Symplectic and almost complex manifolds

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with an appendix by P. Gauduchon

The aim of this chapter is to introduce the basic problems and (soft!) techniques in symplectic geometry by presenting examples—more exactly series of examples—of almost complex and symplectic manifolds: it is obviously easier to understand the classification of symplectic ruled surfaces if you have already heard of Hirzebruch surfaces for instance.

The first § explains why there are so many almost complex structures on a symplectic manifold, an elementary, but very important tool of Gromov's theory. Then, I will explain what the complex ruled surfaces look like (Hirzebruch surfaces) and give some systematic constructions of symplectic manifolds, either by symplectic reduction or by surgery: here I will explain which surgeries cannot work... and which *do* (for instance the ones used in the recent constructions of Gompf). The last § is an appendix, written by P. Gauduchon, in which he gives a construction of an almost complex structure on a certain jet space which will be useful to state and prove Gromov's Schwarz lemma in chapter VII.

I have not tried to give a complete list of references for the classical material of this chapter. Obviously, one should have a look at [1], [19], [21] and [29]. The other references given are either more technical but unavoidable, as is obviously [12], or well known to me. All of this chapter, and specially § 1.1 is an improvement of the lectures given at CIMPA due to the clever and patient help of Emmanuel Giroux and Bruno Sévennec whom I am very pleased to thank.

1. Almost complex structures

The aim of this § is to show that there *are* almost complex structures related to symplectic forms, that there are in fact many of them, but that in some sense not too many: they form a contractible set. I shall also *very* briefly discuss integrability and Kählerness.

An *almost complex structure* on a manifold W is a section J of the bundle $\operatorname{End} TW$ such that $J_x^2 = -\operatorname{Id}_{T_xW}$ for all x in W. For instance, if W is a complex manifold (i.e. with holomorphic change of local coordinates), its tangent space T_xW at any point has a natural structure of complex vector space and multiplication by i is an almost complex structure. It is the situation we are trying to mimic, J plays the role of multiplication by i. Note that J gives to TW the structure of a complex vector bundle, each T_xW being a complex vector space by

$$(a+ib)\cdot\xi = a\xi + bJ\xi.$$

Let X be an oriented surface, endowed with a Riemannian metric, in such a way that we have a notion of rotation by $+\pi/2$ in every tangent plane. The family of all such rotations defines an almost complex structure on X, thus any oriented surface admits almost complex structures. Of course, not any manifold, even of even dimension, even oriented, may be endowed with an almost complex structure. For instance, we shall see below that it is not the case for the 4-sphere.

1.1. The linear case: almost complex structures, almost complex structures related to a non-degenerate skew-symmetric 2-form

In $\mathbf{R}^{2n}$, a *linear* almost complex structure is an endomorphism J such that $J^2 = -\operatorname{Id}$. The linear group $GL(2n,\mathbf{R})$ acts on the space of these structures by conjugation: $g\cdot J = gJg^{-1}$.

1.1.1. Exercise. — Use a J-complex basis to show that this action is transitive and that the stabiliser of a given almost complex structure J_0 is the group of all *complex* (for J_0) automorphisms.

In other words, the set of all almost complex structures can be identified with

$$\mathcal{J}_n = GL(2n,\mathbf{R})/GL(n,\mathbf{C}).$$

In $\mathbf{C}^n$, the standard Hermitian product decomposes into its real and imaginary parts:

$$\langle u,v\rangle = (u,v) - i\omega(u,v)$$

the Euclidean scalar product of $\mathbf{R}^{2n}$ and its symplectic form, respectively. One easily checks that $\omega(u,iv)$ is the Euclidean scalar product and moreover that $\omega(iu,iv) = \omega(u,v)$: multiplication by i is an isometry of ω. We now try to mimic this situation in the following definitions.

Let ω be a non-degenerate skew-symmetric bilinear form on $\mathbf{R}^{2n}$. A linear almost complex structure J is *tamed* by ω if the quadratic form $\omega(x,Jx)$ is positive definite. It is *calibrated* by ω if, moreover, it is an isometry of ω: $\omega(Jx,Jy) = \omega(x,y)$ for all x, $y\in\mathbf{R}^{2n}$.

1.1.2. Exercise. — Show that, in this case, the bilinear form $(x,y)\mapsto \omega(x,Jy) = (x,y)_J$ is a scalar product on $\mathbf{R}^{2n}$, for which one has

$$\begin{cases} E^\perp = (JE)^\circ \\ E^\circ = (JE)^\perp \\ JE\cap E^\circ = 0, \end{cases}$$

$^\circ$ and $^\perp$ denoting orthogonality respectively for ω and $(\ ,\ )_J$.

1.1.3. Exercise. — Let L be any Lagrangian subspace. Check that $JL\in\Lambda_L$ (the space of all Lagrangian subspaces transversal to L) and that it is $L^\perp$ (for $(\ ,\ )_J$). Deduce that there exists a basis $(x_1,\ldots,x_n,Jx_1,\ldots,Jx_n)$ of $\mathbf{R}^{2n}$ which is symplectic.

Let Ω_n be the set of all non-degenerate skew-symmetric bilinear forms on $\mathbf{R}^{2n}$, and, for $\omega\in\Omega_n$, let $\mathcal{J}_c(\omega)$ be the set of almost complex structures calibrated by ω. The symplectic group is denoted by Sp_{2n} ($=\mathrm{Sp}(n,\mathbf{R})$ in the notation of chapter I).

PROPOSITION 1.1.4. — $\Omega_n \cong GL(2n,\mathbf{R})/Sp_{2n}$, $\mathcal{J}_c(\omega)\cong Sp_{2n}/U(n)$.

Proof. — The group $GL(2n,\mathbf{R})$ acts on Ω_n by $(g,\omega)\mapsto\omega(g^{-1}\cdot,g^{-1}\cdot)$. The action is transitive, due to the existence of symplectic bases, and the stabiliser of ω is its group of isometries Sp_{2n}.
Also, if ω is fixed, Sp_{2n} acts on $\mathcal{J}_c(\omega)$ by $(g,J)\mapsto gJg^{-1}$:

$$\begin{aligned} \omega(gJg^{-1}x, gJg^{-1}y) &= \omega(Jg^{-1}x, Jg^{-1}y) && g \text{ is an isometry of } \omega \\ &= \omega(g^{-1}x, g^{-1}y) && J\in\mathcal{J}_c(\omega) \\ &= \omega(x,y) && g^{-1} \text{ is an isometry of } \omega \end{aligned}$$

the action is transitive due to the existence of "complex symplectic bases" (exercise 1.1.3) and the stabiliser of a given J is $Sp_{2n}\cap GL(n,\mathbf{C}) = U(n)$ (see 1.1.5). □

1.1.5. Exercise. — Let $\mathbf{C}^n = \mathbf{R}^{2n}$ be endowed with the standard symplectic structure. Consider the subgroups $O(2n)$, Sp_{2n}, $GL(n,\mathbf{C})$, $U(n)$ and compute the intersections of any two of them.

Denote by $\mathcal{J}_t(\omega)$ the set of all almost complex structure tamed by ω. Consider now $\mathbf{C}^n$ endowed with its canonical structures (J_0, ω and the scalar product). I learned the next proposition from B. Sévennec:

PROPOSITION 1.1.6. — *The map $J\mapsto (J+J_0)^{-1}\circ(J-J_0)$ is a diffeomorphism from $\mathcal{J}_t(\omega)$ (resp. $\mathcal{J}_c(\omega)$) onto the open unit ball in the vector space of matrices (resp. symmetric matrices) S such that $J_0S+SJ_0=0$.*

COROLLARY 1.1.7. — *The spaces $\mathcal{J}_c(\omega)$ and $\mathcal{J}_t(\omega)$ are connected, and even contractible.* □

Remark. — The fact that $\mathcal{J}_c(\omega)$ is contractible is both well-known and easy (see below in exercise 1.1.11 the classical proof). That $\mathcal{J}_t(\omega)$ is contractible is a statement taken from [12]. The original proof of Gromov required some agility in the use of fibrations. The argument of Sévennec we use here is both nicer and more elementary.

Proof of the proposition. — Note first that $J + J_0$ is invertible: if $x \neq 0$, $\omega(x, (J+J_0)(x)) > 0$ thus $\operatorname{Ker}(J+J_0) = \{0\}$ and the map is well defined. Write

$$S = (J+J_0)^{-1} \circ (J - J_0) = (A + \mathrm{Id})^{-1} \circ (A - \mathrm{Id})$$

where $A = J_0^{-1} \circ J$. We now prove that $\|S\| < 1$, that is, that $\|Ax - x\|^2 < \|Ax + x\|^2$ for all $x \neq 0$:

1.1.8. Exercise. — Check that

$$\|Ax + x\|^2 - \|Ax - x\|^2 = 4\omega(x, Jx) > 0.$$

Now let S be a matrix such that $\|S\| < 1$, which implies that $\mathrm{Id} - S$ is invertible and that the endomorphism

$$J = J_0 \circ (\mathrm{Id} + S) \circ (\mathrm{Id} - S)^{-1}$$

is well defined.

1.1.9. Exercise. — Check that J is an almost complex structure ($J^{-1} = -J$) if and only if $J_0 S + S J_0 = 0$ and that it is tamed by ω.

Obviously the map $S \mapsto J$ is the inverse of the one we are considering, so that we have proved the proposition for tamed structures. For calibrated structures, we just have to remark that J is calibrated if and only if S is symmetric:

1.1.10. Exercise. — Check that J is calibrated if and only if for any x, $y \in \mathbf{C}^n$,

$$\omega((\mathrm{Id} - S)x, (\mathrm{Id} - S)y) = \omega((\mathrm{Id} + S)x, (\mathrm{Id} + S)y)$$

and that this is equivalent to

$$\omega(Sx, y) + \omega(x, Sy) = 0.$$

Deduce that J is calibrated if and only if S is symmetric.

This ends the proof of the proposition. □

The next exercise gives "the" classical proof that $\mathcal{J}_c(\omega)$ is contractible.

1.1.11. Exercise. — Let L be a Lagrangian subspace. If $J \in \mathcal{J}_c(\omega)$, $JL \in \Lambda_L$ and $(\ ,\ )_J$ defines a positive definite symmetric bilinear form on L.

1. Conversely, given $L' \in \Lambda_L$ and a scalar product g on L, construct an element of $\mathcal{J}_c(\omega)$: for a non zero vector $x \in L$, consider its g-orthogonal $x^\perp \subset L$ as a subspace of E. Now $(x^\perp)^\circ$ is an $n+1$-dimensional subspace of E. Show that $\dim (x^\perp)^\circ \cap L' = 1$. Define Jx to be the unique vector of this line such that $\omega(x, Jx) = 1$ and check that this defines the required element of $\mathcal{J}_c(\omega)$.
2. Deduce that there is a one-to-one correspondence between $\mathcal{J}_c(\omega)$ and $\Lambda_L \times \mathcal{Q}(L)$, where $\mathcal{Q}(L)$ is the set of all positive definite quadratic forms on L.
3. Prove that Λ_L can be identified with the vector space[1] of all symmetric $n \times n$ matrices.
4. Deduce that $\mathcal{J}_c(\omega)$ is contractible.
5. Consider $\mathcal{C}_n = \{(\omega, J) \mid J \text{ is calibrated by } \omega\}$. Show that $\mathcal{C}_n$ can be identified with the homogeneous space $GL(2n, \mathbf{R})/U(n)$. Show that the first projection $\mathcal{C}_n \to \Omega_n$ is a homotopy equivalence.

1.2. Tamed and calibrated almost complex structures on a symplectic manifold

Let W be a $2n$-dimensional manifold endowed with a symplectic form ω. With its tangent bundle are associated the bundles

$$\begin{array}{rcll} \mathcal{J}(TW) & \longrightarrow & W & \text{with fibre } \mathcal{J}_n \\ \mathcal{J}_c(TW, \omega) & \longrightarrow & W & \text{with fibre } \mathcal{J}_c(\omega) \\ \mathcal{J}_t(TW, \omega) & \longrightarrow & W & \text{with fibre } \mathcal{J}_t(\omega) \end{array}$$

With an obvious definition of an almost complex structure tamed (resp. calibrated) by ω, such an object is nothing other than a *section* of the bundle $\mathcal{J}_t(TW, \omega)$ (resp. $\mathcal{J}_c(TW, \omega)$). These two bundles having contractible fibres, one gets

PROPOSITION 1.2.1. — *The space of almost complex structures tamed (or calibrated) by ω is non empty and contractible.* □

Of course, the existence of a non degenerate 2-form calibrating or taming the complex structure is essential: there *are* oriented even dimensional manifolds which do not admit any almost complex structure, the 4-sphere, for instance (see § 1.5).

There is also a relative version of 1.2.1: you can extend any section already defined along a closed subset of W. It is often[2] used in the following form.

[1] See the appendix to chapter I. There, we also prove that this vector space can be identified with the tangent space of Λ_n.

[2] For instance in the proofs of the classification theorems of Dusa McDuff.

PROPOSITION 1.2.2. — *Let $\Sigma \subset W$ be a symplectic submanifold, and let J_0 be an almost complex structure defined along Σ (i.e. an endomorphism of $TW|_\Sigma$ of square $-\mathrm{Id}$) and tamed (resp. calibrated) by ω. There exists an ω-tamed (resp. ω-calibrated) almost complex structure J on W which extends J_0.* □

1.2.3. Exercise. — If V is an almost complex submanifold of the symplectic manifold W for an almost complex structure calibrated by the symplectic form, show that it is a symplectic submanifold.

Remarks.

1. I must mention here that I have never used the fact that ω is a *closed* form.
2. There can be a lot of symplectic forms taming or calibrating the same almost complex structure. For instance this is the case for all forms $\omega \oplus \lambda\omega$ on $S^2 \times S^2$ (ω a volume form, $\lambda > 0$) and the usual (product) complex structure.
3. It is legitimate to wonder whether it is really useful to consider *tamed* almost complex structures. I know of at least one place where one is forced to use tamed almost complex structures. This is in the study of the so-called *CR*-structures (see the beginning of [7]).

1.3. Integrable complex structures (a few words)

As I have already mentioned, an *integrable* complex structure (that is, the structure of a complex analytic manifold) is the basic example of an almost complex structure.

Of course, there are more examples of almost complex structures than just integrable structures. There even exist almost complex manifolds which do not have any complex structure: this is the case for instance for the connected sum of three (or any odd number—except 1) copies of $\mathbf{P}^2(\mathbf{C})$, see [5] and § 1.5 below.

The way the almost complex structure J is related to the differentiable structure of the manifold is described by the famous Nijenhuis tensor (I understand that it was introduced by Ehresmann, see [21]):

$$N(X,Y) = [JX, JY] - J[X, JY] - J[JX, Y] - [X, Y].$$

In the appendix to this part, P. Gauduchon will use the Nijenhuis tensor to discuss a natural almost complex structure on the space of 1-jets of pseudo-holomorphic mappings between two almost complex manifolds. In this §, we evoke the relation of N with integrability.

1.3.1. Exercise.

1. Check that, if $X \mapsto [X, Y]$ is $\mathbf{C}$-linear (in the sense that it commutes with J), then $N \equiv 0$.
2. Show that N is actually a tensor, that is: $N(X, Y)$ depends only on the values $X(x)$, $Y(x)$ and not really on the vector fields X and Y.
3. Compute $N(X, JX)$ and deduce that, if W is a *surface*, N vanishes identically (on vectors) and everywhere (on W).

1.3.2. Exercise. — For any map $f : \mathbf{R}^{2n} \to \mathbf{C}$ and any vector field X on $\mathbf{R}^{2n}$, recall that $X \cdot f = X \cdot (f_1 + if_2) = X \cdot f_1 + iX \cdot f_2$ so that $f \mapsto X \cdot f$ is a complex linear map.

1. Let $\mathbf{R}^{2n}$ be endowed with an almost complex structure J and suppose f is a *holomorphic* function, that is $df \circ J = i\,df$. Show that $df\,(N(X, Y)) = 0$ for all X, Y.
2. Suppose there exist n holomorphic functions on $\mathbf{R}^{2n}$, which are independent at some point x. Show that N identically vanishes at x.
3. Assume W is a complex manifold and J is its (integrable) complex structure. Show that the Nijenhuis tensor vanishes identically and everywhere on W.

Remarks.

1. What was used in the previous exercise is that an integrable complex structure is one with many holomorphic functions. In general, an almost complex manifold has *no* holomorphic functions at all. On the other hand, it has *a lot* of holomorphic curves (maps $f : \mathbf{C} \to W$ such that $df \circ i = J \circ df$)—they will be the main tool in this book.
2. It is a hard theorem (in its full generality) [27] that the converse is also true. For an introduction to these questions, see [20] for instance.
3. Any almost complex structure on a *surface* is integrable (a result already known to Gauss in the analytic case). In the smooth case, it is of course a consequence of what we just said, but there exist simpler proofs, see [28] for instance.

1.4. Kähler structures

If M is a complex manifold endowed with a *Hermitian* metric, the imaginary part of this metric is skew-symmetric and thus can be viewed as a 2-form ω which calibrates the complex structure J (ω is said to be a $(1,1)$-form). Of course, ω is non-degenerate.

When ω is closed, that is when it is a symplectic form, it is called a *Kähler* form and M is a Kähler manifold.

1.4.1. Exercise. — Calling d', d'' the Dolbeault operators defined by

$$d'f = \sum \frac{\partial f_j}{\partial dz_j} dz_j \qquad d''f = \sum \frac{\partial f_j}{\partial d\bar{z}_j} d\bar{z}_j,$$

consider the function $f : \mathbf{C}^n \to \mathbf{R}$ defined by $f(v) = \log\left(1 + \|v\|^2\right)$. Let ω be the 2-form $\omega = id'd''f$. Show that ω is a Kähler form on $\mathbf{C}^n$.

Kähler manifolds are very common: any complex submanifold of a Kähler manifold is Kähler (for the induced $(1,1)$-form), thus all projective complex algebraic manifolds are Kähler—it suffices to prove that the projective space is, and it is: in this chapter, there will be a lot of definitions of symplectic forms on $\mathbf{P}^n(\mathbf{C})$. All will coincide (up to a scalar factor) and will be Kähler. The next exercise gives a first construction.

1.4.2. Exercise. — Let $\varphi_k : U_k \to \mathbf{C}^n$ $(0 \leq k \leq n)$ be the usual charts in $\mathbf{P}^n(\mathbf{C})$:

$$U_k = \{[x_0, \ldots, x_n] \mid x_k \neq 0\} \qquad \varphi_k([x_0, \ldots, x_n]) = \left(\frac{x_i}{x_k}\right)_{i \neq k} \in \mathbf{C}^n = \left\{(X_k)_{i \neq k}\right\}.$$

Put $\omega_k = \varphi_k^\star \omega$, where ω is the Kähler form of exercise 1.4.1 so that

$$\omega_k = id'd'' \log \frac{\sum |x_i|^2}{|x_k|^2}.$$

Show that, on $U_k \cap U_l$, ω_k and ω_l coincide. Deduce that the ω_k $(0 \leq k \leq n)$ define a Kähler form on $\mathbf{P}^n(\mathbf{C})$.

Kähler manifolds satisfy the Hodge duality, in particular they have $\dim H^{p,q} = \dim H^{q,p}$ (see [11] for instance). Note that it implies that their odd Betti numbers must be even.

It was only in 1976 that Thurston gave the first example of a non-Kähler symplectic manifold: a clever quotient of $\mathbf{R}^4$ by a group preserving the symplectic form... but such that the quotient has $b_1 = 3$ (see [29]). Then in 1984, D. McDuff gave examples of simply connected non-Kähler symplectic manifolds [24] of rather large dimensions. Cleverly combining a very easy construction (see § 5.3 below) with sophisticated examples, Gompf has recently produced 4-dimensional symplectic non-Kähler manifolds which are simply connected [10].

1.5. Some general remarks

As already mentioned, there are many more almost complex structures than just complex integrable structures. In this section, we shall concentrate on oriented 4-manifolds. Let W be such a manifold. It has two obvious topological invariants, its Euler characteristic χ and the signature σ of its intersection form (see chapter IV). Now suppose J is an almost complex structure on W. Then (TW, J) is a complex vector bundle and, as such, it has Chern classes $c_1(J)$, $c_2(J)$. The latter is the Euler class of TW and does not depend on J. The first does depend on J and is related to the topological invariants by

$$\langle c_1^2, [W] \rangle = 3\sigma + 2\chi.$$

For instance if $W = S^4$, $H^2(W) = 0$ thus $\sigma = 0$. On the other hand $\chi = 2$ thus $3\sigma + 2\chi = 4 \neq 0$ and of course there is no element c_1 in $H^2(S^4)$ such that $\langle c_1^2, [S^4] \rangle = 4$, thus S^4 does not admit any almost complex structure[3].

Note also that the mod 2 reduction w of $c_1(J)$ depends only on W: it was shown by Wu in [31] that w is a characteristic element for the quadratic form, that is, it satisfies

$$\forall x \in H^2(W; \mathbf{Z}/2), \qquad w \smile x = x \smile x.$$

It is also a result of Wu in [31] that

PROPOSITION 1.5.1. — *Homotopy classes of almost complex structures on W are in one-to-one correspondence with integral classes $c \in H^2(W; \mathbf{Z})$ which lift w and such that $\langle c^2, [W] \rangle = 3\sigma + 2\chi$.*

As a result one finds as I already mentioned, that there are almost complex structures on the connected sum of 3 copies of $\mathbf{P}^2(\mathbf{C})$, but this manifold has *no* integrable complex structure (and no symplectic form either). Other examples can be found for instance in [5].

1.5.2. Exercise. — Consider the connected sum W of two copies of $\mathbf{P}^2(\mathbf{C})$ both with the canonical orientation. Show that $H^2(W) = H^2(\mathbf{P}^2(\mathbf{C})) \oplus H^2(\mathbf{P}^2(\mathbf{C}))$ and thus is isomorphic to $\mathbf{Z} \oplus \mathbf{Z}$. Compute $\sigma(W)$ and $\chi(W)$. Assume W can be endowed with an almost complex structure J and show that the first Chern class of J must decompose into two odd numbers in the previous splitting of H^2. Deduce that W has no almost complex structure[4].

Of course, two complex structures can be homotopic among almost complex structures without being isomorphic. We shall see in the next § (see § 2) examples of complex structures that have the same c_1 but are not isomorphic.

[3] It is not much harder to prove, using the integrality of the Chern character (see [18] for instance), that the sphere S^{2n} has no almost complex structure, except for $2n = 2$ or 6. It is also a consequence of the results of Wu in [31] that S^6 has an almost complex structure. An explicit one can be described using the Cayley octonions (see [20] for instance).

[4] More or less the same proof gives that if W_1 and W_2 are 4-dimensional manifolds which can be endowed with almost complex structures, then their connected sum cannot (see [5]).

2. Hirzebruch surfaces

In this §, I present the Hirzebruch surfaces [15] (another good reference is [17]). In addition to being relevant examples for the discussion almost complex/complex, they are basic examples for symplectic geometry too: for instance they play a major role in the classification of Hamiltonian S^1-actions [3] and in the work of Dusa McDuff on symplectic 4-dimensional manifolds [26].

2.1. Definition

Consider a complex line bundle L over the projective line $\mathbf{P}^1(\mathbf{C}) = S^2$ and add to it a *section at infinity*, in other words, consider the bundle

$$\mathbf{P}(L \oplus \mathbf{1}) = \left\{(x, l) \mid x \in \mathbf{P}^1(\mathbf{C}), l \text{ is a line in } L_x \oplus \mathbf{C}\right\},$$

a bundle over $\mathbf{P}^1$ with fibre $\mathbf{P}^1$. The zero section is given by $l = 0 \oplus \mathbf{C}$, the section at infinity by $l = L_x \oplus 0$.

If D is any other line bundle over $\mathbf{P}^1$, it is clear that $\mathbf{P}(L \oplus \mathbf{1}) \cong \mathbf{P}((L \oplus \mathbf{1}) \otimes D)$ (by the map $(x, l) \mapsto (x, l \otimes D_x)$) thus the bundles $\mathbf{P}(L \oplus \mathbf{1})$ describe all the $\mathbf{P}(L_1 \oplus L_2)$. Moreover, using $D = L^\star$, one gets an isomorphism $\mathbf{P}(L \oplus \mathbf{1}) \cong \mathbf{P}(L^\star \oplus \mathbf{1})$ which exchanges the zero section and the section at infinity[5].

It is easy to prove that, from the topological viewpoint, all the rank 2 complex vector bundles over $\mathbf{P}^1$ split. This property is still true if one considers *holomorphic bundles*[6] (see [14]).

The isomorphism class of a (topological, or differentiable, or holomorphic) complex line bundle $L \to \mathbf{P}^1$ is well defined by an integer, its Euler class: decompose the sphere into two hemispheres and trivialise the bundle on both. To get the bundle, glue the two trivialisations along the equator S^1 by a map $\varphi : S^1 \to \mathbf{C}^\star$. Identify $\pi_1(\mathbf{C}^\star)$ with $\mathbf{Z}$, associating an integer, the Euler class, to the bundle.

2.1.1. Exercise. — Let B be any orientable surface and let D be a disc in B. Use a decomposition of B as a polygon to show that the complement of D has the homotopy type of a wedge of circles. Let $L \to B$ be a complex line bundle. Show that it is trivialisable both on D and on its complement. Deduce that the isomorphism type of L is given by an integer, its Euler class (see e.g. [4]).

For $k \in \mathbf{Z}$, call $\mathcal{O}(k)$ the bundle over $\mathbf{P}^1$ with Euler class k:

- $\mathcal{O}(0) \to \mathbf{P}^1$ is the trivial bundle.
- $\mathcal{O}(-1) \to \mathbf{P}^1$ is the *tautological* bundle: the fibre at $d \in \mathbf{P}^1$ is the line d of $\mathbf{C}^2$ itself.

[5] Recall that if $L^\star$ is the dual of L, $L \otimes L^\star$ has a canonical trivialisation by $(x, \varphi) \mapsto \varphi(x)$.

[6] The analogous property is definitely false if one replaces $\mathbf{P}^1$ by a Riemann surface of genus > 0, see [2].

- $\mathcal{O}(1)$ is the *canonical* bundle, the dual of the previous one, and
- $\mathcal{O}(k) = \mathcal{O}(k/|k|)^{\otimes |k|}$.

Call, for $k \in \mathbf{N}$, $W_k = \mathbf{P}(\mathcal{O}(-k) \oplus \mathbf{1})$.

2.1.2. Examples.

1. For $k = 0$, $W_0 = \mathbf{P}^1(\mathbf{C}) \times \mathbf{P}^1(\mathbf{C})$.
2. For $k = 1$,

$$W_1 = \left\{(l, d) \mid d \text{ is a line in } \mathbf{C}^2 \text{ and } l \text{ a line in } d \oplus \mathbf{C} \subset \mathbf{C}^2 \oplus \mathbf{C}\right\}.$$

In addition to the structure of a bundle over $\mathbf{P}^1$, this manifold is endowed with a projection π to $\mathbf{P}^2(\mathbf{C})$: to the point (d, l), associate $l \subset \mathbf{C}^3$. Notice that any point in $\mathbf{P}^2$ is the image of a unique point of W_1: d is the projection of l on $\mathbf{C}^2$... except for the line $d = 0 \oplus \mathbf{C}$ which is the image of all points in the $\mathbf{P}^1$ of lines d in $\mathbf{C}^2$.

The projection $\pi : W_1 \to \mathbf{P}^2(\mathbf{C})$ is the *blow up* of $\mathbf{P}^2(\mathbf{C})$ at the point $[0, 0, 1]$. One often denotes W_1 by $\widetilde{\mathbf{P}^2}(\mathbf{C})$.

2.1.3. Exercise. — Consider the standard ball in $\mathbf{P}^2(\mathbf{C})$:

$$B = \left\{[x, y, 1] \mid |x|^2 + |y|^2 \leq R^2\right\}.$$

Check that $\pi^{-1}(B)$ is a tubular neighbourhood of the zero section of $\mathcal{O}(-1)$, its boundary a 3-sphere, and its complement a tubular neighbourhood of the section at infinity.

It is easy to represent W_k as an algebraic submanifold of $\mathbf{P}^1(\mathbf{C}) \times \mathbf{P}^2(\mathbf{C})$:

2.1.4. Exercise. — By definition,

$$W_k = \left\{(\ell, d) \in \mathbf{P}^1(\mathbf{C}) \times \mathbf{P}^N(\mathbf{C}) \mid \ell \subset \mathbf{C}^2, d \subset \ell^{\otimes k} \oplus \mathbf{C} \subset \left(\mathbf{C}^2\right)^{\otimes k} \oplus \mathbf{C}\right\}$$

where $N = \dim (\mathbf{C}^2)^{\otimes k}$. Choose a basis of $\mathbf{C}^2$ and show that

$$W_k = \left\{[a, b], [ua^k, ua^{k-1}b, \ldots, uab^{k-1}, ub^k, v]\right\} \subset \mathbf{P}^1(\mathbf{C}) \times \mathbf{P}^{k+1}(\mathbf{C})$$

(notice that this does *not* mean that $W_k \cong \mathbf{P}^1(\mathbf{C}) \times \mathbf{P}^1(\mathbf{C})$). Similarly, show that the "projection"

$$[z_1, \ldots, z_{k+1}, t] \longmapsto [z_1, z_{k+1}, t]$$

although not well defined on $\mathbf{P}^{k+1}(\mathbf{C})$ defines an embedding

$$W_k = \left\{[a, b], [ua^k, ub^k, v]\right\} \subset \mathbf{P}^1(\mathbf{C}) \times \mathbf{P}^2(\mathbf{C}).$$

Alternatively $W_k = \left\{([a,b],[x,y,z]) \mid a^k y - b^k x = 0\right\}$ and any Kähler form over $\mathbf{P}^1(\mathbf{C}) \times \mathbf{P}^2(\mathbf{C})$ then defines a Kähler form over W_k.

2.2. Topology

In fact, there are only two classes of projectivised $\mathbf{P}^1$-bundles over $\mathbf{P}^1$. Let $W \to S^2$ be such a bundle. Over the hemisphere $S^2_\pm$, W can be trivialised as $S^2_\pm \times \mathbf{P}^1$. The gluing is given by a map $S^1 \to PGL(2,\mathbf{C})$ and there are only two homotopy classes: $\pi_1(PGL(2,\mathbf{C})) \cong \mathbf{Z}/2$.

2.2.1. Exercise. — Projectivising a bundle $L \oplus \mathbf{1}$ may be achieved by a map $\pi_1(\mathbf{C}^\star) \to \pi_1(PGL(2,\mathbf{C}))$. Show that it can be identified with mod 2 reduction $\mathbf{Z} \to \mathbf{Z}/2$. Deduce that $\mathbf{P}(L \oplus \mathbf{1})$ is trivialisable if and only if the Euler class of L is even.

Using the examples above, one then gets

PROPOSITION 2.2.2. — *W_k is diffeomorphic to $S^2 \times S^2$ if k is even and to $\widetilde{\mathbf{P}^2}(\mathbf{C})$ if k is odd.* □

One deduces easily from this the cohomology of W_k, which, of course depends only on k mod 2. It may nevertheless be convenient to express it in terms of k and of the projection π over $\mathbf{P}^1$: from the inclusion $W_k \hookrightarrow \mathbf{P}^1 \times \mathbf{P}^2$, one gets two classes u, $v \in H^2(W_k;\mathbf{Z})$, restrictions of the generator of $H^2(\mathbf{P}^1;\mathbf{Z})$, $H^2(\mathbf{P}^2;\mathbf{Z})$ resp. It is classical (see [18] or the definition of Chern classes in chapter IV for instance) that

PROPOSITION 2.2.3. — *The ring $H^\star(W_k;\mathbf{Z})$ is isomorphic to the ring of polynomials $\mathbf{Z}[u,v]/\left(u^2, v^2 - kuv\right)$.* □

2.2.4. Exercise. — Check that the isomorphism type of this ring actually depends only on k mod 2.

2.3. Symplectic forms

Consider a symplectic form induced by the standard symplectic (Kähler) form on $\mathbf{P}^1 \times \mathbf{P}^2$ (as defined on each factor in 1.4.2 for instance), rescaled such that its cohomology class is $\lambda u + \mu v$ and compute the volumes of the following spheres $i : \mathbf{P}^1 \hookrightarrow W_k$:

- the zero section $i([a,b]) = ([a,b],[0,0,1])$: $i^\star(\lambda u + \mu v) = \lambda x$, it has volume λ,
- the section at infinity $i([a,b]) = ([a,b],[a^k,b^k,0])$: $i^\star(\lambda u + \mu v) = (\lambda + k\mu)x$, it has volume $\lambda + k\mu$,
- the fibre $i([a,b]) = ([1,0],[a,0,b])$: $i^\star(\lambda u + \mu v) = \mu x$, it has volume μ.

(x denotes the generator of $H^2(\mathbf{P}^1;\mathbf{Z})$).

I find it very convenient to represent W_k and the class of the Kähler form under consideration by a trapezium, the lengths of its edges being the volumes of the corresponding spheres as in figure 2.

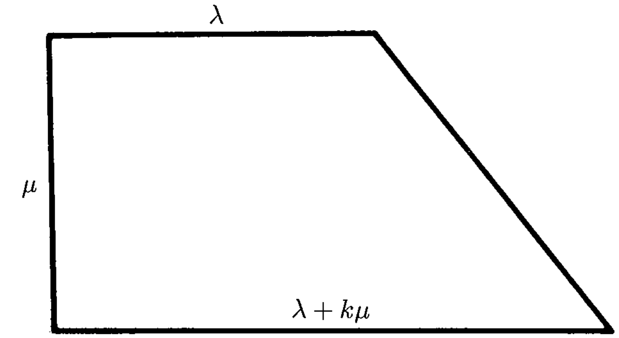

Figure 2

Remark. — In fact, this trapezium is not only a good way to represent the cohomology classes, but it is actually the image of the moment map of a Hamiltonian T^2-action on W_k.

2.3.1. Exercise. — Consider the S^1-action on W_k induced by the action

$$t \cdot ([a,b],[x,y,z]) = ([a,b],[x,y,tz])$$

on $\mathbf{P}^1 \times \mathbf{P}^2$. Check that its fixed points are the sections at zero and at infinity and that $H = \mu |z|^2 / 2\left(|x|^2 + |y|^2 + |z|^2\right)$ is a Hamiltonian for this action. Any $r \in]0,\mu[$ is then a regular value of H and S^1 acts on $H^{-1}(r)$. Check that the quotient sphere $H^{-1}(r)/S^1$ gets a natural "reduced" symplectic structure (see § 4) of volume $\lambda + k(\mu - r)$.

Remark. — This volume is an affine function of r, with slope $\pm k$. This is represented on the trapezium, and is a special case of the Duistermaat-Heckman formula [9]. Note that, for $k > 0$, the volume is *decreasing* when starting from the zero section of $\mathcal{O}(k)$ and *increasing* when starting from the zero section of $\mathcal{O}(-k)$.

2.3.2. Exercise. — Using 2.2.3, show that there does not exist a symplectic form on W_k which gives the same volume to the sections at zero and at infinity ($k \neq 0$).

All these Kähler forms belong to "exotic complex structures" on $S^2 \times S^2$ or $\widetilde{\mathbf{P}^2}(\mathbf{C})$. Fix the parity of k, say k even. The *complex* manifold W_k is $S^2 \times S^2$, with a complex structure which depends on k. The almost complex structures are homotopic due to 1.5.1, but the complex manifolds are not isomorphic (see [15]).

If one agrees to use intersection in homology and the positivity of intersections of complex curves, this result can be made into an exercise: suppose $\varphi : W_k \to W_l$ $(0 \le k \le l)$ is a holomorphic isomorphism. Let Σ_0 be a complex curve[7] with self intersection k in W_k, then $\varphi(\Sigma_0)$ will be a complex curve in W_l. Call (F, S_0) a basis of $H_2(W_l; \mathbf{Z})$ such that F is the class of the fibre and S_0 that of a section such that $S_0 \cdot S_0 = l$.

2.3.3. Exercise. — Write the homology class of $\varphi(\Sigma_0)$ as $aF + bS_0$. Show that $a \ge 0$, $b \ge 0$, and that $k = 2ab + b^2 l$. Deduce that $b = 0$ and $k = 0$. Let Φ be a fibre in W_0. Write the homology class of $\varphi(\Phi)$ as $xF + yS_0$. Use the fact that (Φ, Σ_0) is a basis of $H_2(W_0; \mathbf{Z})$ to prove that $y \ne 0$, then show that $y > 0$ and $x \ge 0$. Compute the self intersection of $\varphi(\Phi)$ and get $x = y = 0$, a contradiction.

3. Coadjoint orbits (of $U(n)$)

The simplest possible example of a compact symplectic manifold is the sphere S^2 endowed with a volume form. Any oriented surface would also work: in dimension 2, a symplectic form is simply a volume form (this is the non degeneracy condition, the condition of being closed is automatically fulfilled).

It turns out that S^2 is the basic example in two families of symplectic manifolds: it is a coadjoint orbit—of $SO(3)$ as well as of $U(2)$—and it is a symplectic reduction.

In this §, I shall briefly explain the symplectic structure of coadjoint orbits and then in §4 the symplectic reduction process (this will also be used to construct Lagrangian submanifolds in chapter X). We will then have at hand a lot of examples and will be in a position to try to manufacture new ones out of these old ones. I shall focus attention on dimension 4, and explain in §5 why taking the connected sum is not a symplectic process while blowing up and down are, and how these remarks can be generalised to give the symplectic fibre sum recently used by Gompf [10] to construct his new examples.

The vector space $\mathbf{C}^n$ is endowed with its canonical Hermitian structure. A matrix A is *Hermitian* if ${}^t\overline{A} = A$.

3.1. Description of the manifolds

Fix a vector $\lambda = (\lambda_1, \ldots, \lambda_n) \in \mathbf{R}^n$ and consider, in the space of all $n \times n$ complex matrices, the subspace

$$\mathcal{H}_\lambda = \{\text{Hermitian matrices with spectrum } \lambda\}.$$

It is well known that Hermitian matrices are diagonalisable in a unitary basis: $A = gDg^{-1}$, D diagonal and $g \in U(n)$. In other words, $\mathcal{H}_\lambda$ is an orbit of the $U(n)$-action by conjugation.

[7] Thinking of W_l, W_k as $\mathbf{P}^1$-bundles over $\mathbf{P}^1$, one can choose S_0 and Σ_0 to be the zero sections.

Call $\mu_1 < \cdots < \mu_k$ the distinct values of the λ_i and n_i the multiplicity of μ_i. It is clear that the stabilisers of the elements of $\mathcal{H}_\lambda$ are conjugate to $U(n_1) \times \cdots \times U(n_k) \subset U(n)$, thus

PROPOSITION 3.1.1. — *The space $\mathcal{H}_\lambda$ is a manifold, diffeomorphic to the homogeneous space $U(n)/U(n_1) \times \cdots \times U(n_k)$.*

Remark. — Note that the diffeomorphism type depends only on the multiplicities of the eigenvalues, not on their values.

To any element in $\mathcal{H}_\lambda$, one can associate the sequence $Q_1, Q_2, \ldots, Q_k$ of its eigenspaces. They are pairwise orthogonal and $\dim Q_i$ is the multiplicity n_i of μ_i. Equivalently, one can consider the *flag*

$$0 \subset P_1 \subset \cdots \subset P_k = \mathbf{C}^n$$

with $P_j = Q_1 \oplus \cdots \oplus Q_j$. The manifold $\mathcal{H}_\lambda$ is a *flag manifold*.

Generic orbits. — Generic (biggest) orbits correspond to simple (multiplicity 1) eigenvalues, the stabiliser type is the torus T^n of diagonal unitary matrices. In this case Q_i is a line and $\mathcal{H}_\lambda$ is the manifold of *complete* flags in $\mathbf{C}^n$,

$$\mathcal{H}_\lambda \cong \{(0 \subset P_1 \subset \cdots \subset P_n = \mathbf{C}^n) \mid \dim P_j = j\}.$$

The dimension of $\mathcal{H}_\lambda$ is $n^2 - n = 2\frac{n(n-1)}{2}$ in this case.

Small orbits. — The small orbits, excepting the trivial case where $k = 1$ and where $\mathcal{H}_\lambda$ is a point, are obtained for 2 distinct eigenvalues μ_1 with multiplicity n_1, and μ_2 with multiplicity n_2. Using again the eigenspaces, one identifies $\mathcal{H}_\lambda$ with the Grassmann manifold $G_{n_1}(\mathbf{C}^n)$ (or $G_{n_2}(\mathbf{C}^n)$) of n_1-planes in $\mathbf{C}^n$, a $2n_1n_2$-dimensional manifold. It is the homogeneous space $U(n)/U(n_1) \times U(n_2)$ or the quotient of the Stiefel manifold $V_{n_1}(\mathbf{C}^n)$ of unitary n_1-frames in $\mathbf{C}^n$ by the group $U(n_1)$.

An interesting subcase is $n_1 = 1$, in which case the Grassmann manifold is the complex projective space $\mathbf{P}^{n-1}(\mathbf{C})$.

3.2. Coadjoint orbits and their symplectic structures

The mapping $h \mapsto ih$ is an isomorphism of real vector spaces from the space $\mathcal{H}$ of all Hermitian matrices onto the space $\mathfrak{u}(n)$ of skew-Hermitian matrices[8].

The assignment $(X, Y) \mapsto \operatorname{tr}(XY)$ defines a (negative) definite ($\mathbf{R}$-)bilinear form on $\mathfrak{u}(n)$, which is invariant under conjugation. This allows us to identify $\mathfrak{u}(n)$ with the dual vector space $\mathfrak{u}(n)^\star$. With these identifications, the $U(n)$-action on $\mathfrak{u}(n)$ or $\mathcal{H}$ by conjugation is the *adjoint* or *coadjoint* action. The flag manifolds $\mathcal{H}_\lambda$ are

[8] It is of course the Lie algebra of the Lie group $U(n)$: the equation ${}^t\overline{A}A = \mathrm{Id}$ gives ${}^t\overline{X} + X = 0$ when you differentiate it at the point Id.

thus described as coadjoint orbits of a Lie group. This is why they have a natural symplectic structure, called the Kirillov structure [19]: in all of this §, you can replace $U(n)$ by a Lie group G and $\mathcal{H}$ or $\mathfrak{u}(n)^\star$ by the dual $\mathfrak{g}^\star$ of its Lie algebra.

First define, for any $h \in \mathcal{H}$, a skew-symmetric bilinear form ω_h on $\mathfrak{u}(n)$ by

$$\omega_h(X,Y) = \operatorname{tr}([X,Y]\,ih) = \operatorname{tr}(X\,[Y,ih]).$$

As $(X,Y) \mapsto \operatorname{tr}(XY)$ is non-degenerate, the kernel of ω_h is the subspace of $\mathfrak{u}(n)$

$$K_h = \{Y \in \mathfrak{u}(n) \mid [Y,ih] = 0\},$$

which is obviously the Lie algebra of the stabiliser of h for the $U(n)$-action.

The tangent space to the orbit $\mathcal{H}_\lambda$ at the point h is the subspace of $\mathcal{H}$ generated by the *fundamental vector fields* of the $U(n)$-action: one considers the orbit map defined by h:

$$\begin{array}{ccc} U(n) & \xrightarrow{f_h} & \mathcal{H} \\ g & \longmapsto & g\cdot h = ghg^{-1} \end{array}$$

and $T_h\mathcal{H}_\lambda$ is the image of $T_1 f_h$, that is of

$$\begin{array}{ccc} \mathfrak{u}(n) & \xrightarrow{T_1 f_h} & \mathcal{H} \\ X & \longmapsto & [X,h] \end{array}$$

in particular, $K_h = \operatorname{Ker} T_1 f_h$, and thus ω_h defines, via $T_1 f_h$, a *non-degenerate* skew-symmetric bilinear form on $T_1 f_h$:

$$\widetilde{\omega}_h([X,h],[Y,h]) = \operatorname{tr}([X,Y]\,ih)$$

for X, $Y \in \mathfrak{u}(n)$ or $[X,h]$, $[Y,h] \in T_h\mathcal{H}_\lambda$.

Thus we get a non-degenerate 2-form $\widetilde{\omega}$ on the orbit $\mathcal{H}_\lambda$. The next claim is, of course, that $\widetilde{\omega}$ is closed. You just need to prove that

$$d\widetilde{\omega}_h([X,h],[Y,h],[Z,h]) = 0 \text{ for all } X,Y,Z \in \mathfrak{u}(n).$$

Call $\underline{X}$ the fundamental vector field associated with X: $\underline{X}_h = [X,h]$. Then

$$\begin{aligned} 3d\widetilde{\omega}(\underline{X},\underline{Y},\underline{Z}) &= \underline{X}\cdot\widetilde{\omega}(\underline{Y},\underline{Z}) - \underline{Y}\cdot\widetilde{\omega}(\underline{X},\underline{Z}) + \underline{Z}\cdot\widetilde{\omega}(\underline{X},\underline{Y}) \\ &\quad + \widetilde{\omega}([\underline{X},\underline{Y}],\underline{Z}) + \widetilde{\omega}([\underline{Y},\underline{Z}],\underline{X}) + \widetilde{\omega}([\underline{Z},\underline{X}],\underline{Y}). \end{aligned}$$

By definition, $\widetilde{\omega}$ is invariant and the vector fields under consideration are fundamental vector fields for the action, thus the first three terms vanish. What is left also vanishes due to the Jacobi identity.

The $\mathcal{H}_\lambda$ are thus compact symplectic manifolds. Note that, although the topology of $\mathcal{H}_\lambda$ depends only on the multiplicities in λ, the symplectic structure does depend on the λ_j.

3.2.1. Exercise. — Assume $n=2$, $k=2$ and $n_1=1$ (thus $\mathcal{H}_\lambda$ is a projective line). Compute $\int_{\mathcal{H}_\lambda}\widetilde{\omega}$ in terms of μ_1 and μ_2.

3.2.2. Exercise. — Consider the group $SO(3)$ of rotations of 3-dimensional Euclidean space. Show that its Lie algebra $\mathfrak{so}(3)$ is the vector space of skew symmetric matrices. Identify it with $\mathbf{R}^3$ by

$$\varphi : (x,y,z) \longmapsto \begin{pmatrix} 0 & -z & y \\ z & 0 & -x \\ -y & x & 0 \end{pmatrix}.$$

Show that φ is an isomorphism of Lie algebras (the Lie algebra structure on $\mathbf{R}^3$ being given by the vector product), that the action of $SO(3)$ by conjugation on $\mathfrak{so}(3)$ corresponds to the action by rotations. Thus the scalar product is invariant, which allows to identify $\mathfrak{so}(3)$ with its dual, and the adjoint action with the coadjoint. Show that the coadjoint orbits are 2-spheres, thus finding once again a family of symplectic structures on S^2.

3.3. Almost complex structures

Once the $\mathcal{H}_\lambda$ are described are flag manifolds of $\mathbf{C}^n$, they clearly have a natural additional structure: they are *complex* manifolds. As above (proposition 3.1.1) the complex diffeomorphism type depends only on the multiplicities. Let us describe the underlying almost complex structure. One can write $\mathfrak{u}(n) = \mathfrak{u}(n_1)\oplus\cdots\oplus\mathfrak{u}(n_k)\oplus\mathfrak{m}$ where $\mathfrak{m}$ is a *complex* vector space. One simply decomposes in blocks a matrix A in an orbit with multiplicities $n_1,\ldots,n_k$:

$$A = \begin{pmatrix} A_1 & X_{1,2} & X_{1,3} & \cdots \\ -{}^t\overline{X_{1,2}} & A_2 & X_{2,3} & \cdots \\ \cdots & & & \\ & & & X_{k-1,k} \\ & & & A_k \end{pmatrix}$$

where $A_i \in \mathfrak{u}(n_i)$ and $X_{i,j}$ is a *complex* matrix. The complex structure on $\mathfrak{m}$ is given by the complex structure of the spaces where the blocks $X_{i,j}$ live, not by the complex structure of the space of big matrices: if

$$X = \begin{pmatrix} 0 & X_{1,2} & X_{1,3} & \cdots \\ -{}^t\overline{X_{1,2}} & \cdots & \cdots & \cdots \\ -{}^t\overline{X_{2,3}} & \cdots & \cdots & \cdots \\ \vdots & \vdots & \vdots & \vdots \end{pmatrix} \in \mathfrak{m}$$

call $j(X)$ the same matrix with each $X_{k,l}$ replaced by $iX_{k,l}$: this is the complex structure of $\mathfrak{m}$.

Of course, $\mathfrak{m}$ is not a Lie subalgebra of $\mathfrak{u}(n)$, but $\mathfrak{g} = \mathfrak{u}(n_1)\oplus\cdots\oplus\mathfrak{u}(n_k)$ is.

Now $\mathfrak{m}$ is a good representant of the tangent space to $U(n)/U(n_1)\times\cdots\times U(n_k)$ at the point image of $I \in U(n)$, and it has a natural complex structure. Everything is invariant enough to define something on $\mathcal{H}_\lambda$.

Fix a diagonal matrix $D = (\mu_1, \dots, \mu_1, \dots, \mu_k, \dots, \mu_k)$ in the orbit $\mathcal{H}_\lambda$ with $\mu_1 < \cdots < \mu_k$ (so that the stabiliser of D is actually $U(n_1) \times \cdots \times U(n_k)$. Any element in $T_d\mathcal{H}_\lambda$ can be writen in a unique way as $[X, D]$ for an $X \in \mathfrak{m}$ (the tangent space to the orbit is generated by fundamental vector fields, unicity comes from the specification $X \in \mathfrak{m}$, this subspace being a supplementary of the Lie algebra of the stabiliser).

Define now $J_D[X, D] = [j(X), D]$ in such a way that J_D is a complex structure on $T_D\mathcal{H}_\lambda$. Let now $h = gDg^{-1}$ be any element in the orbit. Any element in $T_h\mathcal{H}_\lambda$ can be writen uniquely (once g is chosen) as $[gXg^{-1}, h]$ with $X \in \mathfrak{m}$. Put

$$J_h\left[gXg^{-1}, h\right] = \left[gj(X)g^{-1}, h\right]$$

thus defining an almost complex structure J on the orbit $\mathcal{H}_\lambda$.

3.3.1. Exercise. — Show that $\tilde{\omega}(JX, JY) = \tilde{\omega}(X, Y)$: the complex structure is an isometry of $\tilde{\omega}$. Compute $\tilde{\omega}(X, JY)$ in terms of the blocks in X and Y and in terms of the μ_j. Deduce that (if the μ_j are as above) $\tilde{\omega}(X, JY)$ is a Riemannian metrics on $\mathcal{H}_\lambda$. Thus J is calibrated by $\tilde{\omega}$.

4. Symplectic reduction

Symplectic reduction is a very simple and very clever process, due to Marsden and Weinstein [23], already mentionned in chapter I, which can be used for different purposes, such as constructing symplectic quotients, i.e. symplectic manifolds obtained from group actions on large symplectic manifolds (see 4.2), Lagrangian immersions into the reduced symplectic manifolds starting from Lagrangian immersions into the large one or Lagrangian submanifolds of the large symplectic manifold starting from Lagrangian submanifolds of the small one (see chapter X). Here we discuss some classical examples.

4.1. The projective space, dissection

The complex projective space $\mathbf{P}^n(\mathbf{C})$ is the quotient $\mathbf{C}^{n+1} - \{0\}/z \sim \lambda z$ for $\lambda \in \mathbf{C}^\star$, or, what is almost the same, think $S^{2n+1}/z \sim \lambda z$ for $\|\lambda\| = 1$: any line contains unitary vectors! Let us look at that trivial remark more closely: let

$$S_R^{2n+1} = \left\{z \in \mathbf{C}^{n+1} \mid \|z\|^2 = R^2\right\}$$

be the radius R sphere. We are considering the quotient by the S^1 action $(\lambda, z) \mapsto \lambda \cdot z$.

There is a relation between the function $H = \frac{1}{2}\|z\|^2$ (of which S_R^{2n+1} is a level set) and the S^1-action we consider: the fundamental vector field for the action is $\underline{X}_z = iz\ldots$ and this is the symplectic gradient[9] of H:

$$\omega_z(\underline{X}_z, Y) = \operatorname{Im}\langle iz, Y\rangle = \operatorname{Re}\langle z, Y\rangle = dH(Y).$$

[9]Of course, $\mathbf{C}^{n+1}$ is endowed with the standard symplectic form.

What happens is the following: the tangent space to $\mathbf{P}^n(\mathbf{C})$ at the point $[z]$ is the (isomorphic) image of the quotient of $\mathbf{C}^{n+1}$ by the subspace generated by z and iz, in particular it is $\mathbf{C}^{n+1}/\mathbf{C} \cdot z$, thus a complex vector space, which is not surprising: the quotient is a complex manifold.

What happens in the symplectic framework is exactly the same: of course, if you restrict it to S_R^{2n+1}, the symplectic form is degenerate; its kernel at z is generated by iz and it is exactly what one has to kill to get the tangent space to $\mathbf{P}^n(\mathbf{C})$.

4.1.1. Exercise. — Compute the volume of a projective line $\mathbf{P}^1(\mathbf{C}) \subset \mathbf{P}^n(\mathbf{C})$ in terms of R.

4.2. Symplectic reduction: general presentation

Symplectic reduction is a list of consequences of a straightforward linear algebra lemma:

LEMMA 4.2.1. — *If (E, ω) is a symplectic vector space and $F \subset E$ is any coisotropic subspace, then ω defines a non-degenerate 2-form on F/F°.* □

In the $\mathbf{P}^n$ case, $E = \mathbf{C}^{n+1}$, $F = T_z S_R^{2n+1}$, $F^\circ = \mathbf{R} \cdot iz$ and $F/F^\circ = T_{[z]}\mathbf{P}^n(\mathbf{C})$.

4.2.2. Exercise. — Let $\mathbf{R}^{n+1}$ be endowed with its usual Euclidean structure, and $\mathbf{R}^{n+1} \times \mathbf{R}^{n+1}$ be endowed with the symplectic form $\omega((p, q), (p', q')) = p \cdot q' - p' \cdot q$. Consider the Hamiltonian (function) $H(p, q) = \frac{1}{2}\|p\|^2$. Compute the Hamiltonian vector field (symplectic gradient) X_H and its flow φ_t. Show that X_H is complete and write down the corresponding $\mathbf{R}$-action. What are the orbits? Deduce a symplectic structure on the set of oriented affine lines of $\mathbf{R}^{n+1}$. Show that this manifold is diffeomorphic to $T^\star S^n$. What about the symplectic structures?

I present here a very general setting as a list of exercises (see [21]). Let (W^{2N}, ω) be a symplectic manifold (to replace $\mathbf{C}^{n+1}$ endowed with an action of a Lie group G (to replace S^1) with a *moment mapping* $\mu : W \to \mathfrak{g}^\star$ (to replace $H : \mathbf{C}^{n+1} \to \mathbf{R}$) such that

$$\langle T_w\mu(X), Y\rangle = \omega_w(X, \underline{Y}) \text{ for } X \in T_wW, \quad Y \in \mathfrak{g}$$

and such that μ preserves the Poisson brackets: W has a natural Poisson bracket given by its symplectic form, and $\mathfrak{g}^\star$ has also a natural one coming from the Lie bracket on $\mathfrak{g}$ and described in the following exercise.

4.2.3. Exercise. — Let $f, g \in \mathcal{C}^\infty(\mathfrak{g}^\star)$ and put

$$\{f, g\}(\xi) = [df(\xi), dg(\xi)]$$

where $df(\xi)$ is considered as an element of $\mathfrak{g}$ using biduality. Show that this defines a Poisson bracket on $\mathfrak{g}^\star$—that is, a Lie algebra structure on $\mathcal{C}^\infty(\mathfrak{g}^\star)$. Let $\mathcal{O}$ be a coadjoint orbit in $\mathfrak{g}^\star$. Show that the induced Poisson bracket coincides with the one defined by the symplectic structure described in § 3.

4.2.4. Exercise. — Show that the moment map μ is equivariant ($\mathfrak{g}$ is endowed with the coadjoint action).

Assume $\xi \in \mathfrak{g}^\star$ is a regular value of μ. Then $V = \mu^{-1}(\xi)$ is a submanifold of W and

$$\forall x \in V \qquad T_x V = \operatorname{Ker} T_x \mu.$$

Let G_ξ be the stabiliser of ξ for the coadjoint action (it is all of S^1 in the projective space example—as it is G in any example where G is abelian).

4.2.5. Exercise.

1. Show that the G-action on W induces a G_ξ-action on V.

2. Let $j : V \hookrightarrow W$ be the inclusion, then

$$\operatorname{Ker}(j^\star\omega)_x = T_x(G_\xi \cdot x).$$

As ξ is a regular value, we know that the G-action is locally free in the neighbourhood of any $x \in V$. Assume moreover that the G_ξ-action is *free*. Then one can apply lemma 4.2.1 to obtain the unique symplectic form σ on V/G_ξ such that, in the diagram

$$\begin{tikzcd} V \arrow[r, hook, "j"] \arrow[d, "\pi"] & W \\ V/G_\xi \end{tikzcd}$$

$\pi^\star\sigma = j^\star\omega$. The symplectic manifold $(V/G_\xi, \sigma)$ is a *symplectic reduction* of V.

The next § will give some examples.

4.3. Construction of symplectic manifolds

Hirzebruch surfaces, first version. — Consider the hypersurface H in $(\mathbf{C}^2 - 0) \times (\mathbf{C}^3 - 0)$ defined by the equation $a^k y - b^k x = 0$. Being complex, it is symplectic (see 1.2.3). It is endowed with the product action of T^2 by $(s,t)\cdot(a,b,x,y,z) = (sa, sb, tx, ty, tz)$ whose moment mapping is

$$\begin{array}{rcl} \mu : H & \longrightarrow & \mathbf{R}^2 \\ (a,b,x,y,z) & \longmapsto & \left(|a|^2 + |b|^2, |x|^2 + |y|^2 + |z|^2\right) \end{array}$$

if ξ is in the positive quadrant of $\mathbf{R}^2$, $\mu^{-1}(\xi) = (S^3 \times S^5) \cap H$ and the quotient $W_k \subset \mathbf{P}^1 \times \mathbf{P}^2$ appears as a symplectic reduction.

Hirzebruch surfaces, second version. — There is a more economical—in terms of dimensions—way to obtain Hirzebruch surfaces as symplectic reductions, starting from $\mathbf{C}^4$. Consider $\mathcal{U} = (\mathbf{C}^2 - 0) \times (\mathbf{C}^2 - 0) \subset \mathbf{C}^4$, with the action of $\mathbf{C}^\star \times \mathbf{C}^\star$ by

$$(s,t)\cdot(z_1,z_2,z_3,z_4) = \left(s^k t z_1, t z_2, s z_3, s z_4\right). \tag{4.3.1}$$

The quotient is identified to W_k by

$$(z_1,z_2,z_3,z_4) \longmapsto \left([z_3,z_4],[z_3^k z_2, z_4^k z_2, z_1]\right).$$

Let $\mu : \mathbf{C}^4 \to \mathbf{R}^2$ be the moment map for the action of $T^2 \subset \mathbf{C}^\star \times \mathbf{C}^\star$ by (4.3.1):

$$\mu(z_1,z_2,z_3,z_4) = \frac{1}{2}\left(k|z_1|^2 + |z_3|^2 + |z_4|^2, |z_1|^2 + |z_2|^2\right).$$

Let $\xi = (\xi_1, \xi_2) \in \mathbf{R}^2$. Consider $\mu^{-1}(\xi)$. I want to show that, under certain hypotheses on ξ, $\mu^{-1}(\xi)/T^2 = \mathcal{U}/\mathbf{C}^\star \times \mathbf{C}^\star$.

So let $z \in \mathcal{U}$. I am looking for $Z \in \mu^{-1}(\xi)$ in the same $\mathbf{C}^\star \times \mathbf{C}^\star$ orbit and more precisely such that there exist $a, b > 0$ with

$$(Z_1, Z_2, Z_3, Z_4) = \left(a^k b z_1, b z_2, a z_3, a z_4\right).$$

Eliminating b and letting $x = a^2$, it is necessary that the equation

$$\begin{aligned} x^{k+1}|z_1|^2\left(|z_3|^2 + |z_4|^2\right) + x^k\left(k\xi_2|z_1|^2 - \xi_1|z_1|^2\right) \\ + x|z_2|^2\left(|z_3|^2 + |z_4|^2\right) - \xi_1|z_2|^2 = 0 \end{aligned} \tag{4.3.2}$$

has a unique positive root. Then $b^2 = \xi_2/\left(x^k|z_1|^2 + |z_2|^2\right)$ has a unique positive solution when $\xi_2 > 0$. The $k = 0$ case being obvious, assume for simplicity that $k \geq 1$. Call $f(x)$ the left hand side of (4.3.2), and compute the derivatives f' and f''. One gets

$$f''(x) = Ax^{k-2}\left[(k+1)x + \frac{(k-1)(k\xi_2 - \xi_1)}{|z_3|^2 + |z_4|^2}\right]$$

where A is positive. Assume that $k\xi_2 - \xi_1 > 0$, then $f''(x) > 0$ for $x > 0$ and f' is increasing on $[0, +\infty[$. As $f'(0) > 0$ and $f(0) < 0$ there is a unique positive root x:

PROPOSITION 4.3.3. — *If $\xi_1 > 0$, $\xi_2 > 0$ and $k\xi_2 - \xi_1 > 0$ then $\mu^{-1}(\xi)/T^2 = W_k$, and W_k is obtained by symplectic reduction starting from $\mathbf{C}^4$.* □

Remark. — This is a particular case of a more general construction: W_k is a *toric manifold* (see [4] for this construction, which basically comes from [8], although there is a (slight!) mistake in [4] as I learned from Y. Karshon and S. Tolman).

Remark. — We shall see later (chapter X) why it is interesting to have W_k as a symplectic reduction starting from the smallest possible complex vector space.

Grassmannians. — Let us now describe the complex Grassmann manifold $G_n\left(\mathbf{C}^{n+k}\right)$ as a symplectic reduction. It is well known (see 3.1) that it is a quotient of a part of a numerical space: let $V_k\left(\mathbf{C}^{n+k}\right) \subset \left(\mathbf{C}^{n+k}\right)^k$ be the Stiefel manifold of all unitary k-frames in $\mathbf{C}^{n+k}$. It is clear that $G_k\left(\mathbf{C}^{n+k}\right) = V_k\left(\mathbf{C}^{n+k}\right)/U(k)$.

My aim is now to show that this quotient is also a symplectic reduction. Write $\left(\mathbf{C}^{n+k}\right)^k = M_{(n+k)\times k}(\mathbf{C})$, the space of all complex matrices with $n+k$ rows and k columns. This space is endowed with its canonical symplectic form $\omega = \operatorname{Im}\operatorname{tr}\left({}^t\overline{X}Y\right)$. Giving a matrix $A \in M_{(n+k)\times k}(\mathbf{C})$ is equivalent to giving k vectors in $\mathbf{C}^{n+k}$, its columns. These k vectors form a unitary k-frame exactly when ${}^t\overline{A}A = \mathrm{Id} \in M_{k\times k}(\mathbf{C})$, thus $V = V_k\left(\mathbf{C}^{n+k}\right)$ is the "level Id" of

$$\begin{array}{rccc} \mu : & M_{(n+k)\times k}(\mathbf{C}) & \longrightarrow & \mathcal{H} \subset M_{k\times k}(\mathbf{C}) \\ & A & \longmapsto & {}^t\overline{A}A \end{array}$$

where now $\mathcal{H}$ denotes as in § 3 the real vector space of Hermitian matrices.

Of course $U(k)$ acts on $M_{(n+k)\times k}(\mathbf{C})$ by $g \cdot A = gA$. On the other hand, we are not afraid to identify $\mathcal{H}$ with $\mathfrak{u}(k)^\star$. The next assertion is that μ is the moment mapping for that action.

4.3.4. Exercise. — For X in $M_{(n+k)\times k}(\mathbf{C})$ and Y in $\mathfrak{u}(k)$, compute the number $\langle T_A\mu(X), Y\rangle$ and show it is $\pm\omega(X, \underline{Y})$.

We are precisely in the situation of the previous §. Moreover, the stabiliser G_ξ is the whole group $U(k)$ because $\xi = \mathrm{Id}$. The quotient $G_k\left(\mathbf{C}^{n+k}\right)$ is a symplectic reduction starting from a complex vector space.

Remark. — It is not necessary to do the exercises of 4.2 in full generality to get this result. Let A_0 be the matrix of the k first vectors in the canonical basis.

$$T_{A_0}\mu(X) = {}^t\overline{A_0}X + {}^t\overline{X}A_0 = 0 \Leftrightarrow$$

the upper $k \times k$ square in X is a Hermitian matrix. In other words

$$\begin{aligned} T_{A_0}V &= \left\{ X = \begin{pmatrix} X_1 \\ X_2 \end{pmatrix} \middle| X_1 \in \mathfrak{u}(k), X_2 \in M_{n\times k}(\mathbf{C}) \right\} \\ &= \mathfrak{u}(k) \oplus M_{n\times k}(\mathbf{C}) \\ &= \text{isotropic} \oplus \text{symplectic} \end{aligned}$$

is a coisotropic subspace F whose orthogonal is exactly $\mathfrak{u}(k)$, thus $F/F^\circ \cong M_{n\times k}(\mathbf{C})$.

5. Surgery?

A natural idea, to construct new symplectic manifolds out of old ones is to perform surgeries.

5.1. Connected sums

Speculative remarks. — Let us try[10] to make a symplectic connected sum $W = W_1 \# W_2$. A natural idea is to remove a small ball in a Darboux chart of both manifolds and to try to glue what is left along the boundary S^{2n-1}. The reason why you cannot manufacture a symplectic form agreeing with the standard ones outside the sphere is that you are trying to glue together two *outsides* of this sphere. I mean that there is a symplectic way of distinguishing the *inside* from the *outside* of the sphere: there exists a vector field X on $\mathbf{C}^n$, which is transversal to the sphere, and such that $\mathcal{L}_X\omega = \omega$.

5.1.1. Exercise. — Check that the radial field $X(p) = \frac{1}{2}p$ has this property.

5.1.2. Exercise. — If (W, ω) is a symplectic manifold and if X is a vector field such that $\mathcal{L}_X\omega = \omega$, show that X dilates the symplectic form, that is, show that its flow satisfies $\varphi_t^\star\omega = e^t\omega$.

Let us come back to the $(2n-1)$-sphere in $\mathbf{C}^n$. The dilatation is very explicit when $n \geq 2$: the volume of the reduced symplectic manifolds $\mathbf{P}^{n-1}(\mathbf{C})$ is *increasing* in the direction of X.

5.1.3. Exercise. — Let (W, ω) be a symplectic manifold of dimension $2n \geq 4$, and let S be a compact hypersurface. Assume that, in the neighbourhood of S, there exists a vector field X, transversal to S and such that $\mathcal{L}_X\omega = \omega$. Show that $i_X\omega$ defines (a primitive of ω and) a contact form[11] on S. Conversely, assume that α is a contact form on S such that $d\alpha$ is the restriction of ω. Show that a transverse dilating vector field X exists on a neighbourhood of S.

One says that such a hypersuface S has *contact type* (see [30] for this notion... and the solutions of the exercises if needed).

5.1.4. Exercise. — Let Σ be a Riemannian manifold. Endow $T^\star\Sigma$ with the canonical symplectic structure. Show that the sphere bundle (with respect to the given metric) $S(T^\star\Sigma)$ has contact type.

Rigorous statement and proof. — There is *no* symplectic connected sum except for surfaces. Here is a proof (valid except in dimension 6). Suppose (W_1, J_1) and (W_2, J_2) are two almost complex manifolds. Remove balls B_1 and B_2, perform the surgery by gluing the boundaries and assume there exists a J on W such that $J|_{W_i - B_i}$ is homotopic to J_i.

[10] The remarks here are speculative, but the statements in the exercises are rigorous mathematics—see [16].

[11] A *contact form* on a $2n-1$-manifold is a 1-form α such that $\alpha \wedge (d\alpha)^{n-1}$ is a volume form.

Around B_i, TW_i is a trivialisable complex bundle. Fix a trivialisation. The existence of J is equivalent to that of a map $\varphi : S^{2n-1} \longrightarrow GL(n, \mathbf{C})$ to be used to glue together the two trivialisations to construct the (complex) tangent bundle TW. Such a map φ might also be used to glue the trivialisations of the bundles over B_1 and B_2... to give an almost complex structure on the tangent bundle to $B_1 \cup_S B_2 = S^{2n}$.

It is well known that, if $2n \neq 2, 6$, this *cannot* exist. We already mentioned this result[12] in 1.5, where we even proved it for $2n = 4$. It is very easy to make symplectic connected sums of surfaces (and more generally of volume forms) as in the following exercise.

5.1.5. Exercise.

1. Let σ be the usual ("$d\theta$") volume form on S^1, and let $\varphi : \mathbf{R} \to \mathbf{R}$ be a *positive* smooth function such that $\varphi(t) = t^2$ for $|t| > 1$. Check that the 2-form $\omega = \varphi(t)dt \wedge \sigma$ on $\mathbf{R} \times S^1$ is a symplectic form which induces the standard form on $\mathbf{R}^2-$unit disc on *both* sides $]-\infty, -1] \times S^1$ and $[1, +\infty[$ via the diffeomorphisms $(t, z) \mapsto tz$.

2. Deduce a symplectic connected sum of symplectic surfaces. Generalise to volume forms in arbitrary dimensions. Why is there no staightforward generalisation to symplectic forms in higher dimensions?

In dimension 6, any almost complex structure on S^6 allows us to construct almost complex connected sums, but this does not work in the symplectic category, the only proof I know uses pseudo-holomorphic curves (the result is in [12], and a proof, due to D. McDuff is written in [5]).

Now we have spent enough time constructing *non*symplectic manifolds, let us try to understand what actually works. For simplicity, let us concentrate on dimension 4.

5.2. Blowing up and down

Let us come back to the standard ball in Darboux coordinates. Do we know something that we can glue to it, that is, do we know something which is symplectic and whose boundary looks like S^3... but from the inside?

In the spirit of § 2.3, that is, of the Duistermaat-Heckman formula [9], represent the complement of B_1 in W_1 as on the left part of figure 3. The slope is 1 due to the fact that the volume of the 2-sphere obtained by reducing the 3-sphere of radius R (vertical segment) is precisely R^2. It is now clear that we can glue the right part of the same figure, that is, the total space of the tautological line bundle

[12] Of course the speculative consideration of increasing volumes has no significance in dimension 2.

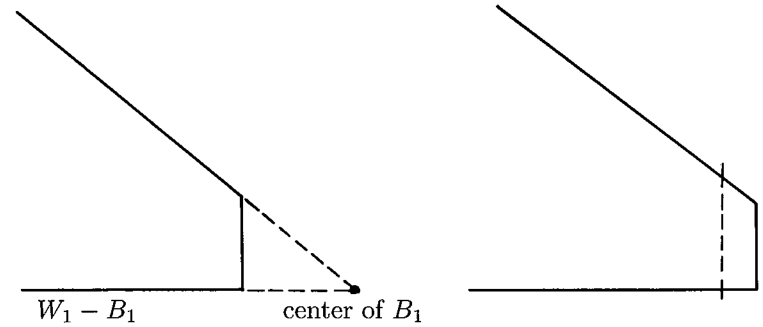

Figure 3

$\mathcal{O}(-1) \to \mathbf{P}^1(\mathbf{C})$: its boundary is actually S^3, and if we endow it with one of the symplectic forms of § 2.3 (The total space of $\mathcal{O}(-1)$ is the complement of the section at infinity in $\mathbf{P}(\mathcal{O}(-1) \oplus 1)$), its boudary is actually seen from where we want. The volume considerations are equivalent to this.

Figure 3 is so accurate that it even shows what we are really allowed to do in the symplectic category: we can glue, provided that the volume λ of the zero section in $\mathcal{O}(-1)$ is less than the square of the radius R of the ball we remove: there must exist an $r > 0$ such that $R^2 = \lambda + r^2$.

This is the *symplectic blowing up* construction of D. McDuff [24], [25]: one replaces a large ball (of radius R) by a (neighbourhood of) a sphere of volume $\lambda < R^2$. Note that the blown up symplectic manifold is *smaller* than the original one. The opposite operation, that is, to replace a sphere with -1 normal bundle (or self intersection) with a large enough ball is the *symplectic blowing down*. This is the process generalised in the symplectic fibre sum used in [10].

5.3. The symplectic fibre sum

In this §, I describe a construction, recently discovered by Gompf [10], although it seems to have been known for a very long time, at least by Gromov (!) as one[13] can see on page 343 of [13].

Suppose that we are given two symplectic manifolds W_1 and W_2 which both contain a symplectic embedded copy of the same surface Σ. We want to try to remove neighbourhoods of Σ_1 and Σ_2 in W_1, W_2 and to glue the complements together to get a symplectic manifold. Of course, we need the boundaries to be diffeomorphic, in other words, the normal bundles (which are symplectic 2-plane or complex line bundles) have to be either isomorphic or anti-isomorphic. For the same reasons as before, the right choice is that they are *anti*-isomorphic. It is more or less obvious. For instance, at the almost complex level we know how to glue together disc bundles of L and L^{-1} over Σ to get a complex manifold, namely $\mathbf{P}(L \oplus 1)$, in which the normal bundle to the zero section is L and that to the section at infinity is L^{-1}. The same

[13] Thanks to E. Giroux for pointing out to me this remark in [13].

gluing map allows us to get an almost complex structure on $W = (W_1 - D(L)) \cup_{S(L)} (W_2 - D(L))$ which is homotopic to the given ones on $W_1 - D(L)$ and $W_2 - D(L)$ (this is the same argument as the use of the sphere in the connected sum problem).

So we are certain to be able to construct an almost complex manifold. What should we do to construct a symplectic one?

As in the blowing up and down process, the symplectic forms should agree near the parts glued together. This is what the *symplectic tubular neighbourhood theorem* allow us to assume[14]: there is a normal form for the symplectic form in a small enough neighbourhood of the submanifold.

In other words, if you understand one example, you understand everything. For instance, if $L \to \Sigma$ is a complex line bundle over the surface Σ (the normal bundle of Σ_1 in W_1), there is a symplectic form on a small enough disc bundle of L which can be used as a model. This is most conveniently defined on a compact version of L, that is on $\mathbf{P}(L \oplus \mathbf{1})$. We already have one if $\Sigma = S^2$, according to 2.3. The next exercise gives an easy and elegant general construction which I learnt from P. Iglesias.

5.3.1. *Exercise.* — Assume L is trivial and construct a symplectic form on $\mathbf{P}(L \oplus \mathbf{1})$. Assume now L non trivial with Euler class k, and endowed with a Hermitian metric. Consider the unit sphere bundle $S(L)$ with its S^1-action by rotations in the fibres.

1. Show that there exists a connection form α on $S(L)$ such that $d\alpha = \pi^*\eta$ for some *volume form* η on Σ (in other words, one can assume that the curvature of α never vanishes) with $\int_\Sigma \eta = k$ (see chapter IV).
2. Consider now the S^1-action on $\mathbf{P}^1(\mathbf{C})$ by $t \cdot [a,b] = [a, tb]$, put
$$H([a,b]) = \mu + \frac{|b|^2}{|a|^2 + |b|^2}$$
(H is a Hamiltonian for the action if $\mathbf{P}^1(\mathbf{C})$ is endowed with the standard symplectic form ω_0). Investigate the kernel of the 2-form
$$\widetilde{\omega} = d(H\alpha) + \omega_0$$
on $S(L) \times \mathbf{P}^1(\mathbf{C})$. Assume $\mu > 0$ and show that $\widetilde{\omega}$ defines a symplectic form ω on the quotient $S(L) \times_{S^1} \mathbf{P}^1(\mathbf{C})$ by the diagonal action.
3. Show that $S(L) \times_{S^1} \mathbf{P}^1(\mathbf{C})$ is actually diffeomorphic to $\mathbf{P}(L \oplus \mathbf{1})$ and compute the volumes of the surfaces sitting inside (fibres and sections) as in 2.3.

[14] In the present case (symplectic submanifold) as well as in all cases where the symplectic form, restricted to the submanifold has constant rank (e.g. symplectic, isotropic, co-isotropic submanifolds) this is the so called Darboux-Weinstein theorem [29], which can be obtained as the classical Darboux theorem (in chapter I) with the help of the path method of Moser. A more general result is due to Giventál (see e.g. [6]) and asserts that if the restrictions of the forms induced on the submanifolds are isomorphic, then they are isomorphic on neighbourhoods.

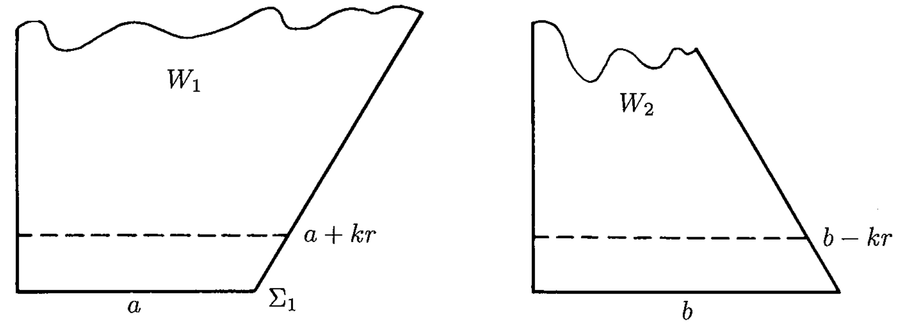

Figure 4

Now look at figure 4 which shows neighbourhoods of Σ_1 in W_1 and Σ_2 in W_2. Call $a = \int_{\Sigma_1} \omega_1$, $b = \int_{\Sigma_2} \omega_2$, and k the Euler class of the normal bundle to Σ_1.

After rescaling one of the symplectic forms, there exists an $r > 0$ such that $a + kr = b - kr$ and we can glue the complementaries W_i' of the neighbourhoods of Σ_i defined by r to get a closed symplectic manifold W (figure 5).

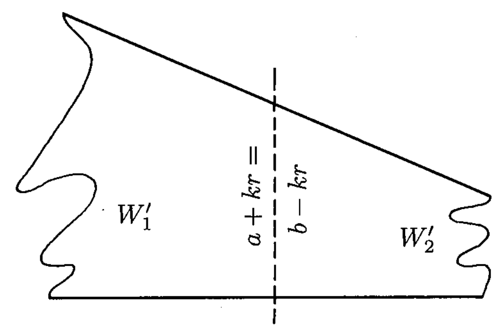

Figure 5

Remarks.

1. Although the diffeomorphism type of W is well defined, the symplectic form might depend on r (not least because of the rescaling).
2. Change the dimensions to get obvious generalisations.
3. Exercise 5.1.4 together with the adjacent remarks about volumes is supposed to explain why one cannot easily replace symplectic by Lagrangian submanifolds: a neighbourhood of a Lagrangian submanifold is isomorphic to a disc bundle of its cotangent bundle, whose boundary has contact type.

5.3.2. Exercise. — Suppose W contains a symplectic sphere Σ with self intersection -1. Using $\mathbf{P}^1(\mathbf{C}) \subset \mathbf{P}^2(\mathbf{C})$, show that one can view the symplectic blowing down of Σ as a symplectic fibre sum.

This construction is used by Gompf in two ways. First of all, using some sophisticated examples (construction of homotopy $K3$-surfaces without complex structure), he gets 4-dimensional simply connected symplectic manifolds which are non Kähler. Second, he constructs, for any finitely presented group G, a 4-dimensional symplectic manifold W with $\pi_1(W) = G$. This result, although new and spectacular (the situation for complex surfaces is quite different) is nevertheless very elementary. It uses only the symplectic fibre sum in the easiest case (submanifolds with *trivial* normal bundles) and the existence of a symplectic manifold V which contains a symplectic 2-torus T with trivial normal bundle and simply connected complement. The construction for $G = \mathbf{Z}$ (no symplectic 4-manifold with $\pi_1 = \mathbf{Z}$ was known before [10]) is given in the following exercise.

5.3.3. Exercise. — Consider a torus (genus 1 surface) $F = \mathbf{R}^2/\mathbf{Z}^2$, with the two circles α ($x_2 = 0$) and β ($x_1 = 0$) so that $\pi_1(F)$ is generated by α and β and that $\pi_1(F)/\langle\beta\rangle = G = \mathbf{Z}$. Consider another torus T^2 with one circle γ ($x_2 = 0$) sitting inside. Set $W = F \times T^2$, endow it with a product symplectic form ω_0.

1. Show that there exists a closed 1-form ρ on F such that $\rho|_\beta$ never vanishes. Let σ be a closed 1-form on T^2 having the same property with respect to γ. Put $\omega_t = \omega_0 + t\,(\rho \wedge \sigma)$. Show that, for $t > 0$ small enough, ω_t is a symplectic form[15] and $\beta \times \gamma$ a symplectic subtorus of (W, ω_t) with trivial normal bundle. Perform the surgery with the manifold V (with the properties stated above) along the tori, to give W_1.

2. Show that for t small enough, $\{z\} \times T^2$ is a symplectic subtorus in W (with trivial normal bundle). Choose z far enough from β, in such a way that $\{z\} \times T^2$ is still a submanifold of W_1. Perform the symplectic fibre sum along it with another copy of V, to give W_2. Show that $\pi_1(W_2) = \mathbf{Z}$.

There do exist complex (Kähler) surfaces with the stated properties, for instance the "rational elliptic surfaces" V obtained by blowing up nine points in general position in $\mathbf{P}^2(\mathbf{C})$, as will be seen from the next exercise.

5.3.4. Exercise. — Choose nine points $p_1, \ldots, p_9$ in $\mathbf{P}^2(\mathbf{C})$.

[15] At this point, there is a small technicality for a more general group G: if G has a presentation with k generators, one can try to begin with a genus k surface F. One then has to kill half of the generators of $\pi_1(F)$, but also all the relations. In order to have an analogue of ω_t, one needs a ρ. For obvious homological reasons, it might be necessary, in order to get one, to increase the genus... and thus the number of elements to kill.

1. Show that, for a generic choice of the points p_i, the set of all cubic curves through these points is a pencil (a complex projective line in the space of all cubic curves). Hint: recall that, in general, ten points define a plane cubic.

2. Assume that the nine points are chosen in this way. Let V be the complex surface obtained by blowing up $\mathbf{P}^2$ at the nine points (so that V is a Kähler manifold). Show that the "map" $\mathbf{P}^2 \to \mathbf{P}^1$ which, to each point p associates the unique cubic of the pencil through p is actually a well defined map $f : V \to \mathbf{P}^1$, almost all of its fibres being smooth cubics.

3. Recall that a smooth plane cubic is diffeomorphic to a torus. Check that almost all the fibres of f are symplectic tori with trivial normal bundles. Show that the fundamental group of the complement of a smooth cubic in $\mathbf{P}^2$ is generated by a small loop in the complex normal line at some point. Deduce that V is simply connected.

Appendix: The canonical almost complex structure on the manifold of 1-jets of pseudo-holomorphic mappings between two almost complex manifolds, by Paul Gauduchon

A.1. Minimal connections on an almost complex manifold

Let (M, J) be an almost complex manifold of (real) dimension $n = 2m$. We denote by N the *Nijenhuis tensor*, or *complex torsion*, defined by:

$$N(X,Y) = [JX, JY] - J[JX, Y] - J[X, JY] - [X, Y], \tag{A.1.1}$$

for all X, Y in T_xM, for all $x \in M$. Recall from § 1.3 that, according to the Newlander-Nirenberg theorem [27], N vanishes identically if and only if the almost complex structure J is integrable.

Let D be a J-linear connection on M, i.e. a $\mathbf{R}$-linear connection preserving the almost complex structure J (for basic facts about connections, see chapter IV). The torsion T of D splits into a J-invariant part, denoted by T^+, and a J-skew-invariant part, denoted by T^-, with respect to the involutive action of J defined by: $T \mapsto T(J.,J.)$. The component T^- in turn splits into two components, denoted respectively by T^{--} and T^{-+}, satisfying the following identities:

$$T^{--}(JX,Y) = T^{--}(X,JY) = -J(T^{--}(X,Y)) \tag{A.1.2}$$

and

$$T^{-+}(JX,Y) = T^{-+}(X,JY) = J(T^{--}(X,Y)). \tag{A.1.3}$$

Then, the following hold (see, e.g. [22]):

- The component T^{--} of the torsion of D is independent of D and is equal to $\frac{1}{4}N$;
- there exists a non-empty (affine) space of J-linear connections on M, of which the torsion is reduced to the component $T^{--} = \frac{1}{4}N$.

Connections of this type will be called *minimal*.

Two minimal J-linear connections D and D' are related by: $D' = D + A$, where A is a 1-form on M with values in the vector bundle of J-linear endomorphisms of the tangent bundle TM; moreover, since D and D' have the same torsion, A is *symmetric*, i.e. satisfies the identity:

(A.1.4) $$A_X Y = A_Y X, \quad \forall X, Y \in T_x M, \quad \forall x \in M.$$

Then, the symmetry of A and the J-linearity of A_X, for any X, imply:

(A.1.5) $$A_{JX} = J \circ A_X, \quad \forall X \in T_x M, \quad \forall x \in M.$$

A.2. The canonical almost complex structure on $J^1(M_1, M_2)$.

Let (M_1, J_1) and (M_2, J_2) be two almost complex manifolds, of (real) dimension n_1 and n_2 respectively. A mapping $f : M_1 \to M_2$ is *pseudo-holomorphic* if $df \circ J_1 = J_2 \circ df$ (df denoting the differential of f).

A.2.1. Exercise. — Denote by N_i the Nijenhuis tensor of (M_i, J_i). If f is a pseudo-holomorphic mapping, check that

$$N_2\left(df(X), df(Y)\right) = df\left(N_1(X, Y)\right).$$

Let $J^1(M_1, M_2)$ denote the manifold of 1-jets of pseudo-holomorphic mappings from (M_1, J_1) into (M_2, J_2), identified with the total space of the (complex) vector bundle E over the product $M_1 \times M_2$, whose fibre $E_{(x_1,x_2)}$ at the point (x_1, x_2) of $M_1 \times M_2$ is the (complex) vector space of $\mathbf{C}$-linear homomorphsims from $T_{x_1}M_1$ to $T_{x_2}M_2$. Chose a J_1-linear connection D^1 on M_1 and a J_2-linear connection D^2 on M_2, both minimal.

These connections induce a $\mathbf{C}$-linear connection, denoted by ∇, acting on the sections of the vector bundle E, defined, for any section a of E, by:

(A.2.2) $$(\nabla_{(X_1,X_2)}a)(Y_1) = \partial_{X_1}(a(Y_1)) + D^2_{X_2}(a(Y_1)) - a(D^1_{X_1}Y_1),$$

for all Y_1 in $T_{x_1}M_1$,with the following significance: in the left hand side, Y_1 denotes any extension (independent of the variable x_2) of the vector Y_1 on the factor $M_1 \times \{x_2\}$; in the first term on the right hand side, the variable x_2 is "frozen", so that $a(Y_1)$ takes its values in the fixed vector space $T_{x_2}M_2$, and ∂_{X_1} denotes the ordinary derivative in the direction of X_1; in the second term of the right hand side, the variable x_1 is "frozen", so that $a(Y_1)$ is considered as a vector field on M_2.

The linear connection ∇ determines (and is determined by) a *horizontal distribution*, denoted by H^∇, on the total space of E, by which we make the following identification, for any vector U tangent at ξ to (the total space of) E:

(A.2.3) $$U = \left(v^\nabla(U), (X_1, X_2)\right), \quad \forall U \in T_\xi E, \xi \in E_{(x_1,x_2)},$$

where:

- $X = (X_1, X_2)$ is the natural projection of U into $T_{(x_1,x_2)}(M_1 \times M_2)$;
- $v^\nabla(U)$ is the *principal part* of U w.r.t. ∇, where v^∇ denotes the (vertical) projection of $T_\xi E$ onto the vertical subspace of $T_\xi E$, identified with the fibre $E_{(x_1,x_2)}$, along the horizontal subspace H^∇_ξ.

For a different choice of (minimal) connections $D'^1 = D^1 + A^1$ and $D'^2 = D^2 + A^2$ on M_1 and M_2, the induced connection ∇' is related to ∇ by: $\nabla' = \nabla + B$, where B is the 1-form on M_1 with values in the vector bundle End E of $\mathbf{C}$-linear endomorphisms of E, equal to:

(A.2.4) $$B_{(X_1,X_2)}\xi = A^2_{X_2} \circ \xi - \xi \circ A^1_{X_1},$$

for all ξ in $E_{(x_1,x_2)} = \mathrm{Hom}_{\mathbf{C}}(T_{x_1}M_1, T_{x_2}M_2)$. It follows readily from (A.2.4) that the principal parts $v^{\nabla'}(U)$ and $v^\nabla(U)$ of a same vector U w.r.t. ∇' and ∇ are related by:

(A.2.5) $$v^{\nabla'}(U) = v^\nabla(U) + A^2_{X_2} \circ \xi - \xi \circ A^1_{X_1}, \quad \forall U \in T_\xi E,$$

where $X = (X_1, X_2)$ is the natural projection of U into $T_{(x_1,x_2)}(M_1 \times M_2)$.

With the help of the connection ∇, we construct an almost complex structure $\mathcal{J}$ on the manifold $J^1(M_1, M_2) = E$, via the identification (A.2.3), by putting:

(A.2.6) $$\mathcal{J}U = (J_2 \circ v^\nabla(U), (J_1X_1, J_2X_2)) \quad \forall U \in T_\xi E, \quad \forall \xi \in E.$$

PROPOSITION A.2.7. — *The almost-complex structure $\mathcal{J}$ is independent of the choice of the (minimal) connections D^1 and D^2.*

Proof. — Let $\mathcal{J}'$ be the almost complex structure on $E = J^1(M_1, M_2)$ built from the connection ∇'. By (A.2.6), $\mathcal{J}'U$ is represented, via the identification (A.2.3) w.r.t. ∇', by the pair $(J_2 \circ v^{\nabla'}(U), (J_1X_1, J_2X_2))$. By (A.2.5), the principal part $v^{\nabla'}(\mathcal{J}'U)$ of $\mathcal{J}'U$ with respect to the initial connection ∇ is equal to

$$J_2 \circ v^{\nabla'}(U) - A^2_{J_2X_2} \circ \xi + \xi \circ A^1_{J_1X_1},$$

equal, by (A.2.5), to

$$J_2 \circ v^\nabla(U) + J_2 \circ (A^2_{X_2} \circ \xi - \xi \circ A^1_{X_1}) - A^2_{J_2X_2} \circ \xi + \xi \circ A^1_{J_1X_1}.$$

By (A.1.5), the latter expression is equal to $J_2 \circ v^\nabla(U)$, i.e. the principal part of $\mathcal{J}U$ w.r.t. ∇. □

Remark. — It follows from Proposition A.2.7 that the almost complex structure $\mathcal{J}$ defined by (A.2.6) on $J^1(M_1 \times M_2)$ is canonical, i.e. only depends on J_1 and J_2.

A.2.8. Exercise. — Check that $\mathcal{J}$ is integrable if and only if J_1 and J_2 are integrable (Hint: compute the Nijenhuis tensor $\mathcal{N}$ of $\mathcal{J}$ by using *distinguished*, vertical and horizontal (w.r.t. ∇), vector fields on E, i.e., respectively, sections of E, considered as vertical vector fields, constant on fibres, and horizontal lifts of vector fields on the basis $M_1 \times M_2$).

Each pseudo-holomorphic map f from (M_1, J_1) into (M_2, J_2) canonically lifts, via its 1-jet, to a map, denoted by $\tilde{f}$, from M_1 into $J^1(M_1, M_2)$. Via the identification $J^1(M_1, M_2) = E$, $\tilde{f}$ is expressed by:

$$\tilde{f}(x) = df_{|T_xM_1}, \quad \forall x \in M_1. \tag{A.2.9}$$

PROPOSITION A.2.10. — *For any pseudo-holomorphic map f from (M_1, J_1) into (M_2, J_2), the canonical lift $\tilde{f}$ is a pseudo-holomorphic map from (M_1, J_1) into $(J^1(M_1, M_2), \mathcal{J})$.*

Proof. — Chose minimal connections D^1 and D^2 on M_1 and M_2 respectively, so that the almost complex structure $\mathcal{J}$ of $J^1(M_1, M_2) = E$ is expressed by (A.2.6) w.r. to the induced connection.

For each point x of M_1 and each vector X of T_xM_1, the vector $d\tilde{f}(X)$, in $T_{\tilde{f}(x)}E$, is represented, via the identification (A.2.3), by the pair:

$$d\tilde{f}(X) = (\nabla_X df, (X, df(X))), \tag{A.2.11}$$

where, for convenience, we still denote by ∇ the $\mathbf{C}$-linear connection induced by D^1 and D^2 on the (complex) vector bundle, over $M_1, T^*M_1 \otimes_{\mathbf{C}} f^*TM_2$ (of which df is a section).

Since the torsion of D^1, resp. D^2, is equal to the Nijenhuis tensor N_1, resp. N_2, of J_1, resp. J_2, the "Hessian" ∇df is symmetric. Indeed, we have:

$$\begin{aligned}(\nabla_X df)(Y) - (\nabla_Y df)(X) &= N_2(df(X), df(Y)) - df(N_1(X,Y)) \\ &= 0,\end{aligned}$$

for the differential df of any pseudo-holomorphic mapping f exchanges the Nijenhuis tensors (exercise A.2.1). It then follows from the $\mathbf{C}$-linearity of df and the connections D^1 and D^2, that ∇df satisfies the relation:

$$(\nabla_{J_1X} df)(Y) = J_2((\nabla_X df)(Y)), \quad \forall X, Y \in T_xM_1, \quad \forall x \in M_1. \tag{A.2.12}$$

By (A.2.11) and (A.2.6), we infer:

$$d\tilde{f}(J_1X) = (J_2 \circ \nabla_X df, (J_1X, J_2df(X)) = \mathcal{J}(d\tilde{f}(X)), \quad \forall X \in T_xM_1, \quad \forall x \in M_1,$$

i.e.

$$d\tilde{f} \circ J_1 = \mathcal{J} \circ d\tilde{f}. \tag{A.2.13}$$

□

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