s^!\mathscr{M}_s)|_{\eta_{\infty'}} \xrightarrow{\sim} \mathscr{F}^{(0,\infty')}(\gamma_s^!\mathscr{M}_s) \cong \tau'^{*}\Phi^{(0,\infty')}(M)$$

is compatible with the Frobenius structures. Tensoring both sides with $L(s)$, we get the lemma. □

5.1.6 Proposition. *Let M be a holonomic F-$\mathscr{D}^{\mathrm{an}}_{\mathscr{S},\mathbb{Q}}$-module, and $\mathfrak{s} \in \mathbb{A}^1_k(\bar{k})$. Recall the isomorphism*

$$\epsilon_{M,\mathfrak{s}} \colon F^*(\Phi^{(\mathfrak{s},\infty')}(M)) \xrightarrow{\sim} \Phi^{(\mathfrak{s},\infty')}(F^*M)$$

in Definition 5.1.3. Then for any $m \in M$, we have

$$\epsilon(1 \otimes \widehat{m}) = \tau'_*(\Upsilon)\cdot \widehat{1 \otimes m}.$$

Here, Υ is the operator of Lemma 5.1.2.

Proof. Since Frobenius structures are compatible with base changes and Υ_π does not depend on the base as well, we may suppose that $\mathfrak{s}$ is a k-rational point by Lemma 2.4.9. From now on, we assume that $s := \mathfrak{s}$ is a rational point. Let M^{can} be the canonical extension of τ_s^*M at s. Let $\epsilon_{M^{\mathrm{can}}}$ be the unique homomorphism making the following diagram commutative.

$$\begin{tikzcd}
F^*(\mathscr{F}^{(s,\infty')}(M^{\mathrm{can}})) \arrow[r, "\sim"] \arrow[d, dotted, "\epsilon_{M^{\mathrm{can}}}"'] & F^*(\Phi^{(s,\infty')}(M)) \arrow[d, "\epsilon_{M,s}"] \\
\mathscr{F}^{(s,\infty')}(F^*M^{\mathrm{can}}) \arrow[r, "\sim", "a"'] & \Phi^{(s,\infty')}(F^*M)
\end{tikzcd}$$

Here the horizontal isomorphisms are induced by the canonical isomorphism $M \cong M^{\mathrm{can}}|_{\eta_s}$. For any $x' \in \Gamma(\widehat{\mathbb{P}}^{(1)}, M^{\mathrm{can}})$, we get

$$\epsilon_{M^{\mathrm{can}}}(1 \otimes \widehat{x'}) = \tau'_*(\Upsilon)\cdot \widehat{1 \otimes x'} \tag{5.1.6.1}$$

by Lemma 5.1.2. Let N be a differential $\mathcal{A}(]r,1[)$-module with Frobenius structure such that $\mathcal{R}\otimes N\cong\mathscr{F}_{\mathrm{naive},\pi}(M^{\mathrm{can}})|_{\eta_\infty}$, and let $\mathscr{M}^{(m)}$ be a stable $\widehat{\mathscr{D}}^{(m)}_{\widehat{\mathbb{A}}^{(1)},\mathbb{Q}}$-module such that $\mathscr{D}^{\dagger}_{\widehat{\mathbb{A}}^{(1)},\mathbb{Q}}\otimes\mathscr{M}^{(m)}\cong M^{\mathrm{can}}|_{\widehat{\mathbb{A}}^{(1)}}$. We note that $N\hookrightarrow\mathscr{F}^{(s,\infty')}(M^{\mathrm{can}})$. There exist an integer n and isomorphisms

$$\mathcal{A}([\omega_n,1[)\otimes N\xrightarrow{\sim}\mathscr{E}^{(n,\dagger)}_{s,\mathbb{Q}}(\mathscr{M}^{(m)}),\quad \mathcal{A}(\big[\omega_{n+h},1\big[)\otimes F^*N\xrightarrow{\sim}\mathscr{E}^{(n+h,\dagger)}_{s,\mathbb{Q}}(F^*\mathscr{M}^{(m)})$$

inducing the stationary phase isomorphisms. We note that the outer square of the diagram

$$\begin{CD}
F^*\mathcal{A}([\omega_n,1[)\otimes N @>{\sim}>> F^*\mathscr{E}^{(n,\dagger)}_{s}(\mathscr{M}^{(m)}) @>>> F^*(\mathscr{F}^{(s,\infty')}(M^{\mathrm{can}}))\\
@V{\sim}VV @V{\epsilon^{(n)}}VV @VV{\epsilon_{M^{\mathrm{can}}}}V\\
\mathcal{A}(\big[\omega_{n+h},1\big[)\otimes F^*N @>>{\sim}> \mathscr{E}^{(n+h,\dagger)}_{s}(F^*\mathscr{M}^{(m)}) @>>{b}> \mathscr{F}^{(s,\infty')}(F^*M^{\mathrm{can}})
\end{CD}$$

is commutative. Let $\epsilon^{(n)}$ be the unique isomorphism making the above diagram commutative. We have $\epsilon^{(n)}(1\otimes\widehat{x})=\tau'_*(\Upsilon)\cdot\widehat{1\otimes x}$ for any $x\in\mathscr{M}^{(m)}$ by the injectivity and 5.1.6.1. We put $c=a\circ b$. By changing m, we may take $n=m$.

Let $x\in M$. We may assume that $x\in(\widehat{\mathscr{D}}^{(m)}_{s,\mathbb{Q}})^{\mathrm{an}}\otimes\mathscr{M}^{(m)}$ by increasing m if necessary. This element can be seen as an element of $\widehat{\mathscr{E}}^{(m,m')}_{s,\mathbb{Q}}\otimes\mathscr{M}^{(m)}$ with some integer $m'\geqslant m$ (cf. 2.2.5). By Remark 2.2.5 (ii), there exists a sequence $\{1\otimes x_k\}$ in $\mathrm{Im}(\mathscr{M}^{(m)}\to\widehat{\mathscr{E}}^{(m,m')}_{s,\mathbb{Q}}\otimes\mathscr{M}^{(m)})$ with $x_k\in\mathscr{M}^{(m)}$ which converges to x in $\widehat{\mathscr{E}}^{(m,m')}_{s,\mathbb{Q}}\otimes\mathscr{M}^{(m)}$ using the topology induced by the $\widehat{\mathscr{E}}^{(m,m')}_{s,\mathbb{Q}}$-module structure. Consider the topology induced by the finite $K_{\widehat{\mathbb{A}}^{(1)}}\{\partial\}^{(m,m')}$-module structure. By Lemma 1.2.3, these topologies are equivalent, and we get that the same sequence converges to x also in this topology. Since

$$F^*\widehat{\mathscr{E}}^{(m,m')}_{s,\mathbb{Q}}(\mathscr{M}^{(m)})\cong K_{\widehat{\mathbb{A}}}\{\partial\}^{(m+h,m'+h)}\otimes_{K_{\widehat{\mathbb{A}}^{(1)}}\{\partial\}^{(m,m')}}\widehat{\mathscr{E}}^{(m,m')}_{s,\mathbb{Q}}(\mathscr{M}^{(m)})$$

by definition, the sequence $\{1\otimes(1\otimes x_k)\}$ in $F^*\widehat{\mathscr{E}}^{(m,m')}_{s,\mathbb{Q}}(\mathscr{M}^{(m)})$ converges to the element $1\otimes x$ using the $K_{\widehat{\mathbb{A}}}\{\partial\}^{(m+h,m'+h)}$-module topology. Since $\epsilon^{(m)}$ is a homomorphism of finite $K_{\widehat{\mathbb{A}}}\{\partial\}^{(m+h,\dagger)}$-modules, it is in particular a continuous homomorphism of topological modules over the noetherian Banach algebra $K_{\widehat{\mathbb{A}}}\{\partial\}^{(m+h,\dagger)}$. Since the topology is separated, we get

$$\begin{aligned}\epsilon^{(m)}(1\otimes\widehat{x})&=\lim_{i\to\infty}\epsilon^{(m)}(1\otimes\widehat{x_i})=\lim_{i\to\infty}\tau'_*(\Upsilon)\cdot(1\otimes x_i)^{\wedge}\\&=\tau'_*(\Upsilon)\cdot\lim_{i\to\infty}(1\otimes x_i)^{\wedge}=\tau'_*(\Upsilon)\cdot(1\otimes x)^{\wedge}.\end{aligned}$$

Now, we get

$$\epsilon_{M,s}(1\otimes\widehat{x})=c(\epsilon^{(m)}(1\otimes\widehat{x}))=c(\tau'_*(\Upsilon)(1\otimes x)^{\wedge})=\tau'_*(\Upsilon)((1\otimes x)^{\wedge})$$

and the proposition follows. □

5.1.7 Definition. Let $\mathscr{M}$ be a holonomic F-$\mathscr{D}^{\dagger}_{\widehat{\mathbb{P}},\mathbb{Q}}(\infty)$-module. Let s be a singularity of $\mathscr{M}$ in $\widehat{\mathbb{A}}$. Take a geometric point $\mathfrak{s} \in \mathbb{A}^1_k(\bar{k})$ sitting over s. We define the Frobenius structure on $\mathscr{F}^{(s,\infty')}(\mathscr{M})$ by using the canonical isomorphism of Lemma 2.4.10

$$\mathscr{F}^{(s,\infty')}(\mathscr{M}) \cong \mathrm{Res}^{K_s}_{K}\bigl(\Phi^{(\mathfrak{s},\infty')}(\tau_{s*}\mathscr{M}|_{S_s})\bigr).$$

The Frobenius structure is well-defined since it does not depend on the choice of $\mathfrak{s}$ by Proposition 5.1.6.

5.1.8 Theorem. *The regular stationary phase isomorphism 4.2.2.1 is compatible with Frobenius structure.*

Proof. To show this, it suffices to show that the following diagram is commutative.

$$\begin{tikzcd}
F^*\mathscr{F}_{\mathrm{naive},\pi}(\mathscr{M})|_{\eta_\infty} \arrow[r, "\sim"] \arrow[d, "\sim"'] & \bigoplus_{s\in S}\tau'^*F^*\mathscr{F}^{(s,\infty)}(\mathscr{M}) \arrow[d, "\sim"] \\
\mathscr{F}_{\mathrm{naive},\pi}(F^*\mathscr{M})|_{\eta_\infty} \arrow[r, "\sim"'] & \bigoplus_{s\in S}\tau'^*\mathscr{F}^{(s,\infty)}(F^*\mathscr{M})
\end{tikzcd}$$

The left vertical arrow is defined by the Frobenius structure of geometric Fourier transform, and the right vertical arrow by Definition 5.1.7. To show that it is commutative, it suffices to show the commutativity for $1 \otimes \widehat{m} \in F^*\mathscr{F}_{\mathrm{naive},\pi}(\mathscr{M})$ for any $m \in \mathscr{M}$. This follows from the description of the vertical isomorphisms in terms of the operator Υ given in Lemma 5.1.2 and Proposition 5.1.6. $\square$

5.2. Explicit calculations of the Frobenius structures on Fourier transforms

To calculate the Frobenius structure of Fourier transforms concretely, the results of the last subsection imply that we only need to determine the differential operator Υ. To calculate this, it suffices to calculate the isomorphism

$$\Phi := \epsilon_{\mathscr{D}^{\dagger}_{\widehat{\mathbb{P}},\mathbb{Q}}(\infty)} : F^*(\mathscr{F}_{\mathrm{naive},\pi}(\mathscr{D}^{\dagger}_{\widehat{\mathbb{P}}^{(1)},\mathbb{Q}}(\infty))) \xrightarrow{\sim} \mathscr{F}_{\mathrm{naive},\pi}(F^*\mathscr{D}^{\dagger}_{\widehat{\mathbb{P}}^{(1)},\mathbb{Q}}(\infty))$$

concretely, which is the goal of this subsection.

5.2.1. Recall the notation of 2.4.1, and consider the following diagram.

$$\begin{tikzcd}
 & \widehat{\mathbb{P}}'' \arrow[dl] \arrow[dr] \arrow[d, "F_{\widehat{\mathbb{P}}''}"] & \\
\widehat{\mathbb{P}} \arrow[d, "F_{\widehat{\mathbb{P}}}"'] & \widehat{\mathbb{P}}''^{(1)} \arrow[dl] \arrow[dr] & \widehat{\mathbb{P}}' \arrow[d, "F_{\widehat{\mathbb{P}}'}"] \\
\widehat{\mathbb{P}}^{(1)} & & \widehat{\mathbb{P}}'^{(1)}
\end{tikzcd}$$

$x \quad y \quad x' \quad y'$

Here, $^{(1)}$ means $\otimes_{R,\sigma}R$, and small letters x, y, x', y' denote the coordinates. The middle vertical morphism $F_{\widehat{\mathbb{P}}''}$ is defined by sending y and y' to x^q and x'^q respectively. By the definition of the morphisms, we note that the diagram is commutative.

To proceed, we will review the construction of the fundamental isomorphism

$$\mathscr{H}^1\mathscr{F}_\pi(\mathscr{D}^\dagger_{\widehat{\mathbb{P}},\mathbb{Q}}(\infty)) \xrightarrow{\sim} \mathscr{F}_{\mathrm{naive},\pi}(\mathscr{D}^\dagger_{\widehat{\mathbb{P}},\mathbb{Q}}(\infty))$$

of Noot-Huyghe 3.2.5.1. Recall that, by definition,

$$p^!(-) = \mathscr{D}^\dagger_{\widehat{\mathbb{P}}''\to\widehat{\mathbb{P}},\mathbb{Q}}(Z) \otimes^{\mathbb{L}}_{p^{-1}\mathscr{D}^\dagger_{\widehat{\mathbb{P}},\mathbb{Q}}(\infty)} (-)[1], \quad p'_+(-) = Rp'_*\left(\mathscr{D}^\dagger_{\widehat{\mathbb{P}}'\leftarrow\widehat{\mathbb{P}}'',\mathbb{Q}}(Z) \otimes^{\mathbb{L}}_{\mathscr{D}^\dagger_{\widehat{\mathbb{P}}'',\mathbb{Q}}(Z)} (-)\right).$$

Consider the Spencer resolution [59, 4.2.1]:

$$\mathscr{D}^\dagger_{\widehat{\mathbb{P}}'',\mathbb{Q}}(Z) \otimes \bigwedge^{\bullet} \mathscr{T}_{\widehat{\mathbb{P}}''/\widehat{\mathbb{P}}} \to \mathscr{D}^\dagger_{\widehat{\mathbb{P}}''\to\widehat{\mathbb{P}},\mathbb{Q}}(Z).$$

Here $\mathscr{T}_{\widehat{\mathbb{P}}''/\widehat{\mathbb{P}}}$ denotes the relative tangent bundle of $\widehat{\mathbb{P}}''$ over $\widehat{\mathbb{P}}$. She showed that we can calculate $\mathscr{F}(\mathscr{D}^\dagger_{\widehat{\mathbb{P}},\mathbb{Q}}(\infty))$ using this resolution, namely, there exists an isomorphism

$$\mathscr{F}(\mathscr{D}^\dagger_{\widehat{\mathbb{P}},\mathbb{Q}}(\infty)) \cong p'_*\left(\mathscr{D}^\dagger_{\widehat{\mathbb{P}}'\leftarrow\widehat{\mathbb{P}}'',\mathbb{Q}}(Z) \otimes_{\mathscr{D}^\dagger_{\widehat{\mathbb{P}}'',\mathbb{Q}}(Z)} \left(\mathscr{L}_{\pi,\mu} \otimes_{\mathcal{O}_{\widehat{\mathbb{P}}'',\mathbb{Q}}(Z)} \mathscr{D}^\dagger_{\widehat{\mathbb{P}}'',\mathbb{Q}}(Z) \otimes \bigwedge^{\bullet} \mathscr{T}_{\widehat{\mathbb{P}}''/\widehat{\mathbb{P}}}\right)\right).$$

Then, she defined a homomorphism

$$p'_*(\mathscr{D}^\dagger_{\widehat{\mathbb{P}}'\leftarrow\widehat{\mathbb{P}}'',\mathbb{Q}}(Z) \otimes_{\mathscr{D}^\dagger_{\widehat{\mathbb{P}}'',\mathbb{Q}}(Z)} (\mathscr{L}_{\pi,\mu} \otimes_{\mathcal{O}_{\widehat{\mathbb{P}}'',\mathbb{Q}}(Z)} \mathscr{D}^\dagger_{\widehat{\mathbb{P}}'',\mathbb{Q}}(Z))) \to \mathscr{F}_{\mathrm{naive},\pi}(\mathscr{D}^\dagger_{\widehat{\mathbb{P}},\mathbb{Q}}(\infty)) \tag{5.2.1.1}$$

and showed that this factors through the geometric Fourier transform. Let us recall how 5.2.1.1 is defined. We identify

$$\mathscr{D}^\dagger_{\widehat{\mathbb{P}}'\leftarrow\widehat{\mathbb{P}}'',\mathbb{Q}}(Z) = \omega^{-1}_{\widehat{\mathbb{P}}'} \otimes (\mathcal{O}_{\widehat{\mathbb{P}}''} \otimes \mathscr{D}^\dagger_{\widehat{\mathbb{P}}'',\mathbb{Q}}(Z) \otimes \omega_{\widehat{\mathbb{P}}''}). \tag{5.2.1.2}$$

The homomorphism 5.2.1.1 sends $\big((\mathrm{d}x')^\vee \otimes 1 \otimes 1 \otimes (\mathrm{d}x \wedge \mathrm{d}x')\big) \otimes (e \otimes P)$ to $\widehat{P}$ where e is the canonical section of $\mathscr{L}_{\pi,\mu}$ (cf. 3.2.1). To verify that this defines a homomorphism, see [59, 5.2.1, *etc.*].

Before starting the calculation, we introduce the Dwork operator. Let $\mathscr{X}$ be a smooth formal scheme possessing a system of global coordinates $\{x_1,\dots,x_d\}$. Let $\{x'_1,\dots,x'_d\}$ be the system of global coordinates of $\mathscr{X}^{(1)}$ induced by pulling-back $\{x_1,\dots,x_d\}$. Assume we have a lifting of the relative Frobenius morphism $\mathscr{X} \to \mathscr{X}^{(1)}$ sending x'_i to x_i^q. Then we put

$$H_{\mathscr{X}} := \frac{1}{q^d} \prod_{1\leqslant i\leqslant d} \sum_{\zeta^q=1} \sum_{k\geqslant 0} (\zeta-1)^k x_i^k \partial_i^{[k]}$$

in $\Gamma(\mathscr{X}, \widehat{\mathscr{D}}^{(h)}_{\mathscr{X},\mathbb{Q}})$. If there is nothing to be confused, we denote $H_{\mathscr{X}}$ by H. We note that even if $\zeta \notin K$, the operator is defined over K, and do not need to extend K to define this operator. For the details, we refer to [42].

By applying $F^*_{\widehat{\mathbb{P}}'}$ to 5.2.1.1, we get the homomorphism

$$F^*_{\widehat{\mathbb{P}}'} p'_*\left(\mathscr{D}^\dagger_{\widehat{\mathbb{P}}'^{(1)}\leftarrow\widehat{\mathbb{P}}''^{(1)},\mathbb{Q}}(Z) \otimes_{\mathscr{D}^\dagger_{\widehat{\mathbb{P}}''^{(1)},\mathbb{Q}}(Z)} (\mathscr{L}^{(1)}_{\pi,\mu} \otimes_{\mathcal{O}_{\widehat{\mathbb{P}}''^{(1)},\mathbb{Q}}(Z)} \mathscr{D}^\dagger_{\widehat{\mathbb{P}}''^{(1)},\mathbb{Q}})\right) \to F^*_{\widehat{\mathbb{P}}'}\mathscr{F}_{\mathrm{naive},\pi}(\mathscr{D}^\dagger_{\widehat{\mathbb{P}}^{(1)},\mathbb{Q}}),$$

where $\mathscr{L}^{(1)}_{\pi,\mu}$ on $\widehat{\mathbb{P}}''^{(1)}$ denotes the base change of $\mathscr{L}_{\pi,\mu}$. From the next paragraph, we start to calculate the Frobenius commutation homomorphism on the source of the homomorphism. For this, we always use the identification 5.2.1.2.

5.2.2. Let us calculate the canonical homomorphism of sheaves on $\widehat{\mathbb{P}}'$

$$\begin{aligned}\phi\colon F^*_{\widehat{\mathbb{P}}'} p'_*\big(\mathscr{D}^\dagger_{\widehat{\mathbb{P}}'^{(1)}\leftarrow\widehat{\mathbb{P}}''^{(1)},\mathbb{Q}}(Z)\otimes_{\mathscr{D}^\dagger_{\widehat{\mathbb{P}}''^{(1)},\mathbb{Q}}(Z)}(\mathscr{L}^{(1)}_{\pi,\mu}\otimes_{\mathcal{O}_{\widehat{\mathbb{P}}''^{(1)},\mathbb{Q}}(Z)}\mathscr{D}^\dagger_{\widehat{\mathbb{P}}''^{(1)},\mathbb{Q}}(Z))\big)\\ \to p'_*\mathscr{D}^\dagger_{\widehat{\mathbb{P}}'\leftarrow\widehat{\mathbb{P}}'',\mathbb{Q}}(Z)\otimes_{\mathscr{D}^\dagger_{\widehat{\mathbb{P}}'',\mathbb{Q}}(Z)}\big(\mathscr{L}_{\pi,\mu}\otimes_{\mathcal{O}_{\widehat{\mathbb{P}}'',\mathbb{Q}}(Z)}\mathscr{D}^\dagger_{\widehat{\mathbb{P}}''\to\widehat{\mathbb{P}},\mathbb{Q}}(Z)\otimes F^*_{\widehat{\mathbb{P}}}\mathscr{D}^\dagger_{\widehat{\mathbb{P}},\mathbb{Q}}(\infty)\big).\end{aligned}$$

In this paragraph, for simplicity, we denote $\mathscr{D}^\dagger_{\widehat{\mathbb{P}},\mathbb{Q}}(Z)$, $\mathscr{D}^\dagger_{\widehat{\mathbb{P}}\leftarrow\widehat{\mathbb{P}}'',\mathbb{Q}}(Z)$ *etc.* by $\mathscr{D}_{\widehat{\mathbb{P}}}$, $\mathscr{D}_{\widehat{\mathbb{P}}\leftarrow\widehat{\mathbb{P}}''}$ *etc.*, and we identify sheaves and its global sections.

In the following, we will compute $\phi(\alpha_0)$ where $\alpha_0 := 1\otimes\big((dy')^\vee\otimes 1\otimes 1\otimes(dy\wedge dy')\big)\otimes(e\otimes 1)$ is a global section of the source of ϕ. First, we get an isomorphism

$$F^*_{\widehat{\mathbb{P}}'}\big(\mathscr{D}_{\widehat{\mathbb{P}}'^{(1)}\leftarrow\widehat{\mathbb{P}}''^{(1)}}\otimes_{\mathscr{D}_{\widehat{\mathbb{P}}''^{(1)}}}(\mathscr{L}^{(1)}_{\pi,\mu}\otimes_{\mathcal{O}_{\widehat{\mathbb{P}}''^{(1)}}}\mathscr{D}_{\widehat{\mathbb{P}}''^{(1)}})\big)\xrightarrow{\sim}\mathscr{D}_{\widehat{\mathbb{P}}'\leftarrow\widehat{\mathbb{P}}''}\otimes_{\mathscr{D}_{\widehat{\mathbb{P}}''}}F^*_{\widehat{\mathbb{P}}''}\big(\mathscr{L}^{(1)}_{\pi,\mu}\otimes_{\mathcal{O}_{\widehat{\mathbb{P}}''^{(1)}}}\mathscr{D}_{\widehat{\mathbb{P}}''^{(1)}}\big).$$

By [4, Proposition 2.5], the element α_0 is sent to

$$\alpha_1 := \big((dx')^\vee\otimes 1\otimes Hx'^{-(q-1)}\otimes(dx\wedge dx')\big)\otimes x^{q-1}x'^{q-1}(e\otimes 1).$$

Now, we get an isomorphism

$$\mathscr{D}_{\widehat{\mathbb{P}}'\leftarrow\widehat{\mathbb{P}}''}\otimes_{\mathscr{D}_{\widehat{\mathbb{P}}''}}F^*_{\widehat{\mathbb{P}}''}\big(\mathscr{L}^{(1)}_{\pi,\mu}\otimes_{\mathcal{O}_{\widehat{\mathbb{P}}''^{(1)}}}\mathscr{D}_{\widehat{\mathbb{P}}''^{(1)}}\big)\xrightarrow{\sim}\mathscr{D}_{\widehat{\mathbb{P}}'\leftarrow\widehat{\mathbb{P}}''}\otimes_{\mathscr{D}_{\widehat{\mathbb{P}}''}}\big(\mathscr{L}_{\pi,\mu}\otimes_{\mathcal{O}_{\widehat{\mathbb{P}}''}}F^*_{\widehat{\mathbb{P}}''}\mathscr{D}_{\widehat{\mathbb{P}}''^{(1)}}\big)$$

using the Frobenius structure of $\mathscr{L}_{\pi,\mu}$. This Frobenius structure $F^*\mathscr{L}_{\pi,\mu}\xrightarrow{\sim}\mathscr{L}_{\pi,\mu}$ sends $1\otimes e$ to $\exp(\pi((xx')-(xx')^q))\cdot e$ as written in [9, (2.12.1)][12]. Using this, α_1 is sent by this isomorphism to

$$\begin{aligned}\alpha_2 := \underbrace{\big((dx'_\bullet)^\vee\otimes 1\otimes Hx'^{-(q-1)}(dx_\bullet\wedge dx'_\bullet)\big)}_{=:A}\\ \otimes x^{q-1}x'^{q-1}\big(\underbrace{\exp(\pi((xx')-(xx')^q))\cdot e}_{=:E}\otimes(1\otimes 1)\big).\end{aligned}$$

Then we get a homomorphism

$$\mathscr{D}_{\widehat{\mathbb{P}}'\leftarrow\widehat{\mathbb{P}}''}\otimes_{\mathscr{D}_{\widehat{\mathbb{P}}''}}\big(\mathscr{L}_{\pi,\mu}\otimes_{\mathcal{O}_{\widehat{\mathbb{P}}''}}F^*_{\widehat{\mathbb{P}}''}\mathscr{D}_{\widehat{\mathbb{P}}''^{(1)}}\big)\to\mathscr{D}_{\widehat{\mathbb{P}}'\leftarrow\widehat{\mathbb{P}}''}\otimes_{\mathscr{D}_{\widehat{\mathbb{P}}''}}\big(\mathscr{L}_{\pi,\mu}\otimes_{\mathcal{O}_{\widehat{\mathbb{P}}''}}F^*_{\widehat{\mathbb{P}}''}\mathscr{D}_{\widehat{\mathbb{P}}''^{(1)}\to\widehat{\mathbb{P}}^{(1)}}\big),$$

which sends α_2 to $\alpha_3 := A\otimes x^{q-1}x'^{q-1}(E\otimes(1\otimes 1\otimes 1))$. Then we have an isomorphism

$$\mathscr{D}_{\widehat{\mathbb{P}}'\leftarrow\widehat{\mathbb{P}}''}\otimes_{\mathscr{D}_{\widehat{\mathbb{P}}''}}\big(\mathscr{L}_{\pi,\mu}\otimes_{\mathcal{O}_{\widehat{\mathbb{P}}''}}F^*_{\widehat{\mathbb{P}}''}\mathscr{D}_{\widehat{\mathbb{P}}''^{(1)}\to\widehat{\mathbb{P}}^{(1)}}\big)\xrightarrow{\sim}\mathscr{D}_{\widehat{\mathbb{P}}'\leftarrow\widehat{\mathbb{P}}''}\otimes_{\mathscr{D}_{\widehat{\mathbb{P}}''}}\big(\mathscr{L}_{\pi,\mu}\otimes_{\mathcal{O}_{\widehat{\mathbb{P}}''}}\mathscr{D}_{\widehat{\mathbb{P}}''\to\widehat{\mathbb{P}}}\otimes F^*_{\widehat{\mathbb{P}}}\mathscr{D}_{\widehat{\mathbb{P}}}\big),$$

which sends α_3 to $\alpha_4 := A\otimes x^{q-1}x'^{q-1}(E\otimes(1\otimes 1)\otimes(1\otimes 1))$. Summing up, the homomorphism ϕ sends

$$1\otimes\big((dx'_\bullet)^\vee\otimes 1\otimes 1\otimes(dx_\bullet\wedge dx'_\bullet)\big)\otimes(e\otimes 1)\mapsto A\otimes x^{q-1}x'^{q-1}(E\otimes(1\otimes 1)\otimes(1\otimes 1)).$$

[12] Note that in [9], our π is equal to $-\pi$ in their notation, and this is why we get $(xx')-(xx')^q$ instead of $(xx')^q-(xx')$.

5.2.3. Let us finish the calculation of Φ. Consider the following commutative diagram

$$\begin{tikzcd}
p'_*\mathscr{D}^\dagger_{\widehat{\mathbb{P}}'\leftarrow\widehat{\mathbb{P}}''}\otimes_{\mathscr{D}^\dagger_{\widehat{\mathbb{P}}''}}\big(\mathscr{L}_{\pi,\mu}\otimes_{\mathcal{O}_{\widehat{\mathbb{P}}''}}\mathscr{D}^\dagger_{\widehat{\mathbb{P}}''\to\widehat{\mathbb{P}}}\big) \arrow[r, "\beta"] \arrow[d, "\alpha"'] & \mathscr{F}_{\mathrm{naive},\pi}(\mathscr{D}^\dagger_{\widehat{\mathbb{P}}}(\infty)) \arrow[d, "\delta"] \\
p'_*\mathscr{D}^\dagger_{\widehat{\mathbb{P}}'\leftarrow\widehat{\mathbb{P}}''}\otimes_{\mathscr{D}^\dagger_{\widehat{\mathbb{P}}''}}\big(\mathscr{L}_{\pi,\mu}\otimes_{\mathcal{O}_{\widehat{\mathbb{P}}''}}\mathscr{D}^\dagger_{\widehat{\mathbb{P}}''\to\widehat{\mathbb{P}}}\otimes F^*_{\widehat{\mathbb{P}}}\mathscr{D}^\dagger_{\widehat{\mathbb{P}}^{(1)}}(\infty)\big) \arrow[r, "\gamma"'] & \mathscr{F}_{\mathrm{naive},\pi}(F^*_{\widehat{\mathbb{P}}}\mathscr{D}^\dagger_{\widehat{\mathbb{P}}^{(1)}}(\infty)),
\end{tikzcd}$$

where we have omitted $\mathbb{Q}$ in subscripts and (Z) to save space. The homomorphisms α and δ are the canonical homomorphisms induced by the homomorphism $\mathscr{D}^\dagger_{\widehat{\mathbb{P}},\mathbb{Q}}(\infty)\to F^*_{\widehat{\mathbb{P}}}\mathscr{D}^\dagger_{\widehat{\mathbb{P}}^{(1)},\mathbb{Q}}(\infty)$, β is nothing but 5.2.1.1, and γ is the homomorphism of Noot-Huyghe 3.2.5.1. By the computation of the last paragraph, we have

$$\Phi(1\otimes\widehat{1})=\gamma(\alpha_4).$$

Since these sheaves are $\mathscr{D}^\dagger_{\widehat{\mathbb{P}}',\mathbb{Q}}(\infty')$-modules, we identify the sheaves with their global sections. Since $\alpha\big(A\otimes x^{q-1}x'^{q-1}(E\otimes(1\otimes 1))\big)=\alpha_4$, all we have to calculate is $(\delta\circ\beta)\big(A\otimes x^{q-1}x'^{q-1}(E\otimes(1\otimes 1))\big)$. Let

$$\exp(\pi(t-t^q))=\sum_{n\geqslant 0}\alpha_n t^n.$$

Then we get

$$\begin{aligned}
&\beta\Big(\big((\mathrm{d}x')^\vee\otimes 1\otimes Hx'^{-(q-1)}\otimes(\mathrm{d}x\wedge\mathrm{d}x')\big)\otimes x^{q-1}x'^{q-1}\\
&\qquad\times\big(\exp(\pi((xx')-(xx')^q))\cdot e\otimes(1\otimes 1)\big)\Big)\\
&\quad=\beta\left(\big((\mathrm{d}x')^\vee\otimes 1\otimes Hx'^{-(q-1)}\otimes(\mathrm{d}x\wedge\mathrm{d}x')\big)\otimes\left(\sum_{n\geqslant 0}\alpha_n(xx')^n x^{q-1}x'^{q-1}\right)\big(e\otimes(1\otimes 1)\big)\right)\\
&\quad=\beta\left(\sum_{n\geqslant 0}\big((\mathrm{d}x')^\vee\otimes 1\otimes(x'^{q-1}x'^n\alpha_n Hx'^{-(q-1)})\otimes(\mathrm{d}x\wedge\mathrm{d}x')\big)\otimes\big(e\otimes(1\otimes x^n x^{q-1})\big)\right)\\
&\quad=\left(\big(x'^{q-1}Hx'^{-(q-1)}\big)^t\cdot\sum_{n\geqslant 0}\alpha_n x'^n\right)\\
&\qquad\cdot\beta\Big(\big((\mathrm{d}x')^\vee\otimes 1\otimes 1\otimes(\mathrm{d}x\wedge\mathrm{d}x')\big)\otimes\big(e\otimes(1\otimes x^n x^{q-1})\big)\Big)\\
&\quad=\left(\big(x'^{q-1}Hx'^{-(q-1)}\big)^t\cdot\sum_{n\geqslant 0}\alpha_n x'^n\right)\cdot(x^n x^{q-1})^\wedge\\
&\quad=\big(x'^{q-1}Hx'^{-(q-1)}\big)^t\cdot\sum_{n\geqslant 0}\alpha_n x'^n\left(\frac{-\partial'}{\pi}\right)^n\left(\frac{-\partial'}{\pi}\right)^{q-1}\cdot\widehat{1}.
\end{aligned}$$

Summing up, we get the following theorem.

5.2.4 Theorem. *Let $\mathscr{M}$ be a coherent $\mathscr{D}^{\dagger}_{\widehat{\mathbb{P}}^{(1)},\mathbb{Q}}(\infty)$-module. We write*

$$\exp(\pi(t - t^q)) = \sum_{n \geqslant 0} \alpha_n t^n$$

with $\alpha_n \in K$. The canonical isomorphism $\Phi\colon F^*_{\widehat{\mathbb{P}}'}(\mathscr{F}_{\mathrm{naive},\pi}(\mathscr{M})) \xrightarrow{\sim} \mathscr{F}_{\mathrm{naive},\pi}(F^*_{\widehat{\mathbb{P}}}\mathscr{M})$ can be described as follows: for any $m \in \Gamma(\widehat{\mathbb{P}}^{(1)}, \mathscr{M})$, we have

$$\Phi(1 \otimes \widehat{m}) = \left(x'^{q-1} \cdot H_{\widehat{\mathbb{P}}'} \cdot x'^{-(q-1)}\right)^t \cdot \sum_{n \geqslant 0} \alpha_n x'^n \left(\frac{-\partial'}{\pi}\right)^n \left(\frac{-\partial'}{\pi}\right)^{q-1} \cdot (1 \otimes m)^{\wedge}.$$

6. A key exact sequence

To show Laumon's formulas, one of the key point was to use the exact sequence appearing in the proof of [50, 3.4.2]. This exact sequence was deduced from an exact sequence connecting nearby cycles and vanishing cycles. Since our definition of local Fourier transforms does not use vanishing cycles, we need some arguments to acquire an analogous exact sequence, which is the main goal of this section.

6.1. Commutation of Frobenius

In this subsection, we show a commutativity result of Frobenius pull-back and microlocalization. This result is used to define a Frobenius structure on microlocalizations defined in the next subsection.

6.1.1. Let $\mathscr{X}$ be an affine formal curve over R. We consider the situation in paragraph 3.1.1. We moreover suppose that $\mathscr{X}$ and $\mathscr{X}'$ possess a global coordinate denoted by x and x' respectively, and that the relative Frobenius homomorphism $\mathscr{X} \to \mathscr{X}'$ sends x' to x^q. For any smooth formal scheme, we can choose such x and x' locally, with the same property. We denote by ∂ and ∂' the differential operators corresponding to x and x' respectively. We denote by X_i and X'_i the reduction of $\mathscr{X}$ and $\mathscr{X}'$ over R_i as usual.

Lemma. *Let i and m be non-negative integers. Let $\Phi\colon \mathscr{D}^{(m+h)}_{X_i} \to F^*\mathscr{D}^{(m)}_{X'_i}$ be the canonical homomorphism of Berthelot [15, 2.5.2.3]. Then $(\partial'^{\langle p^m \rangle_{(m)}})^{p^{i+1}}$ is in the centre of $\mathscr{D}^{(m)}_{X'_i}$ and, for any positive integer l,*

$$\Phi((\partial^{\langle p^{m+h} \rangle_{(m+h)}})^{lp^{i+1}}) = 1 \otimes (\partial'^{\langle p^m \rangle_{(m)}})^{lp^{i+1}}. \qquad (6.1.1.1)$$

Proof. To have lighter notation we will put, for every n and m, $\partial^{\langle p^m \rangle^n_{(m)}} := (\partial^{\langle p^m \rangle_{(m)}})^n$. The case $i = 0$ of the first statement is proven in [13, 2.2.6], and the general case goes similarly.

To finish the proof we show the second claim by induction on i. We put $N_i := p^{i+1}$. When $i = 0$, it has already been verified in [15, 2.2.4.3]. Suppose that the statement is true for i and let us prove it for $i + 1$. By induction hypothesis, for any l, we may write

$$\Phi(\partial^{\langle p^{m+h} \rangle^{lN_i}_{(m+h)}}) = 1 \otimes \partial'^{\langle p^m \rangle^{lN_i}_{(m)}} + \varpi^{i+1} \sum_r f_r^{(l)} \otimes \partial'^{\langle r \rangle_{(m)}} \qquad (6.1.1.2)$$

in $\mathscr{D}^{(m)}_{X'_{i+1}}$, for some $f_r^{(l)} \in \mathcal{O}_{X'_{i+1}}$. By the first part of the lemma, $\partial^{\langle p^{m+h}\rangle_{(m+h)}^{N_i}}$ is contained in the centre of $\mathscr{D}^{(m+h)}_{X_{i+1}}$. For any integer $j > 0$, we get

$$\begin{aligned}
&\partial^{\langle p^{m+h}\rangle_{(m+h)}^{lN_i}} \cdot \Big(1 \otimes \partial'^{\langle p^m\rangle_{(m)}^{lN_i j}} + j\varpi^{i+1}\sum_r f_r^{(l)} \otimes \partial'^{\langle r\rangle_{(m)}}\partial'^{\langle p^m\rangle_{(m)}^{lN_i(j-1)}}\Big)\\
&= \Big(1 \otimes \partial'^{\langle p^m\rangle_{(m)}^{lN_i}}\partial'^{\langle Np^m\rangle_{(m)}^{lN_i j}} + \varpi^{i+1}\sum_r f_r^{(l)} \otimes \partial'^{\langle r\rangle_{(m)}}\partial'^{\langle p^m\rangle_{(m)}^{lN_i j}}\Big)\\
&\quad + j\varpi^{i+1}\sum_r f_r^{(l)} \otimes \partial'^{\langle r\rangle_{(m)}}\partial'^{\langle p^m\rangle_{(m)}^{lN_i j}}\\
&= 1 \otimes \partial'^{\langle p^m\rangle_{(m)}^{lN_i(j+1)}} + (j+1)\varpi^{i+1}\sum_r f_r^{(l)} \otimes \partial'^{\langle r\rangle_{(m)}}\partial'^{\langle p^m\rangle_{(m)}^{lN_i j}}, \qquad (6.1.1.3)
\end{aligned}$$

where $\cdot$ in the first line stand for the left action of $\mathscr{D}^{(m+h)}_{X_{i+1}}$ on $F^*\mathscr{D}^{(m)}_{X'_{i+1}}$. Thus, we get

$$\begin{aligned}
\Phi(\partial^{\langle p^{m+h}\rangle_{(m+h)}^{plN_i}}) &= \partial^{\langle p^{m+h}\rangle_{(m+h)}^{(p-1)lN_i}} \cdot \Big(1 \otimes \partial'^{\langle p^m\rangle_{(m)}^{lN_i}} + \varpi^{i+1}\sum_r f_r^{(l)} \otimes \partial'^{\langle r\rangle_{(m)}}\Big)\\
&= \Big(1 \otimes \partial'^{\langle p^m\rangle_{(m)}^{plN_i}} + p\varpi^{i+1}\sum_r f_r^{(l)} \otimes \partial'^{\langle r\rangle_{(m)}}\partial'^{\langle p^m\rangle_{(m)}^{(p-1)lN_i}}\Big)\\
&= 1 \otimes \partial'^{\langle p^m\rangle_{(m)}^{plN_i}},
\end{aligned}$$

where for the second equality, we used 6.1.1.3 $(p-1)$-times. Thus, the equality 6.1.1.1 holds for $i+1$, and the lemma follows. □

6.1.2. We keep previous notation. Let i and m be non-negative integers. First, let us construct a canonical homomorphism

$$\mathscr{E}^{(m+h)}_{X_i} \to F^*\mathscr{E}^{(m)}_{X'_i} \qquad (6.1.2.1)$$

compatible with Φ, where F^* denotes $\pi^{-1}\mathcal{O}_{X_i}\otimes_{\pi^{-1}\mathcal{O}_{X'_i}}$. Let N be the exponent p^{i+1} in Lemma 6.1.1. Let S_{m+h} be the multiplicative system of $\mathscr{D}^{(m+h)}_{X_i}$ generated by $(\partial^{\langle p^{m+h}\rangle_{(m+h)}})^N$, and S_m be that of $\mathscr{D}^{(m)}_{X'_i}$ generated by $(\partial'^{\langle p^m\rangle_{(m)}})^N$, which is also contained in the centre of $\mathscr{D}^{(m)}_{X'_i}$. For a (non-commutative) ring A, we denote by $A[\zeta]$ the ring of polynomials in the variable ζ such that ζ is in the centre. We know that $S^{-1}_{m+h}\mathscr{D}^{(m+h)}_{X_i} \cong \mathscr{D}^{(m+h)}_{X_i}[\zeta]/((\partial^{\langle p^{m+h}\rangle_{(m+h)}})^N\zeta - 1)$. Define a $\mathscr{D}^{(m+h)}_{X_i}[\zeta]$-module structure on $F^*S_m^{-1}\mathscr{D}^{(m)}_{X'_i}$ by putting

$$\zeta(f \otimes \partial'^{\langle i\rangle_{(m)}}) := f \otimes \partial'^{\langle i\rangle_{(m)}}(\partial'^{\langle p^m\rangle_{(m)}})^{-N}.$$

This structure defines a homomorphism $\mathscr{D}^{(m+h)}_{X_i}[\zeta] \to F^*S_m^{-1}\mathscr{D}^{(m)}_{X_i}$ of left $\mathscr{D}^{(m+h)}_{X_i}[\zeta]$-modules by sending one to $1\otimes 1$. Since $(\partial^{\langle p^{m+h}\rangle_{(m+h)}})^N \cdot (1 \otimes (\partial'^{\langle p^m\rangle_{(m)}})^{-N}) =$

$1 \otimes 1$, the homomorphism factors through $\mathscr{D}^{(m+h)}_{X_i}[\zeta] \to S^{-1}_{m+h}\mathscr{D}^{(m+h)}_{X_i}$, and defines a well-defined homomorphism

$$\alpha\colon S^{-1}_{m+h}\mathscr{D}^{(m+h)}_{X_i} \to F^* S^{-1}_m \mathscr{D}^{(m)}_{X'_i}$$

compatible with Φ and sending $(\partial^{\langle p^{m+h}\rangle_{(m+h)}})^{-N}$ to $(\partial'^{\langle p^m\rangle_{(m)}})^{-N}$. For an integer k, we denote by $(F^*S^{-1}_m\mathscr{D}^{(m)}_{X'_i})_k := \pi^{-1}\mathcal{O}_{X_i} \otimes_{\pi^{-1}\mathcal{O}_{X'_i}} (S^{-1}_m\mathscr{D}^{(m)}_{X'_i})_k$. By the choice of coordinates (cf. 6.1.1), $\alpha(\partial^{\langle 1\rangle_{(m+h)}}) = qx^{q-1} \otimes \partial'^{\langle 1\rangle_{(m)}}$. This implies that, for any integer k, we get

$$\alpha\bigl((S^{-1}_{m+h}\mathscr{D}^{(m+h)}_{X_i})_k\bigr) \subset (F^*S^{-1}_m\mathscr{D}^{(m)}_{X'_i})_{[kp^{-h}]+Np^{m+h}}\,.$$

Thus by taking the completion with respect to the filtration by order, α induces 6.1.2.1.

Let $m' \geqslant m$ be an integer. The homomorphism 6.1.2.1 makes $F^*\mathscr{E}^{(m')}_{X'_i}$ a left $\mathscr{E}^{(m'+h)}_{X_i}$-module. Thus the canonical homomorphism $F^*\mathscr{D}^{(m)}_{X'_i} \to F^*\mathscr{E}^{(m')}_{X'_i}$ of left $\mathscr{D}^{(m+h)}_{X_i}$-modules induces

$$\mathscr{E}^{(m'+h)}_{X_i} \otimes_{\mathscr{D}^{(m+h)}_{X_i}} F^*\mathscr{D}^{(m)}_{X'_i} \to F^*\mathscr{E}^{(m')}_{X'_i}.$$

This is an isomorphism. Indeed, since $F^*\mathscr{D}^{(m)}_{X'_i}$ is flat over $\mathscr{D}^{(m+h)}_{X_i}$, cf. [15, Corollary 2.5.7-(i)], we get that the canonical homomorphism induced by the injective homomorphism $\varpi\mathscr{E}^{(m'+h)}_{X_i} \to \mathscr{E}^{(m'+h)}_{X_i}$,

$$\varpi\mathscr{E}^{(m'+h)}_{X_i} \otimes_{\mathscr{D}^{(m+h)}_{X_i}} F^*\mathscr{D}^{(m)}_{X'_i} \to \mathscr{E}^{(m'+h)}_{X_i} \otimes_{\mathscr{D}^{(m+h)}_{X_i}} F^*\mathscr{D}^{(m)}_{X'_i}$$

is injective. Since $\varpi\mathscr{E}^{(m'+h)}_{X_i} \otimes_{\mathscr{D}^{(m+h)}_{X_i}} F^*\mathscr{D}^{(m)}_{X'_i} \cong \mathscr{E}^{(m'+h)}_{X_{i-1}} \otimes_{\mathscr{D}^{(m+h)}_{X_{i-1}}} F^*\mathscr{D}^{(m)}_{X'_{i-1}}$ is an isomorphism, by the induction on i, it remains to prove the claim for $i = 0$. In this case the verification is straightforward, and left to the reader.

Since $\mathcal{O}_{X_i}$ is free of rank q over $\mathcal{O}_{X'_i}$, F^* commutes with taking inverse limit over i. Thus by taking the inverse limit, we get a homomorphism $\Psi_m\colon \widehat{\mathscr{E}}^{(m+h)}_{\mathscr{X}} \to F^*\widehat{\mathscr{E}}^{(m)}_{\mathscr{X}'}$. For an integer $m' \geqslant m$, this induces a canonical homomorphism

$$\beta_{m'}\colon \widehat{\mathscr{E}}^{(m'+h)}_{\mathscr{X}} \otimes_{\widehat{\mathscr{D}}^{(m+h)}_{\mathscr{X}}} F^*\widehat{\mathscr{D}}^{(m)}_{\mathscr{X}} \to F^*\widehat{\mathscr{E}}^{(m')}_{\mathscr{X}}.$$

This is an isomorphism since both sides are complete with respect to the p-adic filtrations and its reduction over R_i is an isomorphism for any i.

6.1.3. We keep using notation of 6.1.1 and 6.1.2. Let $m' \geqslant m$ be integers. We will first define a homomorphism $\widehat{\mathscr{E}}^{(m+h,m'+h)}_{\mathscr{X}} \to F^*\widehat{\mathscr{E}}^{(m,m')}_{\mathscr{X}'}$ compatible with Ψ_m and $\Psi_{m'}$. Let $\mathscr{E}$ be either $\widehat{\mathscr{D}}^{(m+h)}_{\mathscr{X}}$ or $\widehat{\mathscr{E}}^{(m+h,m'+h)}_{\mathscr{X},0}$ considered as a subgroup of $\widehat{\mathscr{E}}^{(m+h,m'+h)}_{\mathscr{X}}$. For non-negative integers $b \geqslant a$, we consider $\widehat{\mathscr{E}}^{(a,b)}_{\mathscr{X}}$ as a subring of $\widehat{\mathscr{E}}^{(a)}_{\mathscr{X}}$ or $\widehat{\mathscr{E}}^{(b)}_{\mathscr{X}}$ using the canonical inclusions. Then we claim that $\Psi_\bullet(\mathscr{E}) \subset F^*\widehat{\mathscr{E}}^{(m,m')}_{\mathscr{X}'}$ where $\bullet$ is m or m', and $\Psi_m|_{\mathscr{E}} = \Psi_{m'}|_{\mathscr{E}}$. For $\mathscr{E} = \widehat{\mathscr{D}}^{(m+h)}_{\mathscr{X}}$, the claim is nothing but the compatibility of $\Psi_\bullet$ and

Φ. Let us see the claim for $\mathscr{E}=\widehat{\mathscr{E}}^{(m+h,m'+h)}_{\mathscr{X},0}$. By definition, we get $\Psi_m(\partial^{\langle -kp^{m'}\rangle_{(m'+h)}})=\Psi_{m'}(\partial^{\langle -kp^{m'}\rangle_{(m'+h)}})$ for large enough integer k, and thus $\Psi_m(\partial^{\langle -n\rangle_{(m'+h)}})=\Psi_{m'}(\partial^{\langle -n\rangle_{(m'+h)}})$ holds for any positive integer n. By a standard continuity argument, we get the claim.

Since $\widehat{\mathscr{E}}^{(m+h,m'+h)}_{\mathscr{X}}$ is equal to $\widehat{\mathscr{D}}^{(m'+h)}_{\mathscr{X}}+\widehat{\mathscr{E}}^{(m+h,m'+h)}_{\mathscr{X},0}$ in $\widehat{\mathscr{E}}^{(m+h)}_{\mathscr{X}}$, we get the desired homomorphism. Now, we get the following.

Lemma. *The canonical homomorphism*

$$\widehat{\mathscr{E}}^{(m'+h,m''+h)}_{\mathscr{X}}\otimes_{\widehat{\mathscr{D}}^{(m+h)}_{\mathscr{X}}}F^*\widehat{\mathscr{D}}^{(m)}_{\mathscr{X}'}\to F^*\widehat{\mathscr{E}}^{(m',m'')}_{\mathscr{X}'}\tag{6.1.3.1}$$

is an isomorphism.

Proof. Since $F^*\widehat{\mathscr{D}}^{(m)}_{\mathscr{X}'}$ is locally projective over $\widehat{\mathscr{D}}^{(m+h)}_{\mathscr{X}}$, the canonical homomorphism

$$\widehat{\mathscr{E}}^{(m'+h,m''+h)}_{\mathscr{X}}\otimes_{\widehat{\mathscr{D}}^{(m+h)}_{\mathscr{X}}}F^*\widehat{\mathscr{D}}^{(m)}_{\mathscr{X}'}\to\widehat{\mathscr{E}}^{(m'+h)}_{\mathscr{X}}\otimes_{\widehat{\mathscr{D}}^{(m+h)}_{\mathscr{X}}}F^*\widehat{\mathscr{D}}^{(m)}_{\mathscr{X}'}\xrightarrow{\sim}F^*\widehat{\mathscr{E}}^{(m')}_{\mathscr{X}',\mathbb{Q}}$$

is injective. Thus we get the injectivity of 6.1.3.1. Let us see the surjectivity. Since $\widehat{\mathscr{D}}^{(m')}_{\mathscr{X}'}\xrightarrow{\sim}\widehat{\mathscr{E}}^{(m',m'')}_{\mathscr{X}'}/(\widehat{\mathscr{E}}^{(m',m'')}_{\mathscr{X}'})_{-1}$, it suffices to show that the image of 6.1.3.1 contains $(F^*\widehat{\mathscr{E}}^{(m',m'')}_{\mathscr{X}',\mathbb{Q}})_0$. This follows from the fact that $(F^*\widehat{\mathscr{E}}^{(m',m'')}_{\mathscr{X}',\mathbb{Q}})_0=(F^*\widehat{\mathscr{E}}^{(m'')}_{\mathscr{X}',\mathbb{Q}})_0$, and the surjectivity of the homomorphism $\beta_{m''}$. □

6.1.4 Lemma. *Let $\mathscr{M}$ be a coherent $\widehat{\mathscr{D}}^{(m)}_{\mathscr{X}',\mathbb{Q}}$-module. Then there exists a canonical isomorphism*

$$\mathscr{E}^{\dagger}_{\mathscr{X},\mathbb{Q}}\otimes F^*\mathscr{M}\xrightarrow{\sim}F^*(\mathscr{E}^{\dagger}_{\mathscr{X}',\mathbb{Q}}\otimes\mathscr{M}).$$

Proof. Since tensor product is right exact and F^* is exact, it suffices to prove the lemma for $\mathscr{M}=\widehat{\mathscr{D}}^{(m)}_{\mathscr{X}',\mathbb{Q}}$ by the coherence of $\widehat{\mathscr{D}}^{(m)}_{\mathscr{X}',\mathbb{Q}}$. Since inductive limit commutes with tensor product, it suffices to see that the homomorphism $\mathscr{E}^{(m'+h,\dagger)}_{\mathscr{X},\mathbb{Q}}\otimes F^*\widehat{\mathscr{D}}^{(m)}_{\mathscr{X}',\mathbb{Q}}\to F^*\mathscr{E}^{(m',\dagger)}_{\mathscr{X}',\mathbb{Q}}$ is an isomorphism. Since $F^*\widehat{\mathscr{D}}^{(m)}_{\mathscr{X}',\mathbb{Q}}$ is locally projective over $\widehat{\mathscr{D}}^{(m+h)}_{\mathscr{X},\mathbb{Q}}$, this claim follows from Lemma 6.1.3 above by taking the inverse limit over m'', and we conclude the proof. □

6.1.5 Remark. The construction does not depend on the choice of x and x', and Lemma 6.1.4 holds for any smooth formal curve $\mathscr{X}$. Since x' is determined uniquely when x is determined, it suffices to see that the construction only depends on x. This verification is left to the reader.

6.2. An exact sequence

In this subsection, we construct a key exact sequence. We consider the situation in paragraph 2.4.1. First, we use a result of the previous subsection in the following definition.

6.2.1 Definition. Let $\mathscr{M}$ be a holonomic F-$\mathscr{D}^{\dagger}_{\widehat{\mathbb{P}},\mathbb{Q}}(\infty)$-module, and consider $\mathscr{E}^{\dagger}_{0,\mathbb{Q}}\otimes\mathscr{M}:=(\mathscr{E}^{\dagger}_{\mathscr{X},\mathbb{Q}}\otimes\mathscr{M})_{\xi_0}$ (cf. Notation 1.2.1.3). This module is naturally a $\mathscr{D}^{\mathrm{an}}_{\mathscr{S}_0,\mathbb{Q}}$-module by

Corollary 2.2.4. Now, we get the following isomorphism

$$\mathscr{E}^{\dagger}_{0,\mathbb{Q}} \otimes \mathscr{M} \xrightarrow[1\otimes\Phi]{\sim} \mathscr{E}^{\dagger}_{0,\mathbb{Q}} \otimes F^*\mathscr{M} \xrightarrow{\sim} F^*(\mathscr{E}^{\dagger}_{0,\mathbb{Q}} \otimes \mathscr{M}),$$

where the first isomorphism is induced by the Frobenius structure $\Phi\colon \mathscr{M} \xrightarrow{\sim} F^*\mathscr{M}$, and the second by Lemma 6.1.4. This defines a Frobenius structure on $\mathscr{E}^{\dagger}_{0,\mathbb{Q}} \otimes \mathscr{M}$. We denote this F-$\mathscr{D}^{\mathrm{an}}_{\mathscr{S}_0,\mathbb{Q}}$-module by $\mu(\mathscr{M})$.

6.2.2. Recall that, in 1.2.1, we put $E^{\dagger}_{\widehat{\mathbb{A}},\mathbb{Q}} := \Gamma(\mathring{T}^*\widehat{\mathbb{A}}, \mathscr{E}^{\dagger}_{\widehat{\mathbb{A}},\mathbb{Q}})$, $E^{(m,\dagger)}_{\widehat{\mathbb{A}},\mathbb{Q}} := \Gamma(\mathring{T}^*\widehat{\mathbb{A}}, \mathscr{E}^{(m,\dagger)}_{\widehat{\mathbb{A}},\mathbb{Q}})$, $\widehat{E}^{(m,m')}_{\widehat{\mathbb{A}},\mathbb{Q}} := \Gamma(\mathring{T}^*\widehat{\mathbb{A}}, \widehat{\mathscr{E}}^{(m,m')}_{\widehat{\mathbb{A}},\mathbb{Q}})$, *etc.*. We claim that the ring isomorphism of the naive Fourier transform $\iota\colon \Gamma(\widehat{\mathbb{P}}, \mathscr{D}^{\dagger}_{\widehat{\mathbb{P}},\mathbb{Q}}(\infty)) \xrightarrow{\sim} \Gamma(\widehat{\mathbb{P}}', \mathscr{D}^{\dagger}_{\widehat{\mathbb{P}}',\mathbb{Q}}(\infty'))$, $x \mapsto \pi^{-1}\partial'$, $\partial \mapsto -\pi x'$ (cf. 3.2.5), extends to a ring homomorphism $\iota'\colon \mathscr{D}^{\mathrm{an}}_{\mathscr{S}_\infty,\mathbb{Q}}(0) \to E^{\dagger}_{\widehat{\mathbb{A}}',\mathbb{Q}}$, which fits into the following commutative diagram

$$\begin{tikzcd}
\Gamma(\widehat{\mathbb{P}}, \mathscr{D}^{\dagger}_{\widehat{\mathbb{P}},\mathbb{Q}}(\infty)) \arrow[r, "\iota", "\sim"'] \arrow[d, hook] & \Gamma(\widehat{\mathbb{P}}', \mathscr{D}^{\dagger}_{\widehat{\mathbb{P}}',\mathbb{Q}}(\infty')) \arrow[d, hook] \\
\mathscr{D}^{\mathrm{an}}_{\mathscr{S}_\infty,\mathbb{Q}}(0) \arrow[r, dashed, "\iota'"] & E^{\dagger}_{\widehat{\mathbb{A}}',\mathbb{Q}}
\end{tikzcd}$$

where the right vertical injection is

$$\Gamma(\widehat{\mathbb{P}}', \mathscr{D}^{\dagger}_{\widehat{\mathbb{P}}',\mathbb{Q}}(\infty')) \xrightarrow{\text{restriction}} \Gamma(\widehat{\mathbb{A}}', \mathscr{D}^{\dagger}_{\widehat{\mathbb{A}}',\mathbb{Q}}) \xrightarrow{\sim} \Gamma(\mathring{T}^*\widehat{\mathbb{A}}', \pi^{-1}\mathscr{D}^{\dagger}_{\widehat{\mathbb{A}}',\mathbb{Q}}) \hookrightarrow \Gamma(\mathring{T}^*\widehat{\mathbb{A}}', \mathscr{E}^{\dagger}_{\widehat{\mathbb{A}}',\mathbb{Q}}) = E^{\dagger}_{\widehat{\mathbb{A}}',\mathbb{Q}},$$

and the left one is the natural injection sending x to u^{-1} and ∂ to $-u^2\partial_u$, with $u := x^{-1}$ the local parameter of $\mathscr{S}_\infty$. Such a morphism ι' should send u to $\pi\partial'^{-1}$ and ∂_u to $\pi^{-1}\partial'^2 x'$.

Let us construct ι'. We recall that, for every $n, m \geqslant 0$, the integers $q_n^{(m)} \geqslant 0$ and $0 \leqslant r_n^{(m)} < p^m$ are defined by $n = q_n^{(m)} p^m + r_n^{(m)}$. First, we have a continuous homomorphism, for every $m' \geqslant m \geqslant 0$,

$$\iota'_{m'}\colon R[\![u]\!] \to \widehat{E}^{(m,m')}_{\widehat{\mathbb{A}}',\mathbb{Q}}$$

sending u to $\pi\partial'^{-1}$. Indeed, to check that is well-defined, it is enough to note that the sequence $\pi^{p^m\cdot n}/(p^m!)^n$ goes to zero when n goes to infinity. By construction, $\iota'_{m'}$ is independent of m, compatible with natural morphisms changing the level m', and continuous for the (ϖ, u)-adic topology of $R[\![u]\!]$ and $\mathscr{T}_n$-topology (cf. 1.2.2) for $n \gg 0$ on $\widehat{E}^{(m,m')}_{\widehat{\mathbb{A}}',\mathbb{Q}}$. Now we are going to extend this morphism to three different kinds of rings. First, for any integer $r = cp^{m'+1}$, with $c \in \mathbb{N}\backslash\{0\}$, the morphism $\iota'_{m'}$ extends by continuity to a morphism

$$\mathcal{O}_r := R[\![u]\!]\{T\}/(pT - u^r) \to \widehat{E}^{(m,m')}_{\widehat{\mathbb{A}}',\mathbb{Q}} \tag{6.2.2.1}$$

sending T to $\pi^r p^{-1}\partial'^{-r} = (\pi^r/r!)\,(pc)!\,p^{-1}\partial'^{\langle -r\rangle_{(m')}}$. Second, for $s := p^{m+1}$, the morphism $\iota'_{m'}$ extends by continuity to a morphism

$$\mathcal{B}_s := R[\![u]\!]\{Z\}/(u^s Z - p) \to \widehat{E}^{(m,m')}_{\widehat{\mathbb{A}}',\mathbb{Q}} \tag{6.2.2.2}$$

sending Z to $p\pi^{-s}\partial'^{s} = s!\pi^{-s}(p/p!)\partial'^{\langle s\rangle_{(m)}}$. Third, for $m'' \leqslant m-2$ and $m \geqslant 2$, the morphism $\iota'_{m'}$ extends to a continuous morphisms

$$\widehat{\mathscr{D}}^{(m'')}_{\mathscr{S}_\infty} \to \widehat{E}^{(m,m')}_{\widehat{\mathbb{A}}',\mathbb{Q}} \tag{6.2.2.3}$$

sending ∂_u to $\pi^{-1}\partial'^2 x'$. To verify this, it is enough to prove that the sequence $\{\partial_u^{\langle n\rangle_{(m'')}}\}_{n\geqslant 0}$ is sent to a sequence converging to zero in $\widehat{E}^{(m,m')}_{\widehat{\mathbb{A}}',\mathbb{Q}}$. We have

$$\partial_u^{\langle n\rangle_{(m'')}} = \frac{q_n^{(m'')}!}{n!}\partial_u^n \longmapsto Q_n^{(m'')} := \frac{q_n^{(m'')}!}{n!}\frac{(\partial'^2 x')^n}{\pi^n} \in \widehat{\mathscr{D}}^{(m)}_{\widehat{\mathbb{A}}',\mathbb{Q}} \subset \widehat{E}^{(m,m')}_{\widehat{\mathbb{A}}',\mathbb{Q}}.$$

We consider the spectral norm $\|-\|'^{(m)}$ of $\widehat{\mathscr{D}}^{(m)}_{\widehat{\mathbb{A}}',\mathbb{Q}}$, cf. [40, Remark 2.1.4-(ii)], we want to compute $\|Q_n^{(m'')}\|'^{(m)}$. Recall that $\|x'\|'^{(m)} = 1$, $\|\partial'\|'^{(m)} = \omega_0/\omega_m$, where $\omega_m = p^{-1/p^m(p-1)}$; and $|\pi| = \omega_0$. For every integer $n \geqslant 0$, $(\partial'^2 x')^n$ is divisible by $(n!)^2$ in $\mathscr{D}^\dagger_{\widehat{\mathbb{A}}'}$, more precisely we can prove by induction the relation

$$(\partial'^2 x')^n = (n!)^2 \sum_{j=n}^{j=2n} \binom{j}{n}\binom{n+1}{j-n+1}(x')^{j-n}\partial'^{[j]}, \tag{6.2.2.4}$$

where $\partial'^{[j]} = \partial'^j/j!$. Using 6.2.2.4, and denoting by $\sigma(-)$ the sum of p-adic digits, we get

$$\begin{aligned}\|(\partial'^2 x')^n\|'^{(m)} &\leqslant |n!|^2 \sup_{n\leqslant j\leqslant 2n}\left\{\frac{1}{|j!|}\left(\frac{\omega_0}{\omega_m}\right)^j\right\} \\ &= \omega_0^{2n-2\sigma(n)+\inf_{n\leqslant j\leqslant 2n}\left\{\sigma(j)-\frac{j}{p^m}\right\}} \leqslant \omega_0^{2n-2\sigma(n)-\frac{2n}{p^m}},\end{aligned}$$

and finally we have

$$\left\|Q_n^{(m'')}\right\|'^{(m)} = \left\|\frac{q_n^{(m'')}!}{n!}\frac{(\partial'^2 x')^n}{\pi^n}\right\|'^{(m)} \leqslant \omega_0^{q_n^{(m'')}-\sigma(q_n^{(m'')})-\sigma(n)-\frac{2n}{p^m}}.$$

For $m \geqslant 2$ and $m'' \leqslant m-2$, this is converging to zero when n goes to $+\infty$.

Now, since $\widehat{E}^{(m,m')}_{\widehat{\mathbb{A}}',\mathbb{Q}}$ is complete, combining morphisms 6.2.2.1, 6.2.2.2, and 6.2.2.3, we get, for any $m' \geqslant m \geqslant 2$,

$$\mathcal{O}_{p^{m'+1}}\widehat{\otimes}_{R[\![u]\!]}\mathcal{B}_{p^{m+1}}\widehat{\otimes}_{R[\![u]\!]}\widehat{\mathscr{D}}^{(m-2)}_{\mathscr{S}_\infty} \to \widehat{E}^{(m,m')}_{\widehat{\mathbb{A}}',\mathbb{Q}}.$$

Taking the inverse limit on m' and tensoring with $\mathbb{Q}$, we have

$$(\widehat{\mathscr{D}}^{(m-2)}_{\mathscr{S}_\infty,\mathbb{Q}}(0))^{\mathrm{an}} := \left(\varprojlim_{m'}(\mathcal{O}_{p^{m'+1}}\widehat{\otimes}_{R[\![u]\!]}\mathcal{B}_{p^{m+1}}\widehat{\otimes}_{R[\![u]\!]}\widehat{\mathscr{D}}^{(m-2)}_{\mathscr{S}_\infty})\right)_{\mathbb{Q}} \to E^{(m,\dagger)}_{\widehat{\mathbb{A}}',\mathbb{Q}},$$

and finally by a direct limit on m we get the homomorphism ι' we wanted.

Lemma. *Let $\mathscr{M}$ be a holonomic F-$\mathscr{D}^{\dagger}_{\widehat{\mathbb{P}},\mathbb{Q}}(\infty)$-module. Then the F-$\mathscr{D}^{\mathrm{an}}$-module $\mu(\mathscr{F}_{\mathrm{naive},\pi}(\mathscr{M}))$ only depends on $\mathscr{M}|_{\eta_\infty}$.*

Proof. First, we note here that as modules, we have

$$\mathscr{M}|_{\eta_\infty} \cong \mathscr{D}^{\mathrm{an}}_{\mathscr{S}_\infty,\mathbb{Q}}(0) \otimes_{\mathscr{D}^{\dagger}_{\widehat{\mathbb{P}},\mathbb{Q}}(\infty)} \mathscr{M}.$$

Since $\mathscr{M}$ possesses a Frobenius structure, we know that $\mathscr{M}$ is a coherent $\mathscr{D}^{\dagger}_{\widehat{\mathbb{P}},\mathbb{Q}}$-module. Thus, we get

$$\mathscr{D}^{\mathrm{an}}_{\mathscr{S}_\infty,\mathbb{Q}} \otimes_{\mathscr{D}^{\dagger}_{\widehat{\mathbb{P}},\mathbb{Q}}} \mathscr{M} \xrightarrow{\sim} \mathscr{D}^{\mathrm{an}}_{\mathscr{S}_\infty,\mathbb{Q}}(0) \otimes_{\mathscr{D}^{\dagger}_{\widehat{\mathbb{P}},\mathbb{Q}}(\infty)} \mathscr{M}$$

by [31, Theorem 4.1.2]. We denote by

$$\widetilde{\iota}\colon \mathscr{D}^{\mathrm{an}}_{\mathscr{S}_\infty,\mathbb{Q}}(0) \xrightarrow{\iota'} E^{\dagger}_{\widehat{\mathbb{A}},\mathbb{Q}} \hookrightarrow \mathscr{E}^{\mathrm{an}}_{\mathscr{S}_{0'},\mathbb{Q}},$$

the composition of the homomorphism defined in 6.2.2 and the canonical inclusion. We have

$$\begin{aligned}
\mu(\mathscr{F}_{\mathrm{naive},\pi}(\mathscr{M})) &\cong \mathscr{E}^{\mathrm{an}}_{\mathscr{S}_{0'},\mathbb{Q}} \otimes_{\mathscr{D}^{\dagger}_{\widehat{\mathbb{P}}',\mathbb{Q}}(\infty)} \mathscr{F}_{\mathrm{naive},\pi}(\mathscr{M}) \\
&\cong \mathscr{E}^{\mathrm{an}}_{\mathscr{S}_{0'},\mathbb{Q}} \otimes_{\mathscr{D}^{\dagger}_{\widehat{\mathbb{P}}',\mathbb{Q}}(\infty)} (\mathscr{D}^{\dagger}_{\widehat{\mathbb{P}}',\mathbb{Q}}(\infty) \otimes_{\iota,\mathscr{D}^{\dagger}_{\widehat{\mathbb{P}},\mathbb{Q}}(\infty)} \mathscr{M}) \\
&\cong \mathscr{E}^{\mathrm{an}}_{\mathscr{S}_{0'},\mathbb{Q}} \otimes_{\widetilde{\iota},\mathscr{D}^{\mathrm{an}}_{\mathscr{S}_\infty,\mathbb{Q}}(0)} \mathscr{M}|_{\eta_\infty}.
\end{aligned}$$

To see that it is compatible with Frobenius isomorphisms, it suffices to apply Proposition 5.1.6. □

6.2.3. We use the notation of 2.4.1. Let M be a holonomic F-$\mathscr{D}^{\mathrm{an}}_{\mathscr{S},\mathbb{Q}}$-module. Let us define an LF-topology on $\mathscr{E}^{\mathrm{an}} \otimes M$. Let $\mathscr{M}$ be the canonical extension of M at zero. By Corollary 2.2.4, we know that

$$\mathscr{E}^{\mathrm{an}} \otimes M \cong E^{\dagger}_{\widehat{\mathbb{A}},\mathbb{Q}} \otimes \mathscr{M}.$$

Let $\mathscr{M}^{(m)}$ be a stable coherent $\widehat{\mathscr{D}}^{(m)}_{\widehat{\mathbb{A}},\mathbb{Q}}$-module such that $\mathscr{D}^{\dagger}_{\widehat{\mathbb{A}},\mathbb{Q}} \otimes \mathscr{M}^{(m)} \cong \mathscr{M}$. For $m' \geqslant m$, since $E^{(m',\dagger)}_{\widehat{\mathbb{A}},\mathbb{Q}}$ is a Fréchet–Stein algebra, any finitely presented module becomes a Fréchet space. By taking m' to be sufficiently large, we may suppose that $E^{(m',\dagger)}_{\widehat{\mathbb{A}},\mathbb{Q}} \otimes \mathscr{M}^{(m)}$ is finite free over $K_{\widehat{\mathbb{A}}}\{\partial\}^{(m',\dagger)}$ by Corollary 4.2.3. There exist two topologies on $E^{(m',\dagger)}_{\widehat{\mathbb{A}},\mathbb{Q}} \otimes \mathscr{M}^{(m)}$: the Fréchet topology induced from the $E^{(m',\dagger)}_{\widehat{\mathbb{A}},\mathbb{Q}}$-module structure denoted by $\mathscr{T}$, and topology induced from the finite free $K_{\widehat{\mathbb{A}}}\{\partial\}^{(m',\dagger)}$-module structure denoted by $\mathscr{T}'$. Since $(E^{(m',\dagger)}_{\widehat{\mathbb{A}},\mathbb{Q}} \otimes \mathscr{M}^{(m)}, \mathscr{T})$ is a topological $K_{\widehat{\mathbb{A}}}\{\partial\}^{(m',\dagger)}$-module as well, we get that $\mathscr{T}$ and $\mathscr{T}'$ are equivalent by the open mapping theorem. Now, we put on $E^{\dagger}_{\widehat{\mathbb{A}},\mathbb{Q}} \otimes \mathscr{M}$ the inductive limit topology. By the observation above, the inductive limit topology coincides with the $\mathcal{R} \cong \varinjlim_{m'} K_{\widehat{\mathbb{A}}}\{\partial\}^{(m',\dagger)}$-module topology (cf. Lemma 1.3.6). Since $\mathcal{R}$ is an LF-space and the module is finite free over $\mathcal{R}$, we get that the inductive limit-topology is separated and $E^{\dagger}_{\widehat{\mathbb{A}},\mathbb{Q}} \otimes \mathscr{M}$ becomes an LF-space. Thus this defines an LF-space structure on $\mathscr{E}^{\mathrm{an}} \otimes M$. By using Remark 2.2.5 (ii), we note here that this is also a topological $\mathscr{D}^{\mathrm{an}}$-module. Thus, if $\mathscr{E}^{\mathrm{an}} \otimes M$ is finite over $\mathscr{D}^{\mathrm{an}}$, this LF-space topology coincides with the $\mathscr{D}^{\mathrm{an}}$-module topology by the open mapping theorem.

6.2.4 Lemma. *Recall the notation of 3.1.9. Assume that $\mu(\mathscr{M})$ is a* holonomic *F-$\mathscr{D}^{\mathrm{an}}_{\mathscr{S},\mathbb{Q}}$-module. Then the scalar extension homomorphism $\mathscr{M}|_{S_0} \to \mu(\mathscr{M})$ induces an isomorphism*

$$\Phi(\mathscr{M}|_{S_0}) \xrightarrow{\sim} \Phi(\mu(\mathscr{M})).$$

Proof. Let N be a holonomic F-$\mathscr{D}^{\mathrm{an}}$-module such that $M := \mathscr{E}^{\mathrm{an}} \otimes N$ is also a holonomic F-$\mathscr{D}^{\mathrm{an}}$-module. First, we will show that $\mathscr{E}^{\mathrm{an}} \otimes_{\mathscr{D}^{\mathrm{an}}} M \cong M$. Let

$$\alpha\colon M \to \mathscr{E}^{\mathrm{an}} \otimes_{\mathscr{D}^{\mathrm{an}}} M, \quad \beta\colon \mathscr{E}^{\mathrm{an}} \otimes_{\mathscr{D}^{\mathrm{an}}} M \to M,$$

be the homomorphisms mapping $m \mapsto 1 \otimes m$ and $P \otimes m \mapsto Pm$ respectively. Obviously, we get $\beta \circ \alpha = \mathrm{id}$. It suffices to show that α is surjective. Let N^{can} and M^{can} be the canonical extensions of N and M respectively. By Corollary 2.2.4, we get

$$M \cong E^{\dagger}_{\widehat{\mathbb{A}},\mathbb{Q}} \otimes N^{\mathrm{can}}, \quad \mathscr{E}^{\mathrm{an}} \otimes M \cong E^{\dagger}_{\widehat{\mathbb{A}},\mathbb{Q}} \otimes M^{\mathrm{can}}.$$

Using 6.2.3, we endow these with the LF-space topologies (or equivalently the $\mathscr{D}^{\mathrm{an}}$-module topologies). Since these are equipped with $\mathscr{D}^{\mathrm{an}}$-module topologies and α is $\mathscr{D}^{\mathrm{an}}$-linear, α is continuous with these topologies (cf. [19, 3.7.3/2]). To see the surjectivity of α, it suffices to show that it is a homomorphism of $E^{\dagger}_{\widehat{\mathbb{A}},\mathbb{Q}}$-modules. Since ∂ is invertible in $E^{\dagger}_{\widehat{\mathbb{A}},\mathbb{Q}}$, we get that $\alpha(\partial^{-1}m) = \partial^{-1} \otimes m$. This shows that α is linear with respect to the subring E generated by $D^{\dagger}_{\widehat{\mathbb{A}},\mathbb{Q}}$ and ∂^{-1}. Note that E is dense in $E^{\dagger}_{\widehat{\mathbb{A}},\mathbb{Q}}$. Since the target of α is an LF-space, it is separated, and we get the claim.

Now, since the category F-$\mathrm{Hol}(\mathscr{S})$ is abelian by [31, Theorem 7.1.1], we define the F-$\mathscr{D}^{\mathrm{an}}$-modules K, C by the following exact sequence of F-$\mathscr{D}^{\mathrm{an}}$-modules:

$$0 \to K \to \mathscr{M}|_{S_0} \to \mu(\mathscr{M}) \to C \to 0.$$

Here the middle homomorphism is the canonical homomorphism. By the definition of Frobenius structures (cf. 6.2.1), this homomorphism is a homomorphism of modules with Frobenius structures. Take $\mathscr{E}^{\mathrm{an}} \otimes_{\mathscr{D}^{\mathrm{an}}}$, and we get an exact sequence by Remark 2.4.3 and the claim above:

$$0 \to \mathscr{E}^{\mathrm{an}} \otimes K \to \mu(\mathscr{M}) \to \mu(\mathscr{M}) \to \mathscr{E}^{\mathrm{an}} \otimes C \to 0.$$

This shows that $\mathscr{E}^{\mathrm{an}} \otimes K = \mathscr{E}^{\mathrm{an}} \otimes C = 0$. Let N be a holonomic F-$\mathscr{D}^{\mathrm{an}}$-module such that $\mathscr{E}^{\mathrm{an}} \otimes N = 0$. Let N^{can} be the canonical extension of N. We have

$$\mathscr{E}^{\dagger}_{0,\mathbb{Q}} \otimes N^{\mathrm{can}} \cong \mathscr{E}^{\mathrm{an}} \otimes N = 0.$$

This shows that there are no singularities for N at 0, so N is a convergent isocrystal around zero. In particular, we get an isomorphism $N \cong (\mathcal{O}^{\mathrm{an}})^{\oplus n}$ of differential modules for some integer n (note that in the isomorphism, we are forgetting the Frobenius structures). Applying this observation to K and C, we get that these are direct sums of trivial modules. Since $\Phi(\mathcal{O}^{\mathrm{an}}) = 0$ and the functor Φ is exact, we finish the proof. $\square$

6.2.5. Let $\mathscr{M}$ be a holonomic $(F\text{-})\mathscr{D}^{\dagger}_{\widehat{\mathbb{P}},\mathbb{Q}}(\infty)$-module. We define $\mathscr{M}|_{s_\infty} := \mathrm{Hom}_{\partial}(\mathcal{R}_{\mathscr{S}_\infty}, \mathscr{M}|_{\eta_\infty})$. When $\mathscr{M}$ possesses a Frobenius structure, this is a K-vector space with Frobenius structure. There exists a canonical homomorphism

$$\mathcal{R}_{\mathscr{S}_\infty} \otimes_K \mathscr{M}|_{s_\infty} \to \mathscr{M}|_{\eta_\infty}. \tag{6.2.5.1}$$

This homomorphism is injective, and compatible with Frobenius structures if they exist. When this injection is an isomorphism, $\mathscr{M}$ is said to be *unramified at infinity*. This is equivalent to saying that $\mathscr{M}|_{\eta_\infty}$ is a trivial differential module.

Lemma. *Let $\mathscr{M}$ be a holonomic F-$\mathscr{D}^{\dagger}_{\widehat{\mathbb{P}},\mathbb{Q}}(\infty)$-module unramified at infinity. Then the F-$\mathscr{D}^{\mathrm{an}}$-module $\mu(\mathscr{F}_{\mathrm{naive},\pi}(\mathscr{M}))$ is holonomic.*

Proof. Since $\mu(\mathscr{F}_{\mathrm{naive},\pi}(\mathscr{M}))$ depends only on $\mathscr{M}|_{\eta_\infty}$ and the claim does not depend on the Frobenius structure by Lemma 2.1.10, we may suppose that $\mathscr{M}$ is isomorphic to $\mathcal{O}_{\widehat{\mathbb{P}},\mathbb{Q}}(\infty)^{\oplus n}$ where $n = \mathrm{rk}(\mathscr{M})$. In this case, we know that $\mathscr{F}_{\mathrm{naive},\pi}(\mathscr{M})$ is equal to $i_{0'+}(K^{\oplus n})$ by Proposition 3.2.7. Now, $i_{0'+}K|_{\widehat{\mathbb{A}}'} \cong \mathscr{D}^{\dagger}_{\widehat{\mathbb{A}}',\mathbb{Q}}/(x')$, and we may check directly that $\mathscr{E}^{\mathrm{an}}_{0',\mathbb{Q}}/\mathscr{E}^{\mathrm{an}}_{0',\mathbb{Q}}\cdot x'$ is generated over $\mathscr{D}^{\mathrm{an}}_{0',\mathbb{Q}}$ by ∂'^{-1}. Thus the claim follows using Lemma 2.1.10 once again. □

6.2.6 Lemma. *Let $\mathscr{M}$ be a holonomic F-$\mathscr{D}^{\dagger}_{\widehat{\mathbb{P}},\mathbb{Q}}(\infty)$-module which is unramified at infinity. Then we have*

$$\Phi(\mathscr{F}_{\mathrm{naive},\pi}(\mathscr{M})|_{S_{0'}}) = (K^{\mathrm{ur}} \otimes_K \mathscr{M}|_{s_\infty})(1).$$

Proof. By Lemma 6.2.5, we get that $\mu(\mathscr{F}_{\mathrm{naive},\pi}(\mathscr{M}))$ is finite over $\mathscr{D}^{\mathrm{an}}_{0',\mathbb{Q}}$. Thus by Lemma 6.2.4, it suffices to calculate $\Phi(\mu(\mathscr{F}_{\mathrm{naive},\pi}(\mathscr{M})))$. Since $\mu(\mathscr{F}_{\mathrm{naive},\pi}(\mathscr{M}))$ depends only on $\mathscr{M}|_{\eta_\infty}$, we may assume that $\mathscr{M} = \mathscr{M}|_{s_\infty} \otimes_K \mathcal{O}_{\widehat{\mathbb{P}},\mathbb{Q}}(\infty)$. Indeed, its $\mathcal{R}$-module around ∞ is isomorphic to $\mathscr{M}|_{\eta_\infty}$ as differential $\mathcal{R}$-modules with Frobenius structures using the isomorphism 6.2.5.1. In this case, we may use Proposition 3.2.7 to get that

$$\mathscr{F}_{\mathrm{naive},\pi}(\mathscr{M}) \cong i_{0'+}(\mathscr{M}|_{s_\infty})(1).$$

Now, using Proposition 6.2.4 again, it suffices to calculate $\Phi(\mathscr{F}_{\mathrm{naive},\pi}(\mathscr{M})|_{S_{0'}})$, which is nothing but what we stated, and concludes the proof. □

6.2.7 Remark. We believe that $\mu(\mathscr{F}_{\mathrm{naive},\pi}(\mathscr{M}))$ is a holonomic F-$\mathscr{D}^{\mathrm{an}}$-module even when $\mathscr{M}$ is not unramified at infinity, and $\mu(\mathscr{F}_{\mathrm{naive},\pi}(\mathscr{M}))$ is closely related to the $(\infty, 0')$ local Fourier transform of Laumon. However, we do not need this generality in this paper, and we do not seek such functor further.

6.2.8 Proposition. *Let $\mathscr{M}$ be a holonomic F-$\mathscr{D}^{\dagger}_{\widehat{\mathbb{P}},\mathbb{Q}}(\infty)$-module which is an overconvergent isocrystal on a dense open subscheme U of $\mathbb{A}^1_k$, and assume that it is unramified at infinity. We denote by $j : (\widehat{\mathbb{P}}, \mathbb{P}^1_k \setminus U) \to (\widehat{\mathbb{P}}, \{\infty\})$ the canonical morphism of couples. Then there exists an exact sequence of Deligne modules:*

$$\begin{aligned} 0 \to H^1_{\mathrm{rig},c}(U_{K^{\mathrm{ur}}}, \mathscr{M}) &\to \Psi(\mathscr{F}_\pi(\mathscr{M}_!)|_{S_{0'}})(-2) \\ &\to (K^{\mathrm{ur}} \otimes_K \mathscr{M}|_{s_\infty})(-1) \to H^2_{\mathrm{rig},c}(U_{K^{\mathrm{ur}}}, \mathscr{M}) \to 0. \end{aligned} \tag{6.2.8.1}$$

Here, by abuse of language, we denoted $H^i_{\mathrm{rig},c}(U, \mathrm{sp}^*(\mathscr{M})) \otimes K^{\mathrm{ur}}$ by $H^i_{\mathrm{rig},c}(U_{K^{\mathrm{ur}}}, \mathscr{M})$, and $\mathscr{M}_! := j_! j^+ \mathscr{M}$ using the cohomological functors of 3.1.4.

Proof. By 2.1.7.1, there is an exact sequence:

$$\begin{aligned} 0 \to \mathrm{Hom}_{\mathscr{D}^\dagger}(\mathbb{D}\mathscr{F}_\pi(\mathscr{M}_!)|_{S_{0'}}, \mathcal{O}^{\mathrm{an}}_{K^{\mathrm{ur}}}) &\to \mathbb{V}(\mathbb{D}\mathscr{F}_\pi(\mathscr{M}_!)|_{S_{0'}}) \to \mathbb{W}(\mathbb{D}\mathscr{F}_\pi(\mathscr{M}_!)|_{S_{0'}}) \\ &\to \mathrm{Ext}^1_{\mathscr{D}^\dagger}(\mathbb{D}\mathscr{F}_\pi(\mathscr{M}_!)|_{S_{0'}}, \mathcal{O}^{\mathrm{an}}_{K^{\mathrm{ur}}}) \to 0. \end{aligned}$$

Consider the following Cartesian diagram:

$$\begin{tikzcd} (\widehat{\mathbb{P}}, \{\infty\}) \arrow[r, "\iota_0"] \arrow[d, "q_0"'] \arrow[dr, phantom, "\square"] & (\widehat{\mathbb{P}}'', Z) \arrow[d, "p'"] \\ \{0'\} \arrow[r, "i_{0'}"'] & (\widehat{\mathbb{P}}', \{\infty'\}). \end{tikzcd}$$

By [30, Theorem 2.2], we get

$$\mathrm{Ext}^i_{\mathscr{D}^\dagger}(\mathbb{D}\mathscr{F}_\pi(\mathscr{M}_!)|_{S_{0'}}, \mathcal{O}^{\mathrm{an}}_{K^{\mathrm{ur}}}) \cong \mathscr{H}^{i-1}(i^+_{0'}(\mathbb{D}\mathbb{D}\mathscr{F}_\pi(\mathscr{M}_!))) \otimes K^{\mathrm{ur}} \cong \mathscr{H}^{i-1}(i^+_{0'}(\mathscr{F}_\pi(\mathscr{M}_!))) \otimes K^{\mathrm{ur}}$$

for any i. We get the following calculation:

$$\begin{aligned} \mathscr{H}^i(i^+_{0'}(\mathscr{F}_\pi(\mathscr{M}_!))) &\xrightarrow[\sim]{①} \mathscr{H}^i(i^+_{0'}(p'_!(\mu^!\mathscr{L}_\pi \otimes p^!\mathscr{M}_!)))[1][-3] \\ &\xrightarrow[\sim]{②} \mathscr{H}^i(q_{0!}\ \iota^+_0(\mu^!\mathscr{L}_\pi \otimes p^!\mathscr{M}_!))[-2] \\ &\xrightarrow[\sim]{③} \mathscr{H}^i(q_{0!}\iota^+_0(\mu^!\mathscr{L}_\pi \widetilde{\otimes} p^!\mathscr{M}_!))(-2)[-2] \\ &\xrightarrow[\sim]{④} \mathscr{H}^i(q_{0!}\iota^+_0(\mu^+\mathscr{L}_\pi \widetilde{\otimes} p^+\mathscr{M}_!))[2] \\ &\xrightarrow[\sim]{⑤} \mathscr{H}^i(q_{0!}(\iota^+_0\mu^+\mathscr{L}_\pi \widetilde{\otimes} \iota^+_0 p^+\mathscr{M}_!))[-1][2] \\ &\xrightarrow[\sim]{⑥} \mathscr{H}^i(q_{0!}(\mathcal{O}_{\widehat{\mathbb{P}},\mathbb{Q}}(\infty) \widetilde{\otimes} \mathscr{M}_!))[1] \\ &\xrightarrow[\sim]{⑦} \mathscr{H}^i(q_{0!}(\mathscr{M}_!))(1)[1] \xrightarrow[\sim]{⑧} H^{i+2}_{\mathrm{rig},c}(U, \mathscr{M})(2). \end{aligned}$$

Here ① follows from Noot-Huyghe's theorems 3.2.2.2 and 3.2.5.1, ② from the base change Theorem 3.1.6, ③ from 3.1.5.5, ④ from 3.1.5.1 and Lemma 3.2.3, ⑤ from 3.1.5.4, ⑥ from $p \circ \iota_0 = \mathrm{id}_{(\widehat{\mathbb{P}},\{\infty\})}$ and $\iota^+_0\mu^+\mathscr{L}_\pi \cong \mathcal{O}_{\widehat{\mathbb{P}},\mathbb{Q}}(\infty)$ (by construction of $\mu^+\mathscr{L}_\pi$), ⑦ from 3.1.5.5 again, and ⑧ from 3.1.8.3. For the calculation of $\Phi(\mathscr{F}_{\mathrm{naive},\pi}(\mathscr{M}_!)|_{S_{0'}})$, it suffices to apply Lemma 6.2.6, and the fact that $\mathscr{M}_!|_{s_\infty} \cong \mathscr{M}|_{s_\infty}$. □

7. The p-adic epsilon factors and product formula

In this section, we prove the product formula for p-adic epsilon factors and the determinant formula relating the local epsilon factor to the local Fourier transform. We start in section 7.1 by defining the epsilon factors for holonomic modules over a formal disk, then we state the main theorem (the product formula) in section 7.2. Its proof takes the rest of this paper. We begin in section 7.4 by proving it for F-isocrystals with geometrically (globally) finite monodromy: this proof is slightly more technical than the ℓ-adic one and we need some generalities on scalars extension in Tannakian categories, which we collect in section 7.3. We finish the proof of the main theorem in section 7.5 where we give also the determinant formula.

7.1. Local constants for holonomic $\mathscr{D}$-modules

7.1.1. Let us fix some assumptions for this section. Let $\mathbb{F}$ be a finite subfield of k, p^h be the number of elements of $\mathbb{F}$, so that $\mathbb{F}$ is the subfield of k fixed by the h-th absolute Frobenius automorphism σ. Let Λ be a finite extension of $\mathbb{Q}_p$ with residue field $\mathbb{F}$ and absolute ramification index e. We put $K := \Lambda \otimes_{W(\mathbb{F})} W(k)$. It is a complete discrete valuation field with residue field k and ramification index e. We denote by R its ring of integers. We endow K with the endomorphism $\sigma_K = \mathrm{id}_\Lambda \otimes \sigma$. The subfield of K fixed by σ_K is Λ. We say that σ_K is a Frobenius of K of *order* h. Let v_K denote the valuation of K normalized by $v_K(K^*) = \mathbb{Z}$.

We assume (except for section 7.3) that k is finite with $q = p^f$ elements, and that the order h of the Frobenius σ_K divides f, so that $f = ah$ for an integer a. We will see later (cf. Remark 7.2.7) that, in the proofs, it is not restrictive to assume $h = f$ and so $\mathbb{F} = k$ and $K = \Lambda$.

We choose an algebraic closure $\overline{\mathbb{Q}}_p$ of K, and we denote by $\overline{\mathbb{F}}$ its residue field. We choose also a root π of the polynomial $X^{p-1} + p$ and we assume $\pi \in \Lambda$. We recall (cf. 2.4.1) that the choice of π determines a non-trivial additive character $\psi_\pi : \mathbb{F}_p \to K^*$. By composing ψ_π with the trace map $\mathrm{tr}_{k/\mathbb{F}_p} : k \to \mathbb{F}_p$, we get a non-trivial additive character $k \to K^*$ that we denote also by ψ_π.

7.1.2. To fix notation, we review some results collected in [54]. See [54] for more details. We follow the notation and the assumptions of 2.1.1 and 7.1.1. In particular $\mathscr{S} := \mathrm{Spf}(A)$ denotes a formal disk, $\eta_{\mathscr{S}}$ its geometric point, $S = \mathrm{Spec}(A/\varpi A)$ its special fibre, $s = \mathrm{Spec}(k_A)$ the closed point, and $\eta = \mathrm{Spec}(\mathcal{K})$ the generic point of S. We recall (cf. Lemma 2.1.1) that $\mathcal{K} \cong k_A(\!(u)\!)$, for any u in A lifting an uniformizer of $A/\varpi A$. We put $\bar{s} := \mathrm{Spec}(\overline{\mathbb{F}})$, and we choose a geometric algebraic generic point $\bar{\eta}$. We identify $\pi_1(s, \bar{s})$ with $\widehat{\mathbb{Z}}$, by sending any $n \in \widehat{\mathbb{Z}}$ to F^n, where $F\colon x \mapsto x^{-q}$ is the geometric Frobenius in $\pi_1(s, \bar{s})$. Let us denote by $\nu\colon \pi_1(\eta, \bar{\eta}) \to \widehat{\mathbb{Z}}$ the specialization homomorphism and by $W(\eta, \bar{\eta}) := \nu^{-1}(\mathbb{Z})$ (resp. $I_\eta := \ker \nu$) the Weil (resp. inertia) subgroup.

Let $\mathrm{Rep}_{K^{\mathrm{ur}}}(WD(\eta, \bar{\eta}))$ denote the category of Weil–Deligne representations: *i.e.* the category of finite dimensional K^{ur}-vector spaces V endowed with an action $\rho\colon W(\eta, \bar{\eta}) \to \mathrm{Aut}_{K^{\mathrm{ur}}}(V)$ and a nilpotent endomorphism $N\colon V \to V$, satisfying $\rho(g)N\rho^{-1}(g) = q^{\nu(g)}N$, for any $g \in W(\eta, \bar{\eta})$.

On the other hand, p-adic monodromy theorem gives an equivalence of Tannakian categories (nearby cycles)[13]

$$\Psi(-1)\colon F\text{-Hol}(\eta_{\mathscr{S}}) \to \mathrm{Del}_{K^{\mathrm{ur}}}(\pi_1(\eta,\overline{\eta})).$$

For the twist (-1), see 3.1.9. Let $(V,\varphi,N) \in \mathrm{Del}_{K^{\mathrm{ur}}}(\pi_1(\eta,\overline{\eta}))$ be a Deligne module. We can endow V with a linear action $\rho\colon W(\eta,\bar{\eta}) \to \mathrm{Aut}_{K^{\mathrm{ur}}}\big(V\big)$ by putting $\rho(g)(m) := g(\varphi^{a\nu(g)}(m))$, for all $m \in V$ and $g \in W(\eta,\bar{\eta})$. In this way we obtain a Weil–Deligne representation (V,ρ,N) and we denote by

$$L_k\colon \mathrm{Del}_{K^{\mathrm{ur}}}(\pi_1(\eta,\overline{\eta})) \to \mathrm{Rep}_{K^{\mathrm{ur}}}(WD(\eta,\bar{\eta})) \tag{7.1.2.1}$$

this functor of ‘Frobenius linearization’. Since Weil–Deligne representations are linear, we will often implicitly extend the scalars from K^{ur} to $\overline{\mathbb{Q}}_p$. Composing the functors $\Psi(-1)$ and L_k and extending the scalars to $\overline{\mathbb{Q}}_p$, we obtain a faithful exact $\otimes$-functor $\mathrm{WD}\colon F\text{-Hol}(\eta_{\mathscr{S}}) \to \mathrm{Rep}_{\overline{\mathbb{Q}}_p}(WD(\eta,\bar{\eta}))$ (cf. [54, 3.4.4]).

7.1.3. Now, let us introduce local epsilon factors. Langlands [48] defined local epsilon factors extending Tate's definition for rank one case, and in [32], Deligne simplified the construction of the epsilon factors for Weil–Deligne representations. Deligne's definition translates well to free differential $\mathcal{R}_{\mathscr{S}}$-modules with Frobenius structure $F\text{-Hol}(\eta_{\mathscr{S}})$, via the functor WD recalled above. Here, we extend it from $F\text{-Hol}(\eta_{\mathscr{S}})$ to $F\text{-Hol}(\mathscr{S})$ by *dévissage*.

We follow the notation and the assumptions of 2.1.1, 2.1.4, 3.1.9, 7.1.1 and 7.1.2. Let M be a holonomic $F\text{-}\mathscr{D}^{\mathrm{an}}_{\mathscr{S},\mathbb{Q}}$-module, $\omega \in \Omega^1_{\mathcal{K}/k}$ a non-zero meromorphic 1-form, and μ a Haar measure on the additive group of $\mathcal{K}$ with values in K. We denote by $\psi_\pi(\omega)\colon \mathcal{K} \to K^{\mathrm{ur}*}$ the additive character given by $\alpha \mapsto \psi_\pi\big(\mathrm{Tr}_{k/\mathbb{F}_p}(\mathrm{Res}(\alpha\omega))\big)$ (cf. [50, Remarque 3.1.3.6]). The next proposition allows us to define the (local) epsilon factor of the triple (M,ω,μ).

Proposition. *There exists a unique map*

$$\varepsilon_\pi\colon (M,\omega,\mu) \mapsto \varepsilon_\pi(M,\omega,\mu) \in \overline{\mathbb{Q}}_p^*,$$

satisfying the following properties.

(i) *For every exact sequence $0 \to M' \to M \to M'' \to 0$ in $F\text{-Hol}(\mathscr{S})$, we have*

$$\varepsilon_\pi(M,\omega,\mu) = \varepsilon_\pi(M',\omega,\mu)\cdot\varepsilon_\pi(M'',\omega,\mu).$$

(ii) *If M is punctual, i.e. $M = i_+V$ for some φ-K-module V, then*

$$\varepsilon_\pi(M,\omega,\mu) = \det_K(-F;V)^{-1},$$

where $F = \varphi^a$ is the smallest power of the Frobenius φ of V making it linear (cf. see 7.1.1 for the definition of the integer a).

[13] In [54, (3.2.18)] the functor $\Psi(-1)$ was denoted by S, the field of constants K by C, the category of free differential modules over $\mathcal{R}_{\mathscr{S}}$ by $F\text{-Isoc}^{\dagger}_{an}(\eta|C)$ or $\Phi M(\mathcal{R}_{\mathscr{S}})$, and the field $\mathcal{K} = k(\eta)$ by K.

(iii) *If the canonical homomorphism* $j_! j^+ M \to M$ *is an isomorphism, then*

$$\varepsilon_\pi(M, \omega, \mu) = \varepsilon_0(j^+ M(1), \psi_\pi(\omega), \mu)^{-1},$$

where ε_0 *is the local epsilon factor defined in [54, 3.4.4].*

Proof. Immediate by applying the distinguished triangle 3.1.9.1 and [54, 2.19 (2)]. □

In the following, we will always assume $\mu(A/\varpi A) = 1$; to lighten the notation, we put $\varepsilon(M, \omega) := \varepsilon_\pi(M, \omega, \mu)$. Moreover, for a free differential $\mathcal{R}$-module M with Frobenius structure, we define $\varepsilon_0^{\mathrm{rig}}(M, \omega) := \varepsilon_0(M, \psi_\pi(\omega), \mu)$ and $\varepsilon^{\mathrm{rig}}(M, \omega) := \varepsilon(M, \psi_\pi(\omega), \mu)$ using the notation of [54, 3.4.4]. For a complex $\mathcal{C}$ of F-$\mathscr{D}^{\mathrm{an}}_{\mathscr{S},\mathbb{Q}}$-modules with bounded holonomic cohomology, we put

$$\varepsilon(\mathcal{C}, \omega) := \prod_{i \in \mathbb{Z}} \varepsilon(\mathscr{H}^i \mathcal{C}, \omega)^{(-1)^i}.$$

7.1.4 Remark. Let M be an object of F-Hol(η). We define $j_{!+}(M) := \mathrm{Im}(j_! M \to j_+ M)$. Then we have

$$\varepsilon^{\mathrm{rig}}(M, \omega) = \varepsilon_\pi(j_{!+}(M(1)), \omega)^{-1}.$$

This follows from Lemma 3.1.10 and [54, (3.4.5.4)]. For intermediate extensions in a wider context, see [7, 1.4 and 4.3.12].

7.2. Statement of the main result

7.2.1. Let us begin by fixing notation and definitions of global objects. We follow the notation and assumptions of 7.1.1. Let X be a (smooth) curve over k. We denote by C the number of connected components of $X \otimes_k \overline{\mathbb{F}}$ and by g the genus of any of them. Let η_X be the generic point of X, and we choose a geometric point $\bar{\eta}_X$ over η_X. We denote by $|X|$ the set of closed points of X. For any $x \in |X|$, let $\mathfrak{m}_x$ be the maximal ideal of $\mathcal{O}_{X,x}$, k_x its residue field, $i_x : \mathrm{Spec}(k_x) \to X$ the canonical morphism and $K_x := K \otimes_{W(k)} W(k_x)$. Let $\widehat{\mathcal{O}}_{X,x}$ be the completion of $\mathcal{O}_{X,x}$ for the $\mathfrak{m}_x$-adic topology, $\mathcal{K}_x$ the field of fractions of $\widehat{\mathcal{O}}_{X,x}$, $\eta_x = \mathrm{Spec}(\mathcal{K}_x)$ the generic point of $S_x := \mathrm{Spec}(\widehat{\mathcal{O}}_{X,x})$, $\overline{\eta}_x$ (resp. $\bar{x}$) a geometric point over η_x (resp. x). Let us denote by $k(X)$ the field of functions of X and by $\Omega^1_{k(X)/k}$ the module of meromorphic differential 1-forms on X. For every non-zero $\omega \in \Omega^1_{k(X)/k}$ and $x \in |X|$, we denote by $\omega_x \in \Omega^1_{\mathcal{K}_x/k_x}$ the germ of ω at x, $v_x(\omega)$ the order of ω at x.

7.2.2. Let $\overline{X}$ be the smooth compactification of X over k, and $Z := \overline{X} \setminus X$. There exists a smooth formal scheme $\mathscr{X}$ over $\mathrm{Spf}(R)$ such that $\mathscr{X} \otimes_R k \cong \overline{X}$ by [SGA I, Exp. III, 7.4]. The category $F^{(h)}$-$D^b_{\mathrm{hol}}(\mathscr{D}^\dagger_{\mathscr{X},\mathbb{Q}}(Z))$ (cf. 3.1.1) does not depend on the choice of $\mathscr{X}$ up to canonical equivalence of categories using [11, 2.2.1]. We denote this category by $F^{(h)}$-$D^b_{\mathrm{hol}}(X)$ and call it the category of *bounded holonomic* $F^{(h)}$-$\mathscr{D}^\dagger_{X,\mathbb{Q}}$*-complexes.*

Now, let $f : X \to \mathrm{Spec}(k)$ and $\widetilde{f} : (\mathscr{X}, Z) \to (\mathrm{Spf}(R), \emptyset)$ be the structural morphisms. The functors $\widetilde{f}_+$ and $\mathbb{D}_{\mathscr{X},Z}$ (cf. 3.1.4) do not depend on the choice of $\mathscr{X}$ up to equivalences. By abuse of language, these are denoted by f_+ and $\mathbb{D}_X$ respectively.

We note that $f_!$ can also be used, and in the same way, we can consider the functor $j_!\colon F^{(h)}\text{-}D^b_{\mathrm{hol}}(U) \to F^{(h)}\text{-}D^b_{\mathrm{hol}}(X)$ for an open immersion $j\colon U \hookrightarrow X$.

Let R' be a discrete valuation ring finite étale over R, and let $\mathscr{C}$ be an object of $F^{(h)}\text{-}D^b_{\mathrm{hol}}(\mathrm{Spf}(R'))$. The associated σ_K-semi-linear automorphism $\mathscr{C} \xrightarrow{\sim} \mathscr{C}$ (cf. 3.1.2) is denoted by $\varphi_{\mathscr{C}}$. From now on we denote the K-*linear* automorphism $\varphi^a_{\mathscr{C}}$ by F (cf. 7.1.1 for the definition of a).

7.2.3. In [22], Caro defines the L-function of a complex $\mathscr{C}$ in $F^{(h)}\text{-}D^b_{\mathrm{hol}}(X)$. Let us recall the definition. We set[14]

$$L(X,\mathscr{C},t) := \prod_{x\in|X|} \det_{K_x}(1-t^{\deg(x)}F^{\deg(x)};\ i_x^+\mathscr{C})^{-1}$$
$$:= \prod_{x\in|X|}\prod_{r\in\mathbb{Z}} \det_{K_x}(1-t^{\deg(x)}F^{\deg(x)};\ \mathscr{H}^r(i_x^+\mathscr{C}))^{(-1)^{r+1}}.$$

Recall that $F^{\deg(x)} := \varphi^{\deg(x)\,a}$ is the smallest linear power of the Frobenius. Using a result of Etesse–Le Stum, Caro gave the following cohomological interpretation of his L-function (cf. [22, 3.4.1]):

$$L(X,\mathscr{C},t) = \prod_{r\in\mathbb{Z}} \det_K(1-tF;\mathscr{H}^r(f_!\mathscr{C}))^{(-1)^{r+1}}.$$

For careful readers, we remind that in [22], the definition of Frobenius structure of push-forward is re-defined so that it is compatible with adjoints (cf. [22, 1.2.11]). However, this coincides with the usual definition (see [4, Remark 3.12]).

Now, assume that f is proper. Then $f_!$ can be replaced by f_+. By Poincaré duality 3.1.8.4, we get the following functional equation:

$$L(X,\mathscr{C},t) = \varepsilon(\mathscr{C})\cdot t^{-\chi(f_!\mathscr{C})}\cdot L(X,\mathbb{D}_X(\mathscr{C}),t^{-1}),$$

where

$$\varepsilon(\mathscr{C}) := \det(-F;\, f_+\mathscr{C})^{-1} = \prod_{r\in\mathbb{Z}} \det(-F;\mathscr{H}^r(f_+\mathscr{C}))^{(-1)^{r+1}}$$

and F is the smallest linear power of the Frobenius. This invariant is called the *(global) epsilon factor of* $\mathscr{C}$. Finally, for $\mathscr{C}$ in $F\text{-}D^b_{\mathrm{hol}}(X)$, we put $r(\mathscr{C}) := \sum_{i\in\mathbb{Z}}(-1)^i r(\mathscr{H}^i\mathscr{C})$ using the notation of 4.1.1. When $\mathscr{C}$ is a module, $r(\mathscr{C})$ is nothing but the opposite of the generic rank of $\mathscr{C}$.

7.2.4 Remark. Let U be a non-empty open subscheme of a proper curve X, and M be an overconvergent F-isocrystal on U over K. Etesse–Le Stum [38] defined the L-function for M by

$$L_{\mathrm{EL}}(U,M,t) := \prod_{x\in|U|} \det_{K_x}(1-t^{\deg(x)}F^{\deg(x)};\, i_x^*M)^{-1}.$$

[14] The definition of the L-function is slightly different from that of Caro. We have chosen a different convention in order that $L(X, f^+K, t)$ coincides with the L-function of X.

We are able to interpret this global invariant in terms of the global invariant we have just defined in the following way. Let $j: U \hookrightarrow X$ be the open immersion and $f_U: U \to \mathrm{Spec}(k)$ be the structural morphism. Put $\mathscr{C} := j_!(\mathrm{sp}_* M)[-1](-1)$. Then the L-function coincides with that given by Etesse–Le Stum; namely, we get $L(X, \mathscr{C}, t) = L_{\mathrm{EL}}(U, M, t)$. The easiest way to see this might be to use the cohomological interpretation of the two L-functions, and the fact that

$$\mathscr{H}^i(f_+\mathscr{C}) \cong \mathscr{H}^i(f_{U!}(\mathrm{sp}_* M[-1](-1))) \cong H^i_{\mathrm{rig},c}(U, M),$$

where the second isomorphism holds by 3.1.8.3. See also [4, Remark 3.12] for some account.

7.2.5 Theorem (Product Formula)**.** *Let X be a proper (smooth) curve over k, $\mathscr{C}$ a complex in $F^{(h)}\text{-}D^b_{\mathrm{hol}}(X)$ and $\omega \in \Omega^1_{k(X)/k}$ a non-zero meromorphic form on X. We have the following relation between the global and local factors:*

$$\varepsilon(\mathscr{C}) = q^{C(1-g)r(\mathscr{C})} \prod_{x \in |X|} \varepsilon(\mathscr{C}|_{S_x}, \omega_x), \tag{PF}$$

where q denotes the number of elements of k, C denotes the number of geometrically connected components of X, and g is the genus of any of them. We recall that $\mathscr{C}|_{S_x}$ denotes the complex of modules defined by restriction (cf. 2.1.8) from X to its complete trait *S_x at x.*

The proof of the product formula will be given in section 7.5.

7.2.6 Corollary ([54, Conjecture 4.3.5])**.** *Let U be an non-empty open subscheme of a proper curve X, M an overconvergent F-isocrystal on U over K, and $\omega \in \Omega^1_{k(X)/k}$ a non-zero meromorphic form on X. For any $x \in |X|$, let us denote by $M|_{\eta_x}$ the free differential $\mathcal{R}_{K_x}$-module associated with M at η_x. We have*

$$\begin{aligned}\prod_{i=0}^{2} \det_K(-F; H^i_{\mathrm{rig},c}(U, M))^{(-1)^{i+1}} &= q^{C(1-g)\mathrm{rk}(M)} \prod_{x \in |U|} q_x^{v_x(\omega)\mathrm{rk}(M)} \det_M(x)^{v_x(\omega)} \\ &\quad \times \prod_{x \in X \setminus U} \varepsilon_0^{\mathrm{rig}}(M|_{\eta_x}, \omega_x),\end{aligned} \tag{PF*}$$

where $F := \varphi^a$, $q_x := q^{\deg(x)}$, $\det_M(x) := \det_{K_x}(\varphi_{\eta_x}^{a\deg(x)}; i_x^* M)$ and $v_x(\omega)$ denotes the order of ω at x.

Proof. Let us prove that (PF) implies (PF*): we need only to specialize all the factors. Let $j: U \hookrightarrow X$ be the open immersion. We replace $\mathscr{C} := j_!(\mathrm{sp}_* M)[-1](-1)$ in (PF). Then for $x \in |U|$, we get

$$\begin{aligned}\varepsilon(\mathscr{C}|_{S_x}, \omega_x) &= \varepsilon_0^{\mathrm{rig}}(M|_{\eta_x}, \omega_x) \cdot \det_{K_x}(-\varphi^{a\deg(x)}; i_x^* M)^{-1} = \varepsilon^{\mathrm{rig}}(M|_{\eta_x}, \omega_x) \\ &= q_x^{v_x(\omega)\mathrm{rk}(M)} \det_M(x)^{v_x(\omega)},\end{aligned}$$

where the first equality follows from the localization triangle 3.1.9.1 and 3.1.5.1, and the second (resp. third) from [54, (3.4.5.4)] (resp. [54, (2.19-2)]). Finally, for every $x \in X \setminus U$, by definition, we have $\varepsilon(\mathscr{C}|_{S_x}, \omega_x) = \varepsilon_0^{\mathrm{rig}}(M|_{\eta_x}, \omega_x)$. Considering Remark 7.2.4, we get the corollary. $\square$

7.2.7 Remark. (i) Note that in [54], the curve X was assumed to be geometrically connected for simplicity, so that $C = 1$. Since the product formula (PF) is immediate for punctual arithmetic $\mathscr{D}$-modules, the two statements (PF*) and (PF) are equivalent by *dévissage*.

(ii) By definition, the global factor $\varepsilon(\mathscr{C})$ appearing in (PF) and (PF*) does not change if we replace the Frobenius $\varphi_{\mathscr{C}}$ of $\mathscr{C}$ by its smallest linear power $\varphi_{\mathscr{C}}^a$. The same is true for the Weil–Deligne representation $\mathrm{WD}(\mathscr{C}|_{\eta_x})$ and *a fortiori* for the local factors. Replacing $\varphi_{\mathscr{C}}$ by $\varphi_{\mathscr{C}}^a$ is equivalent to assuming $h = f$, thus $\mathbb{F} = k$ and $K = \Lambda$ in 7.1.1.

7.3. Interlude on scalar extensions

In this subsection, we recall a formal way to extend scalars (cf. [35, p. 155]) in a general Tannakian category, with the intention of use in section 7.4. Most of the proofs are formal and they are only sketched. To shorten the exposition, we may implicitly assume that the objects of our categories have elements, so that they are Λ-vector spaces endowed with some extra structures. For a general treatment, see [35]. In this subsection the field k is only assumed to be perfect, whereas in the rest of section 7 it is finite.

Let $\mathcal{A}$ be a Tannakian category over Λ. Assume that the objects of $\mathcal{A}$ have finite length.

7.3.1 Definition. Let Λ' be a field extension of Λ and M an object of $\mathcal{A}$. A Λ'-*structure* on M is a homomorphism of Λ-algebras $\lambda_M\colon \Lambda' \to \mathrm{End}_{\mathcal{A}}(M)$.

Let λ_M (resp. λ_N) be a Λ'-structure on M (resp. N). A morphism $f\colon M \to N$ in $\mathcal{A}$ is said to be *compatible* with the Λ'-structures if for every α in Λ' we have $\lambda_N(\alpha) f = f\lambda_M(\alpha)$. The couples (M, λ_M), where λ_M is a Λ'-structure on an object M in $\mathcal{A}$, form a category, whose morphisms are the morphisms in $\mathcal{A}$ compatible with the Λ'-structures. We denote this category by $\mathcal{A}_{\Lambda'}$. Sometimes, we denote simply by M an object (M, λ_M) in $\mathcal{A}_{\Lambda'}$.

7.3.2. We define an internal tensor product in $\mathcal{A}_{\Lambda'}$ as follows. Let (M_1, λ_1) and (M_2, λ_2) be two objects in $\mathcal{A}_{\Lambda'}$. Since $M_1 \otimes M_2$ has finite length, there exists a smallest sub-object $\iota\colon I \hookrightarrow M_1 \otimes M_2$ such that, for all a in Λ', the image of $\lambda_1(a) \otimes \mathrm{id}_{M_2} - \mathrm{id}_{M_1} \otimes \lambda_2(a)$ factors through ι. We put $M_1 \otimes' M_2 = \mathrm{Coker}(\iota)$. There are two natural Λ'-structures on $M_1 \otimes M_2$, given respectively by the endomorphisms $\lambda_1(a) \otimes \mathrm{id}_{M_2}$ and $\mathrm{id}_{M_1} \otimes \lambda_2(a)$. By construction they induce the same Λ'-structure $\lambda_{M_1 \otimes' M_2}$ on $M_1 \otimes' M_2$. The couple $(M_1 \otimes' M_2, \lambda_{M_1 \otimes' M_2})$ defines an object of $\mathcal{A}_{\Lambda'}$ denoted by $(M_1, \lambda_1) \otimes (M_2, \lambda_2)$. It satisfies the axiom of the tensor product in the category $\mathcal{A}_{\Lambda'}$, which makes $\mathcal{A}_{\Lambda'}$ a Tannakian category.

7.3.3 Example. Let U be a non-empty open subscheme of a proper curve X, $\overline{\eta}$ a geometric point of X. The category $\mathrm{Rep}_{\Lambda}^{\mathrm{fg}}(\pi_1(U, \overline{\eta}))_{\Lambda'}$ of representations with local finite geometric monodromy is equivalent, and even isomorphic, to $\mathrm{Rep}_{\Lambda'}^{\mathrm{fg}}(\pi_1(U, \overline{\eta}))$ as Tannakian category.

Let K be a field as in 7.1.1, and Λ'/K be a finite Galois extension. By construction, the category $\mathrm{Isoc}^\dagger(U, X/K)_{\Lambda'}$ of overconvergent isocrystals with Λ'-structure is equivalent, as Tannakian category, to $\mathrm{Isoc}^\dagger(U, X/\Lambda' \otimes_\Lambda K)$.

Now assume k to be a finite field with $q = p^f$ elements and that the order of the Frobenius is f, so that $\sigma_K = \mathrm{id}_K$. If Λ'/K is totally ramified, then $F\text{-}\mathrm{Isoc}^\dagger(U, X/K)_{\Lambda'}$ is equivalent to $F\text{-}\mathrm{Isoc}^\dagger(U, X/\Lambda')$ as Tannakian category, where $\sigma_{\Lambda'} := \mathrm{id}_{\Lambda'}$. If Λ' is not totally ramified, an overconvergent F-isocrystal with Λ'-structure M' on U over K can be identified with an overconvergent isocrystal on U/Λ', endowed with a 'Frobenius' $\varphi_{M'}$ which is Λ'-linear although only of order f.

7.3.4. Let V be a finite dimensional Λ-vector space and M an object of $\mathcal{A}$. The tensor product $V \otimes_\Lambda M$ is defined canonically in [35, p. 156 and p. 131] as an essentially constant ind-object. In particular, if Λ'/Λ is a finite field extension, the product $\Lambda' \otimes_\Lambda M$ can be endowed with the Λ'-structure induced by the multiplication of Λ', so it belongs to $\mathcal{A}_{\Lambda'}$: we have a functor of *extension of scalars* $\Lambda' \otimes_\Lambda - \colon \mathcal{A} \to \mathcal{A}_{\Lambda'}$. If $a_1, \ldots, a_n$ is a base of Λ' over Λ, then $\Lambda' \otimes_\Lambda M$ is isomorphic to $\oplus_{i=1}^n a_i \Lambda \otimes_\Lambda M$, with an obvious meaning of the latter.

7.3.5. Let $S \colon \mathcal{A} \to \mathcal{B}$ be an additive functor between two categories $\mathcal{A}$ and $\mathcal{B}$ as in section 7.3. It extends to a functor $S_{\Lambda'} \colon \mathcal{A}_{\Lambda'} \to \mathcal{B}_{\Lambda'}$ defined by functoriality as

$$S_{\Lambda'}(M, \lambda) := (S(M), S(\lambda) \colon a \mapsto S(\lambda(a))).$$

By the additivity of S, it is clear that $S_{\Lambda'}$ commutes with the extensions of scalars $\Lambda' \otimes_\Lambda -$. Moreover, if S is compatible with $\otimes$ and right exact, then $S_{\Lambda'}$ is compatible with the inner product $\otimes'$ defined in 7.3.2: this is a consequence of the construction of $\otimes'$, considered that Λ'/Λ is a finite extension and S commutes with finite direct limits and $\otimes$. Finally, if S is an equivalence of Tannakian categories so is $S_{\Lambda'}$.

7.4. Proof for finite geometric monodromy

The goal of this subsection is to prove the product formula in the case of overconvergent F-isocrystals with geometrically finite monodromy, in particular for F-isocrystals which are canonical extensions: see Proposition 7.4.7 and Corollary 7.4.8. Although the proof for overconvergent F-isocrystals with finite monodromy is analogous to that of [50, 3.2.1.7] and it is given in [54, 4.3.15], there are some technical difficulties to show that for *geometrically* finite case, and we treat this case by using the formal scalar extension reviewed in the previous subsection.

In 7.4.1 and 7.4.2, we define local and global 'constants' for generalized isocrystals in $F\text{-}\mathrm{Isoc}^\dagger(U/K)_{\Lambda'}$. They will be seen as elements of the $\overline{\mathbb{Q}}_p$-algebra of functions $\mathrm{Spec}(\Lambda' \otimes_\Lambda \overline{\mathbb{Q}}_p) \to \overline{\mathbb{Q}}_p$. To avoid any confusion with constant functions, we employ the term *factors* instead of 'constants'. For simplicity, we will often assume that the order of Frobenius is f, so that $\Lambda = K$ and $\sigma_K = \mathrm{id}_K$, cf. 7.1.1; we might state the definitions and prove the lemmas in the general case, but this is not needed for proving Proposition 7.4.7.

7.4.1. Assume $h = f$, and so $\Lambda = K$ (cf. 7.1.1). Let Λ'/K be a finite Galois extension, U be a non-empty open subscheme of a proper curve X, and $(M, \lambda) \in F\text{-}\mathrm{Isoc}^\dagger(U, X/K)_{\Lambda'}$.

The Weil–Deligne representation $\mathrm{WD}(M|_{\eta_x})_{\Lambda'}$ is a $(\Lambda'\otimes_K\overline{\mathbb{Q}}_p)$-module with a linear action of ρ_{η_x} and N_{η_x}. For any $\mathfrak{p}\in\mathrm{Spec}(\Lambda'\otimes_K\overline{\mathbb{Q}}_p)$, we denote by $(\mathrm{WD}(M|_{\eta_x})_{\Lambda'})_{\mathfrak{p}}$ the localization of $\mathrm{WD}(M|_{\eta_x})_{\Lambda'}$ at $\mathfrak{p}$; it is stable under ρ_{η_x} and N_{η_x}. Let us define the *local factors* as functions $\mathrm{Spec}(\Lambda'\otimes_K\overline{\mathbb{Q}}_p)\to\overline{\mathbb{Q}}_p$. For any $x\in|X|$, we set:

1. $\mathrm{rk}(M,\lambda)\colon\mathfrak{p}\mapsto\dim_{\overline{\mathbb{Q}}_p}((\mathrm{WD}(M|_{\eta_x})_{\Lambda'})_{\mathfrak{p}})$, which does not depend on x.
2. $\det_{(M,\lambda)}(x)\colon\quad \mathfrak{p}\mapsto\det_{\overline{\mathbb{Q}}_p}(\rho_{\eta_x}(F_x);((\mathrm{Ker}N_{\eta_x})^{I_{\eta_x}})_{\mathfrak{p}})=\det_{\overline{\mathbb{Q}}_p}(\varphi_{\eta_x}^{a\deg(x)};(\overline{\mathbb{Q}}_p\otimes_{K_x}M_{\eta_x}^{\nabla_x})_{\mathfrak{p}})$ (cf. [54, (3.4.5.3)] for the equality).
3. Let $\omega\neq0$ be in $\Omega^1_{k(X)/k}$, and μ_x be the Haar measure on $\mathcal{K}_x$ with values in $\overline{\mathbb{Q}}_p$ normalized by $\mu_x(\widehat{\mathcal{O}}_{X,x})=1$ as usual, and $\psi(\omega_x)\colon\mathcal{K}_x\to\overline{\mathbb{Q}}_p^*$ is also the additive character associated with ω_x (cf. 7.1.3). As already appeared in [32, 6.4], the epsilon factors $\varepsilon_0^{\mathrm{rig}}((M,\lambda)|_{\eta_x},\psi(\omega_x),\mu_x)$ are defined by

$$\mathfrak{p}\mapsto\varepsilon_0^{\mathrm{rig}}((\mathrm{WD}(M|_{\eta_x})_{\Lambda'})_{\mathfrak{p}},\psi(\omega_x),\mu_x).$$

Let us write simply $\varepsilon_0^{\mathrm{rig}}((M,\lambda)|_{\eta_x},\omega_x)$, instead of $\varepsilon_0^{\mathrm{rig}}((M,\lambda)|_{\eta_x},\psi(\omega_x),\mu_x)$.

7.4.2. As in 7.4.1, assume $h=f$, and so $\Lambda=K$. Let (M,λ) be in $F\text{-}\mathrm{Isoc}^\dagger(U,X/K)_{\Lambda'}$. The rigid cohomology groups (with and without supports) of M inherit a Λ'-structure, so that they are F-isocrystals with Λ'-structure on $\mathrm{Spec}(k)$ over K; we denote them respectively by $H^i_{\mathrm{rig}}(U,M)_{\Lambda'}$ and $H^i_{\mathrm{rig},c}(U,M)_{\Lambda'}$ (cf. 7.3.5). They are Λ'-vector spaces endowed with linear Frobenius isomorphisms φ. To shorten the notation, let H denote either $H^i_{\mathrm{rig},c}(U,M)_{\Lambda'}$ or $H^i_{\mathrm{rig}}(U,M)_{\Lambda'}$.

We define $\det(-F;H)$ as the constant function $\mathrm{Spec}(\Lambda'\otimes_K\overline{\mathbb{Q}}_p)\to\overline{\mathbb{Q}}_p$, $\mathfrak{p}\mapsto\det_{\Lambda'}(-F^*;H)$. We denote by $\det(-F^*;H^*_{\mathrm{rig},c}(U,M)_{\Lambda'})$ the product $\prod_{i=0}^2\det(-F;H^i_{\mathrm{rig},c}(U,M)_{\Lambda'})$. From the long exact sequence of rigid cohomology, it follows that $\det(-F^*;H^*_{\mathrm{rig},c}(U,M)_{\Lambda'})$ is multiplicative for short exact sequences and so it is defined on the Grothendieck group of $F\text{-}\mathrm{Isoc}^\dagger(U/K)_{\Lambda'}$.

Remark. In the general case, where $h|f$, the group H is a module over the semi-local ring $\Lambda'\otimes_\Lambda K$ and we endow it with the $(\Lambda'\otimes_\Lambda K)$-linear endomorphism $F=\varphi^a$, where $a=hf^{-1}$. The module H decomposes as $\bigoplus_{\mathfrak{p}\in\mathrm{Spec}(\Lambda'\otimes_\Lambda K)}H_{\mathfrak{p}}$. For every $\mathfrak{p}\in\mathrm{Spec}(\Lambda'\otimes_\Lambda K)$, the localization $H_{\mathfrak{p}}$ is a vector space over the field $K':=\Lambda'\otimes_{\Lambda''}K$ (where $\Lambda'':=\Lambda'\cap K\subset\overline{\mathbb{Q}}_p$ is a finite unramified extension of Λ), and F induces a K'-linear endomorphism of $H_{\mathfrak{p}}$. We define $\det(-F;H)$ as the composition of the canonical map $\mathrm{Spec}(\Lambda'\otimes_\Lambda\overline{\mathbb{Q}}_p)\to\mathrm{Spec}(\Lambda'\otimes_\Lambda K)$, induced by the inclusion $K\subset\overline{\mathbb{Q}}_p$, and the function $\mathrm{Spec}(\Lambda'\otimes_\Lambda K)\to\overline{\mathbb{Q}}_p$ sending $\mathfrak{p}$ to $\det_{K'}(-F;H_{\mathfrak{p}})$. We do not use this remark in the following.

7.4.3. Let us follow the notation of 7.4.1 and 7.4.2. Let us state a variant of the product formula (PF*) for overconvergent F-isocrystals in $F\text{-}\mathrm{Isoc}^\dagger(U,X/K)_{\Lambda'}$. Let U be a non-empty open subscheme of X, $M'=(M,\lambda)$ be in $F\text{-}\mathrm{Isoc}^\dagger(U,X/K)_{\Lambda'}$ and ω a non-zero element of $\Omega^1_{k(X)/k}$. The product formula for M' is the following relation

$$\det\big(-F^*; H^*_{\mathrm{rig},c}(U, M')_{\Lambda'}\big)^{-1} = q^{C(1-g)\mathrm{rk}(M')} \prod_{x\in|U|} q_x^{v_x(\omega)\mathrm{rk}(M')} \det_{M'}(x)^{v_x(\omega)} \times \prod_{x\in X\setminus U} \varepsilon_0^{\mathrm{rig}}(M'_{\eta_x}, \omega_x) \tag{7.4.3.1}$$

between global and local factors associated with M'.

7.4.4. Assume $h = f$ and so $\Lambda = K$. Let U be a non-empty open subscheme of X, and M an overconvergent F-isocrystal on U over K. For any finite Galois extension Λ'/K, we can define an overconvergent F-isocrystal $\Lambda' \otimes_K M$ with Λ'-structure (cf. 7.3.4).

Lemma. *The F-isocrystal M satisfies the product formula (PF*) if and only if $\Lambda' \otimes_K M$ satisfies the product formula with Λ'-structure 7.4.3.1.*

Proof. For an abelian category $\mathcal{A}$, we denote by $\mathbf{Gr}(\mathcal{A})$ its Grothendieck group. In this proof we put $\mathcal{A} = F\text{-Isoc}^\dagger(U, X/K)$. The formula (PF*) (resp. 7.4.3.1) is a relation on the Grothendieck group of $\mathcal{A}$ (resp. $\mathcal{A}_{\Lambda'}$) with values in $\overline{\mathbb{Q}}_p^*$ (resp. in the group of units of the $\overline{\mathbb{Q}}_p$-algebra $\overline{\mathbb{Q}}_p^{\,\mathrm{Spec}(\Lambda'\otimes_K\overline{\mathbb{Q}}_p)}$). Each factors ν appearing in the equality (PF*) (resp. 7.4.3.1) are homomorphisms $\nu\colon \mathbf{Gr}(\mathcal{A}) \to \overline{\mathbb{Q}}_p^*$ (resp. $\nu_{\Lambda'}\colon \mathbf{Gr}(\mathcal{A}_{\Lambda'}) \to \big(\overline{\mathbb{Q}}_p^{\,\mathrm{Spec}(\Lambda'\otimes_K\overline{\mathbb{Q}}_p)}\big)^*$). By the definitions of these factors, cf. (7.4.1–7.4.2), it follows the commutativity of the diagram

$$\begin{array}{ccc}
\mathbf{Gr}(\mathcal{A}) & \xrightarrow{\quad\nu\quad} & \overline{\mathbb{Q}}_p^* \\
\big\downarrow{\scriptstyle \Lambda'\otimes_K -} & & \big\downarrow \\
\mathbf{Gr}(\mathcal{A}_{\Lambda'}) & \xrightarrow{\nu_{\Lambda'}} & \big(\overline{\mathbb{Q}}_p^{\,\mathrm{Spec}(\Lambda'\otimes_K\overline{\mathbb{Q}}_p)}\big)^*,
\end{array}$$

where the right vertical homomorphism maps each element c of $\overline{\mathbb{Q}}_p^*$ to the constant function $\mathrm{Spec}(\Lambda'\otimes_K\overline{\mathbb{Q}}_p) \to \overline{\mathbb{Q}}_p$ of value c. Since this homomorphism is injective we conclude. ☐

7.4.5. We recall that the theorem of Tsuzuki [69, (7.2.2), Theorem 7.2.3] gives an equivalence $G^\dagger\colon F\text{-Isoc}^\dagger(U, X/K)^u \to \mathrm{Rep}^{\mathrm{fg}}_\Lambda(\pi_1(U, \overline{\eta}_X))$, between the categories of unit-root overconvergent F-isocrystals on U over K and continuous Λ-representations of $\pi_1(U, \overline{\eta}_X)$ with local geometrically finite monodromy. We say that a unit-root overconvergent F-isocrystal $M \in F\text{-Isoc}^\dagger(U, X/K)^u$ has *global finite monodromy* if the associated representation $G^\dagger(M)$ factors through a finite quotient of $\pi_1(U, \overline{\eta}_X)$; we say that M has *global geometrically finite monodromy* if the restriction of $G^\dagger(M)$ to $\pi_1(U \otimes_k \overline{\mathbb{F}}, \overline{\eta}_X)$ factors through a finite quotient.

The following lemma extends [54, Theorem 4.3.15] to isocrystals with Λ'-structure.

7.4.6 Lemma. *Assume $h = f$, so that $\Lambda = K$ and $\sigma_K = \mathrm{id}_K$. Let Λ'/K be a finite Galois extension and U be a non-empty open subscheme of X. Let $M' = (M, \lambda)$ be an* *overconvergent F-isocrystal with Λ'-structure on U over K. Assume M is unit-root with global finite monodromy, then the formula 7.4.3.1 is satisfied.*

Proof. By a base change to the algebraic closure k' of k in $k(X)$, we may assume that X is geometrically connected, *i.e.* $C = 1$. If Λ'/K is totally ramified, we may put $\sigma_{\Lambda'} := \mathrm{id}_{\Lambda'}$; then the category $F\text{-Isoc}^\dagger(U, X/K)_{\Lambda'}$ identifies to $F\text{-Isoc}^\dagger(U, X/\Lambda')$, cf. Example 7.3.3, and so we finish by [54, Theorem 4.3.15].

Let us treat the general case. For any representation $\rho\colon \pi_1(U, \bar\eta_X) \to \mathrm{Aut}_K\big(W\big)$ and any closed point $x \in |X|$, we denote by W_{η_x} the representation $\pi_1(\eta_x, \bar\eta_x) \to \pi_1(U, \bar\eta_X) \xrightarrow{\rho} \mathrm{Aut}_K\big(W\big)$. We put $V_{\eta_x} := G^\dagger(M)_{\eta_x}$, and $V'_{\eta_x} := G^\dagger_{\Lambda'}(M')_{\eta_x}$ (cf. 7.3.4) which will be treated as a Λ'-vector space with a linear action of $W(\eta_x, \bar\eta_x)$. Let us start by proving the following statement.

Claim. *The $(\overline{\mathbb{Q}}_p \otimes_K \Lambda')$-module $\mathrm{WD}_{\Lambda'}(M'_{\eta_x})$ is free and for any $\mathfrak{p} \in \mathrm{Spec}(\overline{\mathbb{Q}}_p \otimes_K \Lambda')$, we have $\mathrm{WD}_{\Lambda'}(M'_{\eta_x})_{\mathfrak{p}} = \overline{\mathbb{Q}}_p \otimes_{\Lambda'} V'_{\eta_x}$.*

Proof of the claim. Let us compute $\Psi(-1)_{\Lambda'}(M'_{\eta_x})$ and $\mathrm{WD}_{\Lambda'}(M'_{\eta_x})$. Their monodromy operators N are zero, because M is unit-root. Let us denote by $\underline{\mathrm{D\acute{e}co}}(\widehat{K}^{\mathrm{ur}} \otimes_K V_{\eta_x})$ the sub-K^{ur}-vector space of $\widehat{K}^{\mathrm{ur}} \otimes_K V_{\eta_x}$, spanned by the finite orbits under the action of $\pi_1(\eta_x, \bar\eta_x)$. By [54, 3.3.6], we have $\Psi(-1)(M_{\eta_x}) = \underline{\mathrm{D\acute{e}co}}(\widehat{K}^{\mathrm{ur}} \otimes_K V_{\eta_x})$. Since M has finite monodromy, we get $\Psi(-1)(M_{\eta_x}) = K^{\mathrm{ur}} \otimes_K V_{\eta_x}$ endowed with the diagonal action of $\pi_1(\eta_x, \bar\eta_x)$ (it acts on K^{ur} via the residual action). Hence $\mathrm{WD}(M_{\eta_x}) = \overline{\mathbb{Q}}_p \otimes_K V_{\eta_x}$, where the action of $W(\eta_x, \bar\eta_x)$ is nothing else than the extension by linearity of the action of $W(\eta_x, \bar\eta_x)$. We finish by the equality $\mathrm{WD}_{\Lambda'}(M'_{\eta_x}) = \overline{\mathbb{Q}}_p \otimes_K V'_{\eta_x} = \overline{\mathbb{Q}}_p \otimes_K \Lambda' \otimes_{\Lambda'} V'_{\eta_x}$. ■

To establish the product formula for M', it remains to prove the following relation:

$$\det_{\Lambda'}\big(-F^*; H^*_{\mathrm{rig},c}(U, M')_{\Lambda'}\big)^{-1} = q^{\frac{(1-g)\mathrm{rk}(M)}{|\Lambda':K|}} \prod_{x\in|U|} q_x^{\frac{v_x(\omega)\mathrm{rk}(M)}{|\Lambda':K|}} \det_{\Lambda'}(\rho_{\eta_x}(F_x); (V'_{\eta_x})^{I_{\eta_x}})^{v_x(\omega)} \times \prod_{x\in X\setminus U} \varepsilon_0^{\mathrm{rig}}(V'_{\eta_x}, \omega_x).$$

The proof of this equation works in the same way as the proofs of [54, Theorem 4.3.11 and 4.3.15], by replacing Λ (resp. $\mathrm{rk}(M)$) with Λ' (resp. $\frac{\mathrm{rk}(M)}{|\Lambda':K|}$). □

7.4.7 Proposition. *Let X be a proper curve over k, U be a non-empty open subscheme of X, and M be an overconvergent F-isocrystal on U over K. Assume that M is unit-root with global geometrically finite monodromy. Then M satisfies the product formula (PF*).*

Proof. In this proof we put $\overline{\eta} := \overline{\eta}_X$ for brevity. Let $\rho\colon \pi_1(U, \overline{\eta}) \to \mathrm{Aut}_\Lambda\big(V\big)$ be the representation associated with M by $G^\dagger$, cf. 7.4.5. By assumptions, M has global geometrically finite monodromy: *i.e.* the restriction of ρ to $\pi_1(U \otimes_k \overline{\mathbb{F}}, \overline{\eta})$ factors through a finite quotient I. In particular, the representation ρ factors through a quotient Q of $\pi_1(U, \overline{\eta})$ which is an extension of $\widehat{\mathbb{Z}}$ by the finite group I. By Remark 7.2.7, we may assume that the order of the Frobenius φ_M of M is f, so that $K = \Lambda$ and $\sigma_K = \mathrm{id}_K$.

Let us show how we can reduce to the case of global finite monodromy which is treated in Lemma 7.4.6. The equation (PF*) that we have to prove is a relation in the Grothendieck group of $F\text{-Isoc}^\dagger(U, X/K)$. By Lemma 7.4.4, we can extend scalars to any finite Galois extension Λ'/K. The equivalence $G^\dagger$ above extends to an equivalence of Tannakian categories $G^\dagger_{\Lambda'}\colon F\text{-Isoc}^\dagger(U, X/K)^u_{\Lambda'} \to \mathrm{Rep}^{\mathrm{fg}}_K(\pi_1(U, \overline{\eta}))_{\Lambda'}$ (cf. 7.3.5). We identify $\mathrm{Rep}^{\mathrm{fg}}_K(\pi_1(U, \overline{\eta}))_{\Lambda'}$ with $\mathrm{Rep}^{\mathrm{fg}}_{\Lambda'}(\pi_1(U, \overline{\eta}))$ (cf. Example 7.3.3). The product formula is a relation in the Grothendieck group of $\mathrm{Rep}^{\mathrm{fg}}_{\Lambda'}(\pi_1(U, \overline{\eta}))$; we may assume that the representation is absolutely irreducible. By a classical argument using Schur's lemma (cf. for example [50, Proof of 3.2.1.7] or [32, Variant 4.10.3]), the representation (V', ρ') is isomorphic to $(\tilde{V}, \tilde{\rho}) \otimes_{\Lambda'} (\Lambda', \chi)$, where $\tilde{\rho}\colon \pi_1(U, \overline{\eta}) \to \mathrm{Aut}_{\Lambda'}(\tilde{V})$ factors through a finite quotient and $\chi: \pi_1(U, \overline{\eta}) \to (\Lambda')^*$ is an unramified character. Let $D^\dagger_{\Lambda'}\colon \mathrm{Rep}^{\mathrm{fg}}_K(\pi_1(U, \overline{\eta}))_{\Lambda'} \to F\text{-Isoc}^\dagger(U, X/K)^u_{\Lambda'}$ be a quasi-inverse of $G^\dagger_{\Lambda'}$. Let us put $M' := D^\dagger_{\Lambda'}((V', \rho'))$, $M_1 := D^\dagger_{\Lambda'}((\tilde{V}, \tilde{\rho}))$ and $M_2 := D^\dagger_{\Lambda'}((\Lambda', \chi))$. We have $M' \cong M_1 \otimes' M_2$ in $F\text{-Isoc}^\dagger(U, X/K)^u_{\Lambda'}$ (cf. 7.3.2). By construction, M_1 has global finite monodromy and M_2 is constant as isocrystal, *i.e.*, $M_2 = \iota^* N$, for $N \in F\text{-Isoc}(\mathrm{Spec}(k)/K)^u_{\Lambda'}$ and $\iota\colon U \to \mathrm{Spec}(k)$. Since $\iota^* N$ is a constant isocrystal, we have $H^i_{\mathrm{rig},c}(U, M_1 \otimes' \iota^* N)_{\Lambda'} \cong H^i_{\mathrm{rig},c}(U, M_1)_{\Lambda'} \otimes' N$. By a direct calculation analogous to that of the proof of [54, 4.3.6], we reduce to the case of global finite monodromy, which is proven in Lemma 7.4.6. □

7.4.8 Corollary. *Let M be in $F\text{-Hol}(\eta_{\mathscr{S}_0})$. Then the canonical extension M^{can} satisfies the product formula (PF*).*

Proof. By Kedlaya's filtration theorem [45, 7.1.6], there exists a filtration

$$M = M_0 \supset M_1 \supset \cdots \supset M_s = 0$$

such that the quotient M_i/M_{i+1} is isoclinic for every i. By applying the canonical extension functor 2.1.9, we get an analogous filtration on M^{can}. Considering that the equation (PF*) we have to prove is a relation in the Grothendieck group of $F\text{-Isoc}^\dagger(\mathbb{G}_m, \mathbb{P}^1_k/K)$, we may assume M to be isoclinic of Dieudonné-Manin slope $\lambda \in \mathbb{Q}$. By the definition of Dieudonné-Manin slopes, λ belongs to the discrete subgroup $(\mathrm{rk}(M)eh)^{-1}\mathbb{Z}$. Taking a finite *totally ramified* extension of Λ does not affect the local factors; therefore, by extending Λ to such an extension of degree $\mathrm{rk}(M)$, we may assume that λ belongs to $(eh)^{-1}\mathbb{Z}$. So the isocrystal $K^{(-\lambda)}$ is of rank one by construction. We put $\widetilde{M} := M \otimes K^{(-\lambda)}$. We have $M = \widetilde{M}^{(\lambda)}$, with $\widetilde{M}$ unit-root. By applying [54, Lemme 4.3.6] to $M^{\mathrm{can}} = (\widetilde{M}^{\mathrm{can}})^{(\lambda)}$, we may assume $\lambda = 0$. Since $\widetilde{M}$ is unit-root, $\widetilde{M}^{\mathrm{can}}$ has global geometrically finite monodromy by the very construction of the canonical extension (cf. [29, 2.6 and 2.7]), and we finish by Proposition 7.4.7. □

7.5. Proof of the main result

We use the notation of 2.4.1.

7.5.1 Lemma. *Let E be an overconvergent F-isocrystal on $\mathbb{A}^1_k$. Suppose that it is regular at infinity. Then E is a constant overconvergent F-isocrystal.*

Proof. Let ι denote the structural morphism of $\mathbb{A}^1_k$. By construction of the rigid cohomology [28, (8.1.1)], we have $H^2_{\mathrm{rig}}(\mathbb{A}^1_k, E) = 0$ and, by the GOS formula for rigid cohomology, we get $\dim_K H^0_{\mathrm{rig}}(\mathbb{A}^1_k, E) - \dim_K H^1_{\mathrm{rig}}(\mathbb{A}^1_k, E) = \chi_{\mathrm{rig}}(\mathbb{A}^1_k, E) = \mathrm{rk}(E)$. In particular $h_0 := \dim_K H^0_{\mathrm{rig}}(\mathbb{A}^1_k, E) \geqslant \mathrm{rk}(E)$. By [23, 2.1.2], there is an injection of F-isocrystals $\iota^* H^0_{\mathrm{rig}}(\mathbb{A}^1_k, E) \hookrightarrow E$, so that $h_0 = \mathrm{rk}(E)$ and E is isomorphic to the constant isocrystal $\iota^* H^0_{\mathrm{rig}}(\mathbb{A}^1_k, E)$. □

7.5.2. Let $\mathscr{S} := \mathrm{Spf}(k[\![u]\!])$, $\mathscr{S}' := \mathrm{Spf}(k[\![u']\!])$. We put $\mathcal{K} := k(\!(u)\!)$, $\mathcal{K}' := k(\!(u')\!)$. Let $\pi_{\infty'} : \mathscr{S}_{\infty'} \xrightarrow{\sim} \mathscr{S}'$ sending u' to $1/x'$. For a free differential module M with Frobenius structure on η', using the canonical extension, there exists a canonical overconvergent F-isocrystal on $\mathbb{G}_m$ denoted by $\mathscr{M}$ such that it is tamely ramified at zero and $\mathscr{M}|_{\eta_{\infty'}} \cong \pi^*_{\infty'}(M)$. We denote $\Psi(\mathscr{M}|_{S_1})$ by $\Psi_1(M)$.

On the other hand, suppose moreover that M is of rank one. Using the linearization functor 7.1.2.1, we get a character

$$\chi := (L_k \circ \Psi)(M) \colon G_{\mathcal{K}'} \to K^{\mathrm{ur}*}.$$

For $f' \in \mathcal{K}'^*$, we put

$$M(f') := (\chi \circ \mathrm{rec})(f') \in K^{\mathrm{ur}*},$$

where $\mathrm{rec}\colon \mathcal{K}'^* \to G^{\mathrm{ab}}_{\mathcal{K}'}$ is the reciprocity map normalized *à la* Deligne (*i.e.* it sends uniformizers of $\mathcal{K}'$ to elements of $G^{\mathrm{ab}}_{\mathcal{K}'}$ whose image in G^{ab}_k is the geometric Frobenius F).

When M is regular of rank one, we get

$$\mathrm{tr}(F^*; \Psi_1(M)) = M(-u). \tag{7.5.2.1}$$

This can be seen in exactly the same way as [50, 3.5.2.1], and we leave the details to the reader.

Remark. We need to be careful for the multiplicativity. Namely, given rank one free differential modules M, M', M'' such that $M = M' \otimes M''$, we have

$$M(-1)(f') = M'(-1)(f') \cdot M''(-1)(f').$$

See 7.1.2 for an explanation.

7.5.3 Proposition ([50, Théorème 3.4.2])**.** *Let $U \subset \mathbb{A}^1$ be an open subscheme, and we put $S := \mathbb{A}^1 \setminus U$. Let $\mathscr{M}$ be an overconvergent F-isocrystal of rank r on U which is unramified at infinity. Then we get*

$$\begin{aligned}&\det(R\Gamma_c(U_{K^{\mathrm{ur}}}, \mathscr{M})[1]) \otimes \det(K^{\mathrm{ur}} \otimes_K \mathscr{M}|_{s_\infty})(-r)\\ &\cong \bigotimes_{s \in S} \Psi_1\big(\det\big(\Phi^{(0,\infty')}(j_! \tau_{s*} \mathscr{M}|_{\eta_s})\big)(-\gamma_s - 1)\big)\end{aligned}$$

as Deligne modules, where $\gamma_s = \mathrm{rk}(\mathscr{M}|_{\eta_s}) + \mathrm{irr}(\mathscr{M}|_{\eta_s})$, and we used the notation of 3.1.9.

Proof. Using the notation of Proposition 6.2.8, we get

$$\det(\Psi(\mathscr{F}_{\mathrm{naive}}(\mathscr{M}')|_{S_0})(-2)) \cong \det(H^*_{\mathrm{rig},c}(U_{K^{\mathrm{ur}}},\mathscr{M})[1]) \otimes \det(K^{\mathrm{ur}} \otimes_K \mathscr{M}|_{s_\infty}(-1))$$

as Deligne modules by the same proposition. Since the $G_{\mathcal{K}_0}$-action on the right hand side is unramified (*i.e.* the action of the inertia subgroup is trivial), the left hand side is also unramified. Since $\mathbb{V}$ is an exact functor and commutes with tensor product, we get an equivalence of functors $\det\circ\mathbb{V} \cong \mathbb{V}\circ\det$. On the other hand, for a free differential module M, we get $\mathbb{D}_\eta(\det(M)(-1)) \cong \det(\mathbb{D}_\eta(M(-1)))$. Thus, we get

$$\det(\Psi(\mathscr{F}_{\mathrm{naive}}(\mathscr{M}')|_{S_0})(-2)) \cong \Psi(-1)(\det(\mathscr{F}_{\mathrm{naive}}(\mathscr{M}')|_{S_0}(-1))). \tag{7.5.3.1}$$

We note that the singularity of $\mathscr{F}_{\mathrm{naive}}(\mathscr{M}')$ in $\widehat{\mathbb{A}}$ is only at $0'$ by Corollary 4.1.7, and we showed that $\det(\mathscr{F}_{\mathrm{naive}}(\mathscr{M}'))$ is unramified at $0'$. Thus there exists an overconvergent F-isocrystal $\mathscr{N}$ on $\widehat{\mathbb{A}}$ such that $\mathscr{N}|_{\widehat{\mathbb{A}}'-\{0'\}} \cong \det(\mathscr{F}_{\mathrm{naive}}(\mathscr{M}'))|_{\widehat{\mathbb{A}}'-\{0'\}}$.

Now, we get

$$\begin{aligned}\mathscr{N}|_{\eta_\infty} &= \det\big(\mathscr{F}_{\mathrm{naive}}(\mathscr{M}')|_{\eta_\infty}(-1)\big)\\ &\cong \bigotimes_{s\in S} \tau'^* \det\big(\mathscr{F}^{(s,\infty')}(\mathscr{M}')(-1)\big)\\ &\cong \bigotimes_{s\in S} \det\big(\Phi^{(0,\infty')}(j_!\tau_{s*}\mathscr{M}|_{\eta_s})\big)(-\gamma_s)\otimes L(\delta),\end{aligned}$$

where $\delta := \sum_{s\in S}\gamma_s\cdot\mathrm{tr}(s)$, by the stationary phase formula 5.1.8, Lemma 2.4.10, and Corollary 4.2.4. We note that the differential slope of $\Phi^{(0,\infty')}(j_!\tau_{s*}\mathscr{M}|_{\eta_s})$ is strictly less than one by Corollary 4.2.4 for any $s\in S$. Thus, the differential slope of $\det\big(\Phi^{(0,\infty')}(j_!\tau_{s*}\mathscr{M}|_{\eta_s})\big)$ is also strictly less than one. Since the rank is one, the differential slope of $\det\big(\Phi^{(0,\infty')}(j_!\tau_{s*}\mathscr{M}|_{\eta_s})\big)$ is zero for any $s\in S$ by the Hasse–Arf theorem [25, 14.12]. Thus

$$\mathscr{N}' := \mathscr{N}\otimes\mathscr{L}(-\delta\cdot x')$$

is an overconvergent F-isocrystal on $\widehat{\mathbb{A}}'$, and regular at infinity. By Lemma 7.5.1, $\mathscr{N}'$ is in fact a constant overconvergent F-isocrystal. By 7.5.3.1 and the fact that $\Psi(-1)$ commutes with $\otimes$, the proposition follows. □

7.5.4 Theorem (p-Adic Determinant Formula)**.** *For any free differential $\mathcal{R}_{\mathscr{S}}$-module with Frobenius structure M, we get*

$$\varepsilon_0^{\mathrm{rig}}(M,\mathrm{d}u) = (-1)^\gamma\det(\Phi^{(0,\infty')}(j_!M))(-\gamma-1)(u'),$$

where $\gamma := \mathrm{rk}(M)+\mathrm{irr}(M)$.

Remark. (i) Before proving the theorem, we remark that the right hand side of the equality is multiplicative with respect to short exact sequences by Remark 7.5.2.

(ii) Although the idea of the proof is exactly the same as ([25], Theorem 3.5.11), we include the complete proof of the proposition since there are considerable of minor differences in the quantities appearing (especially the Tate twist $-\gamma-1$ in the statement of the theorem), and we think that it might help the reader to understand the differences with the ℓ-adic case.

Proof. Let V be a σ_K-K-vector space of dimension one. Then the theorem holds for a free differential $\mathcal{R}$-module M if and only if it holds for $M \otimes_K V$. Indeed, by [54, 2.19 (2)], we have

$$\varepsilon_0^{\mathrm{rig}}(M \otimes_K V, \mathrm{d}u) = \varepsilon_0^{\mathrm{rig}}(M, \mathrm{d}u) \cdot \det(F; V)^{\gamma}.$$

On the other hand, we see that

$$\Phi^{(0,\infty')}(j_! M \otimes_K V) \cong \Phi^{(0,\infty')}(j_! M) \otimes_K V$$

by using Proposition 5.1.6. Thus, the claim follows.

First we will treat the case where M is regular. Since both sides are multiplicative, we may assume that M is irreducible. By Kedlaya's slope filtration theorem, we get that M is isoclinic (for Dieudonné-Manin slopes). Since both sides of the equality are stable under base change by a totally ramified extension of Λ, we may assume that the Dieudonné-Manin slope λ of M is in $(eh)^{-1}\mathbb{Z}$. Since the equality is stable under twisting and $(M^{(-\lambda)})^{(\lambda)} \cong M$, it suffices to show the proposition for $M^{(-\lambda)}$, and we may suppose that M is unit-root. Thus, it corresponds to a geometrically finite representation of $G_{\mathcal{K}}$ denoted by ρ. Since M is assumed to be regular, we know that there exists a finite extension k'/k such that ρ is the induced representation of $G_{\mathcal{L}}$ of rank one where $\mathcal{L} := k' \otimes_k \mathcal{K}$. This shows that there exists a finite unramified extension L of K, and a free differential $\mathcal{R}_L$-module with Frobenius structure M_L such that $M \cong f_* M_L$ where $f : \mathcal{R} \hookrightarrow \mathcal{R}_L$ is the canonical finite étale homomorphism. By 2.4.6, we get

$$\Phi^{(0,\infty')}(j_! M) \cong f_*(\Phi^{(0,\infty')}(j_! M_L)).$$

For the calculation of $\varepsilon_0^{\mathrm{rig}}$, use [54, 2.14 (2)]. It remains to show the theorem for M_L, and thus, we may assume that M is of rank one.

In this case we can write $M \cong \mathscr{K}_\alpha|_{\eta_0} \otimes \mathcal{W}$ such that $\mathcal{W}$ is a trivial differential module if we forget the Frobenius structure, and $\mathscr{K}_\alpha$ is the Kummer isocrystal (cf. 3.2.9). Then we may suppose that $\mathcal{W}$ is trivial by the observation at the beginning of this proof. It remains to show the theorem in the $M = \mathscr{K}_\alpha|_{\eta_0} =: \mathcal{F}u_\alpha$ case.

Now, first, assume that $\alpha \notin \mathbb{Z}$. Then $j_! j^+ \mathcal{F}u_\alpha \cong \mathcal{F}u_\alpha$ by Proposition 3.2.9. By the stationary phase formula and the same proposition, we get

$$\Phi^{(0,\infty')}(\mathcal{F}u_\alpha) \cong \tau'^*\big(\mathscr{K}_{1-\alpha} \otimes G(\alpha, \pi)(1)\big)|_{\eta'_\infty} \cong \mathcal{F}u_\alpha \otimes G(\alpha, \pi)(1).$$

Furthermore, we have

$$\Psi(\mathcal{F}u_\alpha) \cong \mathbb{V}((\mathcal{F}u_\alpha)^\vee(-1)) \cong (\mathcal{B} \otimes \mathcal{F}u_\alpha)^{\partial=0}(1)$$

by definition (cf. 3.1.9). Combining these, we get $\Psi\big(\Phi^{(0,\infty')}(\mathcal{F}u_\alpha)\big)(-2) \cong (\mathcal{B} \otimes \mathcal{F}u_\alpha)^{\partial=0} \otimes G(\alpha, \pi)$. The space $(\mathcal{B} \otimes \mathcal{F}u_\alpha)^{\partial=0}$ is the sub-K^{ur}-vector space spanned by $x^{-\alpha} e$ where e is the canonical base of $\mathcal{F}u_\alpha$. Let $\alpha = i \cdot (q-1)^{-1}$, and χ_α be the i-th power of Teichmüller character $k^* \to K^{\mathrm{ur}}$. Then we get

$$\mathrm{tr}(F^*; G(\alpha, \pi)) = -\sum_{x \in k^*} \chi_\alpha(x) \cdot \psi_k(x)$$

by [38, 6.5] using the fact that $H^i_{\mathrm{rig}}(\mathbb{G}_m, \mathscr{K}_\alpha \otimes \mathscr{L}_\pi) = 0$ for $i \neq 1$ (which can be proved by GOS-formula for example). Now, let us treat the case where $\alpha = 0$, and thus $M = \mathcal{R}$. In this case, we get an exact sequence

$$0 \to \delta(1) \to j_!\mathcal{R} \to j_+\mathcal{R} \to \delta \to 0.$$

We get $\Phi^{(0,\infty')}(j_+\mathcal{R}) \cong j_+\mathcal{R}$ by using Proposition 3.2.7 and Theorem 3.2.8. Thus, we obtain $\Phi^{(0,\infty')}(j_!\mathcal{R}) \cong j_+\mathcal{R}(1)$. Combining all of these, we get

$$\det(\Phi^{(0,\infty')}(j_!\mathcal{F}u_\alpha))(-2)(u') = \begin{cases} 1 & \text{if } \alpha = 0 \\ -\chi(-1) \cdot \sum_{x \in k^*} \chi_\alpha(x) \cdot \psi_k(x) & \text{if } \alpha \neq 0. \end{cases}$$

On the other hand, let $\widetilde{\chi} := (\mathrm{rec} \circ \mathrm{WD})(M)$. By using [64, XIV §3, Proposition 8], we get

$$\widetilde{\chi}(x) = \begin{cases} \chi_\alpha^{-1}(x) & x \in k^* \subset \mathcal{K}^* \\ \chi_\alpha(-1) & x = u \\ 1 & x \in 1 + \mathfrak{m} \end{cases}$$

(note that the image of the geometric Frobenius is different, so we need to calculate (x^{-i}, a^{-1}) with $n = q - 1$ in the notation of [64].). This shows that

$$\varepsilon_0^{\mathrm{rig}}(\mathcal{F}u_\alpha, \mathrm{d}u) = \begin{cases} -1 & \text{if } \alpha = 0 \\ \chi(-1) \cdot \sum_{x \in k^*} \chi_\alpha(x) \cdot \psi_k(x) & \text{if } \alpha \neq 0. \end{cases}$$

Thus the theorem follows in this case.

We denote by r the rank of M. By taking the canonical extension, there exists an overconvergent F-isocrystal $\mathscr{M}'$ on $\widehat{\mathbb{P}}^1 \setminus \{0, 1\}$ *regular at* 1, and $\mathscr{M}'|_{\eta_0} \cong \pi_0^*(M)$. We put $\mathscr{M} := \mathscr{D}^\dagger_{\widehat{\mathbb{P}},\mathbb{Q}}(\infty) \otimes_{\mathscr{D}^\dagger_{\widehat{\mathbb{P}},\mathbb{Q}}} \mathscr{M}'$. By Corollary 7.4.8, we get

$$\det(-F;\, H^*_c(\mathbb{A}^1 \setminus \{0,1\}, \mathscr{M}))^{-1} \cdot q^r \cdot \det(-F_\infty;\, \mathscr{M}|_{s_\infty}) = \varepsilon_0^{\mathrm{rig}}(M, -\mathrm{d}u) \cdot \varepsilon_0^{\mathrm{rig}}(\mathscr{M}|_{\eta_1}, -\mathrm{d}x|_{\eta_1}).$$

On the other hand, by Proposition 7.5.3, we get

$$\begin{aligned} &\det(-F;\, H^*_c(\mathbb{A}^1 \setminus \{0,1\}, \mathscr{M}))^{-1} \cdot q^r \cdot \det(-F_\infty;\, \mathscr{M}|_{s_\infty}) \\ &\quad = (-1)^\gamma \det(\Phi^{(0,\infty')}(j_!M))(-\gamma-1)(-u') \cdot (-1)^r \det(\Phi^{(0,\infty')}(j_!\mathscr{M}|_{\eta_1}))(-r-1)(-u') \end{aligned}$$

taking 7.5.2.1 into account (which is applicable because the differential modules $\det(\Phi^{(0,\infty')}(\ldots))$ are regular of rank one). By considering the regular case proven above, we get the theorem. □

7.5.5 Corollary. *Let $U \subset \mathbb{A}^1_k$ be an open subscheme, and M be an overconvergent F-isocrystal of rank r on U which is unramified at ∞. Let $S := \mathbb{A}^1 \setminus U$. Then we get*

$$\det(-F;\, H^*_{\mathrm{rig},c}(U, M))^{-1} \cdot q^r \cdot \det(-F_\infty;\, M_\infty) = \prod_{s \in S} \varepsilon_0^{\mathrm{rig}}(M|_{\eta_s}, -\mathrm{d}x|_{\eta_s}).$$

Proof. The proof is exactly the same as [50, 3.5.2] using Proposition 7.5.3, and we leave the details to the reader. □

7.5.6. Proof of Theorem 7.2.5 The proof is essentially the same as [50, 3.3.2]: it is a reduction to Corollary 7.5.5. We point out a difference from [50]: to prove that the right hand side of the product formula does not depend on the choice of the differential form ω, we proceed by *dévissage* and we use [54, Proposition 4.3.9]. We leave the details to the reader. ■

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